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Mirror-symmetry protected higher-order topological zero-frequency boundary and corner modes in Maxwell lattices

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Maxwell lattices, where the number of degrees of freedom equals the number of constraints, are known to host topologically-protected zero-frequency modes and states of self stress, characterized by a topological index called topological polarization. In this letter, we show that in addition to these known topological modes, with the help of a mirror symmetry, the inherent chiral symmetry of Maxwell lattices creates another topological index, the mirror-graded winding number (MGWN). This MGWN is a higher order topological index, which gives rise to topological zero modes and states of self stress at mirror-invariant domain walls and corners between two systems with different MGWNs. We further show that two systems with same topological polarization can have different MGWNs, indicating that these two topological indices are fundamentally distinct.

Introduction.–Bulk-boundary correspondence is a defining feature of topological states where nontrivial topology of the bulk gives rise to modes localized at the boundary [1, 2]. Early research on topological band theory focused on d-dimensional topological systems with localized states at $(d-1)$-dimensional boundaries (e.g., quantum Hall effect [3], quantum anomalous Hall effect [4], quantum spin Hall effect [5, 6]); this type of topology is now called first-order topology. A new kind of topological states, called higher-order topological states (HOTS), has been proposed in the last five years [7–9]. Here, instead of having $(d-1)$-dimensional topologically protected boundary modes, the d-dimensional n-th order topological system has $(d-n)$-dimensional $(n>1)$ boundary modes. The boundary modes corresponding to $n=d$ and $n=d-1$ are generally called corner and hinge modes, respectively. These higher order states are generally protected by crystalline symmetries such as mirror [10], inversion [11], rotation [12, 13], product of time reversal (TRS) and rotation [9], etc (see [14] for an exhaustive literature survey). Along with realizations in electronic systems, crystalline symmetry protected HOTS have been implemented in mechanical/elastic systems too, offering a class of materials in which elastic energy can be selectively confined to low-dimensional regions [15–20].

One key challenge in the study of HOTS lies in the stability of topological corner modes. For example, in contrast to the quantum Hall effect, where the topological edge modes remain stable for *any* boundary conditions, for a 2D HOTS, unless certain special ingredient is introduced (e.g., a chiral symmetry), the frequency of the topological corner modes is in general not pinned to a particular value. Thus, depending on the microscopic details, such as boundary conditions and disorder near the corners, these topological modes and can disappear into bulk bands [21, 22]. To overcome this challenge, recently, a generalized chiral symmetry was introduced to realize corner modes in an breathing kagome lattice acoustic metamaterial [20], while there are still some open discussions about the topological origin of these modes [22, 23]. Another attempt [24] showed existence of corner modes pinned at zero frequency in an over-constrained system made of rigid quadrilaterals connected by free hinges; however, this can be understood within the framework of boundary obstructed topological phases [25].

In this Letter, we provide a different approach towards HOTS using Maxwell lattices (i.e., lattices with equal numbers of degrees of freedom (DOFs) n_d and constraints n_c [26, 27]), and show that the intrinsic chiral symmetry protected by this counting extends robustness to topological corner modes in this lattices, without requiring any detailed matching at boundaries. As shown by Kane and Lubensky [28], Maxwell systems can be mapped to a superconducting Bogoliubov de Gennes (BdG) Hamiltonian, which naturally has a chiral symmetry. With the BdG Hamiltonian, a first-order topological index, the topological polarization, can be introduced [28], resulting in topologically protected edge modes at zero frequency. We find that in addition to this first-order topological index, a nontrivial higher-order topological index (the MGWN [29–31]) can be introduced to a new class of Maxwell lattices, controlling zero-frequency topological domain-wall/corner modes, with robustness originating from the intrinsic chiral symmetry of the locking of degrees of freedom and constraints in Maxwell lattices.

Kane-Lubensky topological index of Maxwell lattices.– Linear mechanics of lattices made of point masses connected by springs is characterized by the compatibility matrix $\mathbf{C}$ which relates extensions of springs $e_i = C_{ij}u_j$ to the displacements u_i of the point masses. Furthermore, $f_i = C_{ij}^T t_j$ relates the forces f_i on the point masses to the tensions t_i in the springs. In Fourier space, the matrix $\mathbf{C}(\mathbf{q})$ has the size $n_c \times n_d$. The normal mode frequencies of these lattices $\omega^2(\mathbf{q})$ are the eigenvalues of the dynamical matrix $\mathbf{D}(\mathbf{q}) = \mathbf{C}^\dagger(\mathbf{q})\mathbf{C}(\mathbf{q})$ Kane and Lubensky [28] defined a 'square root' of the dynamical matrix,

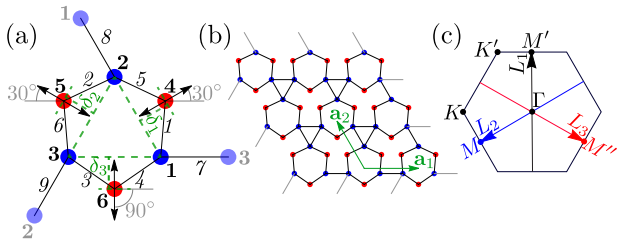


FIG. 1. The mirror symmetric Maxwell lattice. (a) The unit cell consists of three blue and three red point masses enumerated by bold numbers. The blue points can move in both x and y direction whereas the red points can only move along the direction of the corresponding double-directional black arrow. Parameter δ_i $(i = 1, \ldots, 3)$ is the perpendicular distance of point $i+3$ from the line joining points i and $i+1$. The numbers in italics enumerate the springs. The partially transparent blue points are in adjacent unit cells. (b) A 3×3 lattice. The springs shown in grey at the edges are required for periodic boundary condition. The green arrows show the the lattice vectors. (c) First Brillouin zone with the high symmetry points. The non-contractible loop L_i is invariant under mirror reflection when $\delta_i = \delta_{i+1}$.

which in reciprocal space takes the following form:

$$\mathcal{H}(\mathbf{q}) = \begin{pmatrix} \mathbf{0} & \mathbf{C}^\dagger(\mathbf{q}) \\ \mathbf{C}(\mathbf{q}) & \mathbf{0} \end{pmatrix}. \tag{1}$$

For every nonzero eigenvalue $\omega^2(\mathbf{q})$ of $\mathbf{D}(\mathbf{q})$, $\mathcal{H}(\mathbf{q})$ has two eigenvalues $\pm\omega(\mathbf{q})$. The zero modes of $\mathcal{H}(\mathbf{q})$ include nullspace of $\mathbf{C}(\mathbf{q})$ (zero modes – ZMs) and nullspace of $\mathbf{C}^\dagger(\mathbf{q})$ (states of self stress – SSSs), whereas the zero modes of $D(\mathbf{q})$ include the ZMs. Maxwell Calladine theorem [26, 32] dictates that the number of ZMs (n_0) and number of SSSs (n_s) are equal ($n_0 = n_s$) for a Maxwell lattice. The matrix $\mathcal{H}(\mathbf{q})$ has the property that $S\mathcal{H}(\mathbf{q})S = -\mathcal{H}(\mathbf{q})$, where $S = \mathrm{Diag}\{\mathbb{1}, -\mathbb{1}\}$. This property is known as the chiral (or sublattice) (anti)symmetry in the literature. Also, it is easy to check that $\mathcal{H}(\mathbf{q})$ has TRS: $\mathcal{H}(\mathbf{q}) = \mathcal{H}^*(-\mathbf{q})$, where * is complex conjugation. These two symmetries put the matrix $\mathcal{H}(\mathbf{q})$ in BDI class of Altland Zirnbauer classification [33–36]. Along a closed loop l in the Brillouin zone where the spectrum of the matrix is gapped at zero, a topological invariant n_l can be defined: $n_l = \frac{1}{2\pi i}\oint_l d\mathbf{q} \cdot \nabla_{\mathbf{q}} \log\det \mathbf{C}^\dagger(\mathbf{q})$, which controls the number of topological ZMs at an open edge or domain walls.

Mirror-graded winding number.–Interestingly, in mirror symmetric Maxwell lattices, along the mirror invariant lines in the Brillouin zone, the mirror reflection operator $\mathbf{M}(\mathbf{q})$ commutes with the matrix $\mathcal{H}(\mathbf{q})$. Consequently, $\mathbf{M}(\mathbf{q})$ and $\mathcal{H}(\mathbf{q})$ can be simultaneously diagonalized. Since, $\mathbf{M}(\mathbf{q})$ only takes eigenvalues ± 1, using the eigenvectors of $\mathbf{M}(\mathbf{q})$ the matrices $\mathbf{C}(\mathbf{q})$ and $\mathcal{H}(\mathbf{q})$ can be block-diagonalized into odd $(-)$ and even $(+)$ sectors (Supplemental Material (SM) [37] Sec. SM.2-3):

$$\mathbf{C}(\mathbf{q}) = \begin{pmatrix} \mathbf{C}_-(\mathbf{q}) & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_+(\mathbf{q}) \end{pmatrix}, \tag{2a}$$

$$\mathcal{H}(\mathbf{q}) = \begin{pmatrix} \mathbf{0} & \mathbf{C}_-^\dagger(\mathbf{q}) & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_-(\mathbf{q}) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}_+^\dagger(\mathbf{q}) \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_+(\mathbf{q}) & \mathbf{0} \end{pmatrix}. \tag{2b}$$

Now, using $\mathbf{C}_\pm(\mathbf{q})$ we can define a topological invariant in each sector, the MGWNs:

$$\nu_\pm = \frac{1}{2\pi i}\oint_{\mathbf{q}\to\mathbf{q}+\mathbf{G}_m} d\mathbf{q} \cdot \nabla_{\mathbf{q}} \log\det \mathbf{C}_\pm^\dagger(\mathbf{q}), \tag{3}$$

where $\mathbf{G}_m$ is the smallest reciprocal lattice vector along the mirror plane [29–31]. Note that $\nu_+ + \nu_- = n_l$, since in this basis $\det \mathbf{C}(\mathbf{q}) = \det \mathbf{C}_+(\mathbf{q}) \det \mathbf{C}_-(\mathbf{q})$. In other words, the mirror symmetry allows us to split topological polarization into two different topological indices ν_+ and ν_-. This observation expanded the topological classification of Maxwell lattices, and allow us to realize HOTS.

It is worthwhile to highlight that to define a topological index, the Hamiltonian [Eq. (2)] must remain gapped with $\det \mathbf{C} \neq 0$. Because a mirror plane in the momentum space often passes through the Γ point ($k = 0$), it is necessary to gap the acoustic phonon bands at Γ. As will be shown below, this can be achieved by restricting the motion of certain lattice points, which break the translational invariance of the lattice.

The mirror symmetric Maxwell lattice.–We now illustrate one Maxwell lattice that support HOTS. As shown in Fig. 1, each unit cell of this lattice contains 6 point masses with coordinates

$$\mathbf{r}_i = \frac{1}{3}\left(\cos\left(\frac{2\pi i}{3} - \frac{5\pi}{6}\right), \sin\left(\frac{2\pi i}{3} - \frac{5\pi}{6}\right)\right), \tag{4a}$$

$$\mathbf{r}_{i+3} = (\frac{1}{6} + \delta_i)\left(\cos\left(\frac{2\pi i}{3} - \frac{\pi}{2}\right), \sin\left(\frac{2\pi i}{3} - \frac{\pi}{2}\right)\right), \tag{4b}$$

with $i \in \{1, 2, 3\}$. The three points $i = 1, 2$ and 3 can move in both x and y directions, while the rest three are restricted to move along the direction marked by the black arrows shown in Fig. 1(a):

$$\mathbf{u}_i = (u_{ix}, u_{iy}), \tag{5a}$$

$$\mathbf{u}_{i+3} = u_{i+3}\left(\cos\left(\frac{2\pi i}{3} - \frac{\pi}{2}\right), \sin\left(\frac{2\pi i}{3} - \frac{\pi}{2}\right)\right), \tag{5b}$$

for $i \in \{1, 2, 3\}$. Consequently, there are $n_d = 9$ DOFs per unit cell $\{u_{1x}, u_{1y}, \ldots, u_{3y}, u_4, u_5, u_6\}$.

We then repeat this unit cell to form a 2D lattice and connect the mass poits with springs (solid lines in Fig. 1(b)). Here we set the lattice vectors $\mathbf{a}_1 = (1, 0)$ and $\mathbf{a}_2 = \frac{1}{2}(-1, \sqrt{3})$, and the masses of all points and the stiffnesses of all springs are set to 1 for simplicity. Notice that here we have 9 springs per unit cell, which

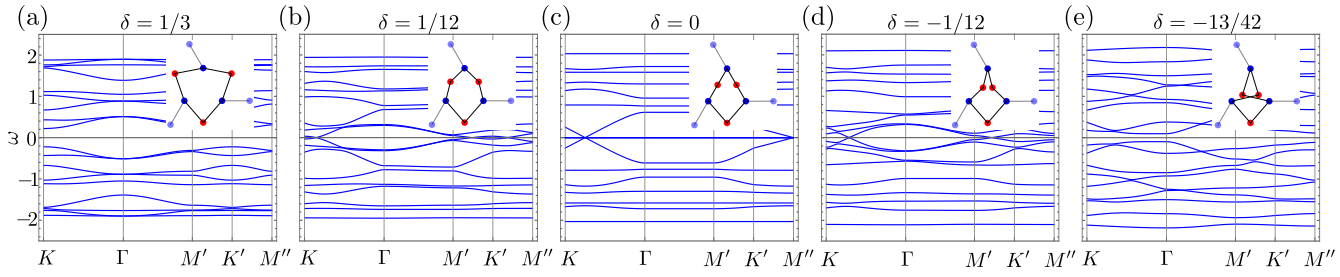


FIG. 2. Spectrum of $\mathcal{H}$ for different values $\delta = \delta_1 = \delta_2$ keeping $\delta_3 = 1/3$. The unit cell corresponding to each configuration is shown inset. Each diagram has $n_c + n_d = 18$ bands. All systems except (c) are gapped along line $\Gamma - M'$ (L_1 in Fig. 1(c)). (c) has four flat bands at zero frequency. The spectrum in (b) and (d) are gapped at $\omega = 0$ along the line $\Gamma - M'$, but not gapped everywhere in the Brillouin zone. Only (a) and (e) are fully gapped at $\omega = 0$ over the entire Brillouin zone.

match the DOFs $n_d = 9$, making the system a Maxwell lattice.

Note that if we set $\delta_i = \delta_{i+1}$, the system is invariant under mirror reflection about the perpendicular bisector of points $i+3$ and $i+4$. The corresponding mirror invariant lines L_i in the reciprocal space (Brillouin zone) are shown in Fig. 1(c). Because all the mirror planes go through Γ, it is important to gap out the phonon bands at Γ to define the topological index. In this setup, this is automatically achieved because points $i = 4, 5, 6$ can only move along the arrow directions, which gaps out the acoustic modes.

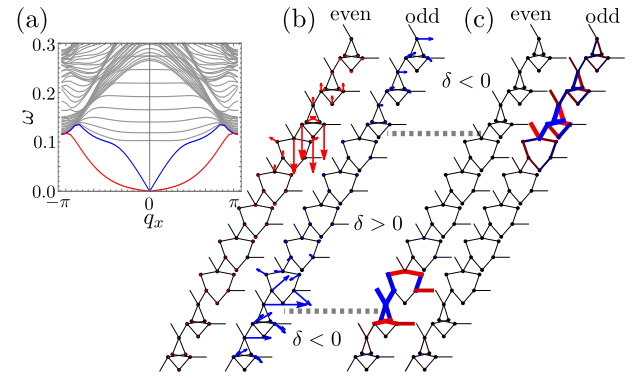


FIG. 3. Spectrum (a), ZMs (b) and SSSs (c) of a supercell consisting of $2N_0$ unit cells among which N_0 in the middle have $\delta = 1/3$ and the other ones have $\delta = -13/42$. Periodic boundary condition is employed in direction $(1/2, \sqrt{3}/2)$, whereas Bloch-periodic boundary condition $\mathbf{u}(\mathbf{x} + (1,0)) = \mathbf{u}(\mathbf{x})e^{iq_x}$ is employed in $(1,0)$ direction. In (a), gray bands are bulk modes whereas the red and the blue bands are localized at the top and bottom domain walls, respectively. The left ZM in (b) is localized at the top domain wall and is even under vertical mirror m_x, whereas the right ZM in (b) is localized at the bottom domain wall and is odd under vertical mirror m_x. The left SSS in (c) is localized at the bottom domain wall and is even under vertical mirror m_x, whereas the right SSS in (c) is localized at the top domain wall and is odd under vertical mirror m_x. The red and blue colors of the bonds in (c) indicate the elongation and compression of the bonds, respectively.

The compatibility matrix $\mathbf{C}(\mathbf{q})$ is given in SM [37] Sec. SM.1. For simplicity we set $\delta_3 = 1/3$ and vary $\delta_1 = \delta_2 \equiv \delta$. In this case, the lattice has one mirror m_x per unit cell with normal in x direction. Along the mirror invariant line $q_x = 0$ (L_1 in Fig. 1(c)), we calculate $\det \mathbf{C}_+^\dagger\big|_{L_1}$, $\det \mathbf{C}_-^\dagger\big|_{L_1}$ and integrate them from $\mathbf{q} = (0, -2\pi/\sqrt{3})$ to $\mathbf{q} + \mathbf{b}_2 = (0, 2\pi/\sqrt{3})$ along path L_1 according to Eq. (3). We find

$$\nu_+ = \begin{cases} 1 & \text{if } \delta > 0, \\ 0 & \text{if } \delta < 0, \end{cases} \quad \nu_- = \begin{cases} 0 & \text{if } \delta > 0, \\ 1 & \text{if } -5/12 < \delta < 0, \\ 0 & \text{if } \delta < -5/12. \end{cases} \quad (6)$$

Clearly, the phases with $\delta > 0$ and $0 > \delta > -5/12$ are distinct w.r.t. the MGWNs but same w.r.t. the Kane-Lubensky index. We will call $\delta > 0$ phase 1, and $0 > \delta > -5/12$ phase 2. In Fig. 2, we show the spectrum of matrix $\mathcal{H}(\mathbf{q})$ for different values of δ. At $\delta = 0$, the DOFs corresponding to points **4** and **5** are perpendicular to the springs connected to them; hence displacements of these points do not change the length of the springs to the linear order. These give two ZMs at every wave-vector $\mathbf{q}$. Then, due to the Maxwell-Calladine index theorem there are two SSSs at every $\mathbf{q}$. Hence, there are 4 flat bands at $\omega = 0$ of the matrix $\mathcal{H}(\mathbf{q})$ for $\delta = 0$. When $\delta \neq 0$, $\omega = 0$ gapped along C_1 line allowing us to define the MGWNs $\nu_\pm$.

In addition to defining the MGWNs, in order to localize ZMs at the junction of two different mirror graded phases, we require the bulk bands to be completely gapped at $\omega = 0$ in addition to the path L_1. We find that phase 1 is fully gapped at $\omega = 0$ over the entire Brillouin zone for $\delta > 5/42$ (Fig. 2(a)), whereas phase 2 is fully gapped for $-5/12 < \delta < -1/6$ (Fig. 2(e)) (see SM [37] SM.4 for details).

Mirror-protected zero frequency edge states.–To examine the bulk-edge correspondence, we create a supercell in Fig. 3 with periodic boundary conditions in both directions, which has domain walls separating $\delta = 1/3$ and $\delta = -13/42$. The domain walls are horizontal – normal to the mirror m_x; hence invariant under reflection about

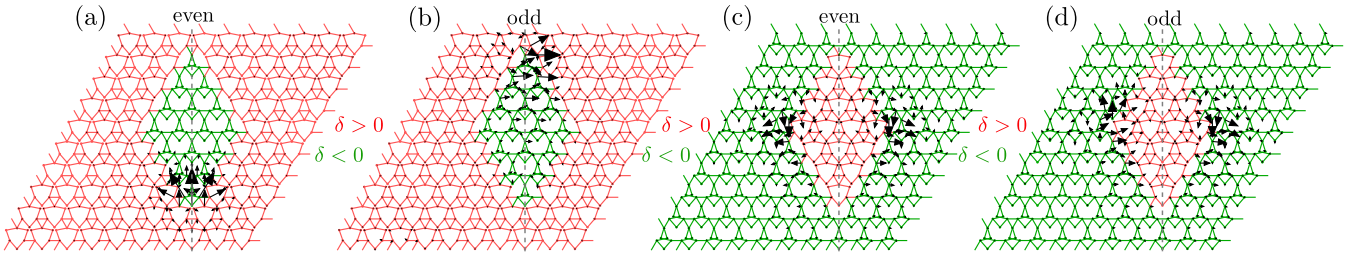


FIG. 4. Corner modes in systems with diamond shaped island of one phase inside the other phase. In each panel, the part of the system in red (green) has $\delta = 1/3$ ($\delta = -13/42$). The vertical grey dashed lines show the line of mirror symmetry; it passes through the the top and bottom corner of the diamond shaped island. We applied periodic boundary conditions in all cases. The black arrows show the displacement field corresponding to the zero modes. In all cases, the zero modes are concentrated at the corners. The corner modes in (a) and (c) are even under the vertical mirror reflection, whereas (b) and (d) are odd under the same reflection.

the mirror m_x. The spectrum of the dynamical matrix $\mathbf{D}(q_x)$ of the system is plotted as a function of surface wave vector q_x. We find two ZMs at $q_x = 0$ (Fig. 3(a)). Since Kane-Lubensky indices of both domains are same: $n^{\delta=1/3} = \nu_+^{\delta=1/3} + \nu_-^{\delta=1/3} = 1 = \nu_+^{\delta=-13/42} + \nu_-^{\delta=-13/42} = n^{\delta=-13/42}$, the ZMs at the domain walls are not given by the Kane-Lubensky index. However, since at $q_x = 0$, matrix $\mathbf{C}(\mathbf{q})$ can be block-diagonalized (Eq. (2)) as discussed above, we can use Eq. (3) on + and − sectors separately. Since matrix $\mathbf{C}(q_x = 0, q_y)$ is block diagonal, the ZMs of each sector are also ZMs of the full system. Hence, at the top and bottom domain walls we get:

$$\text{top wall:} \quad \begin{aligned} \nu_+^< - \nu_+^> &= 1 \Rightarrow \text{ ZM}, \\ \nu_-^< - \nu_-^> &= -1 \Rightarrow \text{ SSS}, \end{aligned} \tag{7a}$$

$$\text{bottom wall:} \quad \begin{aligned} \nu_+^< - \nu_+^> &= -1 \Rightarrow \text{ SSS}, \\ \nu_-^< - \nu_-^> &= 1 \Rightarrow \text{ ZM}, \end{aligned} \tag{7b}$$

where $<$ and $>$ denote phases below and above the domain wall, respectively. It must be emphasized here that because rigid translation is not a zero mode in our lattice, in general such a lattice is not expected to have zero modes and all phonon modes should be gapped. However, at the domain boundary between regions with different topological indices, topological edge modes emerge with frequency pinned to zero by the chiral symmetry.

It is also worthwhile to highlight that these topological zero modes are fundamentally different from the zero modes protected by topological polarization. First of all, they are due to a totally different topological index. Secondly, in contrast to zero modes from topological polarization, the supercell spectrum of which has a flat bands at zero frequency [28], the topological modes here are dispersive. Because the mirror symmetry is broken away from the mirror plane ($q_x \neq 0$), the frequency of the edge modes moves away from zero at $q_x \neq 0$ as shown in Fig. 3. Finally, in contrast to the deformed kagome lattice ([28]) where the SSSs and ZMs are localized on opposite domain walls, in our systems, the ZM and SSS are on the same domain wall. Typically, ZM and SSS cannot be localized on the same domain wall, because they will be lifted to finite frequency in the presence of hybridization between them. In our system, such hybridization is prohibited by the mirror symmetry, because for each domain, its ZM and SSS have opposite mirror parity (even vs odd).

To conclude this section, we would like to point out that this topological index and zero modes can also be characterized by a low-energy continuum theory (SM [37] Sec. SM.5) using a Dirac Hamiltonian and the Jackiw-Rebbi analysis [38, 39].

Mirror-protected corner states.–Mirror symmetric systems in the BDI class where the mirror reflection operator commutes TRS and chiral symmetry operators can have Mirror symmetry protected zero frequency corner modes [10, 40]. To look for such corner states, we create a diamond shaped island of one phase inside a rhombus shaped other phase phase with periodic boundary conditions for the rhombus in both direction (Fig. 4). The top and the bottom corners of the diamond are invariant under a vertical mirror passing through them. In Figs. 4(a-b), we see that when the inner island is $\delta = -13/42$ phase and the outer phase is $\delta = 1/3$, there are zero frequency corner modes localized at the top and the bottom corners, the top (bottom) one being odd (even) under the vertical mirror reflection. The situation is more curious when the inner island is $\delta = 1/3$ and the outer phase is $\delta = -13/42$ (Figs. 4(c-d)). There are still two zero frequency corner modes, one of the odd and the other even under the vertical mirror reflection, but they are both localized at the right and left corners.

The topological nature and the origin of these corner modes can be easily understood using standard approach of HOTS (SM [37] Sec. SM.6). When the domain wall between the two phases is tilted such that the domain wall is not invariant under reflection, the localized states at the domain wall become massive, meaning that the spectrum is gapped at $\omega = 0$. Moreover, two oppositely tilted domain walls have opposite sign of the mass m; the sign of the mass m depends on the sign of the angle of tilt of the domain wall. Therefore, at the corner both δ

(across the domain boundary) and m (along the domain boundary) change sign. As is elaborated in the SM [37] Sec. SM.6, depending on the sign of the mass m, the amplitude of the zero frequency mode ($\sim e^{-mx}$) may either decrease or increases as we move away from the corner point ($x = 0$). If the amplitude increases exponentially as we move away from this corner, it implies that this zero mode is localized at the next corner along the direction of the increasing amplitude. This theory analysis is in perfect agreement with numerical simulations. Furthermore, these corner modes persist even when the corner is not mirror invariant, as long as the bulk structures have mirror symmetry (see SM [37] Sec. SM.7); which implies that this HOTS is "intrinsic" [10, 40].

Conclusions.–In this work we demonstrated how spatial symmetries can protect higher order topological phase in Maxwell frames and gives rise to zero frequency topological edge and corner modes. Furthermore, these edge and corner modes are pinned to zero frequency due to inherent chiral symmetry of Maxwell frames pointed out in [28]. This chiral symmetry is often used as an approximate symmetry in fermionic systems (except in case of superconductors), but in case of Maxwell lattices it is exact. As mentioned earlier, our system falls under the BDI class of Altland-Zirnbauer classification; it has been known in the literature [10, 40] that mirror symmetry that commutes with time reversal and chiral symmetry can protect corner modes in 2-dimensions in this class. To our knowledge, our structure is the first example of this in classical systems. This system should be straightforwardly experimentally realized using hard plastic parts and hinges similar to what was done in [41] for deformed kagome lattice; with the three extra point masses (red points 4-6 in Fig. 1(a)) in our system need to be put on fixed rails such that they can only move along the corresponding rails.

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[4] C.-X. Liu, S.-C. Zhang, and X.-L. Qi, Annual Review of Condensed Matter Physics **7**, 301 (2016).
[5] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[6] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[7] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Science **357**, 61 (2017).
[8] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Phys. Rev. B **96**, 245115 (2017).
[9] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. Parkin, B. A. Bernevig, and T. Neupert, Science advances **4**, eaat0346 (2018).
[10] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. **119**, 246401 (2017).
[11] E. Khalaf, Phys. Rev. B **97**, 205136 (2018).
[12] Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. **119**, 246402 (2017).
[13] G. van Miert and C. Ortix, Phys. Rev. B **98**, 081110 (2018).
[14] B. Xie, H.-X. Wang, X. Zhang, P. Zhan, J.-H. Jiang, M. Lu, and Y. Chen, Nature Reviews Physics **3**, 520 (2021).
[15] H. Fan, B. Xia, L. Tong, S. Zheng, and D. Yu, Phys. Rev. Lett. **122**, 204301 (2019).
[16] H. Wakao, T. Yoshida, H. Araki, T. Mizoguchi, and Y. Hatsugai, Phys. Rev. B **101**, 094107 (2020).
[17] M. Serra-Garcia, V. Peri, R. Süsstrunk, O. R. Bilal, T. Larsen, L. G. Villanueva, and S. D. Huber, Nature **555**, 342 (2018).
[18] J. Attig, K. Roychowdhury, M. J. Lawler, and S. Trebst, Phys. Rev. Res. **1**, 032047 (2019).
[19] H. Xue, Y. Yang, F. Gao, Y. Chong, and B. Zhang, Nature materials **18**, 108 (2019).
[20] X. Ni, M. Weiner, A. Alu, and A. B. Khanikaev, Nature materials **18**, 113 (2019).
[21] M. Proctor, P. A. Huidobro, B. Bradlyn, M. B. de Paz, M. G. Vergniory, D. Bercioux, and A. García-Etxarri, Phys. Rev. Res. **2**, 042038 (2020).
[22] G. van Miert and C. Ortix, npj Quantum Materials **5**, 63 (2020).
[23] M. A. J. Herrera, S. N. Kempkes, M. B. de Paz, A. García-Etxarri, I. Swart, C. M. Smith, and D. Bercioux, Phys. Rev. B **105**, 085411 (2022).
[24] A. Saremi and Z. Rocklin, Physical Review B **98**, 180102 (2018).
[25] E. Khalaf, W. A. Benalcazar, T. L. Hughes, and R. Queiroz, Physical Review Research **3**, 013239 (2021).
[26] J. C. Maxwell, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **27**, 294 (1864).
[27] T. Lubensky, C. Kane, X. Mao, A. Souslov, and K. Sun, Reports on Progress in Physics **78**, 073901 (2015).
[28] C. L. Kane and T. C. Lubensky, Nature Physics **10**, 39 (2014).
[29] T. Neupert and F. Schindler, in *Topological Matter: Lectures from the Topological Matter School 2017* (Springer, 2018) pp. 31–61.
[30] Y. Ren, Z. Qiao, and Q. Niu, Phys. Rev. Lett. **124**, 166804 (2020).
[31] S. Imhof, C. Berger, F. Bayer, J. Brehm, L. W. Molenkamp, T. Kiessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, *et al.*, Nature Physics **14**, 925 (2018).
[32] C. R. Calladine, International journal of solids and structures **14**, 161 (1978).
[33] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[34] A. Kitaev, in *AIP conference proceedings*, Vol. 1134

(American Institute of Physics, 2009) pp. 22–30.
[35] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. Ludwig, New Journal of Physics **12**, 065010 (2010).
[36] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).
[37] "See supplemental material for details,".
[38] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[39] B. A. Bernevig, *Topological insulators and topological superconductors* (Princeton university press, 2013).
[40] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B **97**, 205135 (2018).
[41] D. Z. Rocklin, S. Zhou, K. Sun, and X. Mao, Nature communications **8**, 14201 (2017).