Emergent higher-symmetry protected topological orders in the confined phase of math xmlns="http://www.w3.org/1998/Math/MathML">mrow>mi>U/mi>mo>(/mo>mn>1/mn>mo>)/mo>/mrow>/math> gauge theory

Salvatore D. Pace and Xiao-Gang Wen

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Emergent Higher-Symmetry Protected Topological Orders in the Confined Phase of $U(1)$ Gauge Theory

Salvatore D. Pace and Xiao-Gang Wen
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
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We consider compact $U^\kappa(1)$ gauge theory in 3+1D with a general 2π-quantized topological term $\sum_{I,J=1}^{\kappa} \frac{K_{IJ}}{4\pi}\int_{M^4} F^I \wedge F^J$, where K is an integer symmetric matrix with even diagonal elements and $F^I = \mathrm{d}A^I$. At energies below the gauge charges' gaps but above the monopoles' gaps, this field theory has an emergent $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry, where k_i are the diagonal elements of the Smith normal form of K and $\mathbb{Z}_0^{(1)}$ is regarded as a $U(1)$ 1-symmetry. In the $U^\kappa(1)$ confined phase, the boundary can have a phase whose IR properties are described by Chern-Simons field theory. Such a phase has a $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry that can be *anomalous*. To show these results, we develop a bosonic lattice model whose IR properties are described by this continuum field theory, thus acting as its UV completion. The lattice model in the aforementioned limit has an exact $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. We find that the short-range entangled gapped phase of the lattice model, corresponding to the confined phase of the $U^\kappa(1)$ gauge theory, is a symmetry protected topological (SPT) phase for the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry, whose SPT invariant is $\mathrm{e}^{\mathrm{i}\,\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I \smile B_J + B_I \underset{1}{\smile} \mathrm{d}B_J}\, \mathrm{e}^{\mathrm{i}\,\pi \sum_{I<J} K_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_I \underset{2}{\smile} \mathrm{d}B_J}$. Here, the background $\mathbb{R}/\mathbb{Z}$-valued 2-cochains B_I satisfy $\mathrm{d}B_I = \sum_I B_I K_{IJ} = 0 \mod 1$ and describe the symmetry twist of the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. We apply this general result to a few examples with simple K matrices. We find the non-trivial SPT order in the confined phases of these models and discuss its classifications using the fourth cohomology group of the corresponding 2-group.

I. INTRODUCTION

Symmetry protected topological (SPT) phases of quantum matter are short-range entangled gapped phases whose ground states cannot be adiabatically connected to a trivial product state due to the presence of a symmetry [1–4]. The boundary of a SPT phase is nontrivial because the symmetry is realized anomalously on the boundary. Since the SPT bulk has a trivial intrinsic topological order, the boundary theory by itself (*i.e.* without bulk) is perfectly consistent as a lattice theory, and the 't Hooft anomaly ensures it cannot be in a trivial phase. However, upon turning on background gauge fields of the symmetry, the boundary theory is no longer a physical theory and can only exist as the boundary of an invertible phase (i.e., an SPT). From the prospective of anomaly inflow, the SPT order in the bulk provides a unique characterization of the 't Hooft anomaly on the boundary.

Since their discovery, there have been numerous interesting generalizations of SPT phases. This includes SPT phases with intrinsic topological order in the bulk [5–8], SPT phases with a gapless bulk [9–11], and higher-order SPTs where edge states exist only on a subregion of the boundary [12, 13]. In this paper, we consider the generalization where the SPT order is protected by a higher-symmetry [14–21].

Global symmetries, called 0-symmetries, are symmetries whose transformation acts on a codimension-1 submanifold of spacetime (e.g., all of space), and the charged operators act on a single point in spacetime, creating a 0-dimensional object in space. Higher-symmetry is a generalization that includes p-symmetry, where now the symmetry transformation acts on a codimension-$(p+1)$ submanifold of spacetime and the charged operators act on p-dimensional closed submanifolds [22–26]. A p-symmetry is mathematically described by a $(p+1)$-group[1] [30–32]. Just like global symmetries, higher-symmetries can be spontaneously broken [33], can be anomalous [34], and can be gauged [35].

Generic lattice Hamiltonians do not commute with closed string, membrane, *etc* operators and thus do not have exact higher-symmetries. Instead, the lattice models with exact higher-symmetries are quite special. For instance, the Hamiltonians of many exactly soluble models [36–41] with topological orders [42–44] have exact higher-symmetries. While higher-symmetries are typically not exact UV symmetries, they can nevertheless be *emergent* symmetries occurring in the IR. Intuitively, this is because at high energies the charged p-dimensional closed objects can become open, and $(p-1)$-dimensional excitations living on their boundary explicitly breaks the p-symmetry. At energies smaller than the gap of the excitations that explicitly break a higher-symmetry, the corresponding higher-symmetry can emerge. While emergent 0-symmetries are typically approximate, emergent higher-form symmetries can exactly constraint the IR despite not being UV symmetries [19, 45–53]. In this sense, emergent higher-form symmetries are exact emergent symmetries [53].

[1] Here we will consider only pure p-symmetries, and not the more general p-group symmetries where there are multiple symmetries of different degrees that mix [27–29].

This gives rise to an interesting scenario where some low-energy excitations condense, while the higher-symmetry breaking excitations remain to have a large energy gap. If the condensed phase happens be a short-range entangled state [54] with the exact emergent higher-symmetry, it can be an SPT phase protected by the exact emergent higher symmetry [19]. We denote such a higher SPT phase as an n-SPT phase if it is protected by an n-symmetry.

In particular, suppose a higher-form symmetry emerges at $E < E_{\text{mid-IR}}$, is not spontaneously broken, and is realized anomalously on the boundary. A corresponding nontrivial SPT order could also emerge at $E < E_{\text{mid-IR}}$ and cause the system to be in an SPT phase. The nontrivial bulk SPT order allows an IR observer to turn on background gauge fields due to an anomaly inflow mechanism. The bulk theory for said IR observer would be an invertible topological field theory, called the SPT invariant, which characterizes the SPT order and its universal physical properties (i.e., through a generalized magneto-electric effect [55]). Furthermore, since emergent higher-form symmetries are exact emergent symmetries [53], no local IR measurements could reveal that the high-form symmetry is not exact in the UV.

However, according to a UV observer, the bulk theory would not follow the typical SPT lore since it is the IR degrees of freedom forming an SPT, not the UV degrees of freedom. It is conceivable that a UV observer could directly probe the topological response of the SPT by measuring UV degrees of freedom in a very particular way to couple to the IR degrees of freedom. Nevertheless, there are still direct signs of the emergent SPT order at the boundary, even in the UV. In the context of the SPTs we consider here, the boundary has nontrivial abelian topological order and thus this UV observer could measure the anyon excitations and their nontrivial braiding. The gap of the anyons would be on the scale $E_{\text{mid-IR}}$, and their presence would explicitly break the higher-form symmetries in the UV. However, their braiding would nevertheless reflect the 't Hooft anomaly structure and thus the emergent SPT order.

In this paper, we extend the work presented in Ref. 19 and further investigate this mechanism for creating SPT phases protected by emergent higher-symmetries. In particular, we consider abelian gauge theory in 3+1D which at energies smaller than the gauge charge's and monopole's gap has two exact emergent $U(1)$ 1-symmetries (which we denote as $U(1)^{(1)}$) commonly denoted as the electric and magnetic symmetries. In the strong coupling limit, the gauge theory is in a short-range entangled gapped confined phase where the monopoles condense and the magnetic $U(1)^{(1)}$ symmetry is explicitly broken. However, at energies below the gauge charge gap, the electric symmetry is still present in the confined phase. This gives rise to the aforementioned scenario and the possibility that the confined phase has nontrivial 1-SPT order protected by the emergent electric symmetry.

In fact, here we will show that with 2π-quantized topological terms, the confined phase of abelian gauge theory has nontrivial emergent 1-SPT orders. Usually, a topological term can affect the dynamics of the strong coupling limit in a very non-trivial way, and can make it impossible to calculate the physical properties (such as energy gap) in this limit. However, a 2π-quantized topological terms are much easier to handle [2, 56–58], and we can still determine the strong coupling limit to be a short-range entangled gapped confined phase.

The remaining of this paper is organized as follows. In section II we introduce the notations used in this paper. In section III, we present a simple example of a nontrivial 1-SPT order in the confined phase of 3+1D $\mathbb{Z}_2$ gauge theory. In doing so, we review the cochain lattice field theory formalism and techniques which we use throughout the rest of the paper. Then, in section IV we consider the same scenario but in 3+1D abelian gauge theory with κ-types of $U(1)$ gauge fields and 2π-quantized topological terms. Using the bosonic lattice model we develop, we find that the total emergent electric symmetry $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$ below the gauge charges' gaps and the monopoles' gaps is reduced to $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ at energies above the monopoles' gaps (but still blow the gauge charges' gaps). Subsequently, we find that the confined phase of $U^\kappa(1)$ gauge theory with 2π-quantized topological terms has nontrivial $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-SPT order. We construct the associated 1-SPT invariant and consider some examples.

II. NOTATIONS AND CONVENTIONS

In this paper, we will use the notion of cochain, cocycle, and coboundary, as well as their higher cup product $\underset{k}{\smile}$ and Steenrod square $\mathbb{Sq}^k$. A pedagogical introduction aimed at physicists of chains and cochains along with the cup product $\smile \equiv \underset{0}{\smile}$ and higher cup products $\underset{k}{\smile}$ can be found in the Appendix of Ref. 20. We will abbreviate the cup product $a \smile b$ as ab by dropping $\smile$. We will also use $\overset{n}{=}$ to mean equal up to a multiple of n, and use $\overset{\text{d}}{=}$ to mean equal up to $\text{d}f$ (*i.e.* up to a coboundary). An important identity which we will repeatedly use is that for cochains f_m, h_n,

$$\begin{aligned}\text{d}(f_m \underset{k}{\smile} h_n) &= \text{d}f_m \underset{k}{\smile} h_n + (-)^m f_m \underset{k}{\smile} \text{d}h_n + \\ &(-)^{m+n-k} f_m \underset{k-1}{\smile} h_n + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m.\end{aligned} \tag{1}$$

Furthermore, in this paper we will deal with many $\mathbb{Z}_n$-valued quantities. We will denote them as, for example, $a^{\mathbb{Z}_n}$. However, we will always lift the $\mathbb{Z}_n$-value to $\mathbb{Z}$-value, so the value of $a^{\mathbb{Z}_n}$ has a range from $-\lfloor \frac{n}{2} \rceil$ to $\lfloor \frac{n}{2} \rceil$, where $\lfloor x \rceil$ denotes the integer that is closest to x (if two integers have the same distance to x, we will choose the smaller one, *e.g.* $\lfloor \frac{1}{2} \rceil = 0$). In this case, the expression like $a^{\mathbb{Z}_n} a^{\mathbb{Z}_m}$ makes sense.

III. $\mathbb{Z}_2^{(1)}$ 1-SPT ORDER IN 3+1D THEORIES

In this section, we review one of the simplest ways to realize $\mathbb{Z}_2^{(1)}$ 1-SPT order in 3+1D [19, 32]. Our purpose of doing so is to introduce the formalism we use and warm-up in a simple context before beginning section IV, where things get more involved. We start by considering a twisted $\mathbb{Z}_2$ 2-gauge theory. By considering its confined phase, we then construct a model with $\mathbb{Z}_2^{(1)}$ 1-SPT order by "ungauging" the twisted $\mathbb{Z}_2$ 2-gauge theory. The $\mathbb{Z}_2^{(1)}$ 1-SPT order is exact in this model, but survives elsewhere in the confined phase diagram as an exact emergent $\mathbb{Z}_2^{(1)}$ 1-SPT, existing at energies much smaller than the $\mathbb{Z}_2$ gauge charge gap.

A. Twisted $\mathbb{Z}_2$ 2-gauge theory

To construct a 3+1D bosonic model that realizes $\mathbb{Z}_2^{(1)}$ 1-SPT order, we first construct a local bosonic model with topological order described by a $\mathbb{Z}_2$ 2-gauge theory. We triangulate spacetime $\mathcal{M}^4$ and, working in the Euclidean signature, consider the lattice path integral of cochain fields [59]. The bosonic degrees of freedom for the $\mathbb{Z}_2$ 2-gauge theory are described by a $\mathbb{Z}_2$-valued 2-cochain field $b^{\mathbb{Z}_2}$. As a 2-cochain, $b^{\mathbb{Z}_2}$ is a map from 2-chains to $\mathbb{Z}_2$, as opposed to 1-gauge theory which is described by a 1-cochain field.

Consider the local bosonic model:

$$Z(\mathcal{M}^4, g) = \sum_{b^{\mathbb{Z}_2}} \mathrm{e}^{-\frac{1}{2g}\sum_{ijkl}\left(\mathrm{d}b^{\mathbb{Z}_2}_{ijkl} - 2\lfloor \frac{1}{2}\mathrm{d}b^{\mathbb{Z}_2}_{ijkl}\rceil\right)}, \quad (2)$$

where $\sum_{ijkl}$ sums over all spacetime 3-simplices for a fixed triangulation, and $\sum_{b^{\mathbb{Z}_2}}$ sums over all the 2-cochain field corresponding to the path integral. In the exactly solvable limit $g \to 0$, the path integral becomes

$$Z = \sum_{\mathrm{d}b^{\mathbb{Z}_2} \overset{2}{=} 0} 1, \quad (3)$$

where $\mathrm{d}b^{\mathbb{Z}_2} \overset{2}{=} 0$ means $\mathrm{d}b^{\mathbb{Z}_2} = 0 \mod 2$. Now, Z captures the topological order described by the deconfined phase of pure $\mathbb{Z}_2$ 2-gauge theory. However, we note that in $3+1$D, $\mathbb{Z}_2$ 2-gauge theory is dual to $\mathbb{Z}_2$ 1-gauge theory[2]. Thus, the topological order is also described by $\mathbb{Z}_2$ 1-gauge theory, which is $\mathbb{Z}_2$ topological order. In fact, Eq. (3) is the $3+1$D toric code.

[2] In $\mathbb{Z}_2$ 2-gauge theory in 3+1D, loop excitations carry $\mathbb{Z}_2$ gauge charge while particle excitations carry the $\mathbb{Z}_2$ gauge flux. On the other hand, in $\mathbb{Z}_2$ 1-gauge theory in 3+1D, particles carry the $\mathbb{Z}_2$ gauge charge and loops carrying the $\mathbb{Z}_2$ gauge flux. The duality between $\mathbb{Z}_2$ 2-gauge theory and $\mathbb{Z}_2$ 1-gauge theory in $3+1$D simply switches which excitations are called gauge charges and which are called gauge fluxes.

We now consider the equivalent limit in a twisted $\mathbb{Z}_2$ 2-gauge theory. This is described by a different bosonic model, which is Eq. (3) but with the 1 replaced with the action amplitude $\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(b^{\mathbb{Z}_2})^2}$. Indeed, the path integral is

$$Z(\mathcal{M}^4) = \sum_{\mathrm{d}b^{\mathbb{Z}_2} \overset{2}{=} 0} \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(b^{\mathbb{Z}_2})^2}, \quad (4)$$

where we use the shorthand $(b^{\mathbb{Z}_2})^2 \equiv b^{\mathbb{Z}_2} \smile b^{\mathbb{Z}_2}$ and $\int_{\mathcal{M}^4}$ is a sum over all 4-simplicies of $\mathcal{M}^4$. Note that this action amplitude is correctly invariant under the gauge transformation $b^{\mathbb{Z}_2} \to b^{\mathbb{Z}_2} + 2n^{\mathbb{Z}}$, where $n^{\mathbb{Z}}$ is an arbitrary $\mathbb{Z}$-valued 2-cochain. The twisted $\mathbb{Z}_2$ 2-gauge theory realizes a twisted $\mathbb{Z}_2$ topological order where the $\mathbb{Z}_2$ charges are fermions [59].

B. Lattice model with $\mathbb{Z}_2$ 1-SPT order

We now use the twisted $\mathbb{Z}_2$ 2-gauge theory in Eq. (4) to obtain a local bosonic model that realizes a 1-SPT order protected by the $\mathbb{Z}_2$ 2-group. The classifying space of the $\mathbb{Z}_2$ 2-group is a topological space denoted by $B(\mathbb{Z}_2, 2)$, which satisfies $\pi_2(B(\mathbb{Z}_2, 2)) = \mathbb{Z}_2$ and $\pi_n(B(\mathbb{Z}_2, 2)) = 0$ for $n \neq 2$. The associated symmetry is a $\mathbb{Z}_2$ 1-symmetry, which we denote as $\mathbb{Z}_2^{(1)}$.

The $\mathbb{Z}_2$ 2-gauge theory can be "ungauged" by first parameterizing the dynamical 2-cochain field $b^{\mathbb{Z}_2}$ as

$$b^{\mathbb{Z}_2} = B^{\mathbb{Z}_2} + \mathrm{d}a^{\mathbb{Z}_2}, \quad (5)$$

where $a^{\mathbb{Z}_2}$ is a $\mathbb{Z}_2$-valued 1-cochain field describing the pure 2-gauge fluctuations. However, we now reinterpret the meaning of $B^{\mathbb{Z}_2}$ and $a^{\mathbb{Z}_2}$ by treating $a^{\mathbb{Z}_2}$ as the dynamical field and $B^{\mathbb{Z}_2}$ as a $\mathbb{Z}_2$-valued 2-cocycle background field. This produces a new local bosonic model whose resulting path integral is obtained from the twisted $\mathbb{Z}_2$ 2-gauge theory Eq. (4):

$$Z(\mathcal{M}^4, B^{\mathbb{Z}_2}) = \sum_{a^{\mathbb{Z}_2}} \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(B^{\mathbb{Z}_2} + \mathrm{d}a^{\mathbb{Z}_2})^2}. \quad (6)$$

This path integral is invariant under the gauge transformation

$$a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + \alpha^{\mathbb{Z}_2},$$
$$B^{\mathbb{Z}_2} \to B^{\mathbb{Z}_2} - \mathrm{d}\alpha^{\mathbb{Z}_2}.$$

$B^{\mathbb{Z}_2}$ describes the symmetry-twist of the $\mathbb{Z}_2^{(1)}$ 1-symmetry. Turning off the background symmetry-twist field, and hence ungauging the $\mathbb{Z}_2^{(1)}$ 1-symmetry, the model becomes

$$Z(\mathcal{M}^4, 0) = \sum_{a^{\mathbb{Z}_2}} \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2}, \quad (7)$$

which has an exact $\mathbb{Z}_2^{(1)}$ 1-symmetry is generated by $\mathbb{Z}_2$-valued 1-cocycles $\alpha^{\mathbb{Z}_2}$:

$$a^{\mathbb{Z}_2} \to a^{\mathbb{Z}_2} + \alpha^{\mathbb{Z}_2}, \qquad \mathrm{d}\alpha^{\mathbb{Z}_2} \overset{2}{=} 0. \quad (8)$$

By construction, there is no obstruction to gauging the $\mathbb{Z}_2^{(1)}$ 1-symmetry and therefore the $\mathbb{Z}_2^{(1)}$ 1-symmetry is anomaly-free. This can further be seen from the fact that the path integral $Z(\mathcal{M}^4)$ is invariant under the $\mathbb{Z}_2^{(1)}$ transformation even when $\mathcal{M}^4$ has boundary.

Using that $\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2 = \int_{\partial\mathcal{M}^4} a^{\mathbb{Z}_2}\,\mathrm{d}a^{\mathbb{Z}_2}$, when spacetime $\mathcal{M}^4$ is closed (i.e., $\partial\mathcal{M}^4=\emptyset$) then $\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2 = 0$. Therefore, the action amplitude $\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2} = 1$ and so for a closed spacetime

$$Z(\mathcal{M}^4, 0) = \sum_{a^{\mathbb{Z}_2}} \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2} = 2^{N_e}, \tag{9}$$

where N_e is the number of the edges in the spacetime complex $\mathcal{M}^4$. According to a conjecture presented in Ref. 60, this implies that the ground state of the model Eq. (7) has no topological order (*i.e.* is short range entangled).

Since the ground state has $\mathbb{Z}_2^{(1)}$ 1-symmetry and no topological order, it may have a $\mathbb{Z}_2^{(1)}$ 1-SPT order, which are classified by the fourth cohomology group $H^4(B(\mathbb{Z}_2,2),\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_4$ [18–20, 32]. To see which 1-SPT order is realized, we note that the SPT order is characterized by the volume-independent partition function

$$Z^{\mathrm{top}}(\mathcal{M}^4, B^{\mathbb{Z}_2}) = \frac{Z(\mathcal{M}^4, B^{\mathbb{Z}_2})}{Z(\mathcal{M}^4, 0)}, \tag{10}$$

which is called the SPT-invariant [61–64]. We compute the 1-SPT invariant from Eq. (6), by integrating out $a^{\mathbb{Z}_2}$ for closed spacetime $\mathcal{M}^4$ and for $\mathrm{d}B^{\mathbb{Z}_2} = 0$ mod 2. Using Eq. (9) and the fact that $B^{\mathbb{Z}_2}$ is a cocycle, we find that

$$\begin{aligned} Z^{\mathrm{top}}(\mathcal{M}^4, B^{\mathbb{Z}_2}) &= \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(B^{\mathbb{Z}_2})^2} \\ &= \mathrm{e}^{\mathrm{i}\,\frac{m}{4}2\pi\int_{\mathcal{M}^4}\mathbb{Sq}^2(B^{\mathbb{Z}_2})}\Big|_{m=2}. \end{aligned} \tag{11}$$

Here, the generalized Steenrod square $\mathbb{Sq}^k$ is defined as

$$\mathbb{Sq}^k(c_l) \equiv c_l \underset{l-k}{\smile} c_l + c_l \underset{l-k+1}{\smile} \mathrm{d}c_l, \tag{12}$$

where c_l is any l-cochain. From the above 1-SPT invariant, we see that the model defined by Eq. (7) realizes a $\mathbb{Z}_2^{(1)}$ 1-SPT order that corresponds to $2\in\mathbb{Z}_4$ in the confined phase.

C. Emergent $\mathbb{Z}_2^{(1)}$ 1-SPT order in the confined phase of $\mathbb{Z}_2$ gauge theory

The fact that the theory Eq. (7) has an exact $\mathbb{Z}_2^{(1)}$ symmetry makes it rather special. Indeed, for a typical condensed matter model, the lattice theory would be more like

$$Z[\mathcal{M}^4, g, h] = \sum_{a^{\mathbb{Z}_2}} \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2 - h\sum_{ij}(a^{\mathbb{Z}_2})_{ij}} \times \mathrm{e}^{-\frac{1}{2g}\sum_{ijk}(\mathrm{d}a^{\mathbb{Z}_2})_{ijk} - 2\lfloor\frac{1}{2}(\mathrm{d}a^{\mathbb{Z}_2})_{ijk}\rceil}, \tag{13}$$

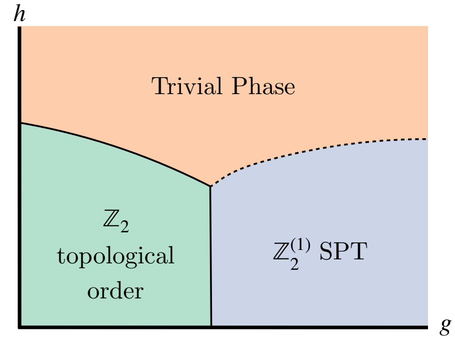

FIG. 1. The schematic phase diagram of the model described by Eq. (13). There is a topologically ordered phase described by the deconfined phase of $\mathbb{Z}_2$ gauge theory (shown in green), and a gapped short-range entangled phase (corresponding to the regions shaded in orange and purple). At energies below the $\mathbb{Z}_2$ gauge charge gap, the confined phase (colored in purple) has an exact emergent $\mathbb{Z}_2^{(1)}$ symmetry, while the symmetry is absent at all energy scales in the trivial phase (colored in orange). At $h=0$, this $\mathbb{Z}_2^{(1)}$ symmetry in the confined phase is exact. Due to the 2π-quantized topological term, the confined phase has a nontrivial SPT order protected by this exact emergent $\mathbb{Z}_2^{(1)}$ symmetry. The SPT invariant describing this 1-SPT order is given by Eq. (11)

where $\sum_{ijk}$ sums over all the triangles and $\sum_{ij}$ sums over all the edges of $\mathcal{M}^4$. This path integral does not have the $\mathbb{Z}_2^{(1)}$ symmetry (8), it is explicitly broken by the h term. Only when $h=0$ does Eq. (13) have the $\mathbb{Z}_2^{(1)}$ symmetry. Thus, at first glance, when $h\neq 0$ this generic model does not realize a $\mathbb{Z}_2^{(1)}$ SPT state since it does not even have the symmetry. However, while the $\mathbb{Z}_2^{(1)}$ symmetry is no longer a UV symmetry, for small $|h|$ the low-energy sectors of the Hilbert space enjoys an *exact emergent* $\mathbb{Z}_2^{(1)}$ symmetry. Indeed, since only the motion of the $\mathbb{Z}_2$ charge excitations can break the $\mathbb{Z}_2^{(1)}$ 1-symmetry (i.e., the h term), a $\mathbb{Z}_2^{(1)}$ symmetry emerges at energies much smaller than the $\mathbb{Z}_2$ charge excitation gap.

When $|h|\ll 1$ and $|g|\ll 1$, the model Eq. (13) realizes the $\mathbb{Z}_2$ topological order described by the deconfined phase of $\mathbb{Z}_2$ gauge theory. As we increase g, it will undergo a phase transition into a confined phase with short-range entanglement. Let us assume this transition is continuous (if it is not, we can modify the model to make the confinement transition nearly continuous). Then, approaching the transition, the $\mathbb{Z}_2$-flux fluctuations are low energy excitations while the $\mathbb{Z}_2$ charge excitations remain to have a large energy gap. This is exactly the scenario for the exact emergent $\mathbb{Z}_2^{(1)}$ symmetry. So in the confined phase (*i.e.* when the $\mathbb{Z}_2$ charge excitations energy gap remains large), the model realizes an 1-SPT order protected by the exact emergent $\mathbb{Z}_2^{(1)}$ 1-symmetry, described by $2\in\mathbb{Z}_4$.

The low-energy effective theory describing the phase with emergent 1-SPT order is Eq. (7). Introducing the Poincaré dual[3] of $\mathrm{d}a^{\mathbb{Z}_2}$, denoted as $\hat{f}$, the lattice action $\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2$ is equal to the intersection number of $\hat{f}$: $\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2 = \#(\hat{f}\cdot\hat{f})$. We note that $\hat{f}$ corresponds to the world sheets of $\mathbb{Z}_2$ flux loops so $\#(\hat{f}\cdot\hat{f})$ is the intersection number of $\mathbb{Z}_2$ flux world sheets in spacetime. The topological term $\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2}$ is therefore

$$\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}(\mathrm{d}a^{\mathbb{Z}_2})^2} = (-1)^{\#(\hat{f}\cdot\hat{f})}. \tag{14}$$

In general, the path integral of a $\mathbb{Z}_2$ gauge theory may or may not contain the topological term $(-1)^{\#(\hat{f}\cdot\hat{f})}$. When the topological term is included, then the confined phase of the $\mathbb{Z}_2$ gauge theory will be a 1-SPT state protected by the exact emergent $\mathbb{Z}_2^{(1)}$ 1-symmetry. However, when the path integral does not include the topological term, then the confined phase of the $\mathbb{Z}_2$ gauge theory will be a trivial SPT state of the exact emergent $\mathbb{Z}_2^{(1)}$ 1-symmetry. Therefore, in a 3+1D $\mathbb{Z}_2$ gauge theory, by adjusting the presence or the absence of the topological term $(-1)^{\#(\hat{f}\cdot\hat{f})}$ (*i.e.* the intersection term for the $\mathbb{Z}_2$-flux world sheet), we can control the presence or the absence of the 1-SPT order protected by the exact emergent $\mathbb{Z}_2^{(1)}$ 1-symmetry in the confined phase.

D. Using confined phases of 3+1D $\mathbb{Z}_n$ gauge theory to realize $\mathbb{Z}_n^{(1)}$ 1-SPT orders for even n

For simplicity, we've presented the above in the $\mathbb{Z}_2$ case, but it can easily be generalized by replacing $\mathbb{Z}_2$ with $\mathbb{Z}_n$, where n is a positive even integer. Indeed, introducing the $\mathbb{Z}_n$-valued 1-cochain field $a^{\mathbb{Z}_n}$, the generalized generic lattice model is

$$Z = \sum_{a^{\mathbb{Z}_n}} \mathrm{e}^{\mathrm{i}\,\pi\frac{m}{n}\int_{\mathcal{M}^4}\mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n}) - h\sum_{ij}(a^{\mathbb{Z}_n})_{ij}} \times \mathrm{e}^{-\frac{1}{ng}\sum_{ijk}(\mathrm{d}a^{\mathbb{Z}_n})_{ijk} - n\lfloor\frac{1}{n}(\mathrm{d}a^{\mathbb{Z}_n})_{ijk}\rceil}, \tag{15}$$

where the topological term is now proportional to $\int_{\mathcal{M}^4}\mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n})$. In Ref. 20, it was shown that $\mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n}+nb) \overset{2n}{=} \mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n})$ and that $\mathrm{e}^{\mathrm{i}\,\pi\frac{m}{n}\int_{\mathcal{M}^4}\mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n})} = 1$ when $\mathcal{M}^4$ is closed. Thus, the inclusion of the topological term $\mathrm{e}^{\mathrm{i}\,\pi\frac{m}{n}\int_{\mathcal{M}^4}\mathbb{Sq}^2(\mathrm{d}a^{\mathbb{Z}_n})}$ does not affect the local dynamics of the model.

As a result, when $|h| \ll 1$ and $|g| \ll 1$, the model (15) realizes the $\mathbb{Z}_n$ topological order described by the deconfined phase of a $\mathbb{Z}_n$ gauge theory. As we increase g, the model will undergo a confinement phase transition. Assuming the transition is continuous, near the transition the $\mathbb{Z}_n$-flux energy gap is much smaller than the gap for the $\mathbb{Z}_n$ charge excitations. So near the transition, the model has an exact emergent $\mathbb{Z}_n^{(1)}$ 1-symmetry, at energies much less than $\mathbb{Z}_n$ charge energy gap. After the confinement transition, the confined phase has 1-SPT order protected by the exact emergent $\mathbb{Z}_n^{(1)}$ 1-symmetry and described by $m \in H^4(B(\mathbb{Z}_n,2);\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{2n}$ for even n[4]. Here $B(\mathbb{Z}_n,2)$ is the classifying space of the $\mathbb{Z}_n$ 2-group describing $\mathbb{Z}_n^{(1)}$ 1-symmetry. It satisfies $\pi_2(B(\mathbb{Z}_n,2)) = \mathbb{Z}_n$ and $\pi_i(B(\mathbb{Z}_2,2)) = 0$, $i \neq 2$. The physical consequence of $\mathbb{Z}_n^{(1)}$ 1-SPT order, such as boundary states, as well as a Hamiltonian description of this phase, was discussed in Ref. 20.

IV. EMERGENT $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times ...$ 1-SPT ORDER IN A 3+1D $U^\kappa(1)$ BOSONIC MODEL

In last section, we saw how $\mathbb{Z}_{2n}^{(1)}$ 1-SPT state can be realized in the confined phase of 3+1D $\mathbb{Z}_n$ gauge theory. Now we investigate more complicated 1-SPT orders which are protected by finite 1-symmetries. In this section, we will construct a 3+1D bosonic model, that corresponds to lattice $U^\kappa(1)$ "gauge theory" with a 2π-quantized topological term. We will show that, due to the topological term, the model has a reduced $\mathbb{Z}_{k_1}^{(1)}\times\mathbb{Z}_{k_2}^{(1)}\times\cdots$ 1-symmetry. We will also show that the confined phase of the $U^\kappa(1)$ gauge theory can have a 1-SPT orders protected by the $\mathbb{Z}_{k_1}^{(1)}\times\mathbb{Z}_{k_2}^{(1)}\times\cdots$ 1-symmetry.

A. 3+1D $U^\kappa(1)$ pure gauge field theory and 2π-quantized topological term

Before we consider the bosonic lattice model, we first consider the corresponding continuum theory. We do so in a timely, but non-rigorous, fashion to see how the results from the lattice theory which we present in the next sections are hinted towards in the continuum theory. It will set the stage for the lattice theory where the formal manipulations are much more involved than those in the field theory.

We consider the theory described by the Euclidean action

$$S = \frac{1}{2g^2}\sum_I \int_{M^4} f^I \wedge * f^I + S_{\text{top}}, \tag{16}$$

where a^I with $I = 1,\ldots,\kappa$ are $\mathbb{R}/\mathbb{Z}$-valued 1-form fields[5],

[3] The Poincaré dual of the $(D-p)$-cycle C with respect to the D-dimensional complex M, denoted as $\hat{C}$, satisfies $\int_C a = \int_M a\,\hat{C}$, where a is any $(D-p)$-cochain.

[4] The lattice model (15) is well defined even for $m \notin \mathbb{Z}$. However, when $m \notin \mathbb{Z}$ it is not clear if the model has a gapped confined phase when $|g| \gg 1$.

[5] Typically the $U(1)$ connection is a map $A: \mathbb{R}^4 \mapsto \mathbb{R}/2\pi\mathbb{Z}$. We define $a = A/2\pi$ to absorb factors of 2π and match the convention used in the bosonic lattice model. In terms of the coupling constant g, the typical $U(1)$ coupling constant is $e = 2\pi g$.

the 2-form curvature $f^I = \mathrm{d}a^I$, and $k_{IJ} \in \mathbb{Z}$. The first term is Maxwell's kinetic term and the second term is the 2π-quantized topological term (topological in the sense that it is independent of the metric)

$$S_{\text{top}} = -2\pi\,\mathrm{i} \sum_{I\le J} k_{IJ} \int_{M^4} f^I \wedge f^J, \tag{17}$$

Furthermore, the quantity $k_{IJ} \int f^I \wedge f^J$ is quantized as an integer when M^4 is closed. Thus, for closed M^4 the action amplitude of the topological term is unity, but for M^4 with a boundary it can have a nontrivial effect.

Since the action depends only on f^I, it is left unchanged by

$$a^I \to a^I + \Gamma^I, \quad \mathrm{d}\Gamma^I = 0. \tag{18}$$

This corresponds to a real symmetry transformation (not a gauge transformation) when $\oint \Gamma^I \neq \mathbb{Z}$. Since there are κ fields a^I (i.e., $I = 1, \cdots, \kappa$), Eq. (18) is associated with κ different $U(1)^{(1)}$ 1-form symmetries: $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$. The associated Noether current can be found by introducing a background symmetry twisted field $\mathcal{B}^I$ in Lorentzian signature and having $\mathrm{d}a^I \to \mathrm{d}a^I - \mathcal{B}^I$. Noting that the conserved current J^I minimally couples to $\mathcal{B}^I$ as $\int \mathcal{B}^I \wedge * J^I$, we find that for the transformation of the Ith field:

$$J^I = \frac{1}{g^2} f^I + 2\pi \sum_J K_{IJ} * f^J, \tag{19}$$

where K_{IJ} is given by

$$K_{II} = 2k_{II}, \qquad K_{IJ} = K_{JI} = k_{IJ}, \quad I < J. \tag{20}$$

The fact that the current is conserved means that $\mathrm{d}^\dagger J^I = 0$, where $\mathrm{d}^\dagger = * \,\mathrm{d}\, *$ in the adjoint of d.

In the above analysis of the symmetry, we consider the field theory without $U(1)$ charges and $U(1)$ monopoles. This $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$ is really an exact emergent 1-form symmetry at energies below the $U(1)$ charge gaps and the monopole gaps. Indeed, at energies above the $U(1)$ gauge charge gaps, terms like $\int a^I \wedge * j^I$ will contribute to the action and this symmetry will be explicitly broken.

Let's now introduce the 1-form $j_m^I = * \,\mathrm{d} f^I$, the Dirac monopole current density associated with the Ith field $a^I$and the Poincaré dual of $* j_m^I$ gives the world-line of the monopole. The continuity equation $\mathrm{d}^\dagger J^I = 0$ then implies that

$$\frac{1}{g^2} \mathrm{d}^\dagger f^I = 2\pi K_{IJ} j_m^J. \tag{21}$$

The effect of the nonzero righthand side is a generalized version of the Witten effect where $U(1)$ monopoles of the Jth field carries K_{IJ} units of the Ith $U(1)$ gauge charge.

The presence of magnetic monopoles complicates things. At energies below the $U(1)$ charge gaps but above the monopole gaps, due to the topological term, the monopoles fluctuations imply $U(1)$ charge fluctuations. This may break the $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$ 1-form symmetry to a smaller symmetry.

In the continuum, monopole configurations can be easily considered by parametrizing the curvature as $f^I = \mathrm{d}\tilde{a}^I + G^I$. The 1-form fields $\tilde{a}^I$ describe the smooth local fluctuations of a^I and satisfy the Bianchi identity $\mathrm{d}(\mathrm{d}\tilde{a}^I) = 0$, while the 2-form fields G^I capture the singular monopole configurations and satisfy $j_m^I = \star \mathrm{d} G^I$. At energies above the monopole gap, the field theory that describes the lattice model instead has the topological term

$$\begin{aligned} S_{\text{top}} = -2\pi\,\mathrm{i} \sum_{I\le J} k_{IJ} \int_{M^4} \mathrm{d}\tilde{a}^I \wedge \mathrm{d}\tilde{a}^J + G^I \wedge G^J \\ - 2\pi\,\mathrm{i} \sum_{I,J} K_{IJ} \int_{M^4} \tilde{a}^I \wedge * j_m^J. \end{aligned} \tag{22}$$

This is equivalent to Eq. (17) up to a boundary term. For all practical purposes, we may treat the density in Eq. (22) as the definition of $f^I \wedge f^J$ for energies above the monopole gap. This distinction is important as the $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$ symmetry of Eq. (17) is broken down to a finite subgroup in Eq. (22), agreeing with the symmetries of the lattice model we study.

Indeed, above the monopole energy gap, Eq. (22) is invariant under the transformation

$$\tilde{a}^I \to \tilde{a}^I + \Gamma^I, \quad \sum_I K_{IJ} \oint_{C^1} \Gamma^I \in \mathbb{Z}, \quad \mathrm{d}\Gamma^I = 0, \tag{23}$$

for any closed 1-submanifold C^1. The additional restriction $\sum_I K_{IJ} \oint_{C^1} \Gamma^I \in \mathbb{Z}$ ensures that the action amplitude $\mathrm{e}^{2\pi\,\mathrm{i} \sum_{I,J} K_{IJ} \int_{M^4} \tilde{a}^I \wedge * j_m^J}$ is invariant since $\oint * j_m^I \in \mathbb{Z}$. We note that this term in Eq. (22) also recovers the Gauss-Witten law Eq. (21). Thus, at a fixed point in spacetime, the values of allowed Γ^I form a rational lattice K^{-1}. So, above the monopole gap the theory has the 1-symmetries $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$, where k_i are the diagonal elements of the Smith normal form for K. Below the monopole gap when j_m^I vanishes, this constraint on Γ^I does not apply so there is instead the aforementioned $U(1)^{(1)} \times U(1)^{(1)} \times \cdots$ symmetries.

Let's now turn on 2-form background fields $\mathcal{B}^I$ that are the flat connections describing the twist of the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ symmetry and satisfy the quantization conditions

$$\sum_I K_{IJ} \oint_{C^2} \mathcal{B}^I \in \mathbb{Z}, \tag{24}$$

for any closed 2-submanifold C^2. We'll work locally at the level of differential forms, ignoring topological subtitles and monopoles. The background fields minimally couple to the dynamical fields a^I by replacing the curvature $\mathrm{d}a^I$ in the Euclidean action by $\mathrm{d}a^I - \mathcal{B}^I$. Making this replacement and taking the $g \to \infty$ limit, the action becomes

$$S = -2\pi\mathrm{i}\sum_{I\le J} k_{IJ}\int_{M^4}(\mathrm{d}a^I - \mathcal{B}^I)\wedge(\mathrm{d}a^J - \mathcal{B}^J). \tag{25}$$

We can use Eq. (25) to find the continuum SPT invariant which describes the 1-SPT order in the confined phase. Indeed, let's consider spacetime M^4 to be closed. Then, since we ignore monopoles and because $\mathrm{d}\mathcal{B}_I = 0$, integrating by parts we can rewrite the Euclidean action as

$$\begin{aligned} S &= -2\pi\mathrm{i}\sum_{I\le J} k_{IJ}\int_{M^4}\mathcal{B}^I\wedge\mathcal{B}^J, \\ &= -\mathrm{i}\pi\sum_{I,J} K_{IJ}\int_{M^4}\mathcal{B}^I\wedge\mathcal{B}^J. \end{aligned} \tag{26}$$

Thus, the path integral Z in this limit is

$$\begin{aligned} Z[M^4,B^I] &= \int D[a^I]\; e^{\mathrm{i}\pi\sum_{I,J}K_{IJ}\int_{M^4}\mathcal{B}^I\wedge\,\mathcal{B}^J}, \\ &= \mathrm{Vol}^\kappa(\mathbb{R}/\mathbb{Z})\; e^{\mathrm{i}\pi\sum_{I,J}K_{IJ}\int_{M^4}\mathcal{B}^I\wedge\,\mathcal{B}^J}, \end{aligned} \tag{27}$$

where we've used that the action amplitude does not depend on the dynamical fields a^I and introduced

$$\mathrm{Vol}^\kappa(\mathbb{R}/\mathbb{Z}) = \int D[a^I].$$

The SPT invariant is given by the volume-independent part of the path integral

$$Z^{\mathrm{top}}(M^4,B^I) = \frac{Z(M^4,B^I)}{Z(M^4,0)}. \tag{28}$$

Therefore, using Eq. (27) we find that in the continuum theory the 1-SPT invariant is

$$Z^{\mathrm{top}}(M^4,B^I) = e^{\mathrm{i}\pi\sum_{I,J}K_{IJ}\int_{M^4}\mathcal{B}^I\wedge\,\mathcal{B}^J}. \tag{29}$$

Thus, without much work we can characterize the 1-SPT order. However, in doing so we ignored nontrivial fibre bundles and magnetic monopoles. In the remainder of this section, we'll regulate this continuum theory by considering a bosonic lattice model whose IR properties are described by the field theory. Using this lattice model, we'll be able to recalculate the SPT invariant more rigorously (see Eq. (75)), and find lattice-dependent terms in addition to one which captures Eq. (29).

B. Lattice Regularization of $U^\kappa(1)$ gauge theory with 2π-quantized topological term

We now regulate the field theory discussed in the previous section by triangulating spacetime. The 1-form fields a^I will be represented by $\mathbb{R}/\mathbb{Z}$-valued 1-cochains $a_I^{\mathbb{R}/\mathbb{Z}}$. There are three key properties that the $U^\kappa(1)$ gauge theory on a lattice must include:

1. Letting $m_I^{\mathbb{Z}}$ be an arbitrary Z-valued 1-cochain, the action amplitude is invariant under $a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + m_I^{\mathbb{Z}}$, even when spacetime $\mathcal{M}^4$ has a boundary;
2. When $\mathcal{M}^4$ is closed, the action amplitude of the 2π-quantized topological term becomes unity;
3. In the smooth field limit (the low energy limit) when $\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \sim \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil$, which implies no monopoles), the action amplitude reduces to its continuum limit $\mathrm{e}^{\mathrm{i}2\pi\int_{M^4}\sum_{I\le J}k_{IJ}f^I\wedge f^J}$.

Regularizing the Maxwell term on the lattice is straight forward, but the 2π-quantized topological term is highly non-trivial. Noting the relationship between the topological term and Chern-Simons theory in the continuum, this motivates us to define the 2π-quantized topological term on the lattice as the derivative of the lattice Chern-Simons action. Indeed, we start with 2+1D $U^\kappa(1)$ Chern-Simons theory on spacetime lattice $\mathcal{B}^3$ obtained in Ref. 65

$$\begin{aligned} Z_{\mathrm{CS}} = \int D[a_I^{\mathbb{R}/\mathbb{Z}}]\; &\mathrm{e}^{\mathrm{i}2\pi\sum_{I\le J}k_{IJ}\int_{\mathcal{B}^3}\mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}}(a_J^{\mathbb{R}/\mathbb{Z}}-\lfloor a_J^{\mathbb{R}/\mathbb{Z}}\rceil)\right)} \\ &\times\mathrm{e}^{\mathrm{i}2\pi\sum_{I\le J}k_{IJ}\int_{\mathcal{B}^3}a_I^{\mathbb{R}/\mathbb{Z}}(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}-\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil)-\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil a_J^{\mathbb{R}/\mathbb{Z}}} \\ &\times\mathrm{e}^{-\mathrm{i}2\pi\sum_{I\le J}k_{IJ}\int_{\mathcal{B}^3}a_J^{\mathbb{R}/\mathbb{Z}}\smile_1\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil}\,\mathrm{e}^{-\sum_I\int_{\mathcal{B}^3}\frac{|\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}-\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil|^2}{g_3}}, \end{aligned} \tag{30}$$

where $a_I^{\mathbb{R}/\mathbb{Z}}$ are the aforementioned $\mathbb{R}/\mathbb{Z}$-valued 1-cochain, the path-integral notation is shorthand for $\int D[a_I^{\mathbb{R}/\mathbb{Z}}] = \prod_{ij,I}\int_{-\frac12}^{\frac12}\mathrm{d}(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$, and $k_{IJ}\in\mathbb{Z}$. This lattice model is rather complicated as it captures the effects of magnetic monopoles. We note that Ref. 65 found that Eq. (30) is invariant under the gauge transformation

$$a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + m_I^{\mathbb{Z}} \tag{31}$$

for any $\mathbb{Z}$-valued 1-cochain $m_I^{\mathbb{Z}}$ even when $\mathcal{B}^3$ has boundary.

The path integral of the 3+1D bosonic model (for spacetime $\mathcal{M}^4$ with or without boundary) is then obtained from Eq. (30) by taking a derivative and setting $g_3\to\infty$. Using the properties of the (higher) cup product, the first line of Eq. (30) vanishes since it is already the d of something, the second line of Eq. (30) becomes

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}2\pi\int_{\mathcal{M}^4}k_{IJ}\,\mathrm{d}\left(a_I^{\mathbb{R}/\mathbb{Z}}(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}-\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil)-\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil a_J^{\mathbb{R}/\mathbb{Z}}\right)} \\ &\quad= \mathrm{e}^{\mathrm{i}2\pi k_{IJ}\int_{\mathcal{M}^4}(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}-\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil)(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}-\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil)}\times \\ &\qquad\mathrm{e}^{\mathrm{i}2\pi k_{IJ}\int_{\mathcal{M}^4}a_I^{\mathbb{R}/\mathbb{Z}}\mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\rceil-\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil a_J^{\mathbb{R}/\mathbb{Z}}}, \end{aligned} \tag{32}$$

and the third line of Eq. (30) becomes

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}2\pi k_{IJ}\int_{\mathcal{M}^4}\mathrm{d}\left(a_J^{\mathbb{R}/\mathbb{Z}}\smile_1\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil\right)} \\ &\quad= \mathrm{e}^{-\mathrm{i}2\pi k_{IJ}\int_{\mathcal{M}^4}\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}}\smile_1\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil}\times \\ &\qquad\mathrm{e}^{\mathrm{i}2\pi k_{IJ}\int_{\mathcal{M}^4}a_J^{\mathbb{R}/\mathbb{Z}}\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil+\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}\rceil a_J^{\mathbb{R}/\mathbb{Z}}}. \end{aligned} \tag{33}$$

Putting this all together and including the lattice Maxwell term $\mathrm{e}^{-\sum_I \int_{\mathcal{M}^4} \frac{|\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}}$, we obtain a 3+1D bosonic model on spacetime lattice with a 2π-quantized topological term

$$Z = \int D[a_I^{\mathbb{R}/\mathbb{Z}}]\ \mathrm{e}^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \\ \times \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil)} \\ \times \mathrm{e}^{\mathrm{i}2\pi \sum_{IJ} K_{IJ} \int_{\mathcal{M}^4} a_I^{\mathbb{R}/\mathbb{Z}} \mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil}\ \mathrm{e}^{-\sum_I \int_{\mathcal{M}^4} \frac{|\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}}, \tag{34}$$

where K_{IJ} is given by

$$K_{II} = 2k_{II}, \qquad K_{IJ} = K_{JI} = k_{IJ}, \quad I < J. \tag{35}$$

Because the lattice Chern-Simons path integral was invariant under the gauge transformation Eq. (31) even when $\mathcal{M}^4$ has boundary, by definition the path integral Eq. (34) is also invariant. Thus, requirement (1) from above is satisfied. Furthermore, since we defined the action as the derivative of something, requirement (2) is also automatically satisfied. Lastly, lets check that Eq. (34) satisfies requirement (3). In the $g \sim 0$ limit, the Maxwell term enforces fluctuations $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil \sim \epsilon$ to be small. Therefore, using that

$$\mathrm{d}\epsilon \sim \mathrm{d}(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil) = \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil, \tag{36}$$

since $\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil \overset{1}{=} 0$ and ϵ is small, this implies that

$$\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil = 0, \tag{37}$$

and hence there are no monopoles. When $a_J^{\mathbb{R}/\mathbb{Z}}$ describes a monopole, it cannot be smooth and thus $\mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil \neq 0$. In fact, $\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil$ is the Poincaré dual of the Dirac monopoles' worldsheets (*i.e.* the trajectory of the Dirac strings of the monopole in spacetime). Thus $\mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil$ is the Poincaré dual of the boundary of the Dirac worldsheet, which is the worldline of the $U(1)$ monopoles.

Therefore, the $g \sim 0$ limit corresponds to the smooth field limit. In this limit, the action amplitude for the topological term in Eq. (34) becomes

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)^2}. \tag{38}$$

Relating the 1-form field a^I to the 1-cochain $(a_I^{\mathbb{R}/\mathbb{Z}})_{ij}$ by

$$\int_i^j a^I = (a_I^{\mathbb{R}/\mathbb{Z}})_{ij} \tag{39}$$

and the 2-form curvature field $f^I = \mathrm{d}a^I$ by

$$\int_{A(ijk)} f^I = (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)_{ijk}, \tag{40}$$

the action amplitude (38) becomes

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)^2} \approx \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{M^4} f^I \wedge f^J}. \tag{41}$$

Therefore, in the smooth field limit (the low energy limit) the 2π-quantized term on the lattice is captured by the continuum field theory and requirement (3) is satisfied.

In the absence of monopoles, Eq. (41) correctly becomes unity on a closed spacetime. For large g, however, due to the presence of monopoles the lattice term $\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil)(\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil)}$ is no longer unity when $\mathcal{M}^4$ is closed and thus is neither 2π-quantized nor topological. Therefore, for large g the low-energy limit of the lattice model may not be described by the continuum topological term (17) since the lattice topological term must be described using all terms in Eq. (34). It's more likely that the low-energy physics of the lattice model for large g, where the highly non-trivial terms in the first and third line of Eq. (34) are included, is better captured by the continuum topological term Eq. (22).

C. 1-symmetries in 3+1D $U^\kappa(1)$ bosonic model

Now that we've introduced the $U^\kappa(1)$ bosonic model, we now focus our attention on studying its symmetries and phase diagram. Firstly, let's review the case when $\mathcal{M}^4$ has no boundary and the topological term vanishes and Eq. (34) becomes Maxwell's theory

$$Z(\mathcal{M}^4) = \int D[a_I^{\mathbb{R}/\mathbb{Z}}]\ \mathrm{e}^{-\sum_I \int_{\mathcal{M}^4} \frac{|\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}}. \tag{42}$$

When $g \sim 0$, the lattice curvature $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}$ fluctuate weakly and so the above model is in a deconfined phase of a compact $U^\kappa(1)$ gauge theory and has a gapless photon excitation. On the other hand, when $g \to \infty$ the model is in a gapped confined phase. Using that $\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}(a_I^{\mathbb{R}/\mathbb{Z}})_{ij} = 1$, the partition function is

$$Z(\mathcal{M}^4) = \int D[a_I^{\mathbb{R}/\mathbb{Z}}] = 1, \tag{43}$$

for any closed spacetime $\mathcal{M}^4$. According a conjecture in Ref. 60, this implies that the gapped confined phase has a trivial topological order.

In what follows, we now consider $\mathcal{M}^4$ with a boundary so the 2π-topological term contributes to the path integral. We'll show that the gapped confined phase now has a 1-SPT order characterized by k_{IJ} (see Fig. 2). This is similar in spirit to section III where in order to get $\mathbb{Z}_2^{(1)}$ SPT order we had to include the twist term Eq. (4).

Regardless the value of g and even on $\mathcal{M}^4$ with boundary, the path integral Eq. (34) is invariant under the transformation

$$a_I^{\mathbb{R}/\mathbb{Z}} \to a_I^{\mathbb{R}/\mathbb{Z}} + \beta_I^{\mathbb{Q}/\mathbb{Z}}, \quad \sum_I \beta_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \in \mathbb{Z}, \quad \mathrm{d}\beta_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0. \tag{44}$$

$\beta_I^{\mathbb{Q}/\mathbb{Z}}$ are $\mathbb{Q}/\mathbb{Z}$-valued 1-cocycles to ensure that the quantities $\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ and $\mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil$ are invariant under the transformation (44). If this were the only requirement, Eq. (44) would correspond to the κ different $U(1)^{(1)}$ 1-symmetries. However, the additional constraint that $\sum_I \beta_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ}$ are $\mathbb{Z}$-valued cochains is required when there are magnetic monopoles to ensure the term $\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^4} \sum_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} K_{IJ} \mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil}$ is invariant. Indeed, under the transformation Eq. (44), this term changes by a phase factor

$$\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^4} \sum_{IJ} \beta_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil},$$

which is 1 provided $\beta_I^{\mathbb{Q}/\mathbb{Z}}$ satisfy $\sum_I \beta_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \in \mathbb{Z}$.

As an integer matrix, K has the following Smith normal form

$$K = U \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & \ddots \end{pmatrix} V, \tag{45}$$

where k_I are integers and U, V are invertible integer matrices. Now the 1-symmetry can be written as

$$\begin{aligned} a_I^{\mathbb{R}/\mathbb{Z}} &\to a_I^{\mathbb{R}/\mathbb{Z}} + \beta_I^{\mathbb{Q}/\mathbb{Z}} = a_I^{\mathbb{R}/\mathbb{Z}} + \sum_J \widetilde{\beta}_J^{\mathbb{Q}/\mathbb{Z}} (U^{-1})_{JI}, \\ \sum_I \beta_I^{\mathbb{Q}/\mathbb{Z}} U_{IJ} k_J &= \widetilde{\beta}_J^{\mathbb{Q}/\mathbb{Z}} k_J \in \mathbb{Z}, \quad \mathrm{d}\widetilde{\beta}_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0. \end{aligned} \tag{46}$$

We see that the 1-symmetry is a $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry generated by the quantized $\widetilde{\beta}_J^{\mathbb{Q}/\mathbb{Z}}$. When $k_I = 0$, $\widetilde{\beta}_I^{\mathbb{Q}/\mathbb{Z}}$ is not quantized and generates $U(1)^{(1)}$ 1-symmetry. The above result remains valid if we regard $\mathbb{Z}_0^{(1)}$ as the $U(1)^{(1)}$ 1-symmetry. We note that, since the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry is valid on the spacetime lattice with or without boundary, the 1-symmetry is anomaly-free.

In addition to giving rise to a finite 1-symmetry, the term $\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^4} \sum_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} K_{IJ} \mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil}$ also causes the $U(1)$ monopoles to be bounded with the $U(1)$ charges. In particular, the unit monopole of the J^{th} $U(1)$ field carries the I^{th} $U(1)$ charge K_{IJ}. This is precisely the lattice version of the generalized Witten effect discussed in the continuum theory (see Eq. (21)).

For large g, these monopole-charge bound states condense which gives rise to a gapped oblique confined phase with $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. We note that the 2+1D lattice $U^\kappa(1)$ Chern-Simons theory (30) also has the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry, which can actually be

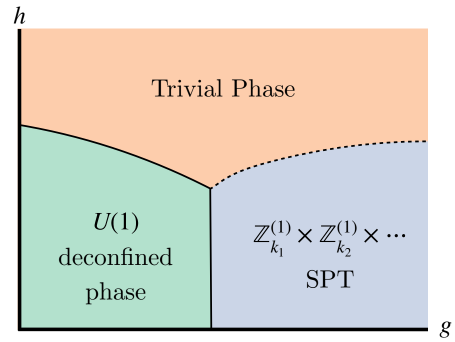

FIG. 2. The schematic phase diagram of the model described by Eq. (34) with the additional term contributing to the action amplitude $\mathrm{e}^{h \sum_{ij,I} a_{I,ij}^{\mathbb{R}/\mathbb{Z}}}$. When $h = 0$, there is no fluctuations of $U(1)$ gauge charge, and the model has an exact $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. For $h \neq 0$, this becomes an exact emergent $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry existing below the energy gaps of $U(1)$ gauge charges, which exists in the green and purple shaded regions. Due to the 2π-quantized topological term, in the confined phase (shown in purple) there is an SPT order protected by the exact emergent $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry whose SPT invariant is given by Eq. (70).

anomalous [65]. Since the 2+1D lattice $U^\kappa(1)$ Chern-Simons theory is the boundary of the $U^\kappa(1)$ model in the gapped confined phase, from the point of view of anomaly inflow [30, 66] the gapped confined phase may have a non-trivial $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-SPT order. Indeed, in the next section we'll show that this confined phase is characterized by the K-matrix and has a 1-SPT order protected by the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. Indeed, the 1-SPT invariant found in the next section is given by Eq. (82).

Before concluding this subsection, we remark that the fact that the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry is exact in the bosonic model is a special feature of the theory. A more generic lattice theory would also include the action amplitude $\mathrm{e}^{h \sum_{ij,I} a_{I,ij}^{\mathbb{R}/\mathbb{Z}}}$ in the path integral, which explicitly breaks the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. However, like in the $\mathbb{Z}_2$ gauge theory case discussed in section III C, for energies below the $U(1)$ gauge charge gaps, there is a region of $h \neq 0$ where the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry is an exact emergent symmetry. In this region, the corresponding 1-SPT order would also affect the low-energy physics and be protected by the exact emergent symmetry (see Fig. 2).

D. Gauging the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \dots$ 1-symmetry

The fact that the boundary Chern-Simons theory has an anomalous $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry [65] means that the bulk theory has 1-SPT order in the large-g

confined phase protected by $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$. In this section, we will characterize the 1-SPT theory in the bulk 3+1D theory by finding the SPT invariant obtained by gauging the 1-symmetry. This subsection contains mostly detailed calculations in order to derive the 1-SPT invariant given by Eq. (82).

Before gauging the symmetry, it's convenient to first slowly turn on addition terms in the action which will not affect the 1-SPT order. In particularly, to the Euclidean lattice action we add

$$S \supset U \sum_{ij,J} \cos\left(2\pi \sum_I (a_I^{\mathbb{R}/\mathbb{Z}})_{ij} K_{IJ}\right). \tag{47}$$

Note that in the $U \to \infty$ limit, this term makes $a_I^{\mathbb{R}/\mathbb{Z}}$ satisfy the quantization condition

$$\sum_I a_I^{\mathbb{R}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0. \tag{48}$$

Crucially, this preserves the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry whose transformation is Eq. (44). Furthermore, when $g \to \infty$ the path integral for any closed spacetime changes smoothly as U changes from 0 to ∞. This is because in this limit the only term in the action is Eq. (47) which is independently defined on each 1-simplex of the spacetime triangulation (i.e., non-interacting). Thus, the $U = 0$ state and the $U \to \infty$ state belong to the same phase and so the $(g, U) = (\infty, \infty)$ phase has the 1-SPT order as the $(g, U) = (\infty, 0)$ phase. By considering the $U \to \infty$ state, the quantization condition turns the $U(1)$ cochain fields $a_I^{\mathbb{R}/\mathbb{Z}}$ into discrete cochain fields $a_I^{\mathbb{Q}/\mathbb{Z}}$[6] satisfying Eq. (48), which allows us to use results and techniques for discrete fields from section III to study the 1-SPT order in the $U(1)$ model.

We now consider the $U \to \infty$ state, which in the strongly-interacting limit $g \to \infty$ the path integral Eq. (34) becomes

$$Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} e^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \rceil} \times \\ e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \rceil)(\mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} \rceil)}, \tag{49}$$

where we have use that for quantized $a_I^{\mathbb{Q}/\mathbb{Z}}$ satisfying Eq. (48),

$$e^{\mathrm{i}2\pi \int_{\mathcal{M}^4} \sum_{IJ} a_I^{\mathbb{R}/\mathbb{Z}} K_{IJ} \mathrm{d}\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil} = 1. \tag{50}$$

As mentioned, just like the original path integral Eq. (34), this path integral Eq. (49) also has the anomaly-free 1-symmetry

$$a_I^{\mathbb{Q}/\mathbb{Z}} \to a_I^{\mathbb{Q}/\mathbb{Z}} + \beta_I^{\mathbb{Q}/\mathbb{Z}}, \quad \sum_I \beta_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \in \mathbb{Z}, \quad \mathrm{d}\beta_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0. \tag{51}$$

Lastly, using that $\lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil \in \mathbb{Z}$ and $\mathrm{d}(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}}) = 0$, we insert unity of the form

$$1 = e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}} \rceil} \times \\ e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{R}/\mathbb{Z}} \rceil) \underset{1}{\smile} \mathrm{d}(\mathrm{d}a_I^{\mathbb{R}/\mathbb{Z}})} \tag{52}$$

into Eq. (49) such that the path integral becomes

$$Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} \rceil) \underset{1}{\smile} \mathrm{d}(\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \rceil)} \times \\ e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \rceil)(\mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} \rceil)}. \tag{53}$$

To determine the SPT order realized by the theory Eq. (49), we gauge the 1-symmetries by first replacing $\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}}$ with $\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - B_I^{\mathbb{Q}/\mathbb{Z}}$ which for convenience we'll denote as

$$b_I^{\mathbb{Q}/\mathbb{Z}} \equiv \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - B_I^{\mathbb{Q}/\mathbb{Z}}, \tag{54}$$

where $B_I^{\mathbb{Q}/\mathbb{Z}}$ is a background symmetry twist field satisfying

$$\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0, \qquad \sum_I B_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0. \tag{55}$$

We use "$\overset{1}{=}$" instead of "=" here since shifting $B_I^{\mathbb{Q}/\mathbb{Z}}$ by a $\mathbb{Z}$-valued 2-cochain corresponds to performing a gauge transformation. After this, the path-integral Eq. (53) of course becomes

$$Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (b_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_J^{\mathbb{Q}/\mathbb{Z}} \rceil) \underset{1}{\smile} \mathrm{d}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}} \rceil)} \times \\ e^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} (b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}} \rceil)(b_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_J^{\mathbb{Q}/\mathbb{Z}} \rceil)}. \tag{56}$$

Note that for $\mathcal{M}^4$ with or without boundary, Eq. (56) is importantly invariant under the gauge transformations

$$\begin{aligned} a_I^{\mathbb{Q}/\mathbb{Z}} &\to a_I^{\mathbb{Q}/\mathbb{Z}} + m_I^{\mathbb{Z}}, \\ b_I^{\mathbb{Q}/\mathbb{Z}} &\to b_I^{\mathbb{Q}/\mathbb{Z}} + n_I^{\mathbb{Z}}, \end{aligned} \tag{57}$$

where $m_I^{\mathbb{Z}} \overset{1}{=} 0$ and $n_I^{\mathbb{Z}} \overset{1}{=} 0$. If we had not inserted Eq. (52) into Eq. (49), the gauged theory would have not been gauge invariant.

The path integral Eq. (56) is a bit cumbersome in its current form and it's hard to see how $\mathcal{M}^4$ being opened or

[6] $a_I^{\mathbb{R}/\mathbb{Z}}$ is renamed as $a_I^{\mathbb{Q}/\mathbb{Z}}$, since the quantized $a_I^{\mathbb{Q}/\mathbb{Z}}$'s, $\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0$, have values in $\mathbb{Q}/\mathbb{Z}$.

closed changes the action amplitude. Thus, let's massage the action amplitude of Eq. (56) a bit to get it in a more enlightening form. First, we consider the first line of Eq. (56). Using that $\mathrm{d}b_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0$ and rewriting

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} (b_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_J^{\mathbb{Q}/\mathbb{Z}}\rceil)\underset{1}{\smile} \mathrm{d}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)} \\ &\quad = \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} \mathrm{d}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)}, \end{aligned} \tag{58}$$

we then can use Eq. (1) and once again $\mathrm{d}b_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0$ to write this as

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} \mathrm{d}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)} \\ &= \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \mathrm{d}\left(b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)\right) - \mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} b_I^{\mathbb{Q}/\mathbb{Z}}} \times \\ &\quad \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil b_J^{\mathbb{Q}/\mathbb{Z}} - b_J^{\mathbb{Q}/\mathbb{Z}}\lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil}. \end{aligned} \tag{59}$$

Next, we consider the second line of Eq. (56). We can use the fact that since $\lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil\lfloor b_J^{\mathbb{Q}/\mathbb{Z}}\rceil \in \mathbb{Z}$, then

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} (b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)(b_J^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_J^{\mathbb{Q}/\mathbb{Z}}\rceil)} \\ &\quad = \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_I^{\mathbb{Q}/\mathbb{Z}} b_J^{\mathbb{Q}/\mathbb{Z}} - b_I^{\mathbb{Q}/\mathbb{Z}}\lfloor b_J^{\mathbb{Q}/\mathbb{Z}}\rceil - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil b_J^{\mathbb{Q}/\mathbb{Z}}}. \end{aligned} \tag{60}$$

Using these simplifications, the gauged model (56) can be rewritten as

$$\begin{aligned} Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_J^{\mathbb{Q}/\mathbb{Z}} b_I^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} b_I^{\mathbb{Q}/\mathbb{Z}}} \\ &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \mathrm{d}\left(b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} \mathrm{d}(b_I^{\mathbb{Q}/\mathbb{Z}} - \lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil)\right)}, \end{aligned} \tag{61}$$

where we have used that $\sum_{IJ} b_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0$,

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} -b_I^{\mathbb{Q}/\mathbb{Z}}\lfloor b_J^{\mathbb{Q}/\mathbb{Z}}\rceil - b_J^{\mathbb{Q}/\mathbb{Z}}\lfloor b_I^{\mathbb{Q}/\mathbb{Z}}\rceil} = 1. \tag{62}$$

Therefore, starting from the $U^\kappa(1)$ bosonic model and gauging the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$, Eq. (61) gives the path integral of the gauged model from which we can find the 1-SPT invariant.

When the spacetime $\mathcal{M}^4$ has no boundary, the total derivative term in Eq. (61) vanishes and the path integral becomes

$$Z(B_I^{\mathbb{Q}/\mathbb{Z}}) = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_J^{\mathbb{Q}/\mathbb{Z}} b_I^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} b_I^{\mathbb{Q}/\mathbb{Z}}}. \tag{63}$$

We note that Eq. (63) is invariant under the following gauge transformation:

$$b_I^{\mathbb{Q}/\mathbb{Z}} \to b_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}\omega_I^{\mathbb{Q}/\mathbb{Z}}, \quad \sum_I \omega_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0. \tag{64}$$

It is straight forward to check that this is indeed the case. When $\partial\mathcal{M}^4 = \emptyset$, the gauge transformation Eq. (64) changes the term $\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_J^{\mathbb{Q}/\mathbb{Z}} b_I^{\mathbb{Q}/\mathbb{Z}}}$ by a factor

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \mathrm{d}\omega_J^{\mathbb{Q}/\mathbb{Z}} b_I^{\mathbb{Q}/\mathbb{Z}} + b_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}\omega_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}\omega_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}\omega_I^{\mathbb{Q}/\mathbb{Z}}} \\ &\quad = \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \omega_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}b_I^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}\omega_I^{\mathbb{Q}/\mathbb{Z}}}. \end{aligned} \tag{65}$$

However, using that $\mathrm{e}^{\mathrm{i}2\pi \sum_{I,J} K_{IJ}\int_{\mathcal{M}^4} \omega_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}b_I^{\mathbb{Q}/\mathbb{Z}}} = 1$ from Eq. (64), and also Eq. (1), we can rewrite Eq. (65) as

$$\mathrm{e}^{-\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \mathrm{d}\omega_I^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}\mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}}. \tag{66}$$

Furthermore, the gauge transformation (64) changes the term $\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} b_I^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}\mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}}$ by a factor

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} \mathrm{d}\omega_I^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}\mathrm{d}b_J^{\mathbb{Q}/\mathbb{Z}}}. \tag{67}$$

Eq. (66) and (67) perfectly cancel each other and, therefore, the action amplitude in Eq. (63) is gauge invariant.

Because the action amplitude is invariant under Eq. (64), it will not depend on the coboundaries $\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}}$ and, therefore, we will be able to evaluate the path integral Eq. (61) when $\partial\mathcal{M}^4 = \emptyset$. Plugging in $b_I^{\mathbb{Q}/\mathbb{Z}}$ and integrating by parts using that $\mathcal{M}^4$ is closed, the path integral Eq. (63) becomes

$$\begin{aligned} Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} -a_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} a_I^{\mathbb{Q}/\mathbb{Z}}} \times \\ &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}\mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}}}. \end{aligned}$$

Now, we can use Eq. (1) to rewrite the terms $a_J^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}}$ and $\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} a_I^{\mathbb{Q}/\mathbb{Z}}$ such that Z becomes

$$\begin{aligned} Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}}} \times \\ &\mathrm{e}^{-\mathrm{i}2\pi \sum_{I,J} K_{IJ}\int_{\mathcal{M}^4} a_I^{\mathbb{Q}/\mathbb{Z}}\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}. \end{aligned}$$

Because the path integral only sums over $a_I^{\mathbb{Q}/\mathbb{Z}}$ satisfying the quantization condition $\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0$ and that $\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0$, the term in the second line of Z becomes unity. Then, using Eq. (1) to rewrite $\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}}$, the path integral becomes

$$\begin{aligned} Z = \sum_{\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0} &\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ}\int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}}\underset{1}{\smile}\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \times \\ &\mathrm{e}^{-\mathrm{i}2\pi \sum_{I,J} K_{IJ}\int_{\mathcal{M}^4} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}\underset{2}{\smile}\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}. \end{aligned}$$

Firstly, note that the action amplitude on the second line is unity since $\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \in \mathbb{Z}$. Additionally, the action amplitude no longer contains the cochains $a_I^{\mathbb{Q}/\mathbb{Z}}$ which the path integral is summing over. Thus, performing the sum we obtain

$$Z = |\det(K)|^{N_e}\ \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}. \tag{68}$$

where N_e is the number of edges in the triangulated spacetime $\mathcal{M}^4$.

E. The 1-SPT invariant

The SPT order is characterized by the volume-independent partition function

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) = \frac{Z(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}})}{Z(\mathcal{M}^4, 0)}. \tag{69}$$

From this, we find that 1-SPT invariant for the 1-SPT state is

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) = \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}, \tag{70}$$

where as a reminder

$$\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0, \qquad \sum_I B_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0. \tag{71}$$

Such a non-trivial 1-SPT invariant Eq. (70) suggests that the 1-SPT order can be non-trivial. We note that, as confirmed in appendix section A, this 1-SPT invariant is correctly gauge invariant.

However, before going on consider some examples of non-trivial 1-SPT invariants (see section IV F), we want to show that any two matrices K and $\widetilde{K}$ related by $\widetilde{K} = U^\top K U$ with $U \in GL(\kappa, \mathbb{Z})$ actually describes the same 1-SPT invariant. We will first try to express the 1-SPT invariant Eq. (70) in terms of only the K-matrix instead of k_{IJ}. In doing so, we'll also find a nice form for the 1-SPT invariant which we can use when considering examples in the next section.

Consider the term $B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}}$ in the SPT invariant Eq. (70). We can first rewrite it as

$$\begin{aligned}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} &= \frac{1}{2}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} - B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} \\ &\quad + \frac{1}{2}\sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}}.\end{aligned}$$

Then using Eq. (1) and the fact that $\mathcal{M}^4$ is closed, this can become

$$\begin{aligned}&\frac{1}{2}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}} + B_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \\ &\quad + \frac{1}{2}\sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}}, \\ &= \frac{1}{2}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}} - B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \\ &\quad + \frac{1}{2}\sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}.\end{aligned}$$

Plugging this expression for $B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}}$ into the action in Eq. (70), we find

$$\begin{aligned}&\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \\ &= \frac{1}{2}\sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \\ &\quad + \frac{1}{2}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}.\end{aligned}$$

We can go further by again using Eq. (1) to rewrite second line of the right hand side and get

$$\begin{aligned}&\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \\ &= \frac{1}{2}\sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \\ &\quad - \frac{1}{2}\sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}}.\end{aligned} \tag{72}$$

Therefore, the 1-SPT invariant that characterizes the 1-SPT order has the form

$$\begin{aligned}Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) &= \mathrm{e}^{\mathrm{i}\,\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \\ &\quad \times \mathrm{e}^{\mathrm{i}\,\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}.\end{aligned} \tag{73}$$

We can recast the relationship between k_{IJ} and K_{IJ}, given by Eq. (35), by treating k_{IJ} as the elements of the upper triangular integer matrix k that satisfies

$$K = k + k^\top. \tag{74}$$

Then, using that $\mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \overset{\mathrm{d}}{=} -\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}$ from Eq. (1), we can replace $\sum_{I\le J} k_{IJ}$ by $\sum_{I<J} K_{IJ}$ in Eq. (73) to obtain

$$\begin{aligned}Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) &= \mathrm{e}^{\mathrm{i}\,\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \\ &\quad \times \mathrm{e}^{\mathrm{i}\,\pi \sum_{I<J} K_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}.\end{aligned} \tag{75}$$

Eq. (75) thus provides a form of the 1-SPT invariant in terms of only the K-matrix. However, due to the sum on the second term only being over $I < J$, it is not covariant.

We see that, from Eq. (73), the 1-SPT invariant is characterized by a pair of integer matrices (K, k). At first glance, due to the k dependence, or equivalently Eq. (75) not being covariant, it appears that the SPT invariant is changed by the transformation $B_I^{\mathbb{Q}/\mathbb{Z}} \to \widetilde{B}_I^{\mathbb{Q}/\mathbb{Z}}$ and $K \to \widetilde{K}$ where

$$\widetilde{B}_I^{\mathbb{Q}/\mathbb{Z}} = (U^{-1})_{IJ} B_J^{\mathbb{Q}/\mathbb{Z}}, \quad \widetilde{K} = U^\top K U, \tag{76}$$

and $U_{IJ} \in GL(\kappa, \mathbb{Z})$. Therefore, it would appear that K and $\widetilde{K}$ do not describe the same 1-SPT invariant.

However, it turns out that K and $\widetilde{K}$ actually do describe the same 1-SPT invariant. To show this, we first show that the 1-SPT invariant is left unchanged when k is replaced by another integer matrix k' (not necessarily upper triangular) such that $K = k' + k'^\top$. The difference $A = k - k'$ is an antisymmetric integer matrix. The respective lattice Lagrangian densities of the 1-SPT invariant Eq. (73) for k and k' (after dividing by 2π) differ by

$$\sum_{I,J} \frac{A_{IJ}}{2} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}.$$

Using that A is antisymmetric, that integer multiples of 2π can be added to the Lagrangian density without changing the path integral, and Eq. (1), this can be rewritten as

$$\sum_{I,J} \frac{A_{IJ}}{2} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} \sum_{I<J} \frac{A_{IJ}}{2} \mathrm{d}\big(- \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{3}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}\big),$$

which vanishes on a closed manifold. Therefore, the two Lagrangian densities differ by only a coboundary term and give the same topological invariants for a closed $\mathcal{M}^4$.

The above result allows us to show that the SPT invariant is unchanged under the transformation Eq. (76). Indeed, we now only need to check the k_{IJ} term in Eq. (73). Let $\bar{k}$ be an integer matrix defined by

$$\bar{k}_{IJ} = \sum_{I' \le J'} (U^\top)_{II'} k_{I'J'} U_{J'J}, \tag{77}$$

such that $\widetilde{K} = \bar{k} + \bar{k}^\top$. Using that, from Eq. (76), $B_I^{\mathbb{Q}/\mathbb{Z}} = U_{IJ} \widetilde{B}_J^{\mathbb{Q}/\mathbb{Z}}$ and plugging it into the second line of Eq. (73), it becomes

$$\mathrm{e}^{\,\mathrm{i}\pi \sum_{I \le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} = \mathrm{e}^{\,\mathrm{i}\pi \sum_{I,J} \bar{k}_{IJ} \int_{\mathcal{M}^4} \mathrm{d}\widetilde{B}_I^{\mathbb{Q}/\mathbb{Z}} \underset{2}{\smile} \mathrm{d}\widetilde{B}_J^{\mathbb{Q}/\mathbb{Z}}}. \tag{78}$$

Let's now introduce the upper triangular integer matrix $\widetilde{k}$ such that $\widetilde{K} = \widetilde{k} + \widetilde{k}^\top$. Using the above result for a closed $\mathcal{M}^4$, the SPT invariant is unchanged by replacing $\bar{k}$ with $\widetilde{k}$. Therefore, the SPT invariant Eq. (73) is unchanged under the transformation Eq. (76).

The fact that K and $\widetilde{K}$ describes the same SPT invariant also allows us to find a convenient expression for the SPT invariant. Indeed, recall that the integer matrix K has the Smith normal form given by Eq. (45). From the above discussion, we see that, without loosing generality, we may transform $K \to U^\top K U$ without changing the SPT invariant and thus may assume K to have the following form

$$K = V^\top \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & \ddots \end{pmatrix} V. \tag{79}$$

The invertible integer matrix V satisfies

$$(V^\top)_{IJ} k_J = k_J V_{JI} = k_I V_{IJ} \quad \text{or} \quad \frac{V_{IJ}}{V_{JI}} = \frac{k_J}{k_I}. \tag{80}$$

Using this expression for K, the 1-SPT invariant in its original form given by Eq. (70) can be rewritten as

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) = \mathrm{e}^{\,\mathrm{i}\pi \sum_I k_I V_{II} \int_{\mathcal{M}^4} B_I^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}}} \times \mathrm{e}^{\,\mathrm{i}2\pi \sum_{I<J} k_I V_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}.$$

Furthermore, the quantization condition on the background cochain field then becomes $\sum_I B_I^{\mathbb{Q}/\mathbb{Z}} k_I V_{IJ} \overset{1}{=} 0$. This can be automatically satisfied if we let $B_I^{\mathbb{Q}/\mathbb{Z}}$ take the form

$$B_I^{\mathbb{Q}/\mathbb{Z}} = k_I^{-1} B_I^{\mathbb{Z}_{k_I}}, \tag{81}$$

where $B_I^{\mathbb{Z}_{k_I}}$ is a $\mathbb{Z}_{k_I}$-valued 2-cocycle and thus satisfies $\mathrm{d}B_I^{\mathbb{Z}_{k_I}} \overset{k_I}{=} 0$. Using this, the 1-SPT invariant for the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry becomes

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Z}_{k_I}}) = \mathrm{e}^{\,\mathrm{i}\pi \sum_I V_{II} k_I^{-1} \int_{\mathcal{M}^4} B_I^{\mathbb{Z}_{k_I}} B_I^{\mathbb{Z}_{k_I}} + B_I^{\mathbb{Z}_{k_I}} \underset{1}{\smile} \mathrm{d}B_I^{\mathbb{Z}_{k_I}}} \times \mathrm{e}^{\,\mathrm{i}2\pi \sum_{I<J} V_{IJ} k_J^{-1} \int_{\mathcal{M}^4} B_J^{\mathbb{Z}_{k_J}} B_I^{\mathbb{Z}_{k_I}} + B_I^{\mathbb{Z}_{k_I}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Z}_{k_J}}}. \tag{82}$$

F. Some Examples of SPT Invariants

In the previous section, we found that $U^\kappa(1)$ gauge theory with a 2π topological term in the confined phase has a non-trivial 1-SPT invariant, Eq. (70). We then massaged the SPT invariant into other forms, such as Eq. (75) and Eq. (82). This suggests that generically there is a phase in the confined phase with non-trivial 1-SPT order which is protected by the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ symmetry discussed in section IV C. Now we will consider same simple examples of different K-matrices and the corresponding 1-SPT order. The first example will have $\kappa = 1$ while the second and third will be $\kappa = 2$.

Example 1

Let's first consider the case where there is only one type of cochain field $a^{\mathbb{R}/\mathbb{Z}}$ so $\kappa = 1$ and the K-matrix would become

$$K = \begin{pmatrix} 2n \end{pmatrix}, \tag{83}$$

with $n \in \mathbb{Z}$. In this case, the 3+1D bosonic model on spacetime lattice Eq. (34) becomes

$$\begin{aligned} Z = \int D[a^{\mathbb{R}/\mathbb{Z}}]\ & \mathrm{e}^{-\int_{\mathcal{M}^4} \frac{|\mathrm{d}a^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil|^2}{g}} \\ & \mathrm{e}^{\mathrm{i}\, 2\pi n \int_{\mathcal{M}^4} (\mathrm{d}a^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil)(\mathrm{d}a^{\mathbb{R}/\mathbb{Z}} - \lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil)} \\ & \mathrm{e}^{\mathrm{i}\, 4\pi n \int_{\mathcal{M}^4} a^{\mathbb{R}/\mathbb{Z}} \mathrm{d}\lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil - \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}\lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil}. \end{aligned} \tag{84}$$

From our previous discussion, this theory has a $\mathbb{Z}_{2n}^{(1)}$ symmetry. Let's see this explicitly. The path integral is invariant under the transformation $a^{\mathbb{R}/\mathbb{Z}} \to a^{\mathbb{R}/\mathbb{Z}} + \frac{1}{2n}\beta^{\mathbb{Z}}$ where $\beta^{\mathbb{Z}}$ is an arbitrary $\mathbb{Z}$-valued 1-cochain satisfying $\mathrm{d}\beta^{\mathbb{Z}} \overset{2n}{=} 0$. The physical part of $\beta^{\mathbb{Z}}$ is defined modulo $2n$ because shifting $\beta^{\mathbb{Z}}$ by $2n$-valued 1-cochain corresponds to shifting $a^{\mathbb{R}/\mathbb{Z}}$ by an integer-valued 1-cochain, which is a gauge transformation. Therefore, this theory indeed has a $\mathbb{Z}_{2n}^{(1)}$ symmetry. When $g \ll 1$, the above bosonic model at low-energies describes the deconfined phase of $U(1)$ gauge field theory. At energies much smaller than the energy gap of the $U(1)$ monopole, $\mathrm{d}\lfloor \mathrm{d}a^{\mathbb{R}/\mathbb{Z}} \rceil = 0$ and the $\mathbb{Z}_{2n}^{(1)}$ symmetry is promoted to an emergent $U(1)^{(1)}$ symmetry.

When $g \gg 1$, the above bosonic model is in a gapped phase with $\mathbb{Z}_{2n}^{(1)}$ 1-symmetry, which corresponds to the confined phase of the $U(1)$ gauge theory. From our general discussion, this gapped phase is an SPT phase protected by the $\mathbb{Z}_{2n}^{(1)}$ 1-symmetry. Indeed, using Eq. (82), this SPT phase is characterized by the 1-SPT invariant

$$\begin{aligned} Z^{\text{top}}(\mathcal{M}^4, B^{\mathbb{Z}_{2n}}) &= \mathrm{e}^{\mathrm{i}\, \frac{2\pi}{4n} \int_{\mathcal{M}^4} B^{\mathbb{Z}_{2n}} B^{\mathbb{Z}_{2n}} + B^{\mathbb{Z}_{2n}} \underset{1}{\smile} \mathrm{d}B^{\mathbb{Z}_{2n}}}, \\ &= \mathrm{e}^{\mathrm{i}\, \frac{2\pi}{4n} \int_{\mathcal{M}^4} \mathbb{Sq}^2(B^{\mathbb{Z}_{2n}})}. \end{aligned} \tag{85}$$

The 3+1D 1-SPT order for the $\mathbb{Z}_{2n}^{(1)}$ 1-symmetry is classified by $H^4(B(\mathbb{Z}_{2n}, 2); \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{4n}$.[20] From the SPT invariant, we find that the 1-SPT order realized by the confined phase is given by $1 \in \mathbb{Z}_{4n}$, and thus is the generator of the SPT orders classified by $H^4(B(\mathbb{Z}_{2n}, 2); \mathbb{R}/\mathbb{Z})$.

Example 2

Let's now consider an example where there are two types of 1-cochain fields $a_1^{\mathbb{R}/\mathbb{Z}}$ and $a_2^{\mathbb{R}/\mathbb{Z}}$, so $\kappa = 2$, and the K-matrix is given by

$$K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{86}$$

We'd like to find the SPT invariant for this K matrix using Eq. (82). This requires us to first find the integers k_1, k_2 and the integer matrix V from Eq. (79). The diagonal elements of the Smith normal form of K are $(k_1, k_2) = (3, 1)$. Thus, by finding k_I we can immediately conclude that the 1-symmetry is $\mathbb{Z}_3^{(1)} \times \mathbb{Z}_1^{(1)} \equiv \mathbb{Z}_3^{(1)}$. However, there does not exist an integer matrix V which will work for this K.

To find the SPT invariant, we can instead consider the matrix

$$\widetilde{K} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}. \tag{87}$$

Since $\widetilde{K} = UKU^\top$, where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}), \tag{88}$$

our results from section IV E show that the SPT order of $\widetilde{K}$ is equivalent to that of the K matrix Eq. (86). Therefore, we now attempt to find the SPT invariant using the same approach but now with $\widetilde{K}$. First note that the diagonal elements of the Smith normal form of $\widetilde{K}$ are still $(k_1, k_2) = (3, 1)$. The $\widetilde{K}$ matrix can be written as

$$\widetilde{K} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \tag{89}$$

and we find the integer matrix V to be

$$V = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

From the k_I and V found for $\widetilde{K}$, we have that

$$(V_{IJ} k_J^{-1}) = \begin{pmatrix} \frac{2}{3} & 1 \\ 1 & 2 \end{pmatrix}. \tag{90}$$

Using Eq. (82), the corresponding 1-SPT invariant for the $\mathbb{Z}_3^{(1)} \times \mathbb{Z}_1^{(1)} \equiv \mathbb{Z}_3^{(1)}$ 1-symmetry is given by

$$\begin{aligned} Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Z}_{k_I}}) = & \mathrm{e}^{\mathrm{i}\, \frac{2\pi}{3} \int_{\mathcal{M}^4} B_1^{\mathbb{Z}_3} B_1^{\mathbb{Z}_3} + B_1^{\mathbb{Z}_3} \underset{1}{\smile} \mathrm{d}B_1^{\mathbb{Z}_3}} \times \\ & \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^4} B_2^{\mathbb{Z}_1} B_1^{\mathbb{Z}_3} + B_1^{\mathbb{Z}_3} \underset{1}{\smile} \mathrm{d}B_2^{\mathbb{Z}_1}} \times \\ & \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^4} B_2^{\mathbb{Z}_1} B_2^{\mathbb{Z}_1} + B_2^{\mathbb{Z}_1} \underset{1}{\smile} \mathrm{d}B_2^{\mathbb{Z}_1}}. \end{aligned} \tag{91}$$

We can now use the fact that the SPT invariant is invariant under the gauge transformation $B_2^{\mathbb{Z}_1} \to B_2^{\mathbb{Z}_1} + m^{\mathbb{Z}}$, where $m^{\mathbb{Z}}$ is a $\mathbb{Z}$-valued 2-cochain, to set $B_2^{\mathbb{Z}_1} = 0$. Doing so, the SPT invariant simplifies to

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Z}_{k_I}}) = \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{3}\int_{\mathcal{M}^4} B_1^{\mathbb{Z}_3} B_1^{\mathbb{Z}_3} + B_1^{\mathbb{Z}_3} \underset{1}{\smile} \mathrm{d}B_1^{\mathbb{Z}_3}}. \tag{92}$$

Therefore, the 1-SPT invariant of the K matrix Eq. (86) is given by Eq. (92). 1-SPT order protected by the 1-symmetry $\mathbb{Z}_3^{(1)}$ is classified by $H^4(B(\mathbb{Z}_3, 2); \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_3$.[20] Therefore, from Eq. (92) the SPT order realized in the confined phase for the K-matrix Eq. (86) is given by $1 \in \mathbb{Z}_3$ and is thus the generator for SPT orders classified by $H^4(B(\mathbb{Z}_3, 2); \mathbb{R}/\mathbb{Z})$.

Example 3

For our final example, let's again consider the scenario where there are $\kappa = 2$ cochain fields $a_I^{\mathbb{Q}/\mathbb{Z}}$, but now where the K-matrix is

$$K = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}, \quad n \in \mathbb{Z}. \tag{93}$$

The diagonal elements of the Smith normal form of this K-matrix are $(k_1, k_2) = (n, n)$. Therefore, the model with this K matrix has a $\mathbb{Z}_n^{(1)} \times \mathbb{Z}_n^{(1)}$ symmetry. Furthermore, this K matrix can be written as

$$K = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{94}$$

Thus, unlike example 2, using the K matrix we start with, there exists the integer matrix

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{95}$$

From this matrix V and from k_I, we find that

$$(V_{IJ} k_J^{-1}) = \begin{pmatrix} 0 & \frac{1}{n} \\ \frac{1}{n} & 0 \end{pmatrix}. \tag{96}$$

Using Eq. (82), the corresponding 1-SPT invariant for the $\mathbb{Z}_n^{(1)} \times \mathbb{Z}_n^{(1)}$ 1-symmetry is given by

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Z}_n}) = \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{n}\int_{\mathcal{M}^4} B_2^{\mathbb{Z}_n} B_1^{\mathbb{Z}_n} + B_1^{\mathbb{Z}_n} \underset{1}{\smile} \mathrm{d}B_2^{\mathbb{Z}_n}}. \tag{97}$$

Thus, we find that the 1-SPT order in the confined phase of $U(1) \times U(1)$ 3+1D gauge theory with K matrix Eq. (93) is a mixed SPT order between the two $\mathbb{Z}_n^{(1)}$ 1-symmetries. In other words, the boundary Chern-Simons theory has a mixed anomaly between two $\mathbb{Z}_n^{(1)}$ 1-symmetries. This Chern-Simons theory describes 2+1D $\mathbb{Z}_n$ topological order. Indeed, the loop operators charged under the two $\mathbb{Z}_n^{(1)}$ 1-symmetries are the loop objects whose open ends correspond to e and m type anyons, respectively. Furthermore, the fact that the e and m anyons have nontrivial mutual statistics is a manifestation of the mixed anomaly between the two $\mathbb{Z}_n^{(1)}$ 1-symmetries.

V. CONCLUSION

In this paper, we have considered 3+1D compact $U^\kappa(1)$ gauge theory with 2π-quantized topological terms. In section IV B, we developed a bosonic lattice model acting as the UV regularization for the continuum theory. Working with this lattice model, we found that at energies much smaller than the gauge charges' gaps but larger than the monopoles' gaps, there is an exact emergent $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. We found that the confined phase of the $U^\kappa(1)$ gauge theory (*i.e.* the symmetric gapped phase of the bosonic model) has non trivial symmetry protected topological (SPT) order which is protected by the exact emergent $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry. We then went on to gauge this symmetry in section IV D and found the corresponding SPT invariant in section IV E. We gave some examples of different K matrices where the confined phases of the $U^\kappa(1)$ gauge theories realizes a $\mathbb{Z}_{2n}^{(1)}$ 1-SPT phase, a $\mathbb{Z}_3^{(1)}$ 1-SPT phase, and a $\mathbb{Z}_n^{(1)} \times \mathbb{Z}_n^{(1)}$ mixed 1-SPT phase.

Note: after the completion of this paper, we noticed the independent work Ref. 55 which studied the emergent 1-symmetry for the $\kappa = 1$ case in a phase where monopoles were only partially condensed.

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Appendix A: The Gauge Invariance of the 1-SPT Invariant

In section IV E of the main text, we found that the 1-SPT invariant for the $\mathbb{Z}_{k_1}^{(1)} \times \mathbb{Z}_{k_2}^{(1)} \times \cdots$ 1-symmetry given by Eq. (70):

$$Z^{\text{top}}(\mathcal{M}^4, B_I^{\mathbb{Q}/\mathbb{Z}}) = \mathrm{e}^{\mathrm{i}\,2\pi \sum_{I \le J} k_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}.$$

Here, $B_I^{\mathbb{Q}/\mathbb{Z}}$, with $I = 1, \ldots \kappa$, are background symmetry twist 2-cochain fields satisfying

$$\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \overset{1}{=} 0, \qquad \sum_I B_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0.$$

In this appendix section, we confirm a claim made in the main text that the above 1-SPT invariant for closed $\mathcal{M}^4$

is invariant under the gauge transformations

$$B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + n_I, \qquad B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}}, \quad \text{(A1)}$$

where n_I are $\mathbb{Z}$-valued 2-cochains and $a_I^{\mathbb{Q}/\mathbb{Z}}$ are $\mathbb{Q}/\mathbb{Z}$-valued 1-cochains satisfying the quantization conditions $\sum_I a_I^{\mathbb{Q}/\mathbb{Z}} K_{IJ} \overset{1}{=} 0$.

First, we'll check the $\mathbb{Z}$-gauge transformation $B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + n_I$, which causes the 1-SPT invariant to change by a factor

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_J B_I^{\mathbb{Q}/\mathbb{Z}} + B_J^{\mathbb{Q}/\mathbb{Z}} n_I} \, \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}n_J}.$$

Assuming that $\partial\mathcal{M}^4 = \emptyset$ and using (1), this can be rewritten as unity:

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_J B_I^{\mathbb{Q}/\mathbb{Z}} + B_J^{\mathbb{Q}/\mathbb{Z}} n_I} \, \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}n_J} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} B_J^{\mathbb{Q}/\mathbb{Z}} n_I} \, \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_J B_I^{\mathbb{Q}/\mathbb{Z}} - B_I^{\mathbb{Q}/\mathbb{Z}} n_J} \\
&\quad \times \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}n_J} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} -\mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} n_J - B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}n_J} \\
&\quad \times \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} + B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}n_J} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} n_J} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}} + n_J \underset{1}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}n_J \underset{2}{\smile} \mathrm{d}B_I^{\mathbb{Q}/\mathbb{Z}}} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} n_I \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} = 1.
\end{aligned}$$

Therefore, the SPT invariant is unchanged by the gauge transformation $B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + n_I$.

Lastly, let's check the gauge transformation $B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}}$, which causes the 1-SPT invariant to change by a factor

$$\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_J^{\mathbb{Q}/\mathbb{Z}} \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}}.$$

Once again, assuming $\partial\mathcal{M}^4 = \emptyset$ and using (1), we can show that this change is equal to unity:

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + B_J^{\mathbb{Q}/\mathbb{Z}} \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} B_J^{\mathbb{Q}/\mathbb{Z}} - \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \\
&\quad \times \mathrm{e}^{\mathrm{i}2\pi \sum_{I\le J} k_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}} \underset{1}{\smile} \mathrm{d}B_J^{\mathbb{Q}/\mathbb{Z}}} \\
&= \mathrm{e}^{\mathrm{i}2\pi \sum_{I,J} K_{IJ} \int_{\mathcal{M}^4} \mathrm{d}a_J^{\mathbb{Q}/\mathbb{Z}} B_I^{\mathbb{Q}/\mathbb{Z}}} = 1
\end{aligned} \quad \text{(A2)}$$

Therefore, the SPT invariant is also unchanged by the gauge transformation $B_I^{\mathbb{Q}/\mathbb{Z}} \to B_I^{\mathbb{Q}/\mathbb{Z}} + \mathrm{d}a_I^{\mathbb{Q}/\mathbb{Z}}$.

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