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Spectral Theorems for Generalized Weyl Nodes with Impurities in a Magnetic Field

Dan Solomon¹ and Ho-Ung Yee¹

¹*Physics Department, University of Illinois at Chicago, Chicago, Illinois 60607, USA**

We prove a few spectral theorems for the density of states of a Weyl node with arbitrary topology. We show that the density of extended states of a Weyl node with random impurity potentials remains gapless in the presence of a magnetic field. Therefore, a magnetic field precludes Anderson localization in Weyl semi-metals, when inter-node transitions are suppressed for smooth enough potentials. We also provide a rigorous quantum mechanical proof of the chiral magnetic effect for arbitrary topology of a Weyl node.

Introduction — Weyl semi-metals, where certain band crossing points feature a topological character which is similar to relativistic massless fermions, provide us with an interesting bridge between condensed matter and high energy physics. While relativistic Weyl fermions [1] obey the constraints from Lorentz invariance, e.g., the alignment of spin and momentum, the quasi-particles in Weyl semi-metals are not encumbered with such restrictions, and can possess more diverse topological character [2, 3]. Some of the most interesting physics stemming from non-trivial topology, such as chiral anomaly [4, 5] and chiral transport phenomena [6–8], therefore find themselves naturally generalized in Weyl semi-metals; a notable example is the chiral magnetic effect [9–12].

In condensed matter systems, impurities due to imperfect lattices are common, and how they affect the spectral and transport properties of Weyl nodes has been a subject of great interest. It has been proposed that as the impurity strength increases there is a quantum phase transition from the semi-metal to the diffusive metal phases [13], where the density of states at the nodal point becomes non-zero [14–20] and a non-Fermi liquid behavior emerges [21]. However, it was also pointed out that the rare region effects could make the transition a cross-over instead [22–26] (see Refs. [27–30] for current status of the issue). For a Dirac semi-metal, i.e., two degenerate Weyl points with opposite topology, a further increase of impurity was shown to cause 3D Anderson metal-insulator transition [31]. Whether there is a similar transition, when each Weyl node is separated in momentum space and inter-node couplings are suppressed for smooth potentials, is an interesting question [32–34]. Obviously, the topology of the Weyl node is at the heart of these questions.

In this work, we hope to shed light on these questions by studying spectral property of a single Weyl node with impurities in a non-zero external magnetic field. A non-trivial interplay of nodal topology and magnetic field is well-known in the physics of chiral anomaly and the chiral magnetic effect [35, 36], which offers a promising way to study the fate of topology in the presence of impurities.

The main spectral theorem and its consequences — We first derive the main spectral theorem that our subsequent discussions will rely upon. We consider a system

described by non-interacting quasi-particle which carries a charge q for electromagnetic interactions with an external vector potential $\mathbf{A}$. We assume that the energy spectrum of quasi-particle is described by a 1-particle Hamiltonian $\hat{H}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$ is the gauge invariant momentum. This includes the lattice models, where $\mathbf{A}$ appears as Wilson lines in the hopping matrix, and $\mathbf{p}$ is the lattice momentum. We also assume the thermodynamic limit of a large volume $V \rightarrow \infty$. This means that we are focusing on the bulk spectral properties and ignoring boundary conditions. Therefore our subsequent discussions and results apply only to the bulk properties of the system.

The primary object we consider is the spectral density in energy, weighted by the current operator, e.g., along the z direction,

$$\rho_J(\varepsilon) \equiv \lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr} \left(\hat{J}_z \delta(\hat{H} - \varepsilon) \right), \quad (1)$$

where the current operator is defined by $\hat{J}_z = -\frac{\partial \hat{H}(\mathbf{A})}{\partial A_z} = q \frac{\partial \hat{H}}{\partial p_z}$, with a constant, auxiliary value of the z component of the vector potential $\mathbf{A}$, and $\delta(x)$ is the Dirac's delta function, or any smooth function that is sufficiently narrow around $x = 0$ with unit area. Our subsequent results do not depend on the details of the shape of this function. Because of $\delta(\hat{H} - \varepsilon)$, only the energy eigenstates with the eigenvalues close to ε contribute to $\rho_J(\varepsilon)$. This makes sure that $\rho_J(\varepsilon)$ is finite and well-defined, even in the case where the spectrum of $\hat{H}$ has no lower or upper bounds in energy. Since it can be shown in general that $\langle \hat{J}_z \rangle = 0$ for a bound state, $\rho_J(\varepsilon)$ captures only the unbound extended states in the continuum.

Our main theorem is that $\rho_J(\varepsilon)$ is a constant, independent of ε :

$$\text{Theorem I: } \rho_J(\varepsilon) = C_J \text{ (constant)}. \quad (2)$$

We prove by showing that $\frac{\partial}{\partial \varepsilon} \rho_J(\varepsilon) = 0$, under reasonable assumption of finiteness of the density of states in an energy interval under consideration,

$$\rho(\varepsilon) \equiv \lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr} \left(\delta(\hat{H} - \varepsilon) \right). \quad (3)$$

We have $\frac{\partial}{\partial \varepsilon} \rho_J(\varepsilon) = -\lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr} \left(\hat{J}_z \delta'(\hat{H} - \varepsilon) \right)$, and using the definition of $\hat{J}_z = -\frac{\partial \hat{H}}{\partial A_z}$, we arrive at $\frac{\partial}{\partial \varepsilon} \rho_J(\varepsilon) =$

$\frac{\partial}{\partial A_z} \rho(\varepsilon, \mathbf{A})$, where the dependence on $\mathbf{A}$ is implicit in the Hamiltonian $\hat{H}(\hat{\pi})$. We now utilize the following observation: for any eigenstate $\psi(\mathbf{x})$ of energy ε of the Hamiltonian $\hat{H}(\hat{\pi})$, the state $e^{i\alpha \cdot \mathbf{x}/\hbar} \psi(\mathbf{x})$ with a constant α is an eigenstate of the Hamiltonian $\hat{H}(\hat{\pi} - \alpha)$ with the same energy. This corresponds to a shift of $\mathbf{A} \rightarrow \mathbf{A} + \alpha/q$, which implies that the spectrum of $\hat{H}$, and hence the spectral density $\rho(\varepsilon)$, remains invariant under a constant shift of $\mathbf{A}$. This proves that $\frac{\partial}{\partial \varepsilon} \rho_J(\varepsilon) = \frac{\partial}{\partial A_z} \rho(\varepsilon, \mathbf{A}) = 0$.

Although the theorem and its proof is simple, its implications are rich. Suppose that the quasi-particles are fermions, and the system has a chemical potential μ at zero temperature, and all eigenstates of energy $\varepsilon \leq \mu$ are occupied. The current density, i.e., the current per unit volume, of the system is then given by $j_z = \int_{-\infty}^{\mu} \rho_J(\varepsilon) d\varepsilon$, which can be shown by

$$j_z = \frac{1}{V} \sum_{\alpha} \langle \psi_{\alpha} | \hat{J}_z | \psi_{\alpha} \rangle \Theta(\mu - \varepsilon_{\alpha}), \quad (4)$$

where $|\psi_{\alpha}\rangle$ are the energy eigenstates with energy ε_{α} , labeled by α , and the fact that $\rho_J(\varepsilon)$ can be written as

$$\rho_J(\varepsilon) = \frac{1}{V} \sum_{\alpha} \langle \psi_{\alpha} | \hat{J}_z | \psi_{\alpha} \rangle \delta(\varepsilon_{\alpha} - \varepsilon). \quad (5)$$

The theorem then implies that $j_z = C_J \int_{-\infty}^{\mu} d\varepsilon$, which is infinite, unless $C_J = 0$.

Although the appearance of infinity for j_z looks troublesome, it is in fact expected by the following reason. Suppose the spectrum of $\hat{H}$ has an upper or lower bound. If ε is outside of these bounds, there is no state to contribute to $\rho_J(\varepsilon)$, and $\rho_J(\varepsilon) = 0$ for such ε . The theorem then implies that $\rho_J(\varepsilon) = 0$ for all ε , i.e., $C_J = 0$. Therefore, a non vanishing C_J is possible only if the spectrum has no upper or lower bounds. An example we will discuss in more detail later is provided by the spectrum of a relativistic Weyl fermion. We will also consider an exception to the statement, when the density of states $\rho(\varepsilon)$ is allowed to possess a non-integrable singularity in ε . The necessity of unbounded spectrum for non vanishing $\rho_J(\varepsilon)$ can also be understood by considering $N(\varepsilon) \equiv \lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr} \left(\Theta(\varepsilon - \hat{H}) \right)$, where $\Theta(x)$ is the step function, which counts the number of states whose energy is less than ε . The similar steps as above shows $\rho_J(\varepsilon) = \frac{\partial}{\partial A_z} N(\varepsilon) = 0$, since the energy spectrum, and hence $N(\varepsilon)$, do not depend on A_z . What saves a non vanishing $\rho_J(\varepsilon)$ for the unbounded spectrum is that $N(\varepsilon)$ is infinite and ill-defined.

The above discussion leads to the following corollary of the theorem, which is an alternative proof of the Bloch's theorem [37, 38]: *The system with a lower bound in spectrum has no persistent current in any thermodynamic ensemble.* The proof relies on the fact that any thermodynamic ensemble is defined by the occupation number of 1-particle states, which depends only on its energy

eigenvalue ε . For example, a grand canonical ensemble of temperature T and chemical potential μ gives the current density,

$$j_z = \frac{1}{V} \sum_{\alpha} \langle \psi_{\alpha} | \hat{J}_z | \psi_{\alpha} \rangle n_F(\varepsilon_{\alpha}) = \int_{-\infty}^{\infty} n_F(\varepsilon) \rho_J(\varepsilon) d\varepsilon, \quad (6)$$

where $n_F(x) = (1 + e^{(x-\mu)/(k_B T)})^{-1}$. A lower bound on the spectrum implies $C_J = 0$, i.e., $\rho_J(\varepsilon) = 0$ for all ε .

When the spectrum has no bounds, and $C_J \neq 0$, a proper way to proceed is to first define the ground state (or vacuum state) of the system, e.g., the state at $T = \mu = 0$ where all 1-particle states of $\varepsilon \leq 0$ are occupied, and the current density j_z is measured with respect to the value of the ground state. This gives, for example, in the grand canonical ensemble, $j_z = \int_{-\infty}^{\infty} (n_F(\varepsilon) - \Theta(-\varepsilon)) \rho_J(\varepsilon) d\varepsilon = C_J \mu$, which is linear in μ , and more interestingly, independent of T . This leads to the corollary: *A persistent current in the grand canonical ensemble, if it is non vanishing, is strictly linear in μ and independent of T .* The theorem and its consequences do not assume any details of the Hamiltonian or external conditions.

The theorem implies a strong robustness of $\rho_J(\varepsilon)$ under any reasonable perturbation. Since $\rho_J(\varepsilon) = C_J$ is independent of ε , the constant C_J is a property of the 1-particle states with arbitrarily large energy ε , and therefore its value is robust under any perturbation that could affect only the states with a finite ε . This is a non-trivial conclusion, since a perturbation in general may affect the states of small ε in significant ways, e.g., the density of states and the current expectation values will be modified substantially for small ε . Yet, we find that $\rho_J(\varepsilon)$ should remain the same for *all* energy ε , since it should be the same for $\varepsilon \rightarrow \infty$. We will establish this robustness more rigorously in the following discussions.

A toy example — To illustrate the theorem and its consequences in a simple example, we consider a 1-dimensional system described by the Hamiltonian

$$\hat{H} = f(\hat{p}_z) + V_0(z) \equiv \hat{H}_0 + V_0(z), \quad (7)$$

where $f(\hat{p}_z)$ is an arbitrary function on the momentum operator $\hat{p}_z = -i\hbar \partial_z$, and $V_0(z)$ is a potential. Let us ignore the potential for a moment, and consider the spectrum of $\hat{H}_0$, which is parametrized by the momentum eigenvalue p_z of the eigenstate $\psi_{p_z}(z) = e^{-ip_z z/\hbar}$, i.e., $\varepsilon(p_z) = f(p_z)$. The density of states in momentum space is $\frac{L}{2\pi\hbar} dp_z$, where L is the size of the system. Then, the density of states in energy is $\rho(\varepsilon) \equiv \frac{1}{L} \text{Tr} \left(\delta(\hat{H}_0 - \varepsilon) \right) = \frac{1}{2\pi\hbar} \int dp_z \delta(f(p_z) - \varepsilon) = \frac{1}{2\pi\hbar} \sum_{\alpha} \frac{1}{|f'(p_{\alpha})|}$, where p_{α} are the solutions of $f(p_{\alpha}) = \varepsilon$. The current operator is $\hat{J}_z = q f'(\hat{p}_z)$, and our current weighted spectral density is $\rho_J(\varepsilon) = \frac{1}{L} \text{Tr} \left(\hat{J}_z \delta(\hat{H}_0 - \varepsilon) \right) = \frac{q}{2\pi\hbar} \sum_{\alpha} \frac{f'(p_{\alpha})}{|f'(p_{\alpha})|}$. Although $\rho(\varepsilon)$ depends on both ε and the shape of $f(p_z)$,

it is easy to see that $\rho_J(\varepsilon)$, which counts the number of signed crossings of the curve $f(p_z)$ at energy ε , is constant in ε . It coincides with a well-known topological index of 1-dimensional Hamiltonian.

When the potential is present, p_z is no longer a good quantum number. The problem is still solvable exactly if $\hat{H}_0 = \hat{p}_z$. The energy eigenstates are $\psi_\varepsilon(z) = e^{\frac{i}{\hbar}(p_z z - \int_{z_0}^z V_0(z') dz')}$, with the spectrum $\varepsilon(p_z) = p_z$. The current operator is an identity $\hat{J}_z = q \frac{\partial \hat{H}}{\partial p_z} = q$, and we find $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar}$, which is constant, independent of the potential $V_0(z)$. For a general $f(p_z)$ without bounds, i.e., $|f(p_z)| \rightarrow \infty$ as $p_z \rightarrow \pm\infty$, we can find the eigenstates of large energy in the eikonal approximation, $\psi_{p_z}(z) \sim e^{\frac{i}{\hbar}(p_z z + \phi(z))}$, where $\phi(z) = -\frac{1}{f'(p_z)} \int_{z_0}^z V_0(z') dz'$, and the spectrum is $\varepsilon(p_z) = f(p_z)$ which defines p_z given ε . The small parameter for the approximation is $V_0(z)/f(p_z) \rightarrow 0$ as $p_z \rightarrow \infty$, and the approximation becomes exact in $\varepsilon \rightarrow \infty$ limit. The current operator is $\hat{J} = q f'(\hat{p}_z)$, with the expectation value $\langle \psi_{p_z} | \hat{J}_z | \psi_{p_z} \rangle = q f'(p_z)$, up to the same corrections of $V_0(z)/f(p_z) \rightarrow 0$ in the eikonal limit. We then find that $\rho_J(\varepsilon)$ is given by the same expression, $\frac{q}{2\pi\hbar} \sum_\alpha \frac{f'(p_\alpha)}{|f'(p_\alpha)|}$, as $\varepsilon \rightarrow \infty$, and hence for all ε according to our theorem, irrespective of the potential $V_0(z)$.

As a corollary, our result proves *the absence of Anderson localization in 1-dimensional systems if the kinetic operator dispersion relation behaves at large momentum as $f(p_z) \sim p_z^{2n+1}$ where $n = 0, 1, 2, \dots$* , since we have $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar} \neq 0$, which implies the absence of a gap in the spectrum of extended states with random potentials. In particular, the $n = 0$ case corresponds to a relativistic 1D Weyl node.

Discussion on a counter example — In this short digression, we discuss the following counter example in 1-dimensions: $\hat{H} = \varepsilon_0 \tanh(\hat{p}_z)$, where ε_0 is a constant. For the interval $|\varepsilon| < \varepsilon_0$, the spectrum is a monotonically increasing function on p_z , and we have $\rho_J(\varepsilon) = \frac{q}{2\pi\hbar}$, while outside of the interval, no states exist and $\rho_J(\varepsilon) = 0$. What causes the discontinuity in $\rho_J(\varepsilon)$ across $\varepsilon = \pm\varepsilon_0$ is the infinite number of states accumulated around the energy $\pm\varepsilon_0$. The density of states is $\rho(\varepsilon) = \frac{1}{2\pi\hbar \tanh'(\tanh^{-1}(\varepsilon/\varepsilon_0))} = \frac{1}{2\pi\hbar(1-(\varepsilon/\varepsilon_0)^2)}$, which is non integrable at $\pm\varepsilon_0$. The theorem applies only when the density of states is a finite function in the energy interval of interest.

In the following, we discuss our main application of the Theorem I; we will compute the constant $C_J = \rho_J(\varepsilon)$ for a Weyl node of arbitrary topology in the presence of a magnetic field and impurities.

A clean Weyl node in a magnetic field — Let us first ignore impurities for a while, and compute C_J for a clean Weyl node in the presence of a magnetic field. Our main technique is based on the following observation: since

$\rho_J(\varepsilon) = C_J$ is independent of ε , we can express C_J by

$$C_J = \frac{1}{\sqrt{\pi}M} \int_{-\infty}^{\infty} d\varepsilon \rho_J(\varepsilon) e^{-\frac{\varepsilon^2}{M^2}} = \frac{1}{\sqrt{\pi}M} \frac{1}{V} \text{Tr} \left(\hat{J}_z e^{-\frac{\hat{H}^2}{M^2}} \right), \quad (8)$$

for any $M > 0$. The $M \rightarrow 0$ limit reproduces the definition of $\rho_J(\varepsilon)$, while $M \rightarrow \infty$ limit is similar to the common strategy in the proof of the index theorems.

Before considering the most general form of a Weyl node, let us consider a concrete example of the simplest (relativistic) Weyl Hamiltonian in the presence of a constant magnetic field $\mathbf{B} = B\hat{z}$, where $\hat{z}$ is the unit vector in the z -direction,

$$\hat{H} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} = \sigma_x \hat{p}_x + \sigma_y (\hat{p}_y - qBx) + \sigma_z \hat{p}_z, \quad (9)$$

where σ_i are the Pauli matrices, and we work in the Landau gauge $\mathbf{A} = (0, Bx, 0)$. In this case we are able to compute the right-hand side of Eq.(8) exactly for any M to find that it is indeed M -independent, which demonstrates the validity of our Theorem I. We have $\hat{H}^2 = \hat{H}_0 + \hat{H}_I$, where $\hat{H}_0 = (\hat{p}_x)^2 + (\hat{p}_y - qBx)^2 + (\hat{p}_z)^2$ and $\hat{H}_I = -\hbar q B \sigma_z$. Since $\hat{J}_z = q \sigma_z$, the Eq.(8) becomes,

$$C_J = \frac{q}{\sqrt{\pi}M} \frac{1}{V} \text{Tr} \left(\sigma_z e^{-\frac{1}{M^2}(\hat{H}_0 - \hbar q B \sigma_z)} \right). \quad (10)$$

To evaluate this first note that $\hbar q B \sigma_z$ commutes with $\hat{H}_0$ and that $e^{\frac{q\hbar B \sigma_z}{M^2}} = \cosh\left(\frac{q\hbar B}{M^2}\right) + \sigma_z \sinh\left(\frac{q\hbar B}{M^2}\right)$. Taking the spin trace we obtain

$$C_J = \frac{q}{\sqrt{\pi}M} \frac{2 \sinh\left(\frac{q\hbar B}{M^2}\right)}{V} \text{Tr} \left(e^{-\frac{\hat{H}_0}{M^2}} \right). \quad (11)$$

To evaluate the remaining trace in position-momentum space, we note that $\hat{H}_0$ is the classical Landau-level problem for a relativistic scalar particle. The eigenfunctions are given by $|\psi_{n,p_y,p_z}\rangle = e^{\frac{i}{\hbar}(p_y y + p_z z)} |\phi_{n,p_y}\rangle$, where $|\phi_{n,p_y}\rangle$ are the 2D Landau wave functions, and the eigenvalues are $E_{n,p_z} = 2q\hbar B(n + 1/2) + p_z^2$, with the degeneracy per unit transverse area $g_\perp = \frac{qB}{2\pi\hbar}$. From these we obtain

$$\text{Tr} \left(e^{-\frac{\hat{H}_0}{M^2}} \right) = \frac{qVB}{(2\pi\hbar)^2} \int_{-\infty}^{+\infty} dp_z e^{-\frac{p_z^2}{M^2}} \sum_{n=0}^{\infty} e^{-\frac{2q\hbar B}{M^2}(n + \frac{1}{2})}, \quad (12)$$

which is evaluated to be $\frac{qVB\sqrt{\pi}M}{(2\pi\hbar)^2} \frac{1}{2 \sinh(q\hbar B/M^2)}$. Using this in Eq.(11), we indeed find the M -independent result,

$$C_J = q^2 B / (2\pi\hbar)^2. \quad (13)$$

We now consider the most general form of a Weyl node in the presence of a constant magnetic field $\mathbf{B} = B\hat{z}$,

$$\hat{H} = \mathbf{F}(\hat{\boldsymbol{\pi}}) \cdot \boldsymbol{\sigma}, \quad (14)$$

where $\mathbf{F}(\hat{\boldsymbol{\pi}})$ is an arbitrary vector valued function in $\hat{\boldsymbol{\pi}}$. We will use Eq.(8) in $M \rightarrow \infty$ limit to compute C_J , since

it is not possible to solve the eigenvalue problem exactly. Using $[\hat{\pi}_x, \hat{\pi}_y] = i\hbar qB$ and $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$, we have $\hat{H}^2 = \mathbf{F}^2(\hat{\boldsymbol{\pi}}) - \hbar qB \left(\frac{\partial \mathbf{F}}{\partial \pi_x} \times \frac{\partial \mathbf{F}}{\partial \pi_y} \right) \cdot \boldsymbol{\sigma} \equiv \hat{H}_0 + \hat{H}_I$. In obtaining this we neglected the terms arising from position-momentum commutators, since those terms possess less powers of momentum than the above leading term. As we will see, a finite contribution in $M \rightarrow \infty$ limit comes only from $\mathbf{p} \rightarrow \infty$ region, and the terms with less powers of momentum become irrelevant compared to the leading term. Following similar steps in the proofs of index theorems, we invoke the heat kernel expansion

$$e^{-\frac{\hat{H}^2}{M^2}} = e^{-\frac{\hat{H}_0}{M^2}} \left(1 - \frac{1}{M^2} \int_0^1 dt e^{t\frac{\hat{H}_0}{M^2}} \hat{H}_I e^{-t\frac{\hat{H}_0}{M^2}} \right) + \mathcal{O}\left(\frac{1}{M^4}\right). \quad (15)$$

Since $\hat{J}_z = q \left(\frac{\partial \mathbf{F}}{\partial p_z} \right) \cdot \boldsymbol{\sigma}$, the first non vanishing trace in spin space appears with the second term in the expansion, and we find

$$C_J = \frac{2\hbar q^2 B}{\sqrt{\pi} M^3} \frac{1}{V} \text{Tr} \left(e^{-\frac{\mathbf{F}^2(\hat{\boldsymbol{\pi}})}{M^2}} \left(\frac{\partial \mathbf{F}}{\partial \hat{\pi}_x} \times \frac{\partial \mathbf{F}}{\partial \hat{\pi}_y} \right) \cdot \frac{\partial \mathbf{F}}{\partial \hat{p}_z} \right), \quad (16)$$

up to sub-leading corrections from position-momentum commutators. We then evaluate the remaining trace in momentum basis, and in large M limit the trace is dominated by large momentum $\mathbf{p}$, for which $\boldsymbol{\pi}$ can be replaced by $\mathbf{p}$ up to sub-leading corrections, and we arrive at

$$C_J = \frac{2\hbar q^2 B}{\sqrt{\pi} M^3} \int \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} e^{-\frac{\mathbf{F}^2(\mathbf{p})}{M^2}} \left(\frac{\partial \mathbf{F}}{\partial p_x} \times \frac{\partial \mathbf{F}}{\partial p_y} \right) \cdot \frac{\partial \mathbf{F}}{\partial p_z}. \quad (17)$$

Noting that $\left(\frac{\partial \mathbf{F}}{\partial p_x} \times \frac{\partial \mathbf{F}}{\partial p_y} \right) \cdot \frac{\partial \mathbf{F}}{\partial p_z}$ is the signed Jacobian for the map $\mathbf{p} \rightarrow \mathbf{F}(\mathbf{p})$, which allows us to perform a change of variables $\mathbf{p} \rightarrow \mathbf{F}(\mathbf{p})$ in the above integration, we find

$$C_J = \frac{2\hbar q^2 B}{\sqrt{\pi} M^3} N_{\mathbf{F}} \int \frac{d^3 \mathbf{F}}{(2\pi\hbar)^3} e^{-\mathbf{F}^2/M^2} = \frac{q^2 B}{(2\pi\hbar)^2} N_{\mathbf{F}}, \quad (18)$$

where $N_{\mathbf{F}} \in \pi_2(S^2) = \mathbf{Z}$ is the signed winding number of the map $\mathbf{p} \rightarrow \mathbf{F}(\mathbf{p})$ at asymptotic infinity, $S_\infty^2 \rightarrow S_\infty^2$, which defines the topology of the Weyl node with the Hamiltonian $\hat{H} = \mathbf{F}(\hat{\mathbf{p}}) \cdot \boldsymbol{\sigma}$ [39]. The integer $N_{\mathbf{F}}$ counts the flux of the Berry's curvature on a Fermi surface around the Weyl node, and also appears in the chiral anomaly relation near the Weyl node. $N_{\mathbf{F}}$ is essentially the number of times the $\mathbf{F}$ -space at infinity is covered in the map $\mathbf{p} \rightarrow \mathbf{F}(\mathbf{p})$. As an example, $N_{\mathbf{F}} = N$ for $F^\pm = (p^\pm)^N$ and $F_z = p_z$, where $F^\pm = F_x \pm iF_y$ and $p^\pm = p_x \pm ip_y$. Only the asymptotic region of S_∞^2 at infinity in both $\mathbf{p}$ and $\mathbf{F}$ spaces matters, since the surviving contribution in $M \rightarrow \infty$ limit comes only from this region as seen in the $\mathbf{F}$ integration in Eq.(18).

We can interpret the above result in the following interesting way. In the Landau gauge the eigenstates of $\hat{H}$ can be written as $\psi(\mathbf{x}) = \psi_{p_z}^{(n)}(x - p_y/(qB)) e^{\frac{i}{\hbar}(p_y y + p_z z)}$, with good quantum numbers (p_y, p_z) . The eigenvalue

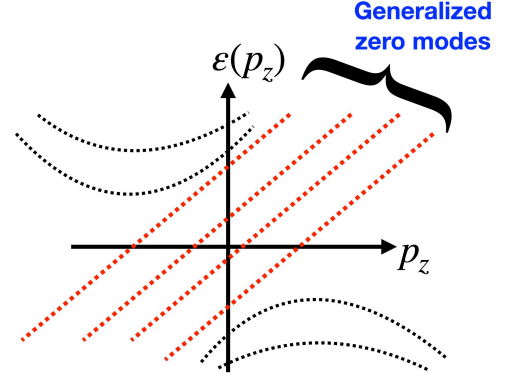


FIG. 1. A schematic example of spectral curves, $\varepsilon_{(n)}(p_z)$.

equation takes the form $\hat{H}_{p_z} \psi_{p_z}^{(n)}(x) = \varepsilon_{(n)}(p_z) \psi_{p_z}^{(n)}(x)$, where $\hat{H}_{p_z} = \mathbf{F}(\pi_x \rightarrow \hat{p}_x, \pi_y \rightarrow -qBx, \pi_z \rightarrow p_z) \cdot \boldsymbol{\sigma}$ is a Hamiltonian in the 1-dimensional space of x , parametrized by p_z , and $\hat{p}_x = -i\hbar \partial_x$. The non negative integer n is the discrete label for the spectral curves, $\varepsilon_{(n)}(p_z)$, as a function of the continuous parameter p_z , which may well be called the generalized Landau levels. The energy spectrum do not depend on p_y , and we have the usual density of states per unit transverse area, $g_\perp = \frac{qB}{2\pi\hbar}$. A typical shape of the spectral curves is depicted in FIG. 1. Using the Feynman-Hellmann theorem, the current expectation value is $\langle \psi_{p_z}^{(n)} | \hat{J}_z | \psi_{p_z}^{(n)} \rangle = q \langle \psi_{p_z}^{(n)} | \frac{\partial \hat{H}_{p_z}}{\partial p_z} | \psi_{p_z}^{(n)} \rangle = q \frac{\partial \varepsilon_{(n)}(p_z)}{\partial p_z}$, which implies that for each n , the contribution to $\rho_J(\varepsilon)$ is equal to that from a 1-dimensional problem with a Hamiltonian $\hat{H}_n \equiv \varepsilon_{(n)}(\hat{p}_z)$, i.e., $\frac{q}{2\pi\hbar} \sum_\alpha \frac{\varepsilon'_{(n)}(p_\alpha)}{|\varepsilon'_{(n)}(p_\alpha)|}$, that we discussed as a toy example.

A non vanishing contribution comes only from the curves with no lower or upper bounds in energy, which we may call the generalized zero modes. We then conclude that $\rho_J(\varepsilon) = g_\perp \frac{q}{2\pi\hbar} N_0 = \frac{q^2 B}{(2\pi\hbar)^2} N_0$, where N_0 is the signed total number of zero modes of the Hamiltonian $\hat{H}_{p_z}$, which is a topological property of $\hat{H}_{p_z}$. Our result for C_J then proves the relation $N_0 = N_{\mathbf{F}}$.

For a special case of

$$\hat{H} = \mathbf{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) \cdot \boldsymbol{\sigma}_\perp + \hat{p}_z \sigma_z, \quad (19)$$

where $(\hat{\boldsymbol{\pi}}_\perp, \boldsymbol{\sigma}_\perp)$ have only the transverse components in (x, y) directions, we have $\hat{H}^2 = \hat{D}_\perp^2 + \hat{p}_z^2$, where $\hat{D}_\perp \equiv \mathbf{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) \cdot \boldsymbol{\sigma}_\perp$ is a generalized Dirac operator in 2-dimensions. Our expression for C_J then factorizes as

$$C_J = \frac{1}{\sqrt{\pi} M} \frac{1}{V} \text{Tr} \left(\hat{J}_z e^{-\hat{H}^2/M^2} \right) = \frac{q}{2\pi\hbar} \frac{1}{V_\perp} \text{Tr} \left(\sigma_z e^{-\hat{D}_\perp^2/M^2} \right), \quad (20)$$

where we performed the trace over z -dimension, and the last trace is defined only in the transverse 2-dimensions. Noting that $\{\hat{D}_\perp, \sigma_z\}_+ = 0$, the trace coincides with the Atiyah-Singer index of the Dirac operator $\hat{D}_\perp$. Our result then gives a generalized version of the Atiyah-Singer

index theorem,

$$\text{Index}(\hat{D}_\perp) = \frac{qB}{2\pi\hbar} N_{\mathbf{F}} V_\perp = \frac{q}{2\pi\hbar} N_{\mathbf{F}} \int_{R_\perp^2} F_2, \quad (21)$$

where $N_{\mathbf{F}} \in \pi_1(S^1) = \mathbf{Z}$ is the winding number of the map $\mathbf{p}_\perp \rightarrow \mathbf{F}_\perp(\mathbf{p}_\perp)$ at asymptotic infinity, i.e., $S_\infty^1 \rightarrow S_\infty^1$, which defines the topology of the Dirac operator $\hat{D}_\perp$.

Application to the chiral magnetic effect — Our result implies that in the system described by a Hamiltonian $\hat{H} = \mathbf{F}(\hat{\boldsymbol{\pi}}) \cdot \boldsymbol{\sigma}$, there is a non vanishing current density along the direction of the magnetic field in grand canonical ensemble,

$$j_z = C_J \mu = \frac{q^2 B}{(2\pi\hbar)^2} N_{\mathbf{F}} \mu, \quad (22)$$

which is the chiral magnetic effect, that is generalized to arbitrary topological number $N_{\mathbf{F}}$. As shown in Ref.[39], the same topological number also appears in the chiral anomaly relation, $\partial_\mu j^\mu = \frac{q^3}{(2\pi\hbar)^2} N_{\mathbf{F}} (\mathbf{E} \cdot \mathbf{B})$, where j^μ is the charge current density in relativistic notation. This affirms the connection between the chiral magnetic effect and chiral anomaly in the most general case of topology.

In real Weyl semi-metals, as in all condensed matter systems, the global spectrum has a lower bound, and our Theorem I dictates that there is no net persistent current in any thermodynamic ensembles. This is consistent with the Nielsen-Ninomiya theorem that the summation of $N_{\mathbf{F}}$ for all Weyl nodes vanishes [40]. Instead, one may consider a quasi equilibrium state, where each Weyl node, labeled by α , with a non vanishing $N_{\mathbf{F}_\alpha}$, has its own effective chemical potential μ_α , and $j_z = \frac{q^2 B}{(2\pi\hbar)^2} \sum_\alpha N_{\mathbf{F}_\alpha} \mu_\alpha$ may not vanish. These quasi equilibrium states would make sense, only if inter-node transitions are slow enough.

A Weyl node with impurities and a magnetic field — In this section we explicitly examine whether C_J is robust under perturbations, by considering several different perturbations and showing that C_J is indeed independent of these perturbations. Although these examples do not exhaust all possible perturbations, we expect that C_J is independent of many other reasonable perturbations. We summarize here the main elements and ideas of our proofs (see the Supplemental Material [41] for full details).

We examined the Hamiltonians with the general form, $\hat{H}_\lambda = \boldsymbol{\sigma} \cdot \mathbf{F}(\hat{\boldsymbol{\pi}}) + \lambda \hat{V}$, where $\hat{V}$ is the perturbation and λ is a parameter which controls the magnitude of the perturbation. The perturbations we consider have a general form $\hat{V}(\mathbf{x}) = V_0(\mathbf{x}) + \mathbf{V}(\mathbf{x}) \cdot \boldsymbol{\sigma}$ where $\mathbf{V}$ is a vector valued function representing a spin-dependent potential, and V_0 is a spin-independent potential. Recall that in the Landau gauge we have $\hat{\boldsymbol{\pi}} = (\hat{p}_x, \hat{p}_y - qBx, \hat{p}_z)$. The following

cases have been examined;

- (1) $\hat{H}_{1\lambda} = \boldsymbol{\sigma}_\perp \cdot \mathbf{F}_\perp(\hat{\boldsymbol{\pi}}_\perp) + \sigma_z \hat{p}_z + \lambda \hat{V}(x, y)$
- (2) $\hat{H}_{2\lambda} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \lambda \hat{V}(x, y, z)$
- (3) $\hat{H}_{3\lambda} = [\sigma_x \hat{\pi}_x^{n_x} + \sigma_y \hat{\pi}_y^{n_y} + \sigma_z \hat{\pi}_z^{n_z}] + \lambda V_0(x, y, z)$

where the $n_i \geq 1$ ($i = x, y, z$) in Case 3 are positive integers. Note that in Case 1, $\hat{V}(x, y)$ is independent of z , and the momentum along the z direction, p_z , is a good quantum number, which makes the proof particularly simple. This proof is presented in Section 1 of the Supplemental Material [41].

Our main technique for the proofs is again the $M \rightarrow \infty$ limit and the heat kernel expansion. As an example, we present here an outline of the main analysis for Case 2. We first write $\hat{H}_{2\lambda}^2 = \hat{\mathbf{p}}^2 + \hat{H}_c$ where $\hat{H}_c = \hat{H}_{2\lambda}^2 - \hat{\mathbf{p}}^2$, and do the heat kernel expansion for C_J treating $\hat{H}_c/M^2$ as a perturbation. Taking the trace in momentum basis, and using the fact that in the large M limit we can ignore position-momentum commutators, the exponential term in the expression for C_J can be Taylor expanded as $e^{-\frac{\mathbf{p}^2}{M^2}} e^{-\frac{\hat{H}_c}{M^2}} = e^{-\frac{\mathbf{p}^2}{M^2}} \left(1 - \frac{\hat{H}_c}{M^2} + \frac{\hat{H}_c^2}{2!M^4} + \dots\right)$. From this we obtain $C_J = \sum_{m=0}^\infty C_{Jm}$ where C_{Jm} is given by

$$\frac{(-1)^m q}{M^{(2m+1)} m! \sqrt{\pi}} \frac{1}{V} \int \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \int d^3 \mathbf{x} e^{-\frac{\mathbf{p}^2}{M^2}} \text{Tr}_{\text{spin}} \left(\sigma_z \hat{H}_c^m \right). \quad (23)$$

Since we are interested in the dependence of C_J on $\hat{V}$, we only consider the terms that include $\hat{V}$. The terms with odd powers of $\mathbf{p}$ drop out upon integration, and many terms vanish by spin trace. After this it is found that in the limit $M \rightarrow \infty$ the only terms that remain are C_{J1} and C_{J2} . The $\hat{V}$ dependent parts of these terms are non-zero, however they cancel in the sum $C_{J1} + C_{J2}$ quite non-trivially, proving that C_J is indeed independent of $\hat{V}$. Further details are given in Section 2 of the Supplemental Material [41]. Similar steps are used to prove that C_J is independent of V_0 for the Case 3 (see Section 3 of the Supplemental Material [41] for details).

The independence of C_J on the potential is a non-trivial fact, if viewed naively in quantum mechanics. The constant C_J as expressed in Eq.(8) is a complicated expression when expanded as a power series of $\hat{V}$. To obtain the claimed result requires precise cancellation of all terms containing $\hat{V}$. To demonstrate explicitly that these cancellations do indeed occur, we considered the simple case where $\hat{H}_{1a\lambda} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\pi}} + \lambda \hat{V}(x, y)$, and showed that the terms in C_J up to second order of $\hat{V}$ do cancel out exactly for any value of M . This non-trivial calculation is provided in the Section 4 of the Supplemental Material [41].

Implication for the spectral density — We discuss one immediate consequence of our results on the spectral density of a Weyl node in a magnetic field and impurities. We first state the following spectral theorem for the Hamiltonians with $\left| \frac{\partial \mathbf{F}(\hat{\boldsymbol{\pi}})}{\partial \hat{p}_z} \right| \leq C$ for all normalized states, where

C is a non-zero positive constant,

Theorem II : $\rho(\varepsilon) \geq |C_J|/(qC)$, for all ε . (24)

An example is $\hat{H}_0 = \mathbf{F}_\perp(\hat{\pi}_\perp) \cdot \boldsymbol{\sigma}_\perp + \hat{p}_z \sigma_z$, where $|\frac{\partial \mathbf{F}(\hat{\pi})}{\partial \hat{p}_z}| = 1$. The proof is based on $|\langle \hat{J}_z \rangle| = q|\langle \frac{\partial \mathbf{F}}{\partial \hat{p}_z} \cdot \boldsymbol{\sigma} \rangle| \leq q|\langle \frac{\partial \mathbf{F}(\hat{\pi})}{\partial \hat{p}_z} \rangle| \leq qC$, and we have $|\rho_J(\varepsilon)| = \frac{1}{V} |\text{Tr}(\hat{J}_z \delta(\hat{H} - \varepsilon))| \leq qC \frac{1}{V} \text{Tr}(\delta(\hat{H} - \varepsilon)) = qC\rho(\varepsilon)$. The theorem implies that the energy spectrum has no gap if $\rho_J(\varepsilon) = C_J$ is non vanishing, which is the case for a Weyl node in a magnetic field and impurities.

Discussion — Our result implies the absence of a bulk spectral gap for a single Weyl node, as the impurity strength varies, when an external magnetic field is present. Whether this behavior is smooth in $B \rightarrow 0$ limit, or there is a discontinuity at $B = 0$ is an interesting question to study. In real Weyl semi-metals, different Weyl nodes with opposite topological numbers are separated in momentum space, and the inter-node mixings can occur for sufficiently strong short-ranged potentials, which could invalidate our conclusion based on a single Weyl node. The robustness of C_J for a single Weyl node shows that the chiral magnetic effect is not affected by any random impurities, as long as inter-node transitions are suppressed. This should be ultimately related to the robustness of chiral anomaly [42, 43] that is in effect near each Weyl node.

The same constant C_J is also responsible for the chiral energy transfer along a magnetic field, $T_{0z} = \int_{-\infty}^{\infty} d\varepsilon (f(\varepsilon) - \Theta(-\varepsilon)) \frac{1}{V} \text{Tr}(\hat{v}_z \hat{H} \delta(\hat{H} - \varepsilon)) = \frac{1}{q} \int d\varepsilon (f(\varepsilon) - \Theta(-\varepsilon)) \varepsilon \frac{1}{V} \text{Tr}(\hat{J}_z \delta(\hat{H} - \varepsilon)) = \frac{C_J}{q} \int d\varepsilon (f(\varepsilon) - \Theta(-\varepsilon)) \varepsilon = \frac{C_J}{q} \left(\frac{\mu^2}{2} + \frac{\pi^2 T^2}{6} \right) = \frac{qBN_F}{(2\pi\hbar)^2} \left(\frac{\mu^2}{2} + \frac{\pi^2 T^2}{6} \right)$, where we used $\hat{J}_z = q\hat{v}_z$ with the velocity operator $\hat{v}_z = \frac{\partial \hat{H}}{\partial p_z}$. The chiral energy transfer is equivalent to the chiral vortical effect in time-reversal invariant systems [44].

Our Theorem I is general enough to be applicable to interacting multi-particle systems and quantum field theories of charged particles, as it only relies upon gauge invariance. Moreover, the idea can easily be generalized to produce many similar versions. For example, for any gauge invariant operator $\hat{\mathcal{O}}$ which is independent of $\hat{\pi}_z$ and commutes with $\hat{J}_z$, the weighted spectral density $\rho_{\mathcal{O}}(\varepsilon) = \lim_{V \rightarrow \infty} \frac{1}{V} \text{Tr}(\hat{\mathcal{O}} \hat{J}_z \delta(\hat{H} - \varepsilon))$ is independent of ε , and hence robust under perturbations. It is an interesting speculation whether this line of thinking might lead to a useful definition of topology for interacting multi-particle systems, **since these objects can be defined quite generally independent of the presence of interactions among multi-particles in the system.**

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* dsolom2@uic.edu, hyee@uic.edu

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