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Phys. Rev. B **108**, 115139 — Published 19 September 2023

DOI: [10.1103/PhysRevB.108.115139](https://doi.org/10.1103/PhysRevB.108.115139)

# Symmetric Mass Generation of Kähler-Dirac Fermions from the Perspective of Symmetry-Protected Topological Phases

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The Kähler-Dirac fermion, recognized as an elegant geometric approach, offers an alternative to traditional representations of relativistic fermions. Recent studies have demonstrated that symmetric mass generation (SMG) can precisely occur with two copies of Kähler-Dirac fermions across any spacetime dimensions. This conclusion stems from the study of anomaly cancellation within the fermion system. Our research provides an alternative understanding of this phenomenon from a condensed matter perspective, by associating the interacting Kähler-Dirac fermion with the boundary of bosonic symmetry-protected topological (SPT) phases. We show that the low-energy bosonic fluctuations in a single copy of the Kähler-Dirac fermion can be mapped to the boundary modes of a  $\mathbb{Z}_2$ -classified bosonic SPT state, protected by an inversion symmetry universally across all dimensions. This implies that two copies of Kähler-Dirac fermions can always undergo SMG through interactions mediated by these bosonic modes. This picture aids in systematically designing SMG interactions for Kähler-Dirac fermions in any dimension. We present the exact lattice Hamiltonian of these interactions and validate their efficacy in driving SMG.

## I. INTRODUCTION

Symmetric mass generation (SMG) [1–8] is a novel mechanism for gapless fermions to acquire a many-body excitation gap without spontaneous symmetry breaking. It occurs in a fermion system that is (i) free of any quantum anomaly [9–13], ensuring no obstruction towards trivial gapping; yet (ii) the symmetry is restrictive enough to rule out any fermion bilinear mass, thus gapping can only be achieved by multi-fermion condensation.

The SMG mechanism has attracted much research interest in both high-energy and condensed matter physics [14–31]. In condensed matter physics, the study of SMG was initially driven by the classification problem of interacting fermionic symmetry-protected topological (SPT) phases [32–53]. Some fermionic SPT classifications reduce from  $\mathbb{Z}$  to  $\mathbb{Z}_N$  in the presence of interactions [1, 2, 54–64]. Consequently, the gapless fermion edge states of  $N$  copies of interacting fermionic SPT states can be trivially gapped by the SMG mechanism without condensing any symmetry-breaking fermionic bilinear mass. In high-energy physics, the SMG mechanism was originally proposed as a solution to regularize chiral fermions on the lattice [3, 65–75], circumventing the Nielsen-Ninomiya theorem [76–78] and solving the fermion doubling problem by introducing non-perturbative interaction effects.

The SMG can occur in Majorana fermions in any spacetime dimension [60]. One intriguing feature is that it only occurs when the fermion flavor number is a multiple of a certain "magic number"  $N$ . This number  $N$  is determined by the anomaly cancellation condition (or the gapping condition) and varies with the spacetime dimension in a rather complicated pattern as enumerated in Tab. I for Majorana/Weyl fermions. However, it was recently discovered [74, 79, 80] that the SMG of Kähler-

Dirac fermions [81, 82] enjoys a strikingly simple flavor number requirement: every two copies of Kähler-Dirac fermions can always be gapped by SMG regardless of the spacetime dimension. The Kähler-Dirac fermion provides an alternative representation of gapless relativistic fermions, utilizing antisymmetric tensor fields as opposed to traditional spinor fields. This approach was initially conceived as an extension of the Kogut-Susskind staggered fermion [83]. It possesses an elegant geometric construction on the lattice and can be systematically defined in the flat spacetime of any dimension.

TABLE I. The required fermion flavor number  $N$  in each spacetime dimension  $D$  for SMG to occur.

$D = d + 1$	1	2	3	4	5	6	7	8
Majorana	8	8	16	16	32	32	64	64
Kähler-Dirac	2	2	2	2	2	2	2	2

The goal of this work is to explore the deeper reason behind the universal flavor number requirement for the SMG of Kähler-Dirac fermions. Although this flavor number requirement can be (and has been) obtained through anomaly cancellation analysis [74, 79, 80], here we would like to provide a more condensed-matter-oriented explanation. We first establish the connection that a single copy of interacting Kähler-Dirac fermion in any spacetime dimension  $D$  can be mapped to an  $O(D + 1)$  sigma model (NL $\sigma$ M) with a Wess-Zumino-Witten (WZW) term, describing the boundary of a bosonic SPT phase [60, 84] in the  $D$ -dimensional bulk. Then, we show that this bulk bosonic SPT phase is universally  $\mathbb{Z}_2$ -classified under the spatial inversion symmetry  $\mathbb{Z}_2^P$ , regardless of the spacetime dimension  $D$ . Therefore, every two copies of Kähler-Dirac fermion correspond to the boundary of a trivial SPT phase, which has no obstruction towards trivial gapping, thus allowing SMG to

occur.

Our method offers a unified theoretical framework for SMG of Majorana and Weyl fermions across all spatial dimensions, apart from the calculation of the anomaly-free condition. Furthermore, we provide an extensive exploration of the second quantization of the Kähler-Dirac equation, illustrating how fermions hop on the lattice and giving a very simple geometric interpretation to it.

## II. LATTICE MODEL

### A. Spacetime Lattice and Fermionic Fields

As illustrated in Fig. 1, the Kähler-Dirac fermion in flat spacetime can be defined on a hypercubic lattice equipped with the metric  $\eta^{\mu\nu}$ , where  $\eta^{\mu\nu}$  could correspond to either the Euclidean or Minkowski metric. A  $D$ -dimensional hypercubic lattice is defined by a set of orthonormal lattice vectors  $e^\mu$  ( $\mu = 0, 1, 2, \dots, d = D - 1$ ) with an inner product structure  $e^\mu \cdot e^\nu = -\eta^{\mu\nu}$ , such that each lattice point can be represented as a linear combination of these lattice vectors with integer coefficients, given by  $x = \sum_{\mu=1}^D m_\mu e^\mu$  with  $m_\mu \in \mathbb{Z}$ . Unlike conventional lattice fermion models, where fermion modes are defined on lattice points (0-chains) exclusively, in the lattice model of Kähler-Dirac fermions, each cell on the lattice is associated with a fermion mode. This approach avoids the doubling problem, ensuring that in the continuum limit of the lattice model, wherein fermionic fields are defined on forms, the same number of copies of Dirac fermions is retained.

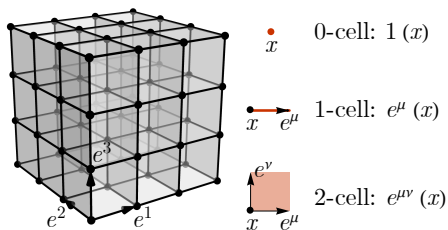


FIG. 1. A hypercubic lattice in  $D = 3$  dimensional flat spacetime, with a few examples of  $p$ -simplices. In fact, the cells aren't triangles; the simplex refers to a singular simplex in mathematics.

On the hypercubic lattice, the co-chain is isomorphic to chain, and we denote a basis of co-chain as  $e^\mu(x)$  as the dual of  $e_\mu(x)$ , which satisfies  $e^{x;\mu}(e_{x';\mu'}) = \delta_{x,x'} \delta_{\mu,\mu'}$ .

The cell centers form a refined lattice with each unit cell containing  $2^D$  points. The fermion field  $\psi_\mu(x)$  transforms as a rank- $p$  anti-symmetric tensor under point group rotations of the lattice. The field  $\psi_\mu(x)$  can pair up with the  $p$ -cell  $e^\mu(x)$  to define the following fermionic

field on the lattice:

$$\Psi(x) = \sum_{\mu} \psi_\mu(x) e^\mu(x), \quad (1)$$

which collects all fermion fields on the  $2^D$  cells within the unit cell labeled by  $x$ . As a result,  $\Psi(x)$  has  $2^D$  complex components correspondingly.

Similar to differential forms, we can define the fundamental operations of the exterior product  $\wedge$  and the interior product  $\iota$  for cells. The exterior product  $\wedge$  is defined as

$$e^\mu(x) \wedge e^{\mu'}(x') = \text{sgn}(\mu \cup \mu') \delta(x - x') e^{\mu \cup \mu'}(x), \quad (2)$$

where  $\mu \cup \mu'$  denotes the union of the two index sets  $\mu$  and  $\mu'$ , sorted in ascending order.  $\text{sgn}(\mu \cup \mu')$  represents the sign of the permutation required to sort the sequence  $\mu_1 \mu_2 \dots \mu_p \mu'_1 \dots \mu'_p$  into ascending order. If  $\mu$  and  $\mu'$  share at least one common index,  $\text{sgn}(\mu \cup \mu') = 0$  is assigned.

The interior product  $\iota$  is defined similarly

$$e^\mu(x) \iota e^{\mu'}(x') = \text{sgn}(\mu/\mu') \delta(x - x') e^{\bar{\mu}}(x), \quad (3)$$

We assume  $\mu \subset \mu'$  and  $\text{sgn}(\mu/\mu') = (-1)^p \text{sgn}(\mu \bar{\mu}) \text{sgn}(\mu')$ , where  $\bar{\mu}$  only contains elements doesn't belong to  $\mu$  but belongs to  $\mu'$ . Indeed, from a mathematical perspective, a lattice constitutes a particular type of chain, which makes the rigorous definition of interior product  $\iota$  and exterior product  $\wedge$  on a general lattice challenging and the true definition is cap product  $\cap$  and cup product  $\cup$  (details could be found in appendix.D). However, for hypercubic lattices, the computational rules remain consistent, enabling the straightforward use of exterior and interior product notations. This specificity streamlines the mathematical manipulations and contributes to the practical convenience of our model.

### B. Lagrangian Formulation

In this section, we discuss the operations of Kähler-Dirac fermions on a hypercubic lattice, as presented in the related literature [85].

The Kähler-Dirac operator, as a type of Dirac operator, acts on differential forms on the spacetime manifold as a "square root" of the Laplace operator  $\Delta = \partial^\mu \partial_\mu$ . In continuous spacetime, one can define the exterior derivative  $d = \partial_\mu dx^\mu \wedge$  and its adjoint  $\delta \equiv d^\dagger = \partial_\mu dx^\mu \iota$ . They satisfy  $d^2 = \delta^2 = 0$  and are related by the Hodge dual as  $\delta = (-1)^p \star^{-1} d \star$  when acting on  $p$ -forms. The Kähler-Dirac operator  $K$  is simply defined as

$$K = d - \delta = \partial_\mu dx^\mu \wedge - \partial_\mu dx^\mu \iota, \quad (4)$$

such that  $K^2 = (d - \delta)^2 = \star d \star d + d \star d \star = 2\Delta$ , where  $\Delta$  is the Laplace operator. This is consistent with the

property that the square of the Dirac operator produces the Laplace operator.

To define the Kähler-Dirac operator on the hypercubic lattice, two lattice difference operators should be introduced: the forward difference  $\partial_\mu^+ = T(e^\mu) - 1$  and the backward difference  $\partial_\mu^- = -(\partial_\mu^+)^\dagger = 1 - T(-e^\mu)$ , where  $T(e^\mu)$  denotes the translation operator that translates the field it acts on by the lattice vector  $e^\mu$ . More explicitly,

$$\partial_\mu^+ \psi_\nu(x) = \psi_\nu(x + e^\mu) - \psi_\nu(x), \quad (5)$$

$$\partial_\mu^- \psi_\nu(x) = \psi_\nu(x) - \psi_\nu(x - e^\mu). \quad (6)$$

With this notation, the lattice Kähler-Dirac operator  $K$  can be expressed as

$$K = \partial_\mu^- e^\mu \wedge - \partial_\mu^+ e^\mu \iota. \quad (7)$$

The lattice Kähler-Dirac operator  $K$  approaches its continuum limit in Eq. (4) when  $e^\mu \rightarrow dx^\mu$  and  $\partial_\mu^\pm \rightarrow \partial_\mu$ .

The lattice model of Kähler-Dirac fermions is then described by the following action

$$S = \sum_x \bar{\Psi} (\partial_\mu^- e^\mu \wedge - \partial_\mu^+ e^\mu \iota - m) \Psi, \quad (8)$$

where  $\Psi = \Psi_\mu e^\mu$  is the fermion field defined in Eq. (1).  $\bar{\Psi} = \Psi^\dagger e^0 (\wedge - \iota)$  symbolizes the conjugation of  $\Psi$ , and  $\bar{\Psi}$  is an independent field in the action. Taking the saddle point equation  $\delta S / \delta \bar{\Psi} = 0$  yields the Kähler-Dirac equation for fermions on a lattice:

$$(\partial_\mu^- e^\mu \wedge - \partial_\mu^+ e^\mu \iota - m) \Psi(x) = 0, \quad (9)$$

where  $m$  parameterizes the fermion mass. This describes  $2^{\lceil D/2 \rceil}$  copies of Dirac fermions in flat spacetime, where  $\lceil x \rceil$  represents the ceiling function. For instance, when  $d + 1 = 4$ , and the dimension of the Hilbert space,  $\dim(\mathcal{H}) = 2^4 = 16$ , can be decomposed into invariant subspaces  $\bigoplus_{a=1}^4 \mathcal{V}^a$ . The Kähler-Dirac equation implies  $(i\partial_\mu \gamma^\mu - m)\Phi^a = 0$ ,  $\Phi^a \in \mathcal{V}^a$ , which suggests that Kähler-Dirac fermions correspond to four copies of Dirac fermions in  $(3 + 1)d$  [86].

The analogy of exterior derivative  $d$  and its conjugation on lattice are  $d = \partial_\mu^- e^\mu \wedge$  and  $\delta = \partial_\mu^+ e^\mu \iota$ . In the context of the continuum limit,  $d$  and  $\delta$  correspond to nothing other than the differential operator  $d$  and co-differential  $\delta = (-1)^p \star^{-1} d \star$  acting on  $p$ -forms. By employing these operators, the action of Kähler-Dirac fermions can be formulated in a more geometric manner:

$$S = \int d^D x \bar{\Psi} (d - \delta - m) \Psi \quad (10)$$

### C. Hamiltonian Picture

We now provide a Hamiltonian formulation for Kähler-Dirac fermions following the process of second quantization.

At the single-particle level, we can formulate a Dirac-like equation similar to the Kähler-Dirac equation up to basis transformations. To obtain a Hamiltonian picture, we need to take a continuum limit in the time direction and keep the space components discrete in Minkowski spacetime. The equation of motion can be expressed as follows:

$$i\partial_0 \Psi(x) = i(\partial_i^- e^i \wedge - \partial_i^+ e^i \iota + m e^0 (\wedge - \iota)) \Psi(x), \quad (11)$$

where  $i \neq 0$ .

An equivalent form of the equation of motion is

$$i\partial_0 \Psi(x) = (i(d' - \delta') + im dx^0 (\wedge - \iota)) \Psi(x). \quad (12)$$

In these equations,  $d'$  and  $\delta'$  represent the restrictions of  $d$  and  $\delta$  on  $d$ -dimensional space, respectively. Here,  $d' e^{x;\mu} = \sum_{i=1}^d \text{sgn}(i \cup \mu) (e^{x-e^i; i \cup \mu} - e^{x+i; i \cup \mu})$  and  $\delta' e^{x;\mu} = \sum_{i=1}^d \text{sgn}(i/\mu) (e^{x;i/\mu} - e^{x+e^i; i/\mu})$ . The strength of this formulation lies in its capacity to maintain simplicity in the geometric interpretation of space components. Since we are interested in the case when the physical mass  $m = 0$  where SMG can happen, the Hamiltonian is further simplified as  $H = i(d' - \delta')$ . The second quantization of this Hamiltonian gives a tight-binding-like model:

$$H = \sum_{x,\mu} \left( \sum_i i \text{sgn}(i/\mu) (\psi_{x;i/\mu}^\dagger - \psi_{x+e^i; i/\mu}^\dagger) \psi_{x;\mu} + \text{h.c.} \right), \quad (13)$$

where the first summation  $\sum_{x;\mu}$  runs over all simplexes on an equal time slice, and the second summation  $\sum_i$  runs over the basis of space  $e^i$ . The annihilation operator is denoted as  $\psi_{x;e^\mu}$ , which nullifies fermions existing on the cell  $e^\mu(x)$ . The annihilation and creation operators satisfy  $\{\psi_{x;\mu}^\dagger, \psi_{x';\mu'}\} = \delta(x - x') \delta_{\mu,\mu'}$ , where  $x$  and  $x'$  belong to the same equal time slice. After the second quantization, each simplex is attributed with a fermion.

In order to simplify this equation, we define the annihilation operator on a chain rather than on a simplex [87], as  $\psi_{\sigma_1 + \sigma_2} = \psi_{\sigma_1} + \psi_{\sigma_2}$ , where  $\sigma_1$  and  $\sigma_2$  are distinct simplexes. Employing this configuration, we can express the Hamiltonian in a more geometric fashion:

$$H = i \sum_\sigma (\psi_\sigma^\dagger \psi_{\partial'\sigma} - \psi_{\partial'\sigma}^\dagger \psi_\sigma), \quad (14)$$

where the summation runs over all simplexes  $\sigma$  on an equal time slice, and  $\partial' = \sum_{i \neq 0} \text{sgn}(i \cup \mu) (e_\mu(x + e^i) - e_\mu(x))$  is the restriction of the boundary on the equal time slice, which may be a sum of different simplexes. It becomes evident that hopping occurs exclusively between a simplex and the restriction of the boundary of that simplex.

### III. NL $\sigma$ M FROM KÄHLER-DIRAC FERMIONS

In Section II B, we discuss the inclusion of Kähler-Dirac fermions, which comprise  $2^{\lceil D/2 \rceil}$  distinct copies of

Dirac fermions in the context of flat spacetime. This inherent redundancy enables the fermionic model to be effectively mapped onto a Nonlinear Model (NLM) through the coupling with an  $O(d+2)$  bosonic field [60]. In this section, we demonstrate the procedure for coupling the Kähler-Dirac fermions with the bosonic field, leading to the emergence of an NLM.

The bosonic field plays a crucial role in facilitating interactions among distinct flavors of Kähler-Dirac fermions. Prior studies have established that the flavor symmetry of Kähler-Dirac fermions in Euclidean space corresponds to the conformal symmetry group  $SO(1, D+1)$  [88], which is large enough to support the NLM.

To determine the additional  $(d+2)$  gamma matrices, we will briefly review the origins of the known gamma matrices. These matrices are derived from the square root of the Laplacian:

$$\sqrt{\partial_\mu \partial^\mu} = \gamma^\mu \partial_\mu. \quad (15)$$

Next, we consider the square root of  $-\partial_\mu \partial^\mu$ , as the square root of an operator is not necessarily linearly dependent on the square root of the negation of the same operator. To be more specific:

$$\sqrt{-\Delta} \neq \sqrt{-1} \sqrt{-\Delta}. \quad (16)$$

We can verify that  $\sqrt{-\Delta} = d + \delta$ , since  $(d + \delta)^2 = -(-d\delta - \delta d)$ , and we define it as  $\tilde{K} = d + \delta$  or the dual Kähler-Dirac operator. Upon choosing a basis, this operator can be decomposed into  $i\partial_\mu \tilde{\gamma}^\mu$  where  $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = -2\delta^{\mu\nu}$  in Euclidean spacetime. This operator  $\tilde{K}$  is linearly independent of  $K$  and generates the additional  $(d+1)$  gamma matrices. However, we still need to verify whether the new matrices mutually anti-commute with the original gamma matrices generated by  $K$ . It is straightforward to verify that  $\{K, \tilde{K}\} = \{d - \delta, d + \delta\} = \{d, \delta\} - \{\delta, d\} = 0$ , and since  $\partial_\mu$  are linearly independent and anti-commute with each other, we can demonstrate that  $(\gamma^\mu, \tilde{\gamma}^\mu), \mu = 1, 2, \dots, d+1$  extend the Clifford algebra to  $\mathcal{Cl}_{d+1, d+1}$ . Similar to the case of  $K$  we can represent  $\tilde{K}$  through the exterior product  $\wedge$  and the interior product  $\iota$ :

$$\tilde{K} = \partial_\mu (\wedge + \iota) dx^\mu, \quad \mu = 1, 2, \dots, d+1. \quad (17)$$

However,  $\gamma^\mu$  and  $\tilde{\gamma}^\mu$  do not encompass all matrices that mutually anti-commute. The structure of  $\mathcal{Cl}_{d+1, d+1}$  enables the definition of one additional chiral operator  $\Gamma$  as the highest-grade pseudo-scalar in  $\mathcal{Cl}_{d+1, d+1}$ , which serves as a generator of chiral symmetry and anti-commutes with all gamma matrices. In general, the Clifford algebra,  $\Gamma$  is the product of all other independent gamma matrices  $\prod_\mu \gamma_\mu \prod_\mu i\tilde{\gamma}_\mu$ , but calculating this product geometrically in our discussion proves to be challenging. Our approach involves identifying a geometric operator that acts on tensors and verifying that it anti-commutes with both  $K$  and  $\tilde{K}$ . Luckily, the geometric

interpretation of  $\Gamma$  is quite simple. It turns a  $p$ -chain into the same  $p$ -chain up to a sign  $(-1)^p$ :

$$\begin{aligned} \Gamma : \quad C^p &\longrightarrow C^p \\ e^\mu(x) &\longrightarrow (-1)^p e^\mu(x), \end{aligned} \quad (18)$$

where  $\mu = \mu_1 \mu_2 \dots \mu_p$ . In the continuum limit,  $\Gamma$  acts on  $p$ -forms as  $dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \rightarrow (-1)^p dx_1^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$ .

$\Gamma$  and  $\tilde{\gamma}^\mu$  don't appear in the original Kähler-Dirac equation and transfer as a spinor representation of  $SO(1, d+1)$ . To couple the Kähler-Dirac field with  $n_\mu$  and derive the nonlinear sigma model (NL $\sigma$ M) on the lattice, we need to replace the translation operator  $T(e_\mu)$  in  $\tilde{K}$  with  $e^{in_\mu}$  and approximate  $e^{in_\mu} - 1$  to  $in_\mu$ . Consequently, we obtain the following equation:

$$((\partial_\mu^- + in_\mu)e^\mu \wedge - (\partial_\mu^+ - in_\mu)e^\mu \iota + iN\Gamma)\Psi(x) = 0 \quad (19)$$

And the action is:

$$S = \sum_x \bar{\Psi}(x) ((\partial_\mu^- + in_\mu)e^\mu \wedge - (\partial_\mu^+ - in_\mu)e^\mu \iota + iN\Gamma)\Psi(x) \quad (20)$$

Also, we could write down the tight binding like model for this fermionic NL $\sigma$ M without physical mass term  $m$ .

$$\begin{aligned} H = \sum_{x, \mu} &\left( \left( \sum_{\mu=0}^d \sum_{i=1}^d i(-1)^{i/\mu} (\psi_{x, i/\mu}^\dagger - \psi_{x+e_i, i/\mu}^\dagger) \psi_{x, \mu} \right. \right. \\ &\left. \left. + n_\mu (-1)^{\mu/\mu} \psi_{x, \mu/\mu}^\dagger \psi_{x, \mu} + h.c. \right) + N(-1)^p \psi_{x, \mu}^\dagger \psi_{x, \mu} \right) \end{aligned} \quad (21)$$

#### IV. KÄHLER-DIRAC FERMIONS AS BSPT

$(\Gamma, \tilde{\gamma}_\mu)$  transforms as the spinor representation of  $O(d+2)$ , and the coupling between  $(N, n_\mu)$  and  $(\Gamma, \tilde{\gamma}_\mu)$  makes the order parameters select a specific direction, and the symmetry group transitions from  $O(d+2)$  to  $O(d+1)$ . This suggests that the effective theory of this model is the quotient space  $\frac{O(d+2)}{O(d+1)} \cong S^{d+1}$  NL $\sigma$ M. Since  $\pi_{d+1}(S^{d+1}) \cong \mathbb{Z}$  is nontrivial, this NL $\sigma$ M permits a topological theta term. This analysis can be corroborated by integrating out the fermionic degrees of freedom, and readers can consult reference [89] for further details regarding the field theory.

$$S_{d+1} = \int d^{d+1}x \left( \frac{1}{g^2} \partial_\mu n^a \partial^\mu n_a + i \frac{\theta}{\Omega_{d+1}} \epsilon_{ab\dots n^a} \partial_1 n^b \partial_2 n^c \dots \right) \quad (22)$$

$$\mu = 1, 2, \dots, d+1, \quad a, b, c, \dots = 0, 1, 2, \dots, d+1, \quad \theta = 2k\pi$$

Next, we need to determine which symmetry prevents the  $\theta$  term from being trivialized. On a lattice, our focus should be directed towards certain discrete subgroups of the orthogonal group. Specifically, the Kähler-Dirac equation possesses an inversion symmetry, denoted

as  $\mathbb{Z}_{2,P}$ , which protects  $\mathbb{Z}_2$  SPT phases. The action of the inversion symmetry is represented by the operator  $\mathcal{P} = \mathcal{I}e^0(\Lambda - \iota)$ , where 0 labels the time direction and  $(e^0(\Lambda - \iota))^2 = -1$ ,  $\mathcal{I}\partial^\mu\mathcal{I}^{-1} = -\partial^\mu$ , thus  $\mathcal{P}^2 = -1$ . The transformation acting on our model can be expressed as follows:

$$\begin{cases} N \rightarrow -N \\ n_\mu \rightarrow -n_\mu \\ \Psi \rightarrow e^0(\Lambda - \iota)\Psi \end{cases} \quad (23)$$

In the continuum limit with a certain basis, this symmetry can be represented as  $\mathcal{P} = \mathcal{I}dx^0(\Lambda - \iota) = \mathcal{I}\gamma^0$ . With this understanding in place, we are now prepared to discuss the classification of bosonic SPT phases. It is crucial to note that two copies of Kähler-Dirac fermions can be smoothly connected to a trivial state without breaking any symmetry. This conclusion is substantiated by the existence of an interlayer coupling, given by [84]:

$$S_{CP} = - \int d^D x A N_1 N_2 - B n_{1,\mu}^\dagger n_{2,\mu} \quad (24)$$

where 1, 2 denote different layers. Consequently, the effective field theory for the combined field  $n$  exhibits a vanishing theta term due to the cancellation of layers.

This coupling of bosonic field can help us derive the SMG interaction of fermions. On the meanfield level, the bosonic field  $N \sim \bar{\Psi}\Gamma\Psi$  and  $n^\mu \sim \bar{\Psi}e^\mu(\Lambda + \iota)\Psi$ . As a result, the interaction of  $S_{CP}$  can be ported as the following four fermions interaction:

$$\begin{aligned} S_{CP} = & - \int d^D x A \bar{\Psi}_1 \Gamma \Psi_1 \bar{\Psi}_2 \Gamma \Psi_2 \\ & - B \bar{\Psi}_1 e^\mu(\Lambda + \iota) \Psi_1 \bar{\Psi}_2 e^\mu(\Lambda + \iota) \Psi_2 \end{aligned} \quad (25)$$

In the Hamiltonian picture, we could get the SMG interaction by projecting bosonic field in equation (21) out. The coupling can be expressed as:

$$\begin{aligned} H_{CP} = & \int d^d x A \Psi_1^\dagger \Gamma \Psi_1 \Psi_2^\dagger \Gamma \Psi_2 - B \Psi_1^\dagger e^\mu(\Lambda + \iota) \Psi_1 \Psi_2^\dagger e^\mu(\Lambda + \iota) \Psi_2 \\ = & \sum_{\mu, \mu', \mu, x} A (-1)^{p+p'} \psi_{1,x,\mu}^\dagger \psi_{1,x,\mu} \psi_{2,x,\mu'}^\dagger \psi_{2,x,\mu'} \\ & - B \psi_{1,x,\mu/\mu}^\dagger \psi_{1,x,\mu} \psi_{2,x,\mu/\mu'}^\dagger \psi_{2,x,\mu'} \end{aligned} \quad (26)$$

where  $A \sim \langle \Psi_1^\dagger \Gamma \Psi_1 \Psi_2^\dagger \Gamma \Psi_2 \rangle$  and  $B \sim \langle \Psi_1^\dagger e^\mu(\Lambda + \iota) \Psi_1 \Psi_2^\dagger e^\mu(\Lambda + \iota) \Psi_2 \rangle$  come from four fermions condensing and  $p, p'$  is the length of  $\mu, \mu'$ . In the strong coupling limit, when  $H_{CP}$  dominates, we can demonstrate that the model possesses a unique and symmetric ground state. We proceed under the assumption that  $A \gg B$ , treating  $B \Psi_1^\dagger e^\mu(\Lambda + \iota) \Psi_1 \Psi_2^\dagger e^\mu(\Lambda + \iota) \Psi_2$  as a perturbation. It is important to note that  $H_{CP}$  does not intermix fields from different  $x$  variables, thus it can be understood

within the framework of 0+1 dimensional quantum mechanics. The ground state of  $H_0 = \int d^d x A \Psi_1^\dagger \Gamma \Psi_1 \Psi_2^\dagger \Gamma \Psi_2$  is two-fold degenerate, expressed as:

$$\begin{aligned} |\Psi^+\rangle &= |\Psi_{\text{even}}^1\rangle |\Psi_{\text{odd}}^2\rangle \\ |\Psi^-\rangle &= |\Psi_{\text{odd}}^1\rangle |\Psi_{\text{even}}^2\rangle \end{aligned} \quad (27)$$

Here,  $|\Psi_{\text{even}}^{1/2}\rangle = \prod_{x,\sigma \in \text{even cells}} \psi_{\sigma,x}^\dagger |Vac^{1/2}\rangle$  and  $|\Psi_{\text{odd}}^{1/2}\rangle = \prod_{x,\sigma \in \text{odd cells}} \psi_{\sigma,x}^\dagger |Vac^{1/2}\rangle$ , where  $|Vac^{1/2}\rangle$  is the vacuum state of sublattice 1 or 2.  $|\Psi_{\text{even}}^{1/2}\rangle$  and  $|\Psi_{\text{odd}}^{1/2}\rangle$  label states in which all even or odd simplexes of sublattice 1 or 2 are occupied by fermions.

And since  $H_1$  contains hopping from even dimensional simplexes and odd dimensional simplexes.  $\langle \Psi_+ | H_1 | \Psi_- \rangle \sim B$ . The energy degeneracy of  $|\Psi_\pm\rangle$  is removed and our model process a unique symmetric ground state:

$$\begin{aligned} |\Psi_{\text{gnd}}\rangle &= \frac{1}{\sqrt{2}} (|\Psi^+\rangle - |\Psi^-\rangle) \\ &= \frac{1}{\sqrt{2}} (|\Psi_{\text{even}}^1\rangle |\Psi_{\text{odd}}^2\rangle - |\Psi_{\text{odd}}^1\rangle |\Psi_{\text{even}}^2\rangle) \end{aligned} \quad (28)$$

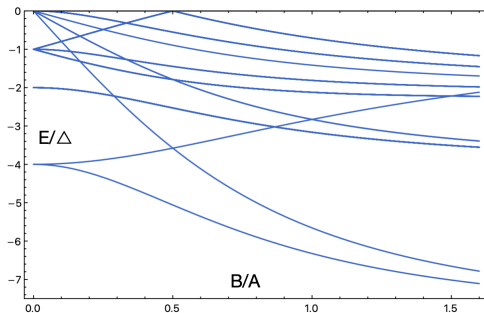


FIG. 2. Many-body energy levels in strong coupling limit of the single site quantum mechanics in 1+1d by exact diagonalization (256 $\times$ 256 matrix). X axis is  $\frac{B}{A}$  and Y axis is  $\frac{E}{\Delta}$ , where  $\Delta = \sqrt{A^2 + B^2}$ . When  $\frac{B}{A} = 0$ , the ground state has two fold degeneracy as expected. Otherwise, the many-body gap never closes.

Away from the perturbative limit, we can numerically check that the excitation gap never close as we tune the  $B/A$  ratio, as shown in Fig. 2. This implies that in the strong interaction limit, the two copies of the Kähler-Dirac fermion can indeed be gapped by the interaction without spontaneous symmetry breaking or topological ordering.

## V. CLASSIFICATION OF FERMIONIC SPT WITH $\mathbb{Z}_2^P$ SYMMETRY

In this section, our focus has been on the classification of fermionic SPT phases possessing inversion symmetry

$\mathbb{Z}_2^P$ . It is established that any two copies of  $(d-1)+1$ -dimensional massless Kähler-Dirac fermions, which serve as the boundary states of  $d+1$ -dimensional Kähler-Dirac fermions, can acquire a gap symmetrically. Consequently, our main objective is to determine the number of copies of root boundary fermions present in Kähler-Dirac fermions. The boundary state of a fermionic SPT phase in  $d+1$  dimensions is characterized by the Hamiltonian  $H = i\partial_i\gamma^i$ , where  $i = 1, 2, \dots, d-1$ , and  $\gamma_i$  corresponds to the real Clifford algebra  $Cl_{d-1,0}$ . The dimension of the real irreducible representation of  $Cl_{d-1,0}$  is denoted as  $dim_{\mathbb{R}}Cl_{d-1,0}$ . It is essential to note that Kähler-Dirac fermions comprise  $2^d$  independent flavor fermions and have a real dimension of  $dim_{\mathbb{R}}(\mathcal{H}) = 2dim_{\mathbb{C}}(\mathcal{H}) = 2 \times 2^d = 2^{d+1}$ . Thus,  $(d-1)+1$ -dimensional Kähler-Dirac fermions contain  $\nu$  copies of the root boundary fermions, where

$$\nu = \frac{2^{d+1}}{dim_{\mathbb{R}}Cl_{d-1,0}}. \quad (29)$$

Since  $\nu$  copies of our root state are Kähler-Dirac fermions, and two copies of Kähler-Dirac fermions are trivial, the corresponding fermionic SPT phases are  $\mathbb{Z}_{2\nu}$  classified.

Furthermore, the bulk state of SPT phases also undergoes SMG [60? ]. In the absence of interaction, the bulk is characterized by different topological numbers for distinct topological phases, and the bulk is gapless at the phase transition point. If the boundary state acquires mass symmetrically, the bulk does not undergo a phase transition either, and the original gapless phase transition point symmetrically acquires a mass gap as well. The parameter  $\nu$  can be calculated from the bulk perspective as follows:

$$\nu = \frac{2^{d+2}}{dim_{\mathbb{R}}Cl_{d,1}} \quad (30)$$

where  $2^{d+2}$  is the dimension of bulk Kähler-Dirac fermions, and the root fermions described by  $i\partial_\mu\gamma^\mu\psi = 0$  in Minkowski spacetime represent  $Cl_{d,1}$ .

## VI. CONCLUSIONS

In conclusion, we have explained the symmetric mass generation (SMG) of Kähler-Dirac fermions by extending the algebraic structure of the Kähler-Dirac field and

defining a new operator,  $\tilde{K}$ . Our study shows that Kähler-Dirac fermions correspond to bosonic symmetry protected topological (SPT) phases, specifically  $\mathbb{Z}_2$  SPT phases protected by  $\mathbb{Z}_2^P$  symmetry. Constructing an  $O(D+2)$  nonlinear sigma model (NL $\sigma$ M) with a topological term, we have unveiled connections between interacting Kähler-Dirac fermions and condensed matter physics, as well as applications in classifying interacting fermionic SPT phases.

TABLE II. The classification of  $\mathbb{Z}_2^P$  fermionic SPT phases. Where  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{H}$  denote real numbers, complex numbers and quaternions and the notation  $\mathbb{F}(N)$  denotes the algebra of  $N \times N$  matrices over  $\mathbb{F}$ . And  $dim_{\mathbb{F}}(N) = Ndim_{\mathbb{F}}$ . The result of classification in  $d+8$  dimension comes from Bott periodicity.

d	$Cl_{d-1,0}$	$Cl_{d,1}$	$\nu$	classification
1	$\mathbb{R}$	$\mathbb{R}(2)$	4	$\mathbb{Z}_8$
2	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	4	$\mathbb{Z}_8$
3	$\mathbb{R}(2)$	$\mathbb{R}(4)$	8	$\mathbb{Z}_{16}$
4	$\mathbb{C}(2)$	$\mathbb{C}(4)$	8	$\mathbb{Z}_{16}$
5	$\mathbb{H}(2)$	$\mathbb{H}(4)$	8	$\mathbb{Z}_{16}$
6	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	8	$\mathbb{Z}_{16}$
7	$\mathbb{H}(4)$	$\mathbb{H}(8)$	16	$\mathbb{Z}_{32}$
8	$\mathbb{C}(8)$	$\mathbb{C}(16)$	32	$\mathbb{Z}_{64}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
d+8	$Cl_{d-1} \otimes \mathbb{R}(16)$	$Cl_{d,1} \otimes \mathbb{R}(16)$	$16\nu_d$	$\mathbb{Z}_{32\nu_d}$

Moreover, our approach to constructing bosonic SPT phases has proven to be applicable to lattice models. We demonstrate that the boundary of these fermions consists of massless Kähler-Dirac fermions. Utilizing this relationship, we have explored the bulk boundary correspondence and discussed the classification of  $\mathbb{Z}_2$  fermionic SPT. Our findings reveal that only an even number of Kähler-Dirac fermion copies can symmetrically acquire a mass gap, regardless of the spacetime dimensions in which they are defined. To be specific, eight copies of massless Dirac fermions can symmetrically gap in (3+1)d space and time. This work broadens our understanding of symmetric mass generation, providing valuable insights into the topological properties of interacting fermionic systems.

## ACKNOWLEDGMENTS

This research is supported by the NSF Grant DMR-2238360.

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## Appendix A: Review of Continuum Kähler-Dirac Fermions

In this section, we present a concise overview of the principle of Kähler-Dirac fermions and the fundamental notations of differential geometry. The Kähler-Dirac equation generalizes the Dirac equation and was originally proposed by Kähler, who demonstrated that fermions with spin 1/2 can be formulated from differential forms. To better comprehend the Kähler-Dirac equation, we need to establish some geometrical prerequisites.

Firstly, we introduce the notations of differential and dual-differential. The operator  $d$  maps  $p$ -forms to  $p + 1$ -forms. The Hodge dual of a  $k$ -form  $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$  is an  $(n-k)$  form that corresponds to the complementary set  $\bar{I} = \bar{i}_1 < \dots < \bar{i}_{n-k}$ . This can be expressed as:

$$\star dx^I = \text{sgn}(\sigma(I)) t dx^{\bar{I}} \quad (\text{A1})$$

where  $\text{sgn}(\sigma(I))$  is the sign of permutation  $\sigma(I) = i_1 \dots i_k \bar{i}_1 \dots \bar{i}_{n-k}$  and  $t = \Pi_i(dx^i \iota dx^i)$ . Using the identity that applying the Hodge star twice leaves a  $p$  form unchanged up to a sign, i.e.,  $\star \star \omega^p = (-1)^{p(n-p)} \omega^p$ , where  $\eta = 1$  for Euclidean spacetime and  $\eta = -1$  for Minkowski spacetime. We can define the inverse of Hodge star as  $\star^{-1} = (-1)^{p(n-p)} \star$ . The notation of Hodge dual helps us define the dot product on the lattice. The interior product of two  $p$ -forms  $\omega$  and  $\eta$  is given by  $(\omega, \eta) = \int_{\mathcal{M}} \bar{\omega} \wedge \star \eta$ , where  $\bar{\omega}$  is complex conjugation of  $\omega$ . On manifolds with this interior product structure, we can find the adjoint operator of the differential operator,  $(\omega, d\eta) = (\delta\omega, \eta)$ , where  $\delta = (-1)^p \star^{-1} d \star$  satisfies this property. Since the differential operator satisfies  $d^2 = 0$ , the dual differential also has the corresponding property  $\delta^2 = 0$ .

Now, we can define the Kähler-Dirac equation in continuum Euclidean space and time:

$$(d - \delta - m)\Phi = 0 \quad (\text{A2})$$

where  $\Phi$  denotes the superposition of different differential forms  $\Phi = \sum_{I_H} \phi_{I_H} dx^{I_H}$ , and  $K = d - \delta$  is the Kähler operator. The Kähler-Dirac equation generalizes the Dirac equation since  $K^2 = (d - \delta)^2 = -(d\delta + \delta d)$  is the Laplace operator on manifolds. And  $(d - \delta)^\dagger = -(d - \delta)$  is an anti-symmetric operator which can be decomposed into  $\partial_\mu \gamma^\mu$ , where  $\gamma^\mu$  are  $d + 1$  mutually anti-commuting gamma matrices .

Moreover, the Kähler-Dirac equation can be written using the notation of Clifford product:

$$(\partial_\mu \vee^- dx^\mu - m)\Phi = 0 \quad \mu = 1, 2 \dots, d + 1 \quad (\text{A3})$$

where  $\vee^- = \wedge - \iota$  denotes Clifford product on vector space and  $\iota$  is the interior product, where  $dx^\mu \iota dx^\nu = \eta^{\mu\nu}$ . If we represent Clifford product in the space of differential forms, we can get the following corresponding relation  $\vee^- dx^\mu \rightarrow \gamma^\mu$ , where  $\gamma^\mu$  satisfy  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

is representation of Clifford algebra  $\mathcal{Cl}(d+1)$  in  $d+1$  dimensional flat spacetime.

It's straightforward to get the Kahler-Dirac equation in the presence of gauge field by replaced the derivative  $\partial^\mu$  by covariant derivative  $\partial^\mu - ieA^\mu$

$$((\partial_\mu - ieA_\mu) \nabla^- dx^\mu - m)\Phi = 0 \quad \mu = 1, 2 \dots, d+1 \quad (\text{A4})$$

## Appendix B: Domain wall Fermions of Kähler-Dirac Fermions

To discuss the anomaly inflow of Kähler-Dirac fermions, we need to indentify what the boundary sates of Kähler-Dirac fermions look like. We consider domain wall fermion in  $d+1$  dimension defined on manifold  $M^{d-1} \times R$  with coordinates  $(x^a, z)$ ,  $a = 1, 2 \dots d-1$ . Mass term  $M(z)$  is function of  $z$  and changes it sign at  $z = 0$  and we except massless fermions would appear on the boundary

$$(\partial_t \nabla^- dt + \partial_a \nabla^- dx^a + \partial_z \nabla^- dz - M(z))\psi(x^a, z, t) = 0 \quad (\text{B1})$$

where  $M(z) > 0$  when  $z > 0$  and  $M(z) < 0$  when  $z < 0$  and we choose metric  $\eta^{\mu\nu} = (-, +, \dots, +)$ .  $\psi$  can be decomposed into  $\psi(x^a, z, t) = \chi(x^a, t) \wedge \phi(z)$ , where  $\chi(x^a)$  only contain forms without  $dz$  and don't doesn't on  $z$ . And it gives:

$$\begin{aligned} \partial_z \nabla^- dz \phi(z) &= M(z) \phi(z) \\ (\partial_t \nabla^- dt + \partial_a \nabla^- dx^a) \chi(x^a) &= 0 \end{aligned} \quad (\text{B2})$$

since  $(\nabla^+ dz)^2 = 1$  in Minkowski space and time and the eigenstate  $i \nabla^+ dz$  with eigenvalue 1 is normalized state localized near the boundary. This equation described a exponentially localized domain wall fermion. And the freedoms the constraint of different forms on surface. And  $\chi(x^a)$  satisfied the equation of motion of massless  $(d-1) + 1$  dimensional Kähler-Dirac fermions.

## Appendix C: Continuum Kähler Ddirac fermions in $2D$

In this appendix, we present the explicit procedure for calculating the Kähler-Dirac model in  $2D$ . Any forms can be represented as  $\Phi = \phi + \phi_1 dx^1 + \phi_2 dx^2 + \phi_{12} dx^1 \wedge dx^2$ . Moreover,  $d\Phi = \partial_1 \phi dx^1 + \partial_2 \phi dx^2 + (\partial_1 \phi_2 - \partial_2 \phi_1) dx^1 \wedge dx^2$ . In the basis  $1, dx^2, dx^1, dx^1 \wedge dx^2$ , we can employ the

vector  $\begin{pmatrix} \phi \\ \phi_2 \\ \phi_1 \\ \phi_{12} \end{pmatrix}$  to represent all combinations of forms.

And  $\star\Phi = \phi_{12} - \phi_1 dx^2 + \phi_2 dx^1 + \phi dx^1 \wedge dx^2$  In the chosen

basis, both  $d$  and the Hodge star  $\star$  can be expressed as:

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 \\ 0 & \partial_1 & -\partial_2 & 0 \end{pmatrix}, \star = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (\text{C1})$$

By making use of the relation  $\star^{-1} = (-1)^{p(n-p)} \eta \star$ , we can demonstrate that  $\delta = -\star d\star$  in the two-dimensional case. Upon performing some matrix calculations, we establish that  $\delta$  is the Hermitian conjugation of  $d$ .

$$\begin{aligned} \delta &= -\star d\star = \\ &= - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 \\ 0 & \partial_1 & -\partial_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \partial_2 & \partial_1 & 0 \\ 0 & 0 & 0 & \partial_1 \\ 0 & 0 & 0 & -\partial_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{C2})$$

And then we can check whether  $K$  and  $\tilde{K}$  can be decomposed into gamma matrixes.

$$K = (d - \delta) = \begin{pmatrix} 0 & -\partial_2 & -\partial_1 & 0 \\ \partial_2 & 0 & 0 & -\partial_1 \\ \partial_1 & 0 & 0 & \partial_2 \\ 0 & \partial_1 & -\partial_2 & 0 \end{pmatrix} = -i\partial_1 \sigma^{20} - i\partial_2 \sigma^{32} \quad (\text{C3})$$

$$\tilde{K} = d + \delta = \begin{pmatrix} 0 & \partial_2 & \partial_1 & 0 \\ \partial_2 & 0 & 0 & \partial_1 \\ \partial_1 & 0 & 0 & -\partial_2 \\ 0 & \partial_1 & -\partial_2 & 0 \end{pmatrix} = \partial_1 \sigma^{10} + \partial_2 \sigma^{31} \quad (\text{C4})$$

## Appendix D: Products on Lattice

Though we use the idea of exterior product and interior product on lattice, but in fact cap product and cup product are mathematical rigorous definition of products on lattice. We introduce these ideas in this appendix. More mathematic details can be found in Ref[90]. Using the rigorous definition, we can define Kähler-Dirac equation on arbitrary lattice whose co-chain is isomorphic to chain.

First, our field is grassmanian number value co-chain rather than simplex on hypercubic lattice, which maps a cell to a grassmanian number. But the co-lattice and lattice is isomorphic for hypercubic lattice, so we don't bother distinguish it in the above discussion. On the hypercubic lattice, the co-chain is isomorphic to chain, and we denoted a basis of co-chain as  $\omega^\mu(x)$  as the dual of cell  $e^\mu(x)$  which satisfied  $\omega^\mu(x)(e_{\mu'}(x')) = \delta_{x,x'} \delta_{\mu,\mu'}$ . The cup product  $\cup$  is the generalization of the exterior

TABLE III. In the accompanying table, we delineate the manifestations of the geometric quantities  $K$ ,  $\tilde{\gamma}$ , and  $\Gamma$  within our carefully selected basis of Euclidean spacetime. This basis has been chosen to facilitate the representation of all geometric quantities as tensor products of Pauli matrices. The specific purpose of this choice is to ensure that  $K$  can be depicted as a tensor product of Pauli matrices, corroborated by confirming that  $K$  has a real representation. The procedure to select this basis is rooted in a systematic approach, dividing the differential forms into two distinct categories. The initial categorization is based on the presence or absence of  $dx^1$ ; forms that contain  $dx^1$  comprise the second group, while those without are part of the first group. This process is iterated within each group, further segregating them into subgroups based on whether they contain  $dx^2$ . This procedure is replicated until the basis is unequivocally determined.

d+1	K	$\Gamma$	$\tilde{\gamma}_i$	Basis
1	$-i\partial_1\sigma^2$	$\sigma^3$	$i\sigma^1$	$1, dx^1$
2	$-i\partial_1\sigma^{20} - i\partial_2\sigma^{32}$	$\sigma^{33}$	$i\sigma^{10}, i\sigma^{31}$	$1, dx^2, dx^1, dx^1 \wedge dx^2$
3	$-i\partial_1\sigma^{200} - i\partial_2\sigma^{320} - i\partial_3\sigma^{332}$	$\sigma^{333}$	$i\sigma^{100}, i\sigma^{310}, i\sigma^{331}$	$1, dx^3, dx^2, dx^2 \wedge dx^3, dx^1, dx^1 \wedge dx^3, dx^1 \wedge dx^2, dx^1 \wedge dx^2 \wedge dx^3$
4	$-i\partial_1\sigma^{2000} - i\partial_2\sigma^{3200} - i\partial_3\sigma^{3320} - i\partial_4\sigma^{1000}$	$\sigma^{3333}$	$i\sigma^{3310}, i\sigma^{3320}, i\sigma^{3331}, i\sigma^{2000}$	...
...	...	...	...	...

product  $\wedge$  acting on co chain, and is determined by the following equation:

$$\omega^{x;\mu} \cup \omega^{x';\mu'} = (-1)^{\mu \cup \mu'} \delta^{x,x'} \omega^{x;\mu \cup \mu'} \quad (D1)$$

where  $(-1)^{\mu \cup \mu'}$  represents the sign of permutation  $(\mu_1, \mu_2, \dots, \mu'_1, \mu'_2, \dots)$ , with  $\mu_1 = (\mu_1 < \mu_2 < \dots < \mu_p)$ ,  $\mu_1 < \mu_2 < \dots < \mu_p$  and  $\mu_2 = (\mu'_1, \mu'_2, \dots)$ ,  $\mu'_1 < \mu'_2 < \dots < \mu'_p$ .

Although the cap product is initially defined between co-chain and chain, the application of Pontrjagin duality permits the definition of the interior product of co-chain  $\omega^{x;\mu_1 \dots \mu_k} \cap \omega^{x';\mu'_1 \dots \mu'_k} = \eta^{\mu_1, \mu'_1} \dots \eta^{\mu_p, \mu'_p} \delta_x^x$ . The cap product  $\cap$  can be considered as the Hermitian conjugation of the cup product  $\cup$ , adhering to the relation  $(\omega_1, \rho \cup \omega_2) = (\rho \cap \omega_1, \omega_2)$ :

$$\omega^{x;\mu} \cap \omega^{x';\mu'} = (-1)^{\mu/\mu'} \delta^{x,x'} \omega^{x;\mu/\mu'} \quad (D2)$$

By introducing the concepts of cap product  $\cap$  and cup product  $\cup$ , the lattice Kähler-Dirac fermions can be further simplified and made analogous to the conventional definition of continuum Kähler-Dirac fermions through Clifford product. We shall not delve into the mathematical details of products on co-chain but will only demonstrate how to compute them on a square lattice. More comprehensive information regarding algebraic topology. Utilizing the notation, above we can reformulate the Kähler-Dirac equation on a square lattice as:

$$(\partial_\mu^- e^\mu \cup + \partial_\mu^+ e^\mu \cap - m) \Phi(x) = 0 \quad (D3)$$

where  $e^\mu = \sum_x e^{x;\mu}$  is summation of 1 co-chain on the square lattice.

Deriving the continuum Kähler-Dirac fermions from the aforementioned equation is rather straightforward. In the continuum model, there is no distinction between  $\partial^{\pm, \mu}$ , and the 1 co-chain  $e^\mu$  should be replaced by the 1-form  $dx^\mu$ :

$$(\partial_\mu dx^\mu (\wedge - \iota) - m) \Phi(x) = 0 \quad (D4)$$

Here, the exterior product  $\wedge$  and interior product  $\iota$  are the direct analogs of cup product  $\cup$  and cap product  $\cap$ , respectively.  $\Phi$  is the superposition of distinct differential forms, denoted as  $\Phi(x) = \sum_{I_H} \phi_{I_H} dx^{I_H}$ . Furthermore, we consistently represent  $\wedge - \iota$  as the Clifford product  $\nabla^-$ .

TABLE IV. In the following table, we present a dictionary between operators on manifolds and their counterparts on lattices. Our analysis reveals a one-to-one correspondence between these operators, demonstrating that our discussion can be equally applicable to both lattice and manifold models in subsequent sections of this paper.

Continuum or hypercubic lattice	General lattice
Tensor $\mathcal{T}_V(p, 0)$	p chain $C_p$
Hodge star $\star$	Hodge star $\star$
differential form $\Lambda^p$	co-chain (cells) $C^p$
exterior product $\wedge$	cup product $\cup$
Interior product $\iota$	cap product $\cap$
exterior differential d	co-boundary d
dual differential $\delta$	dual co-boundary $\delta$
partial $\partial_\mu$	backward/forward partial $\partial_\mu^\pm$