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Kinetics of information scrambling in correlated metals: disorder-driven transition from shock-wave to FKPP dynamics

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Quenched disorder slows down the scrambling of quantum information. Using a bottom-up approach, we formulate a kinetic theory of scrambling in a correlated metal near a superconducting transition, following the scrambling dynamics as the impurity scattering rate is increased. Within this framework, we rigorously show that the butterfly velocity v is bounded by the light cone velocity $v_{\rm lc}$ set by the Fermi velocity. We analytically identify a disorder-driven dynamical transition occurring at small but finite disorder strength between a spreading of information characterized at late times by a discontinuous shock wave propagating at the maximum velocity $v_{\rm lc}$, and a smooth traveling wave belonging to the Fisher or Kolmogorov-Petrovsky-Piskunov (FKPP) class and propagating at a slower, if not considerably slower, velocity v. In the diffusive regime, we establish the relation $v^2/\lambda_{\rm FKPP} \sim D_{\rm el}$ where $\lambda_{\rm FKPP}$ is the Lyapunov exponent set by the inelastic scattering rate and $D_{\rm el}$ is the elastic diffusion constant.

Information scrambling refers to the efficient spreading and loss of information throughout an extended many-body system. Its characterization is fundamental to the foundations of quantum chaos and has implications in the development quantum computing technologies. The picture that has emerged from the exact or numerical solutions to a variety of classical and quantum thermalizing models with local interactions is a ballistic spreading of information at a so-called butterfly velocity. In dual-unitary circuits, the information scrambling occurs precisely on the light rays propagating at the maximum velocity allowed by causality [1, 2]. In random quantum circuits made of Haar-distributed unitaries, scrambling dynamics was related to classical growth processes with slower ballistic fronts that broaden either diffusively in $d = 1$, or according to fluctuations governed by the Kardar-Parisi-Zhang universality class in $d = 2$ and $d = 3$ [3, 4].

Alongside these minimal models, as well as classical [5–8], semi-classical, large N or holographic models [9–12], it is essential to address those questions in realistic situations where exact or numerical solutions are out of reach. Aleiner, Faoro and Ioffe articulated those questions in the larger framework of electronic transport by deriving a quantum kinetic equation for out-of-time-ordered correlators (OTOCs) within a so-called many-world Keldysh formalism [13]. They argued that the scrambling dynamics in metals with either phonon or Coulomb interactions are governed by FKPP equations, resulting in smooth non-broadening scrambling fronts propagating at a butterfly velocity set by the Fermi velocity, $v_{\rm F}$. Interestingly, it was argued that the presence of disorder would significantly reduce this butterfly velocity [13–16].

Building on the approach of Ref. [13], we worked out the kinetics of quantum information scrambling in a paradigmatic model of clean interacting metals in the vicinity of a superconducting phase transition, where electron-electron interactions are dominated by superconducting fluctuations [17]. We found scrambling fronts that travel at $v_{\rm F}$ but do not belong to the FKPP universality class. Remarkably, their late-time spatial profiles develop a shock-wave discontinuity at the boundary of the light cone.

In this Letter, we investigate the impact of impurity scattering on the dynamics of the scrambling front. Momentum relaxation is indeed significant in metals with an elastic timescale typically in the tens of femtoseconds, *i.e.* much shorter than the scattering time due to electronic interactions, especially at low temperatures. We first derive an effective set of two coupled partial differential equations (PDEs) governing the scrambling dynamics at large scales. Then, we analytically elucidate how the traveling-wave solutions that develop at late times are affected by the presence of disorder, from the clean case to the diffusive regime. We find that the shock-wave phenomenology found in the clean case is robust against weak disorder. However, we unravel a dynamical phase transition that causes the scrambling kinetics to abruptly conform to the FKPP class when the disorder strength exceeds a critical value, see Fig. 1. We fully characterize the information scrambling in the FKPP regime by working out the profiles of the traveling fronts and their butterfly velocities.

Model. We consider a system of interacting electrons in $d \geq 2$ dimensions that are subject to both elastic and inelastic scattering. The former is due to static non-magnetic impurities and defects. The latter is due to electron-electron interaction in the Cooper channel. This choice is guided by the relatively straightforward treatment of superconducting fluctuations within the random

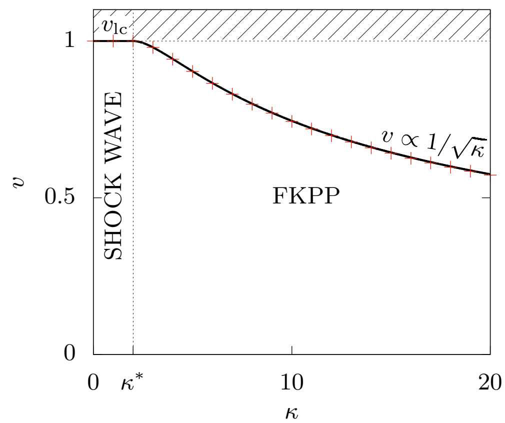

FIG. 1. Quantum butterfly velocity v as a function of the disorder strength κ. For $\kappa < \kappa^*$, the front propagates at the light cone velocity $v_{\rm lc} = 1$ in units of $v_{\rm F}/\sqrt{d}$. For $\kappa > \kappa^*$, the solid line is the FKPP prediction made in Eq. (12). The red marks are numerical results obtained by solving Eqs. (5) for $d = 1$ up to times $\tau = 3000$ ($\gamma = 1$ *i.e.* $\kappa^* := 1 + \gamma = 2$).

phase approximation. However, our approach can be extended to other models with different interactions as long as a quasi-particle description is valid. For concreteness, we have in mind the Hamiltonian

$$H = \sum_{\boldsymbol{k},\,\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}\boldsymbol{k}',\,\sigma} V_{\boldsymbol{k}-\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}'\sigma} + U \sum_{\boldsymbol{k}\boldsymbol{k}'\boldsymbol{k}'',\,\sigma} c^\dagger_{\boldsymbol{k}\sigma} c^\dagger_{-\boldsymbol{k}+\boldsymbol{k}'\bar\sigma} c_{\boldsymbol{k}''+\boldsymbol{k}'\bar\sigma} c_{-\boldsymbol{k}''\sigma}. \tag{1}$$

The operator $c^\dagger_{\boldsymbol{k}\sigma}$ creates a fermion with spin $\sigma =\uparrow$ or $\downarrow$ ($\bar\sigma =\downarrow, \uparrow$) and momentum $\boldsymbol{k}$ in the Brillouin zone. $\epsilon_{\boldsymbol{k}}$ is the dispersion relation. Electronic energies are measured relative to the chemical potential and $E_{\rm F}$ is the Fermi energy. For simplicity, we shall assume a spherical Fermi surface, *i.e.* $\epsilon_{\boldsymbol{k}} = 0$ when $k \to k_{\rm F}$.

The attractive interaction $U < 0$ facilitates superconductivity. In dimensions $d > 2$, this model exhibits a finite-temperature phase transition towards a superconducting phase associated with the spontaneous breaking of the $U(1)$ symmetry. In $d = 2$, it is replaced by a BKT transition with quasi-long-range order [18]. Here, we work near criticality in the normal phase of the BCS-type superconductor and in the regime of validity of the Ginzburg-Levanyuk criterion, where the superconducting fluctuations are sizable but weakly interacting [19, 20].

The disordered potential $V(\boldsymbol{x})$ is assumed to be short-ranged and Gaussian-distributed, with covariance $\langle V(\boldsymbol{x})V(\boldsymbol{x}')\rangle - \langle V(\boldsymbol{x})\rangle^2 = g\delta(\boldsymbol{x}-\boldsymbol{x}')$ with $g > 0$ [21]. We work far from localization regimes, where the disorder can be treated in a classical fashion, *i.e.* not accounting for coherent effects between scattering trajectories. In practice, the impurity scattering is treated with the Born approximation to second order in g [22].

We formulate the kinetic theory of quantum information scrambling by starting from the quantum kinetic equation on the many-world distribution functions $F_{\alpha\beta}(t, \boldsymbol{x}; \omega, \boldsymbol{k})$, where the indices $\alpha, \beta \in \{u, d\}$ span two replicated worlds (up and down) [13, 17]. The intraworld components $\alpha = \beta$ correspond to the standard electronic distribution functions. We concentrate on the interworld components $F_{\alpha\neq\beta}$ which are directly related to four-point OTOCs and to the growth of operators. In the gradient approximation, valid when the microscopic scales set by $\hbar/E_{\rm F}$ and $1/k_{\rm F}$ are much shorter than the spatiotemporal variations of $F_{\alpha\beta}$, the dynamics of the latter are governed by a non-linear partial-integrodifferential equation reading

$$[\partial_t + \boldsymbol{v}_{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}] F_{\alpha\beta} = I_{\alpha\beta}, \tag{2}$$

with the velocity $\boldsymbol{v}_{\boldsymbol{k}} := \boldsymbol{\nabla}_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}}$ and where the collision integral $I_{\alpha\beta}$ is a non-linear functional of the $F_{\alpha\beta}$'s collecting contributions from the disordered potential and the electronic interactions. The kinetic theory is considerably simplified by working (i) in terms of the first two components of a partial-wave expansion in $\boldsymbol{k}$, (ii) on-shell, *i.e.* $\omega \to \epsilon_{\boldsymbol{k}}$, and (iii) near the Fermi surface, *i.e.* $k \to k_{\rm F}$. This amounts to working with the ansatz

$$F_{\alpha\beta}(t, \boldsymbol{x}; \omega, \boldsymbol{k}) \mapsto \epsilon_{\alpha\beta} \left[\phi(t, \boldsymbol{x}) + \boldsymbol{u}_{\boldsymbol{k}} \cdot \boldsymbol{\phi}_1(t, \boldsymbol{x})\right], \tag{3}$$

with $\epsilon_{du} = -\epsilon_{ud} = 1$, $\boldsymbol{u}_{\boldsymbol{k}} := \boldsymbol{k}/k$, and where ϕ and $\boldsymbol{\phi}_1$ are the isotropic and first anisotropic corrections, respectively, to the on-shell interworld distribution function. The validity of the above ansatz is discussed for the clean case in Ref. [17], and the partial-wave truncation is all the more accurate in the presence of impurity scattering as it reduces momentum anisotropy. Following the steps detailed in Sect. A of Ref. [23], the kinetic equation can be brought to a set of coupled non-linear PDEs reading

$$\begin{cases} \partial_t \phi + \frac{v_{\rm F}}{d} \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{\phi}_1 = \phi(\phi^2 - 1)/\tau_{\rm sc} \\ \partial_t \boldsymbol{\phi}_1 + v_{\rm F} \boldsymbol{\nabla}_{\boldsymbol{x}} \phi = \boldsymbol{\phi}_1(\gamma\phi^2 - 1)/\tau_{\rm sc} - \boldsymbol{\phi}_1/\tau_{\rm el}. \end{cases} \tag{4}$$

This represents an effective kinetic theory of scrambling in terms of the two fields $\phi(t, \boldsymbol{x})$ and $\boldsymbol{\phi}_1(t, \boldsymbol{x})$. The dimensionless parameter γ tunes the distance to the superconducting transition: $\gamma = 1$ corresponds to criticality and $0 < \gamma < 1$ to off-critical regimes in the normal phase. $\tau_{\rm el} \propto 1/g$ is the elastic timescale due to scattering on the disordered potential and $\tau_{\rm sc}$ is the timescale set by the inelastic scattering on the superconducting fluctuations (Cooperons). These parameters of the model (4) depend in a non-trivial fashion on those of the original microscopic model (1) and have to be understood as renormalized quantities resulting from a complex cross-feed. For example, disorder is known to enhance the inelastic scattering rate $1/\tau_{\rm sc}$ at low temperatures [24]. The PDEs (4) have a "correlated-world" solution, $\phi = 1$ and $\boldsymbol{\phi}_1 = 0$, corresponding to both replicated worlds evolving coherently and is expected to be unstable for chaotic systems.

$\phi = \boldsymbol{\phi}_1 = 0$ is the "uncorrelated-world" solution, corresponding to a total loss of coherence between worlds.

At $\tau_{\rm el} \to \infty$, we recover the clean model studied in Ref. [17]. In the non-interacting limit, $\tau_{\rm sc} \to \infty$, and in the diffusive regime (at late times) where $\partial_t \boldsymbol{\phi}_1$ can be neglected relative to $\boldsymbol{\phi}_1/\tau_{\rm el}$, the above coupled PDEs reduce to a simple diffusion equation: $\partial_t \phi - D_{\rm el} \boldsymbol{\nabla}_{\boldsymbol{x}}^2 \phi = 0$ with the elastic diffusion coefficient $D_{\rm el} := v_{\rm F}^2 \tau_{\rm el}/d$. In this Drude limit, the correlated-world solution $\phi = 1$ is stable against local perturbations, expressing the absence of quantum information scrambling when only disorder is present. In the generic case with both elastic and inelastic scattering, we study how the scrambling dynamics evolve as the dimensionless disorder coupling constant $\kappa := \tau_{\rm sc}/\tau_{\rm el} \geq 0$ is increased from the clean to the diffusive metal [25]. The analysis is simplified by rescaling time and space, $\tau := t/\tau_{\rm sc}$ and $\boldsymbol{X} := \boldsymbol{x}/\ell_{\rm sc}$ with $\ell_{\rm sc} := v_{\rm F}\tau_{\rm sc}/\sqrt{d}$, together with $\boldsymbol{\phi}_1 \mapsto \sqrt{d}\, \boldsymbol{\phi}_1$. The above PDEs become

$$\begin{cases} \partial_\tau \phi + \boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \boldsymbol{\phi}_1 = \phi(\phi^2 - 1) \\ \partial_\tau \boldsymbol{\phi}_1 + \boldsymbol{\nabla}_{\boldsymbol{X}} \phi = \boldsymbol{\phi}_1(\gamma\phi^2 - 1) - \kappa \boldsymbol{\phi}_1. \end{cases} \tag{5}$$

The initial conditions are taken as local and spherically symmetric perturbations to the correlated-world solution:

$$\begin{cases} \phi(\tau = 0, \boldsymbol{X}) = 1 - \delta\phi_0(X) \\ \boldsymbol{\phi}_1(\tau = 0, \boldsymbol{X}) = 0, \end{cases} \tag{6}$$

with the radial coordinate $X := \|\boldsymbol{X}\|$ and a perturbation $0 \leq \delta\phi_0(X) \ll 1$ which is non-vanishing on small support of radius R_0 and $\delta\phi_0(X > R_0) = 0$. This guarantees that subsequent dynamics reduce to an effective one-dimensional problem for $\phi(\tau, X)$ and $\phi_1(\tau, X)$ along the radial direction. The late-time dynamics marginally depend on the choice of $\delta\phi_0(X)$.

The instability of the correlated-world solution against local perturbations is expected to generate a transient state where both solutions ($\phi \simeq 1$ and $\phi \simeq 0$) are separated by a domain wall, a front, located on a sphere of growing radius. The profile and the motion of this front determine the dynamics of the scrambling of quantum information. At late times, the front is located far from the origin and it is governed by the $d = 1$ version of the PDEs (5) where X is now the radial coordinate and ϕ_1 is the radial component of $\boldsymbol{\phi}_1$ [17]. In Sect. B of Ref. [23], we provide rigorous proof that this growth is bounded by a maximal velocity set by the Fermi velocity, $v_{\rm lc} = 1$ in units of $v_{\rm F}/\sqrt{d}$. This ensures that the dynamics strictly take place within a causal light cone where $v_{\rm lc}$ acts as the effective speed of light. This motivates us to look for traveling fronts propagating at a constant velocity $v \leq v_{\rm lc}$ and located at $m_\tau \sim v\tau$ by assuming

$$\begin{cases} \phi(\tau, m_\tau + z) \overset{\tau\to\infty}{\longrightarrow} f(z) \\ \phi_1(\tau, m_\tau + z) \overset{\tau\to\infty}{\longrightarrow} f_1(z). \end{cases} \tag{7}$$

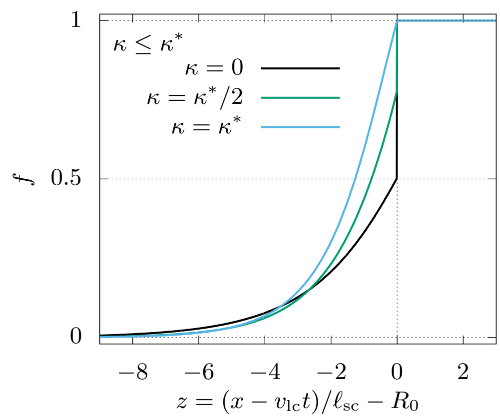

FIG. 2. Radial profiles $f(z)$ of the late-time Fermi shock wave traveling at the maximal velocity $v_{\rm lc}$, computed exactly from the coupled PDEs (5) for $\kappa = 0$, $\kappa^*/2$ and κ^* ($\gamma = 1$ *i.e.* $\kappa^* = 2$.) The offset R_0 is the radius of the initial condition. The front thickness is of the order of $\ell_{\rm sc} := v_{\rm F}\tau_{\rm sc}/\sqrt{d}$ in the original units.

Fermi shock-wave dynamics. In the spirit of Ref. [17], we first look for traveling-wave solutions propagating at the maximum velocity $v = v_{\rm lc} = 1$ and that are discontinuous at the boundary of the light cone. We leave the full computation of the front profile to Sect. B of Ref. [23]. Here, we simply focus on extracting the discontinuity by parametrizing the near-front geometry as

$$\begin{cases} f(z < 0, |z| \ll 1) \simeq L + z/\xi, & f(z > 0) = 1 \\ f_1(z < 0, |z| \ll 1) \simeq -M - z/\xi_1, & f_1(z > 0) = 0, \end{cases}$$

where L, M, ξ, and ξ_1 are positive parameters to be determined, and $m_\tau = \tau + R_0$. We find a critical disorder strength

$$\kappa^* = 1 + \gamma \tag{8}$$

separating two distinct solutions. For $\kappa < \kappa^*$, we find a solution of Eqs. (5) with a traveling discontinuity from $f(0^-) = L$ to $f(0^+) = 1$, with

$$L(\kappa < \kappa^*) = \frac{\sqrt{1 + 4\kappa^*(1 + \kappa)} - 1}{2\kappa^*} \tag{9}$$

and $M = 1 - L$. We find finite values of ξ, indicating that the thickness of the front is controlled by $\ell_{\rm sc}$. This generalizes the Fermi shock-wave dynamics identified in Ref. [17] to the weakly-disordered case. We illustrate this shock-wave profile in Fig. 2. When $\kappa \to \kappa^*$ from below, the discontinuity closes continuously, $L \to 1$, but the slope remains discontinuous. For $\kappa > \kappa^*$, $L = 1$ is the only solution and the finite slope left of the front brutally vanishes, $f'(0) = 0$, signaling the sudden death of the Fermi shock wave.

FKKP dynamics. Inspired by the FKPP equation proposed in Ref. [13], we look for smooth traveling-wave solutions that belong to the FKPP class, with exponential tails ahead of the front of the form

$$\begin{cases} f(z \gg 1) & \simeq 1 - Az\exp(-\mu z) \\ f_1(z \gg 1) & \simeq -Bz\exp(-\mu z), \end{cases} \tag{10}$$

where A, B, and the spatial decay rate μ are positive parameters. To reveal the hidden FKPP nature of the PDEs (5), we reformulate them in terms of $\delta\phi := 1 - \phi$ and $\boldsymbol{\phi}_1$ which are expected to be small and slowly varying ahead of the front, as per Eq. (10). Standard algebra detailed in Sect. D of Ref. [23] yields, for any $\kappa \geq 0$,

$$\left[\partial_\tau^2 + (\kappa - \kappa^*)\partial_\tau - \boldsymbol{\nabla}_{\boldsymbol{X}}^2\right]\delta\phi = 2(2+\kappa-\kappa^*)\,\delta\phi - \mathrm{NL}, \tag{11}$$

where we collected the non-linear terms under the symbol NL. The term in $\partial_\tau\delta\phi$ is odd under time reversal and its prefactor changes sign at $\kappa = \kappa^*$. When $\kappa > \kappa^*$, it can be loosely interpreted as a dissipative term, bringing Eq. (11) to a standard FKPP fashion, whereas it acts as a drive when $\kappa < \kappa^*$. The term $\partial_\tau^2\delta\phi$ can be interpreted as inertia which, to the best of our knowledge, has not been discussed in the broader context of FKPP. Following a standard FKPP analysis detailed in Sect. D2 of Ref. [23], we find that there are no traveling-wave solutions of the form (10) when $\kappa < \kappa^*$. However, for $\kappa > \kappa^*$, we now find a front propagating at the velocity, in units of $v_\mathrm{F}/\sqrt{d}$,

$$v(\kappa > \kappa^*) = 2\sqrt{2}\frac{\sqrt{2+\kappa-\kappa^*}}{4+\kappa-\kappa^*} \leq v_\mathrm{lc}. \tag{12}$$

In the limit $\kappa \to \kappa^*$, we recover $v \to v_\mathrm{lc}$, consistently with the Fermi shock wave. v decreases monotonously with increasing disorder strength, and $v \propto 1/\sqrt{\kappa} \to 0$ when $\kappa \to \infty$. In Fig. 1, we compare the front velocities extracted from the numerical solutions of the PDEs (5) to the FKPP prediction in Eq. (12). The agrement is excellent. In Sect. D4 of Ref. [23], using results of Refs. [26–35], we provide a meticulous numerical analysis that further demonstrates, beyond any reasonable doubt, the FKPP nature of the front dynamics as soon as $\kappa > \kappa^*$.

Interestingly, the early-time dynamics governed by the linearized version of Eq. (11) are characterized by an exponential growth of the spatially-integrated perturbation $M(\tau) := \int \mathrm{d}\boldsymbol{X}\,\delta\phi(\tau, \boldsymbol{X}) = M(0)\exp(2\tau)$ where the Lyapunov exponent $\lambda = 2$ does not depend on whether the shock-wave or the FKPP regime is governing the scrambling dynamics. In the FKPP regime, the growth of $\delta\phi$ ahead of the front is also exponential, as per Eq. (10), and the scrambling dynamics can be characterized by a rate $\lambda_\mathrm{FKPP} = v\mu = 4 + 8/(\kappa - \kappa^*)$ where μ is computed in Sect. D2 of Ref. [23]

Diffusive regime. We now delve into the overdamped regime $\kappa \gg 1$ where the inertial term of Eq. (11) may be neglected. Keeping only the leading-order terms the RHS (see the details in Sect. D3 of Ref. [23]), the scrambling dynamics are now governed by, back in terms of ϕ

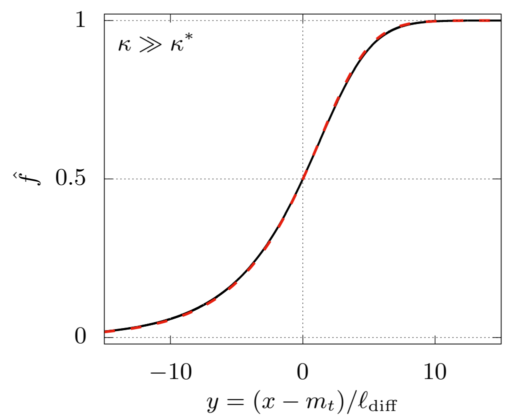

FIG. 3. Radial profile of the late-time FKPP front at $\kappa \gg 1$, solution of Eq. (13) traveling at the butterfly velocity v given in Eq. (14). $m_t \sim vt$ is the location of the front. The dashed line is the numerical result obtained by solving Eqs. (5) with $\kappa = 60$ and $\gamma = 1$ up to time $\tau = 30$. The front thickness is of the order of $\ell_\mathrm{diff} := v_\mathrm{F}\sqrt{\tau_\mathrm{sc}\tau_\mathrm{el}/2d} \ll \ell_\mathrm{sc}$.

and in the original units,

$$\partial_t\phi - D_\mathrm{el}\boldsymbol{\nabla}_{\boldsymbol{x}}^2\phi = \phi(\phi^2 - 1)/\tau_\mathrm{sc}, \tag{13}$$

where $1/\tau_\mathrm{sc}$ is the inelastic scattering rate and $D_\mathrm{el} := v_\mathrm{F}^2\tau_\mathrm{el}/d$ is the elastic diffusion coefficient. The dependence on both $\boldsymbol{\phi}_1$ and γ has dropped. This FKPP equation is the non-integrable Newell-Whitehead equation which was first studied in the context of non-linear fluid mechanics [36]. Similar equations with diffusive terms were recently put forward to describe the dynamics of scrambling of quantum information [10, 13, 37–39]. The traveling-wave solutions of Eq. (13) propagate at a butterfly velocity, in the original units,

$$v = 2\sqrt{\frac{2}{d}}\sqrt{\frac{\tau_\mathrm{el}}{\tau_\mathrm{sc}}}v_\mathrm{F} \ll v_\mathrm{F}. \tag{14}$$

Incidentally, this yields the relation

$$v^2/\lambda_\mathrm{FKPP} = 2D_\mathrm{el} \tag{15}$$

which concretely connects scrambling dynamics on the one side to a measurable transport quantity on the other side [9, 40–42]. The front profile is conveniently computed by now measuring space in units of $l_\mathrm{diff} := v_\mathrm{F}\sqrt{\tau_\mathrm{sc}\tau_\mathrm{el}/2d} \ll \ell_\mathrm{sc}$. The rescaled front profile $\hat{f}(y) := f(y\,l_\mathrm{diff})$ is the solution to $2\hat{f}'' + 4\hat{f}' = \hat{f}(1 - \hat{f}^2)$ with $\hat{f}(-\infty) = 0$, $\hat{f}(\infty) = 1$, and we can require $\hat{f}(0) = 1/2$. It is a smooth monotonous function that we represent in Fig. 3. The agreement with the numerical solutions of the PDEs (5) computed in the diffusive regime ($\kappa = 60$) is excellent.

Discussion. In the Fermi shock-wave regime, $\kappa < \kappa^*$, it is still to be clarified whether the scrambling front discontinuity could be smoothened by corrections to the gradient approximation. In the FKPP regime, $\kappa > \kappa^*$, we found that disorder can considerably reduce the butterfly velocity, corroborating the results of Refs. [14, 15]: a realistic ratio at room temperatures $\kappa \approx 10^4$ yields $v \sim 10^{-2} v_{\mathrm{F}} \sim 10^4$ m/s, on par with the typical phonon-mediated sound velocity in metals. Contrary to the shock-wave velocity, it cannot strictly be seen a (slower) effective speed of light since the small tail ahead of the front, while providing a quantum-chaotic exponential regime controlled by the inelastic scattering rate, surreptitiously undermines the causality of the light-cone structure. In both the Fermi shock-wave and the FKPP regimes, we found a sharp butterfly front. This is similar to what has been reported for models with a large local Hibert space such as the $O(N)$ or the SYK models [9, 10], but different from the diffusively broadening fronts obtained for random quantum circuits [3, 4]. The precise conditions under which strong quantum fluctuations and strong disorder could generate relevant perturbations to the FKPP dynamics are still to be elucidated.

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