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Anomalous Bilayer Quantum Hall Effect

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In parallel to the condensed-matter realization of quantum Hall (Chern insulators), quantum spin Hall (topological insulators) and fractional quantum Hall (fractional Chern insulators) effects, we propose that bilayer flat band (FB) lattices with one FB in each layer constitute solid-state analogues of bilayer quantum Hall (BQH) system, leading to anomalous BQH effect without magnetic field. By exact diagonalization of a bilayer Kagome lattice Hamiltonian, as a prototypical example, we demonstrate the stabilization of excitonic condensate Halperin's (1, 1, 1) state at the total filling $\nu_T = 1$ of the two FBs. Furthermore, by tuning the inter-layer tunneling and distance between the Kagome layers at $\nu_T = 2/3$, we show phase transitions among Halperin's (3, 3, 0), spin-singlet (1, 1, 2), and particle-hole conjugate of Laughlin's $1/3$ states, as previously observed in conventional BQH systems. Our work opens a new direction in the field of FB physics by demonstrating bilayer FB materials as an attractive avenue for realizing exotic anomalous BQH states including non-Abelian anyons.

Condensed-matter realization of quantum Hall effect (QHE), first proposed by Haldane [1], has been one of the most significant scientific breakthroughs that instigated the origin of topological states of matter. Subsequently, topological insulators exhibiting quantum spin Hall effect was proposed by Kane-Mele [2, 3] and Bernevig-Hughes-Zhang [4], further fostering the field of topological materials [4, 5]. Of particular latest interest is the interplay between topology and electron interaction, inherent to a topological flat band (FB). 2D lattices hosting Chern FB [6–9] have been shown to stabilize fractional QH (FQH) effect without magnetic field, manifesting a fractional Chern insulator (FCI) as a condensed matter analog of FQH effect [10–12]. The anyonic statistics followed by emergent excitations of FQH systems [13–15] make them appealing due to the potential application of non-Abelian anyons in topological quantum computation [16, 17].

Beyond single-component FQH, on the other hand, multi-component FQH effects offer a richer and more exotic quantum phase diagram, with extra degrees of freedom allowing for additional tuning parameters [18–20]. Bilayer Quantum Hall (BQH) systems, such as GaAs quantum wells [21–24], are spectacular examples of multi-component FQH effect. Either a wide quantum well or a double quantum well can be mapped onto two layers of two-dimensional electron gas separated by a finite distance (Fig. 1(a)) forming the two-component FQH setup, leading to exotic multi-component quantum Hall plateaus [21–24]. In addition, BQH systems are known to host a plethora of Abelian and non-Abelian FQH phase transitions owing to exceptional tunability of parameters, such as inter-layer distance and tunneling strength, which has generated a lot of theoretical and experimental interest [25–30].

In parallel to anomalous QH and FQH effects, the concept of anomalous BQH effect is quite appealing since it could lead to tunable exotic phases, including excitonic

superfluid [31, 32], and non-Abelian anyons with 2-body interactions at $\nu_T < 1$ [26, 28] while its realization in single-component FCIs requires strong n -body interaction ($n > 2$) [33–35] or $\nu_T > 1$ [36, 37]. So far, however, the material realization of BQH effect without a magnetic field has remained elusive. Some recent works [38–40] discussed the possibility of stabilizing multi-component bosonic/fermionic FQH states in a single FB partially filled by spin-up and -down electrons. However, this is extremely hard, if not impossible, to do experimentally; also, the tunability of inter-component interaction and tunneling strength, an essential element of BQH systems, is entirely lacking. It is also worth noting that some earlier works [41–43] illustrated the stabilization of multi-component FQH states in higher Chern number FBs, although still lacking in tunability and experimental feasibility.

In this Letter, we propose a *direct solid-state analogue* of the BQH effect, to open the door for its materials realization by exploiting a two-layer FB system whose layer index of the two separate layers is easily accessible in experiments, in contrast to spin index within the same band. We use bilayer Kagome lattice (Fig. 1(b)) as a prototypical example, which is one of the most widely studied 2D lattices and material systems hosting topological FBs [7, 44–47]. Excitingly, a recent experiment has observed the topological FB in a *monolayer* breathing-Kagome lattice [48], and bilayer-Kagome-lattice materials have also been experimentally synthesized as candidates to host quantum spin liquids [49, 50]. In addition, topological FBs are proposed in multi-layer heterostructures [51, 52], as well as in metal organic frameworks [53–56].

Importantly, using layer degree of freedom instead of spin, the inter-component interactions and tunneling strength become highly tunable. The individual FB layers can be separated by a tunneling barrier [57] (Fig. 1(c)), then one may access different phase regions

by systematically tuning the tunneling barrier materials and interlayer distance, and at the weak tunneling limit with a high barrier one can change the distance and hence interlayer Coulomb interaction independently [23, 58]. This is very crucial for realizing different exotic Abelian/non-Abelian phase transitions hosted in anomalous BQH systems as we propose, which directly mimic the conventional BQH systems fabricated using double quantum wells [23]. Here, using exact diagonalization (ED) method, we demonstrate characteristic signatures of anomalous BQH states in a bilayer Kagome lattice. We have calculated topological degeneracy of ground-state manifold on a torus, non-trivial Chern matrix and excitonic off-diagonal long-range order, to prove the stabilization of excitonic condensate states at total filling $v_T = 1$ of two FBs, and further illustrate the potential of this lattice in stabilizing exotic tunable anomalous BQH states at $v_T = 2/3$.

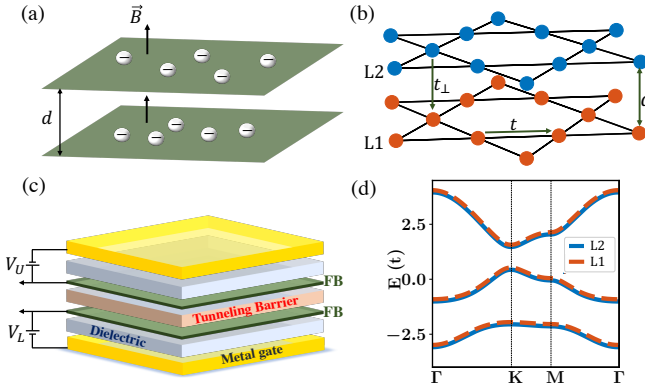


FIG. 1. (a) Schematic of BQH setup containing two layers of 2D electron gas separated by a distance d and a tunneling barrier. (b) Schematic of bilayer Kagome lattice with blue and orange colors indicating atoms in layer L1 and L2 respectively. (c) Schematic of possible device configuration that can be used to realize anomalous BQH. $V_{U(L)}$ represent gate voltages for upper(lower) layer. FB material is indicated by green layer with a tunneling barrier in between. (d) Single-particle band structure of bilayer Kagome lattice at $\lambda = 0.3t$ and $t_{\perp} = 0$. The two FBs belonging to individual layers are depicted in blue and orange color respectively.

We use a nearest-neighbor (NN) tight-binding model of bilayer Kagome lattice (Fig. 1(b)) with spin-orbit coupling (SOC) that conserves the out-of-plane spin-component [44], leading to the kinetic part of total Hamiltonian in the absence of inter-layer tunneling,

$$H_{kin} = \sum_{\sigma} [-t \sum_{\langle i,j \rangle \alpha} c_{i\sigma\alpha}^{\dagger} c_{j\sigma\alpha} + i\lambda \sum_{\langle i,j \rangle \alpha\beta} \frac{2}{\sqrt{3}} (\mathbf{r}_{ki} \times \mathbf{r}_{kj}) \cdot \boldsymbol{\tau}_{\alpha\beta} c_{i\sigma\alpha}^{\dagger} c_{j\sigma\beta}] \quad (1)$$

where t is the NN hopping integral, λ represents SOC strength, $\mathbf{r}_{ki}$ and $\mathbf{r}_{kj}$ are the two NN unit vectors pointing from atomic site k to i and j respectively with i, j ,

and k forming a triangle in the Kagome lattice (see Fig. S1 in supplementary material (SM) [59]), $\alpha(\beta)$ are spin indices, $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ with τ_i being the i^{th} Pauli's matrix, and $\sigma(\sigma')$ represents each layer of the bilayer lattice. Electron interactions are described by an extended Hubbard model including NN intra-layer (V) and direct inter-layer ($V_{\perp}$) interactions. $V_{\perp}$ depends on the inter-layer distance d as shown in Fig. 1(b) which can be varied easily, rendering the tunability of inter-component interactions. We also include tunneling between the layers ($t_{\perp}$) in the interaction part of the Hamiltonian which is given as,

$$H_{int} = \sum_{\sigma, \sigma'} [V \sum_{\langle r, r' \rangle > intra} n_{r\sigma} n_{r'\sigma'} + V_{\perp} \sum_r n_{r\sigma} n_{r\sigma'} + t_{\perp} \sum_r c_{r\sigma}^{\dagger} c_{r\sigma'}] \quad (2)$$

where $n_{r\sigma} = c_{r\sigma}^{\dagger} c_{r\sigma}$ is the electron density operator. The spin indices are suppressed. Throughout this work we set $V = 2t$, which is smaller than the single-particle gap separating the FBs (Fig. 1(b)), while a distance dependence of $V_{\perp} = 1.5t/d$ is assumed. Note that d (hence, $V_{\perp}$) and $t_{\perp}$ can be systematically varied in general and independently in bilayer FB setup at weak tunneling limit as mentioned above [23, 58].

Single-particle band structure is shown in Fig. 1(d). With $t_{\perp} = 0$, the two layer-indexed FBs have the same Chern number which exactly match the two Landau levels in each layer of conventional BQH system. We use a slightly high SOC ($\lambda = 0.3t$) to isolate the bottom FBs from other bands. This procedure is widely used in realizing FCI where a high flatness ratio of FB is desirable [44]. It is worth mentioning that such conditional parameters can be realized using Floquet band engineering in real materials [47, 60]. To study the effect of interactions, we exactly diagonalize the full Hamiltonian ($H = H_{kin} + H_{int}$) in reciprocal space with interactions projected to the two FBs of finite lattice containing total number of $N_s = 6 \times N_x \times N_y$ sites. The total filling factor is given by $v_T = N_e / (N_x \times N_y)$, where N_e is the number of electrons in the system. Under periodic boundary condition, we implement translational symmetry and diagonalize the Hamiltonian in each momentum sector $q = (2\pi k_x / N_x, 2\pi k_y / N_y)$ with k_x and k_y being the integers, labelled as (k_x, k_y) . One can also assign a pseudospin notation to each FB and define $S_z = (N_{\uparrow} - N_{\downarrow})/2$, where $N_{\uparrow(\downarrow)}$ is the number of electrons in FB belonging to upper (lower) layer. S_z is a good quantum number when $t_{\perp} = 0$. Detailed methodology of ED for multiple bands can be found in earlier work [61] (also see SM Sec. I [59]).

First, we study the $v_T = 1$ case. Multi-component FQH systems are described by Halperin's (m,m,n) wavefunctions [18]. In the following, we only use integers (m,m,n) to refer to these states. In a BQH setup with negligible $t_{\perp}$, the $v_T = 1$ plateau is described by (1,1,1) state featuring a non-degenerate ground state on two tori

[40]. In Fig. 2(a) we plot the energy spectra for $S_z = 0$, or the case of balanced layers with $N_\uparrow = N_\downarrow$, at a finite $d = 0.3a$ and negligible $t_\perp$. There is clearly presence of a non-degenerate ground state separated from excited states in two systems of different sizes, which are shown to confirm the convergence of our results. For all subsequent calculations the system size of 4×3 is used. $(1, 1, 1)$ state on two tori should belong to the total momentum sector $(\sum_{i=1}^N k_x^i, \sum_{i=1}^N k_y^i)$, where $N = N_x \times N_y$, implying all the reciprocal points are occupied; for the 4×3 system, it is the $(2, 0)$ sector, as shown in Fig. 2(a), implying a non-trivial topology of the ground state with a binding of particle-hole between two layers.

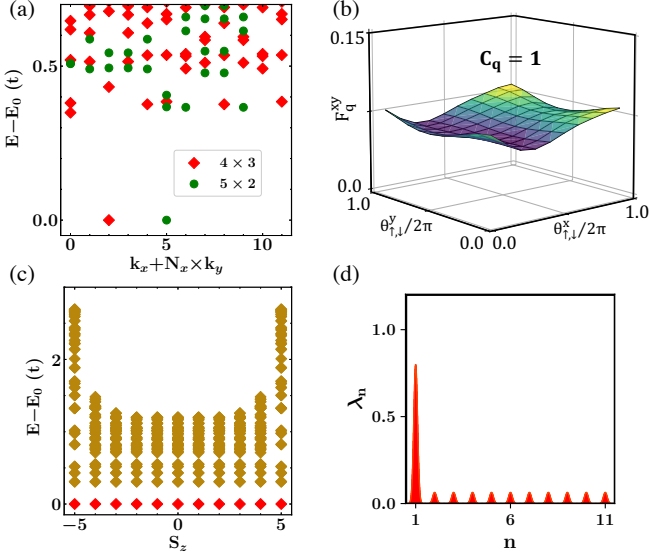


FIG. 2. (a) Momentum-resolved spectra of H at $d = 0.3a$, $t_\perp = 0$, and $S_z = 0$ for two system sizes, $N_s = 6 \times 5 \times 2$ and $N_s = 6 \times 4 \times 3$ denoted by green circles and red diamonds respectively. (b) Berry curvature for total charge Chern number indicated by C_q . (c) Energy spectra of H at $d \rightarrow 0$ and $t_\perp = 0$ with varying S_z for $N_s = 6 \times 4 \times 3$. Red (yellow) diamonds denote the ground (excited) states. (d) Eigenvalues of $\rho^{(2)}$ for (b) plotted in descending order.

Beyond the energy spectra, we extract the Chern number matrix $C = \begin{bmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{bmatrix}$ for the two-component FQH system, which contains information for quantized Hall conductance [27, 38–40]. For Abelian multi-component FQH states, the diagonal and off-diagonal elements of C are related to intra- and inter-component Hall transports respectively. Each component of C is defined under twisted boundary conditions with $\theta_{\uparrow(\downarrow)}^{x(y)}$ being the twist angle in layer $\uparrow(\downarrow)$ in $x(y)$ direction [38–40, 59, 62]. With $\theta_\uparrow^x = \theta_\downarrow^x = \theta^x$, and $\theta_\uparrow^y = \theta_\downarrow^y = \theta^y$, charge Hall conductance is related to $C_q = \sum_{\sigma,\sigma'} C_{\sigma,\sigma'}$. For the single non-degenerate ground state at $v_T = 1$, we numerically calculate Berry curvatures on a 11×11 mesh in the phase space and obtain a topological invariant $C_q = 1$ with a well-defined smooth Berry curvature as plotted in

Fig. 2(b), further confirming the non-trivial topology of $(1, 1, 1)$ state.

A fundamental global property of the Halperin’s $(1, 1, 1)$ ground state at $v_T = 1$ is the binding of electrons in one layer to the empty states (holes) in the other layer. Such interlayer excitons form a coherent superfluid which has been observed in the counter-flow transport measurements. In fact, bilayer quantum Hall systems are the only physical systems where this exotic transport property of excitonic superfluid has been observed [31, 32, 57], albeit under a strong magnetic field. Next, we show excitonic properties of the ground state in bilayer Kagome lattice at $v_T = 1$ without magnetic field. In Fig. 2(c) we show the energy spectra of H for $v_T = 1$ with negligible d and $t_\perp$ at multiple S_z which determines the relative number of electrons in each FB. The degeneracy of states with varying S_z indicates that there is no energy cost for electrons to tunnel in between the two layers, a signature of spontaneous symmetry breaking, also known as the “which layer” uncertainty [32]. This leads to a huge interlayer tunneling current at zero energy as observed in double quantum well BQH experiments [31]. At a finite d , the state with $S_z = 0$ becomes energetically stable due to anisotropy in interactions (see SM Sec. II [59]).

We further illustrate excitonic coherence order by calculating eigenvalues of excitonic reduced two-body density matrix [61], $\rho^{(2)}(k, k'; \bar{k}, \bar{k}') = \langle \Psi | \psi_2^\dagger(k) \psi_1(k') \psi_1^\dagger(\bar{k}') \psi_2(\bar{k}) | \Psi \rangle$, where $\psi_{1(2)}^\dagger(k)$ creates an electron in FB belonging to layer 1(2) at reciprocal lattice point k , and $|\Psi \rangle$ is the many-body ground-state wavefunction. The eigenvalues of two-body density matrix are the occupations of bosonic natural orbitals. Hence, the presence of one large eigenvalue signifies Bose-Einstein condensation. In Fig 2(d), we plot the normalized eigenvalues (λ_n), i.e., eigenvalues divided by the total number of excitons in the system (for a system size 4×3 , it is 6) in descending order, so that the presence of one large, normalized eigenvalue of order 1 concretely shows the presence of excitonic superfluid. At $d \sim 0.3a$, there is indeed one large eigenvalue ~ 0.8 implying 80% of excitons are in the same natural orbital. We note that if the interlayer distance is reduced further, the condensation factor is enhanced as expected (Fig. S2 in SM [59]). This concretely shows the presence of a symmetry-broken condensate order parameter pointing to the possible experimental observation of excitonic superfluid in counter-flow transport measurements without magnetic field in bilayer FB lattices.

Next, we focus on filling factor $v_T = 2/3$, in order to illustrate the tunability of anomalous BQH states as realized in bilayer Kagome system. At $v_T = 2/3$ and a relatively large $d = 2.25a$ between the two layers, we plot the energy spectrum for balanced layers ($S_z = 0$), as depicted in Fig. 3(a). The individual layers are decoupled when d is large and should stabilize $(3, 3, 0)$ state [28], which is simply the bilayer Laughlin’s state. We, first, use topological degeneracy of ground-state manifold on a torus to identify the non-trivial character of this state

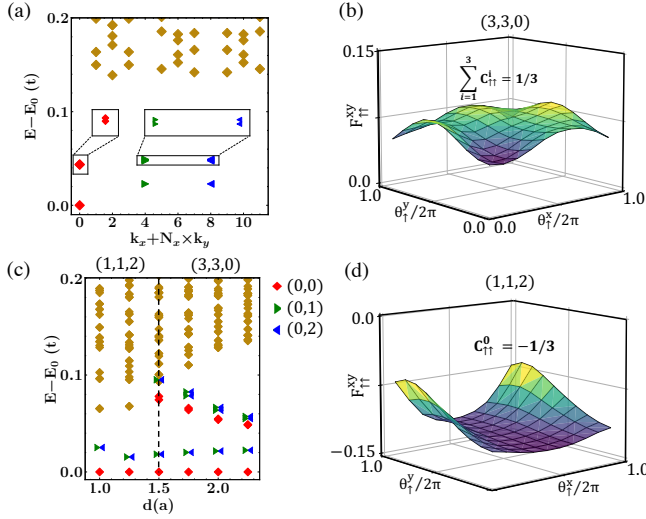


FIG. 3. (a) Energy spectra of H at $d = 2.25a$ and $t_{\perp} = 0$ for $N_s = 6 \times 4 \times 3$ system size. Red diamonds (0,0), blue (0,1) and green (0,2) triangles denote the 9-fold degenerate ground-state manifold while yellow denotes the excited states. Inset shows the zoomed-in view of ground-state manifold. (b) Total intra-component Berry curvature ($C_{\uparrow\uparrow}$) of the 3 quasi-degenerate states at (0,0) for system in (a). (c) d -driven phase transition from (1,1,2) state to (3,3,0) state. (d) Intra-component Berry curvature ($C_{\uparrow\uparrow}$) for the single state at (0,0) of the 3-fold degenerate ground-state manifold of (1,1,2) calculated at $d = 1.0a$.

[63–65]. Inter-component correlations in (m, m, n) state, in general, can be studied using K -matrix, $K = \begin{bmatrix} m & n \\ n & m \end{bmatrix}$, as formulated within the field theory framework [66, 67]. Topological degeneracy of ground state for a system with $v_T = p/q$ on a torus is given by the determinant of K -matrix, $\det K = qN'$, where N' is an integer describing different translations of individual center-of-mass (COM) of a component, while q determines the overall COM degeneracy. In the case of (3,3,0) state at $v_T = 2/3$, and a 4×3 system size, the three degenerate states corresponding to $N' = 3$ should lie at (0,0) [61]. Taking COM degeneracy into account there should be additional 3-fold degeneracy for each of these states as depicted in Fig. 3(a).

To further illustrate this state to be topologically ordered, we calculate the C matrix (which should be $= K^{-1}$ [27]) for the 9 quasi-degenerate ground states of (3,3,0) state. For the three states at (0,0), we found that the intra-component Berry curvature is well-defined and smooth (Fig. 3(b)), from which we obtain a C matrix element, $\sum_{i=1}^3 C_{\uparrow\uparrow}^i = 1$, while the off-diagonal element of C matrix vanishes ($\sum_{i=1}^3 C_{\uparrow\downarrow}^i = 0$). Similarly, for all the 9 states in momentum sectors (0,0), (0,1) and (0,2), we obtain $\sum_{i=1}^9 C_{\uparrow\uparrow}^i = 3$ and $\sum_{i=1}^9 C_{\uparrow\downarrow}^i = 0$. The other two elements are related to these two by symmetry $c_{\uparrow} \rightarrow c_{\downarrow}$.

This finally gives,

$$C = \begin{bmatrix} C_{\uparrow\uparrow} & C_{\uparrow\downarrow} \\ C_{\downarrow\uparrow} & C_{\downarrow\downarrow} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad (3)$$

which is inverse of the K -matrix for (3,3,0) state, proving the topological characteristic of this state. In addition, we calculate the spectral flow of these 9 states under flux insertion, confirming fractional charge Hall conductance (Fig. S3(a) and (b) [59]).

Next, we show a d -driven transition from spin-singlet (1,1,2) state at small d to (3,3,0) state at large d in Fig. 3(c). When d is small, the $v_T = 2/3$ anomalous BQH resembles the usual $v_T = 2/3$ FQH state with spin, i.e., for small d the pseudospin can be directly mapped onto electronic spin, which stabilizes (1,1,2) state with 3-fold degeneracy on a torus [27]. In Fig. 3(c) we show a transition from the 3- to 9-fold degenerate ground state, as d increases, with gap closing at $d = 1.5a$. The latter ($d > 1.5a$) is identified as (3,3,0) state having the C matrix as shown in Eqn. 3; the former ($d < 1.5a$) is characterized with $C = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$, as expected for (1,1,2) state, numerically calculated from the inter- and intra-component Berry curvature (Fig. 3(d)). We note that the energy spread of the ground-state manifold in Fig. 3(c) is comparable to the many-body gap near the phase transition point. This is not an artifact of the finite-size effect but signifies a topological phase transition involving gap closing and reopening with different numbers of states in the ground-state manifold on two sides of the transition point.

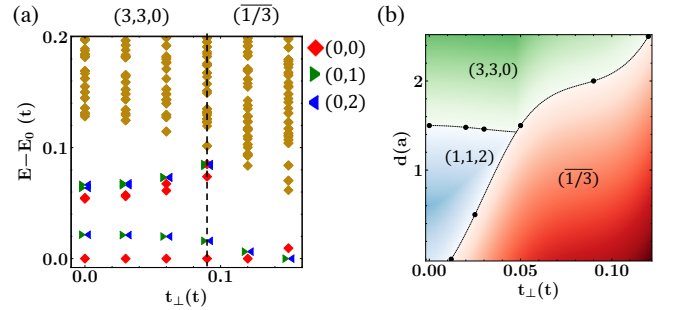


FIG. 4. (a) $t_{\perp}$ -driven phase transition at $v_T = 2/3$, $d = 2.0a$, $S_z = 0$ for $N_s = 6 \times 4 \times 3$ system size. (b) Phase diagram for bilayer Kagome lattice at $v_T = 2/3$ in the phase space of d and $t_{\perp}$ between the two Kagome layers

There is also a possibility of $t_{\perp}$ -driven transition at $v_T = 2/3$ [28]. In Fig. 4(a) we show a transition from (3,3,0) state to particle-hole conjugate of Laughlin's $1/3$ state ($\overline{1/3}$) at $d = 2.0a$. At weak $t_{\perp}$, (3,3,0) state is identified using the C matrix as described earlier. At strong $t_{\perp}$, the bilayer effectively behaves as a single layer due to strong correlations and stabilizes the $\overline{1/3}$ state with 3-fold COM degeneracy at (0,0), (0,1), and (0,2), which is also seen in Fig. 4(a), similar to BQH system [32]. Hence, in the phase space of $t_{\perp}$ and d , we identify three phases:

(3, 3, 0) state at large d and weak $t_{\perp}$; (1, 1, 2) state at small d and weak $t_{\perp}$; $1/3$ state at small d and strong $t_{\perp}$, as shown by the phase diagram in Fig. 4(b). The phase transition boundary from the 3-fold degenerate (1, 1, 2) to $1/3$ state is obtained using gap-closing and reopening points (Fig. S3(c) [59]). This phase diagram for bilayer FB lattice is in excellent agreement with the one found in conventional BQH system [28].

In addition to the exceptional tunability of bilayer FB lattices, an important aspect of realizing the exotic anomalous BQH states in such systems is their experimental feasibility. Recently, FCI was experimentally identified in twisted bilayer graphene over hexagonal boron nitride substrate under a weak magnetic field [68] and in a twisted bilayer transition metal dichalcogenide without magnetic field [69]. A promising way to realize anomalous BQH states is to construct two such bilayers separated by a tunneling barrier (Fig. 1(b)). Moreover, Floquet engineering of isolated FBs have been shown as a promising route towards realizing FCIs in 2D materials, for example, graphene [60], and a metal-organic 2D

monolayer with Kagome bands [47]. This approach can be generalized to bilayers of such FB materials. One possibility is using Floquet engineering to isolate two FBs of bilayers such as superatomic graphene lattice whose single-layer structure has already been experimentally synthesized using molecular building blocks of triangulene [70] [also see SM Sec. VI [59]]. In addition, the striking tunability of anomalous BQH states, as illustrated here for bilayer Kagome lattice, could also lead to material realization of non-Abelian anyonic states without magnetic field. We observe some preliminary signatures of such exotic states [SM sec V [59]] whose further investigation has been left for future work.

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- [1] F. D. M. Haldane, Model for a quantum hall effect without landau levels: Condensed-matter realization of the "parity anomaly", *Physical review letters* **61**, 2015 (1988).
 - [2] C. L. Kane and E. J. Mele, Quantum spin hall effect in graphene, *Physical review letters* **95**, 226801 (2005).
 - [3] C. L. Kane and E. J. Mele, Z₂ topological order and the quantum spin hall effect, *Physical review letters* **95**, 146802 (2005).
 - [4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Quantum spin hall effect and topological phase transition in hgte quantum wells, *science* **314**, 1757 (2006).
 - [5] C.-Z. Chang, J. Zhang, X. Feng, J. Shen, Z. Zhang, M. Guo, K. Li, Y. Ou, P. Wei, L.-L. Wang, *et al.*, Experimental observation of the quantum anomalous hall effect in a magnetic topological insulator, *Science* **340**, 167 (2013).
 - [6] E. J. Bergholtz and Z. Liu, Topological flat band models and fractional chern insulators, *International Journal of Modern Physics B* **27**, 1330017 (2013).
 - [7] Z. Liu, F. Liu, and Y.-S. Wu, Exotic electronic states in the world of flat bands: From theory to material, *Chinese Physics B* **23**, 077308 (2014).
 - [8] J. Wang, J. Cano, A. J. Millis, Z. Liu, and B. Yang, Exact landau level description of geometry and interaction in a flatband, *Physical review letters* **127**, 246403 (2021).
 - [9] B. Estienne, N. Regnault, and V. Crépel, Ideal chern bands are landau levels in curved space, *arXiv preprint arXiv:2304.01251* (2023).
 - [10] D. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Fractional quantum hall effect in the absence of landau levels, *Nature communications* **2**, 389 (2011).
 - [11] N. Regnault and B. A. Bernevig, Fractional chern insulator, *Physical Review X* **1**, 021014 (2011).
 - [12] P. J. Ledwith, A. Vishwanath, and D. E. Parker, Vortexability: A unifying criterion for ideal fractional chern insulators, *arXiv preprint arXiv:2209.15023* (2022).
 - [13] F. D. M. Haldane, Fractional quantization of the hall effect: a hierarchy of incompressible quantum fluid states, *Physical Review Letters* **51**, 605 (1983).
 - [14] B. I. Halperin, Statistics of quasiparticles and the hierarchy of fractional quantized hall states, *Physical Review Letters* **52**, 1583 (1984).
 - [15] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional statistics and the quantum hall effect, *Physical review letters* **53**, 722 (1984).
 - [16] G. Moore and N. Read, Nonabelions in the fractional quantum hall effect, *Nuclear Physics B* **360**, 362 (1991).
 - [17] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Non-abelian anyons and topological quantum computation, *Reviews of Modern Physics* **80**, 1083 (2008).
 - [18] B. I. Halperin, Theory of the quantized hall conductance, *Helvetica Physica Acta* **56**, 75 (1983).
 - [19] X.-G. Wen, Continuous topological phase transitions between clean quantum hall states, *Physical review letters* **84**, 3950 (2000).
 - [20] M. Barkeshli and X.-G. Wen, Bilayer quantum hall phase transitions and the orbifold non-abelian fractional quantum hall states, *Physical Review B* **84**, 115121 (2011).
 - [21] Y. Suen, L. Engel, M. Santos, M. Shayegan, and D. Tsui, Observation of a $\nu = 1/2$ fractional quantum Hall state in a double-layer electron system, *Physical review letters* **68**, 1379 (1992).
 - [22] J. Eisenstein, G. Boebinger, L. Pfeiffer, K. West, and S. He, New fractional quantum hall state in double-layer two-dimensional electron systems, *Physical review letters* **68**, 1383 (1992).
 - [23] Y. Suen, H. Manoharan, X. Ying, M. Santos, and M. Shayegan, Origin of the $\nu = 1/2$ fractional quantum hall state in wide single quantum wells, *Physical review letters* **72**, 3405 (1994).
 - [24] J. Shabani, Y. Liu, M. Shayegan, L. Pfeiffer, K. West,

- and K. Baldwin, Phase diagrams for the stability of the $\nu = 1/2$ fractional quantum Hall effect in electron systems confined to symmetric, wide GaAs quantum wells, *Physical Review B* **88**, 245413 (2013).
- [25] D. Sheng, L. Balents, and Z. Wang, Phase diagram for quantum hall bilayers at $\nu = 1$, *Physical review letters* **91**, 116802 (2003).
- [26] W. Zhu, Z. Liu, F. Haldane, and D. Sheng, Fractional quantum Hall bilayers at half filling: Tunneling-driven non-Abelian phase, *Physical Review B* **94**, 245147 (2016).
- [27] I. McDonald and F. Haldane, Topological phase transition in the $\nu = 2/3$ quantum Hall effect, *Physical Review B* **53**, 15845 (1996).
- [28] S. Geraedts, M. P. Zaletel, Z. Papić, and R. S. Mong, Competing Abelian and non-Abelian topological orders in $\nu = 1/3 + 1/3$ quantum Hall bilayers, *Physical Review B* **91**, 205139 (2015).
- [29] V. Crépel, B. Estienne, and N. Regnault, Variational ansatz for an abelian to non-abelian topological phase transition in $\nu = 1/2 + 1/2$ bilayers, *Physical review letters* **123**, 126804 (2019).
- [30] Q. Hu, T. Neupert, and G. Wagner, Single-parameter variational wavefunctions for quantum hall bilayers, *arXiv preprint arXiv:2303.12825* (2023).
- [31] J. Eisenstein and A. MacDonald, Bose–einstein condensation of excitons in bilayer electron systems, *Nature* **432**, 691 (2004).
- [32] J. Eisenstein, Exciton condensation in bilayer quantum hall systems, *Annu. Rev. Condens. Matter Phys.* **5**, 159 (2014).
- [33] Y.-F. Wang, H. Yao, Z.-C. Gu, C.-D. Gong, and D. Sheng, Non-abelian quantum hall effect in topological flat bands, *Physical review letters* **108**, 126805 (2012).
- [34] Y.-L. Wu, B. A. Bernevig, and N. Regnault, Zoology of fractional chern insulators, *Physical Review B* **85**, 075116 (2012).
- [35] M. Greiter, X.-G. Wen, and F. Wilczek, Paired hall state at half filling, *Physical review letters* **66**, 3205 (1991).
- [36] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. English, Observation of an even-denominator quantum number in the fractional quantum hall effect, *Physical review letters* **59**, 1776 (1987).
- [37] K. Pakrouski, M. R. Peterson, T. Jolicoeur, V. W. Scarola, C. Nayak, and M. Troyer, Phase diagram of the $\nu = 5/2$ fractional quantum hall effect: effects of landau-level mixing and nonzero width, *Physical Review X* **5**, 021004 (2015).
- [38] T.-S. Zeng, W. Zhu, and D. Sheng, Two-component quantum hall effects in topological flat bands, *Physical Review B* **95**, 125134 (2017).
- [39] T.-S. Zeng and W. Zhu, Chern-number matrix of the non-Abelian spin-singlet fractional quantum hall effect, *Physical Review B* **105**, 125128 (2022).
- [40] T.-S. Zeng, D. Sheng, and W. Zhu, Quantum hall effects of exciton condensate in topological flat bands, *Physical Review B* **101**, 195310 (2020).
- [41] Y.-F. Wang, H. Yao, C.-D. Gong, and D. Sheng, Fractional quantum hall effect in topological flat bands with chern number two, *Physical Review B* **86**, 201101 (2012).
- [42] Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Läuchli, Fractional chern insulators in topological flat bands with higher chern number, *Physical review letters* **109**, 186805 (2012).
- [43] M. Trescher and E. J. Bergholtz, Flat bands with higher chern number in pyrochlore slabs, *Physical Review B* **86**, 241111 (2012).
- [44] E. Tang, J.-W. Mei, and X.-G. Wen, High-temperature fractional quantum hall states, *Physical review letters* **106**, 236802 (2011).
- [45] Y. Zhou, G. Sethi, H. Liu, Z. Wang, and F. Liu, Excited quantum anomalous and spin hall effect: dissociation of flat-bands-enabled excitonic insulator state, *Nanotechnology* **33**, 415001 (2022).
- [46] G. Sethi, Y. Zhou, L. Zhu, L. Yang, and F. Liu, Flat-band-enabled triplet excitonic insulator in a diatomic kagome lattice, *Physical Review Letters* **126**, 196403 (2021).
- [47] H. Liu, G. Sethi, D. Sheng, Y. Zhou, J.-T. Sun, S. Meng, and F. Liu, High-temperature fractional quantum hall state in the floquet kagome flat band, *Physical Review B* **105**, L161108 (2022).
- [48] M. Pan, X. Zhang, Y. Zhou, P. Wang, Q. Bian, H. Liu, X. Wang, X. Li, A. Chen, X. Lei, *et al.*, Growth of mesoscale ordered two-dimensional hydrogen-bond organic framework with the observation of flat band, *Physical Review Letters* **130**, 036203 (2023).
- [49] J. Ni, Q. Liu, Y. Yu, E. Cheng, Y. Huang, Z. Liu, X. Wang, Y. Sui, and S. Li, Ultralow-temperature heat transport in the quantum spin liquid candidate $\text{Ca}_{10}\text{Cr}_7\text{O}_{28}$ with a bilayer kagome lattice, *Physical Review B* **97**, 104413 (2018).
- [50] A. Balodhi and Y. Singh, Synthesis and pressure and field-dependent magnetic properties of the kagome-bilayer spin liquid $\text{Ca}_{10}\text{Cr}_7\text{O}_{28}$, *Physical Review Materials* **1**, 024407 (2017).
- [51] L. Balents, C. R. Dean, D. K. Efetov, and A. F. Young, Superconductivity and strong correlations in moiré flat bands, *Nature Physics* **16**, 725 (2020).
- [52] Y.-H. Zhang, D. Mao, Y. Cao, P. Jarillo-Herrero, and T. Senthil, Nearly flat chern bands in moiré superlattices, *Physical Review B* **99**, 075127 (2019).
- [53] T. Kambe, R. Sakamoto, K. Hoshiko, K. Takada, M. Miyachi, J.-H. Ryu, S. Sasaki, J. Kim, K. Nakazato, M. Takata, *et al.*, π -conjugated nickel bis (dithiolene) complex nanosheet, *Journal of the American Chemical Society* **135**, 2462 (2013).
- [54] W. Jiang, X. Ni, and F. Liu, Exotic topological bands and quantum states in metal–organic and covalent–organic frameworks, *Accounts of Chemical Research* **54**, 416 (2021).
- [55] X. Ni, H. Li, F. Liu, and J.-L. Brédas, Engineering of flat bands and dirac bands in two-dimensional covalent organic frameworks (COFs): relationships among molecular orbital symmetry, lattice symmetry, and electronic-structure characteristics, *Materials Horizons* **9**, 88 (2022).
- [56] Z. Liu, Z.-F. Wang, J.-W. Mei, Y.-S. Wu, and F. Liu, Flat chern band in a two-dimensional organometallic framework, *Physical review letters* **110**, 106804 (2013).
- [57] J. Li, T. Taniguchi, K. Watanabe, J. Hone, and C. Dean, Excitonic superfluid phase in double bilayer graphene, *Nature Physics* **13**, 751 (2017).
- [58] N. Kumada, D. Terasawa, Y. Shimoda, H. Azuhata, A. Sawada, Z. Ezawa, K. Muraki, T. Saku, and Y. Hiramaya, Phase diagram of interacting composite fermions in the bilayer $\nu = 2/3$ quantum hall effect, *Physical review letters* **89**, 116802 (2002).
- [59] See supplementary material at where we have discussed computational methods, spectral flow analysis, prelimi-

- nary signature of possible non-abelian state at $\nu_T = 2/3$, and material realization of ABQH effect, which also includes refs. [71–73].
- [60] A. G. Grushin, Á. Gómez-León, and T. Neupert, Floquet fractional chern insulators, *Physical review letters* **112**, 156801 (2014).
 - [61] G. Sethi, M. Cuma, and F. Liu, Excitonic condensate in flat valence and conduction bands of opposite chirality, *Physical Review Letters* **130**, 186401 (2023).
 - [62] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized hall conductance as a topological invariant, *Physical Review B* **31**, 3372 (1985).
 - [63] R. Tao and Y.-S. Wu, Gauge invariance and fractional quantum hall effect, *Physical Review B* **30**, 1097 (1984).
 - [64] F. Haldane, Many-particle translational symmetries of two-dimensional electrons at rational landau-level filling, *Physical review letters* **55**, 2095 (1985).
 - [65] I. McDonald, Degeneracy of multicomponent quantum hall states satisfying periodic boundary conditions, *Physical Review B* **51**, 10851 (1995).
 - [66] X.-G. Wen and A. Zee, Classification of Abelian quantum hall states and matrix formulation of topological fluids, *Physical Review B* **46**, 2290 (1992).
 - [67] B. Blok and X.-G. Wen, Effective theories of the fractional quantum hall effect at generic filling fractions, *Physical Review B* **42**, 8133 (1990).
 - [68] Y. Xie, A. T. Pierce, J. M. Park, D. E. Parker, E. Khalaf, P. Ledwith, Y. Cao, S. H. Lee, S. Chen, P. R. Forrester, *et al.*, Fractional chern insulators in magic-angle twisted bilayer graphene, *Nature* **600**, 439 (2021).
 - [69] J. Cai, E. Anderson, C. Wang, X. Zhang, X. Liu, W. Holtzmann, Y. Zhang, F. Fan, T. Taniguchi, K. Watanabe, *et al.*, Signatures of fractional quantum anomalous hall states in twisted mote2 bilayer, *arXiv preprint arXiv:2304.08470* (2023).
 - [70] A. Delgado, C. Dusold, J. Jiang, A. Cronin, S. G. Louie, and F. R. Fischer, Evidence for excitonic insulator ground state in triangulene kagome lattice, *arXiv preprint arXiv:2301.06171* (2023).
 - [71] A. W. Sandvik, Computational studies of quantum spin systems, in *AIP Conference Proceedings*, Vol. 1297 (American Institute of Physics, 2010) pp. 135–338.
 - [72] B. N. Parlett and D. S. Scott, The lanczos algorithm with selective orthogonalization, *Mathematics of computation* **33**, 217 (1979).
 - [73] G. Kresse and J. Furthmüller, Efficient iterative schemes for ab initio total-energy calculations using a plane-wave basis set, *Physical review B* **54**, 11169 (1996).