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Hydrodynamics in long-range interacting systems with center-of-mass conservation

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In systems with a conserved density, the additional conservation of the center of mass (dipole moment) has been shown to slow down the associated hydrodynamics. At the same time, long-range interactions generally lead to faster transport and information propagation. Here, we explore the competition of these two effects and develop a hydrodynamic theory for long-range center-of-mass-conserving systems. We demonstrate that these systems can exhibit a rich dynamical phase diagram containing subdiffusive, diffusive, and superdiffusive behaviors, with continuously varying dynamical exponents. We corroborate our theory by studying quantum lattice models whose emergent hydrodynamics exhibit these phenomena.

Hydrodynamic theories are coarse-grained descriptions of the flow of conserved densities. They help us understand large-scale behavior and universality without knowing how these emerge from microscopic details [1, 2]. The price for this is the introduction of unknown macroscopic parameters such as diffusivity or viscosity, which are difficult to compute from first principles [3–8]. Understanding how hydrodynamics emerges in quantum systems is of great interest, and recent progress has been made in this direction [9–13]. It is also of interest to understand what kinds of hydrodynamic behavior can possibly arise in quantum many-body systems, including both chaotic and integrable models [14–30], as well as models with symmetries or kinetic constraints [31–37].

For highly-excited quantum chaotic lattice models, the default expectation is often that standard diffusion will emerge. However, the dynamics can be systematically slowed down by imposing further constraints on the allowed microscopic transitions such that higher moments of the density distribution are conserved. A common example of this is the subdiffusion that results from conserving the center of mass (COM), also known as the dipole moment [38–50]. This can occur in "tilted" systems with a strong linear potential [51–54], as realized in cold-atom setups [55–57], and is relevant also for quantum Hall systems [58–61]. The additional conservation law can not only modify the hydrodynamics but can result in a frozen phase where the state space is fragmented into dynamically disconnected sectors [62–71]. While we focus on the highly-excited dynamics of such systems, there has also been interest in their low-temperature equilibrium properties [72–84].

In contrast to constrained dynamics, faster dynamics can be achieved in systems with long-range interactions [85, 86]. Interactions that decay as a power law of the distance between particles—including Coulomb, dipolar, and van der Waals—are ubiquitous, and can nowadays be explored using, e.g., ultracold atoms [87–90], polar molecules [91–93], or trapped ions [94]. In such systems, Lieb-Robinson bounds can become superballistic, and hydrodynamics superdiffusive, depending on the power-law exponent and the dimensionality of the underlying lattice [95–105].

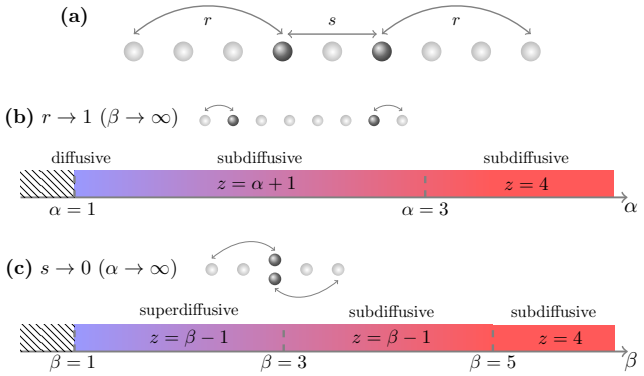


FIG. 1. **(a)** We consider models involving long-range interactions and COM conservation. The dynamics are governed by pairs of equal and opposite currents (or the underlying "pair hopping" processes depicted) acting on lattice sites separated by the distances $s \geq 0$ and $r \geq 1$ (the "interaction" and "hopping" distances). The strengths of the currents are suppressed by power laws $(s+1)^{-\alpha} r^{-\beta}$, so α controls the range of interactions and β the range of hopping. [**(b)**, **(c)**] Schematic cuts through the dynamical phase diagram for the limiting cases $r \to 1$ ($\beta \to \infty$) and $s \to 0$ ($\alpha \to \infty$).

Given their competing effects, it is natural to examine the interplay between higher-order conservation laws and long-range interactions, however, this problem has only been briefly touched on in the literature [39]. Here, we bridge the gap and consider a class of models where "hops" are long-range but always occur in pairs such that the COM is conserved [Fig. 1(a)]. We develop a hydrodynamic theory of such models and map out its phase diagram [Fig. 1(b),(c)]. More specifically, our model is characterized by two power-law exponents, α and β, that control the suppression of pair-hopping amplitudes. We calculate the dynamical exponent z, governing long-wavelength relaxation, as a function of these two exponents, as well as the scaling functions for density-density correlations. When α and β are large enough, we re-

cover the known $z = 4$ subdiffusion of short-range COM-conserving systems in one dimension. When α or β is small enough, z continuously varies. The scaling function of the correlations also continuously varies with the exponents. We find consistent results when examining a less-tractable quantum spin-1 model (a long-range version of the systems studied in Ref. [64, 65]). We also discuss the relevance of our findings to long-range many-body quantum systems in a tilted potential.

Setup. As a starting point, we consider a 1D quantum spin model

$$H = \sum_x \sum_{s,r} J_{sr} \left(S^+_x S^-_{x+r} S^-_{x+r+s} S^+_{x+s+2r} + \text{h.c.} \right), \quad (1)$$

where $S^\pm_x$ are raising and lowering operators at site x, and J_{sr} depend on the distances $s \geq 0$ and $r > 0$. The Hamiltonian conserves the magnetization $\sum_x S^z_x$ and its dipole moment $\sum_x x S^z_x$, i.e., the "center of mass" of the S^z density. Eq. (1) can be understood as a generalization of the short-range "pair-hopping" models of Refs. [64, 65, 67]. When those systems thermalize, they exhibit subdiffusion with dynamical exponent $z = 4$ and density correlations that are nonmonotonic in space [38–40]. Here, we are interested in how the hydrodynamics is affected by long-range interactions, particularly for J_{sr} with power-law decay.

Since directly computing the emergent hydrodynamics of quantum models is generally intractable, here we also introduce a solvable hydrodynamic model that is inspired by Eq. (1). We consider a density $n_x \in \mathbb{R}$ which is slightly perturbed from its uniform equilibrium value. The dynamics of the model are governed by pairs of equal but opposite currents that displace density from positions $x+r$ and $x+r+s$ to positions x and $x+s+2r$ (and vice versa), where $s \geq 0$ and $r \geq 1$ [Fig. 1(a)]. Such "pair currents" occur in superposition for all s and r, but the magnitudes of each are driven by the density at the four positions involved. The analog to Fick's law is that each pair current is driven with strength proportional to the curvature of n_x on those four sites [38]. The proportionality for a given s and r is denoted C_{sr} and it encodes the suppression of such pair currents as a function of s and r. The equation of motion for the density profile is

$$\begin{aligned} \dot{n}_x = \sum_{r,s} C_{sr} [&- (n_{x-2r-s} - n_{x-r-s} - n_{x-r} + n_x) \\ &+ (n_{x-r-s} - n_{x-s} - n_x + n_{x+r}) \\ &+ (n_{x-r} - n_x - n_{x+s} + n_{x+s+r}) \\ &- (n_x - n_{x+r} - n_{x+s+r} + n_{x+s+2r})]. \end{aligned} \quad (2)$$

The four lines correspond to the ways in which four sites with spacings s and r can intersect site x. As a simple tractable choice, we will focus on the separable form

$$C_{sr} = (s+1)^{-\alpha} r^{-\beta}, \quad (3)$$

where α and β are tuning parameters. Later, we will also discuss the nonseparable form of C_{sr} that arises in strongly tilted systems.

While this classical hydrodynamic model is our main focus, we expect that our results qualitatively carry over to the transport behavior of appropriate quantum systems. Indeed, after solving our hydrodynamic model, below we numerically study the dynamics generated by Eq. (1) for small spin-1 chains and obtain behavior consistent with our hydrodynamic theory if the pair-hopping amplitudes in the Hamiltonian are chosen as $J_{sr} \sim \sqrt{C_{sr}}$, in accordance with Fermi's Golden Rule [101], cf. Fig. 4.

Hydrodynamic Theory. Equation (2) is a linear equation with a basis of decaying plane-wave solutions $n_x = e^{ikx - \gamma_k t}$, where the decay rates are $\gamma_k = \sum_{s,r} 16 C_{sr} \sin^2\left(\frac{kr}{2}\right) \sin^2\left(\frac{kr+ks}{2}\right)$. We are interested in the small-k behavior of γ_k as a function of α and β. In that limit, the lattice approximates a continuum such that

$$\gamma(k) = \int_1^\infty \int_0^\infty \frac{16 \sin^2\left(\frac{kr}{2}\right) \sin^2\left(\frac{kr+ks}{2}\right)}{(s+1)^\alpha r^\beta} ds dr \,. \quad (4)$$

At small enough k we expect $\gamma(k) \propto k^z$ where z is a dynamical exponent that could dependent on α and β. It is then useful to define an effective exponent

$$\hat{z}(k) = \frac{d \log \gamma(k)}{d \log k}. \quad (5)$$

To get an immediate picture of the model's behavior, in Fig. 2 we numerically evaluate $\log_2(\gamma_{2k}/\gamma_k) \cong \hat{z}$ on the (α, β)-plane [Fig. 2(a)], and along two cuts through it [Fig. 2(c),(d)]. We identify three phases: **(A)** $\alpha > 3$ and $\beta > 5$. The power laws are sufficiently short-ranged that $z = 4$, consistent with the known behavior in finite-range COM-conserving systems in one dimension [38–40, 55]. **(B)** $\alpha > \beta - 2$ and $\beta < 5$. The long-range "hopping" dominates. The result is a dynamical exponent $z \in [0, 4]$ that continuously varies with β but not α. This is similar to Refs. [100, 101] where the hydrodynamics of models with long-range hopping (but no higher-moment conservation laws) was studied. In particular, once $\beta \leq 3$ such that $z \leq 2$, the hydrodynamics of our center-of-mass conserving model is equivalent to the model in [100, 101]. **(C)** $\beta > \alpha + 2$ and $\alpha < 3$. The long-range "pairing" dominates, and hopping is short ranged. This again leads to a continuously varying dynamical exponent, but this time it varies with α and not β. Notably, the fastest relaxation possible in this phase is diffusive ($z = 2$). This results from the effective loss (locally) of the COM conservation that follows from allowing the paired short-ranged currents to be far away from each other.

While we focus on $\alpha, \beta > 1$ to maintain locality of both interaction and hopping distances, here we briefly comment on the cases where $\alpha \leq 1$ or $\beta \leq 1$ (cf. hatched areas in Fig. 1). For $\alpha \leq 1$, the two left sites and two right

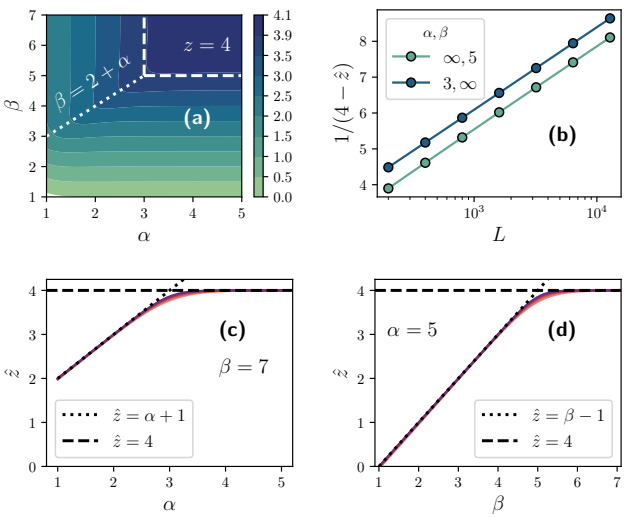


FIG. 2. The dynamical exponent z. **(a)** Contour plot of $\log_2(\gamma_{2k}/\gamma_k) \cong \hat{z}$ on the (α, β)-plane, numerically evaluated using $k = 2\pi/3200$. The dashed box marks $z = 4$ subdiffusion. The dotted line along $\beta = 2 + \alpha$ marks where the behavior of z changes from $z = \beta - 1$ to $z = \alpha + 1$. **(b)** Demonstration of the boundary behavior $\hat{z}(k) = 4 - \log(1/k)^{-1}$ where the horizontal axis is $L = 2\pi/k$, the lattice sizes used in the estimates. The vertical shift between the two curves is from different subleading additive contributions. **[(c),(d)]** Cuts along the top and right edges of the contour plot. Colors (light to dark) represent estimates using $k \geq 2\pi/3200$, i.e., lattices of size $L \leq 3200$.

sites in the pair-hopping process can be arbitrarily far away from each other. Thereby, center-of-mass conservation is effectively lost and the system behaves as a system with only charge conservation and hopping governed by β. In contrast, for $\beta \leq 1$, the hops can be arbitrarily long-ranged such that the system effectively becomes an all-to-all model where hydrodynamics breaks down.

These phases and boundaries can be studied in more detail by analyzing Eq. (4): At large α and β, the integral is dominated by $s, r \ll k^{-1}$ and we can expand the sin functions to leading order which yields $\gamma(k) \propto k^4$ as expected, i.e., $z = 4$. The resulting integral is only convergent when $\alpha > 3$ and $\beta > 5$, consistent with Fig. 2.

For $\alpha = \infty$ ($s = 0$), the asymptotic small-k behavior of Eq. (4) is

$$\gamma(k) = \begin{cases} (\beta - 5)^{-1} k^4 & \beta > 5 \\ k^4 \log(1/k) & \beta = 5 \\ (2^\beta - 8)\Gamma(1 - \beta) \sin\left(\frac{\pi\beta}{2}\right) k^{\beta - 1} & \beta < 5 \end{cases}, \quad (6)$$

where $\Gamma(\cdot)$ is the Gamma function. Thus, for $\beta > 5$ the long-wavelength behavior is characterized by dynamical exponent $z = 4$, while for $\beta < 5$ it is $z = \beta - 1$, in accordance with Fig. 2. At the phase boundary $(\alpha, \beta) = (\infty, 5)$, the dynamics incurs a logarithmic correction that corresponds to $\hat{z}(k) = 4 - \log(1/k)^{-1}$. This scaling of $\hat{z}$ at the phase boundary is demonstrated in Fig. 2(b).

For $\beta = \infty$ ($r = 1$), we similarly get

$$\gamma(k) = \begin{cases} (\alpha - 3)^{-1} k^4 & \alpha > 3 \\ k^4 \log(1/k) & \alpha = 3 \\ 2[-\Gamma(1 - \alpha)] \sin\left(\frac{\pi\alpha}{2}\right) k^{\alpha + 1} & \alpha < 3 \end{cases}, \quad (7)$$

that is, the behavior changes from $z = 4$ to $z = \alpha + 1$ at $\alpha = 3$, and at the boundary we obtain the same logarithmic correction.

In contrast, across $\beta = 2 + \alpha$ [dotted line in Fig. 2(a)] there is a qualitative change of behavior, from $z = \beta - 1$ to $z = \alpha + 1$, but the effective z converges to a constant without a log correction. The dashed and dotted boundaries in Fig. 2(a) are therefore distinct in that respect. It should be noted that the boundaries between the "phases" are not continuous phase transitions with a diverging length scale, so there is no criticality of that type.

Distinct spatial correlations develop as a result of the various different z values. In Fig. 3 we show the late-time density profile resulting from evolving $n(x, 0) = \delta_{x,0}$, and compare with our theoretical understanding. At late enough times, the relaxation is governed by $\gamma(k) \cong Dk^z$, so $n(x, t) \cong (Dt)^{-1/z} F_z(x/(Dt)^{1/z})$, where D is the generalized diffusion constant (depends on α and β) and the only parameter controlling the scaling function $F_z(\cdot)$ is the value of z governing the long-wavelength relaxation. More specifically, we have $F_z(\xi) = \int_{-\infty}^{\infty} e^{i\kappa\xi - |\kappa|^z} \frac{d\kappa}{2\pi}$. In Fig. 3 we test this scaling, with z set to its predicted asymptotic value in each panel, and find excellent agreement. When $z = 4$ the sign of correlations is positive at smaller distances, negative at intermediate distances, and positive again at larger distances [Fig. 3(d)-(f)]. At $z = 3$, the correlations no longer become positive again at larger distances; they approach zero from below [Fig. 3(c)]. At $z = 2$, we regain standard Gaussian diffusion at long times [Fig. 3(b)], which does not have any negative lobe in the spatial profile. Finally, at $z = 1$ we have a heavy-tailed Lorentzian profile [Fig. 3(a)].

Dynamics of a quantum spin-1 chain. Having established the properties of our classical hydrodynamic model, we now turn to less-tractable microscopic many-body quantum systems. Given a quantum spin Hamiltonian H as in Eq. (1) with couplings J_{sr}, transition rates $|\langle a|H|b\rangle|^2$ between two spin configurations $|a\rangle$ and $|b\rangle$ will scale as $\propto J_{sr}^2$ due to Fermi's Golden rule [101]. Thus, in order to compare the transport properties of H to our hydrodynamic theory, it is natural to choose $J_{sr} \propto \sqrt{C_{sr}}$ [103, 105, 106]. More specifically, we study Eq. (1) for spin-1 and set $J_{sr} = 1$ if $s + 2r \leq 3$, while $J_{sr} \propto s^{-\alpha/2} r^{-\beta/2}$ for terms of longer range. This choice of J_{sr} avoids signatures of strong Hilbert-space fragmentation and, under $\alpha, \beta \to \infty$, recovers the pair-hopping model of Ref. [64].

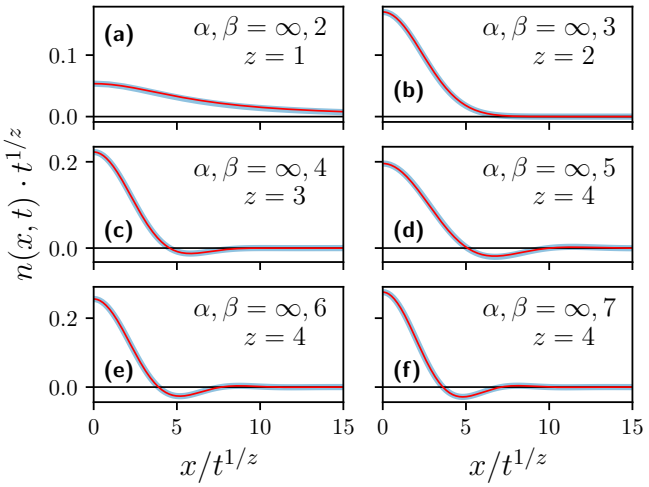


FIG. 3. Scaled density profiles obtained from time evolving $n(x,0) = \delta_{x,0}$ under Eq. (2) (red), and comparison with $F_z(\xi)$. The system size is $L = 3200$ and the time is $t \sim \gamma^{-1}_{2\pi/L}$ for each panel. Position x extends from $-\frac{L}{2}$ to $\frac{L}{2}$ but we show $x \geq 0$. We set $\alpha = \infty$, and β varies over panels (a)-(f). The annotated values of z are used for the scaling of the axes. The slowly vanishing logarithmic correction to $z = 4$ [$\hat{z}(k) = 4 - \log(1/k)^{-1}$] that is applicable to panel (d) is small enough to ignore. Data at other late times collapses well onto the same curves, but we show curves from one time for clarity. Data is in red, and $D^{-1/z}F_z$ is plotted in faint blue for comparison, where D is extracted from $n(0,t) = Dt^{1/z}$, but these are directly behind the data curves due to excellent agreement.

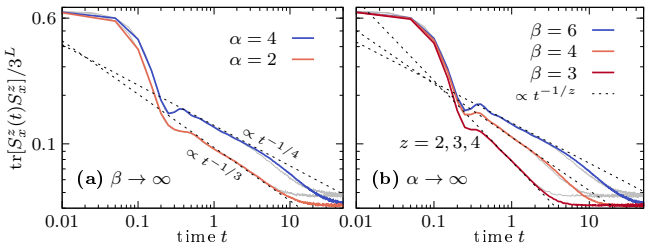


FIG. 4. Infinite-temperature autocorrelation function for a long-range COM-conserving spin-1 chain as in Eq. (1) with $J_{sr} \propto s^{-\alpha/2} r^{-\beta/2}$ and system size $L = 16$. **(a)** Varying α for $\beta \to \infty$. **(b)** Varying β for $\alpha \to \infty$. The dashed curves indicate power laws $\propto t^{-1/z}$ with z set according to the prediction of our hydrodynamic theory. The thin gray curves are data for smaller $L = 14$.

In Fig. 4, we show the infinite-temperature autocorrelation function $\mathrm{tr}[S^z_x(t)S^z_x]/3^L$, obtained using quantum typicality [107, 108], for chains with $L = 16$. The specific site x is irrelevant due to periodic boundaries. We consider the limiting cases of $\beta \to \infty$, i.e., short hopping distance r [Fig. 4 (a)], and $\alpha \to \infty$, i.e., short pairing distance s [Fig. 4 (b)]. In all cases, we find a hydrodynamic tail $\propto t^{-1/z}$, where the value of the dynamical exponent z is consistent with the prediction of our hydrodynamic theory [Fig. 1]. Given the small systems in the quantum case, a more detailed comparison is challenging and the regime of smaller α, β is not accessible. Nevertheless, the data in Fig. 4 substantiates that our hydrodynamic model indeed correctly captures the dynamics of generic quantum systems with long-range pair-hopping processes.

Quantum systems in a tilted lattice. Our study of the hydrodynamics in long-range models with COM conservation is also motivated by potential experimental realizations. Consider, e.g., a long-range spin-1/2 XY model in a "tilted" potential,

$$H_{\mathrm{XY}} = \sum_{i<j} \frac{J}{|i-j|^\nu}(\sigma^+_i\sigma^-_j + \sigma^-_i\sigma^+_j) + F\sum_i i n_i \ , \quad (8)$$

where F is the tilt strength, $\nu > 1/2$ the power-law exponent, and $n_i = (\sigma^z_i + \mathbb{1})/2$. Such models with power-law interactions can nowadays be achieved in experiments with Rydberg atoms, polar molecules, or trapped ions, at least for certain values of ν, e.g., Refs. [100, 109, 110].

When F/J is large, the model has a long-lived conservation of a dressed version of $F\sum_i i n_i$. As we show in the SM [111] using a Schrieffer-Wolff (SW) transformation, the dynamics in the strong-tilt regime are governed by pair-hopping processes $\sigma^+_i\sigma^-_{i+r}\sigma^-_{i+r+s}\sigma^+_{i+2r+s}$ (and h.c.). These terms occur with amplitude

$$J_{\mathrm{eff}} \propto \frac{J^3}{F^2}\frac{1}{r^{\nu+1}(r+s)^{\nu+1}}\left(\frac{1}{s^\nu} - \frac{1}{(2r+s)^\nu}\right) \ , \quad (9)$$

and can be understood as third-order processes via, e.g., combining $\sigma^+_{i+2r+s}\sigma^-_{i+r+s}$, $\sigma^+_{i+r+s}\sigma^-_{i+r}$, and $\sigma^+_i\sigma^-_{i+r+s}$, with the intermediate states being off-shell by energies Fr and $F(r+s)$ [111]. Therefore, strongly-tilted long-range models are indeed governed by pair-hopping analogous to our hydrodynamic model. However, we note that Eq. (9) is not separable and depends only on one tuning exponent ν.

Similar to our results in the context of Fig. 2, we can make predictions about the hydrodynamics of the strongly-tilted XY model by studying the small-k behavior of the decay rates $\gamma(k)$, but now for a hydrodynamic model with the more complicated pair-current strengths $C_{sr} \propto J^2_{\mathrm{eff}}$. Doing so, we find that $z = 4$ subdiffusion in fact remains stable even for originally long-range systems with small $\nu = 1/2$. This stability occurs because the exponents in the denominator of J^2_{eff} are larger due to the pair-hopping processes being generated at third order. Realizing the dynamical regimes with continuously-varying z found for our simplified model in Fig. 1 thus seems challenging in strongly tilted systems; see also the SM [111].

Conclusion. Using a hydrodynamic model inspired by pair-hopping, we have investigated the interplay of long-range interactions and COM conservation on the dissipative hydrodynamics of 1D systems. We have also corroborated the predictions of our theory using simula-

tions of small spin-1 chains. When the interactions decay sufficiently quickly, the dynamics are subdiffusive with dynamical exponent $z = 4$, consistent with earlier work. As the power-law interactions are made longer-ranged, our model enters a phase where z, and the associated spatial profile of density correlations, vary continuously. One possible extreme for this is when the "pairing distance" of the pair currents is long-range but the "hopping distance" is short-range, where transport can become as fast as diffusive ($z = 2$). Another extreme is where the pairing distance is short and the hopping distance is long, then z can be made arbitrarily small.

While we have discussed the connections and differences of our pair-hopping model to potential experimental realizations in strongly-tilted systems, we note that in the case of short-range systems, Ref. [55] observed and explained subdiffusive relaxation at long wavelengths even with weak lattice tilts. Indeed, as we numerically illustrate in the SM [111], taking F to be large appears not to be necessary to study the interplay of long-range interactions and COM conservation. Nonetheless, it appears that the new regimes of $z < 4$ found in this work will be difficult to realize using tilted systems. How to realize these regimes in experimental systems is therefore an interesting question for future research.

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Note added. While finalizing this manuscript we became aware of related independent works by J. Gliozzi *et al.* [112] and Ogunnaike *et al.* [113] that will appear in the same arXiv posting. These contain generalizations of our results for higher-moment conservation laws and higher dimensions.

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