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# Crystalline Responses for Rotation-Invariant Higher-Order Topological Insulators

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Two-dimensional higher-order topological insulators can display a number of exotic phenomena, such as half-integer charges localized at both corners and disclination defects. In this paper, we analyze these phenomena, focusing on the paradigmatic example of the quadrupole insulator with $C_4$ rotation symmetry, and present a topological field theory description of the mixed geometry-charge responses. Our theory provides a unified description of the corner and disclination charges in terms of a physical geometry (which encodes disclinations), and an effective geometry (which encodes corners). We extend this analysis to interacting systems, and predict the response of fractional quadrupole insulators, which exhibit charge $e/2(2k+1)$ bound to corners and disclinations.

Higher-order topological insulators (HOTIs) are forms of crystalline topological phases that host gapped surfaces, which are connected by anomalous corner or hinge modes. The first example of a HOTI was the 2D model constructed in Ref. 1, which was dubbed the topological quadrupole insulator (QI). The QI is protected by $C_4$ lattice rotation symmetry, and is characterized by having half-integer corner charges when defined on a lattice with boundary. The QI has vanishing electric polarization, but, due to its corner modes, harbors a non-vanishing $xy$ quadrupole moment. Additionally, when chiral symmetry is present, the QI has zero energy corner modes as well. In addition, it has been shown that this insulating phase is sensitive to the presence of $\pi/2$ disclinations on which half-integer charges are bound[2]. After this initial development, the family of HOTIs has expanded to include other 2D HOTIs with corner modes, and various 3D HOTIs with either hinge modes or corner modes[3–10]. There have also been a number of experimental realizations of HOTIs[11–20].

Despite the rapid advances, our understanding of HOTIs still pales in comparison to our understanding of (first-order) topological insulators. While a number of detailed works analyze and classify HOTIs based on their symmetries[21–24], the field theoretic understanding of HOTI responses is incomplete[25–27] . Historically topological response theories have been a powerful tool with which to probe the physics of topological insulators, and, in turn, topological insulators have helped to provided new contexts in which one can realize topological field theories[28,29]. Motivated by this, we consider the charge response properties of the QI with $C_4$ rotation symmetry, especially its corner and disclination bound charge responses. Although we only consider the $C_4$ symmetric QI here, our analysis can be straightforwardly generalized to other $C_n$ symmetric HOTIs that have a Dirac fermion description in the continuum. We also discuss extensions to HOTIs protected by magnetic point groups[6] in the supplementary material[30] (see, also, Ref. 31 therein).

To analyze the $C_4$ symmetric QI, we consider a 4-band model of fermions on a square lattice at half filling. The Bloch Hamiltonian is:

$$\begin{aligned} h^q(\vec{k}) =& \sin(k_x)\Gamma^1 + \sin(k_y)\Gamma^2 + \Delta_1\Gamma^3 \\ &+ \tfrac{1}{\sqrt{2}}\left[\Delta_2 + \cos(k_x) - \cos(k_y)\right]\Gamma^4 \\ &+ \tfrac{1}{\sqrt{2}}\left[\Delta_3 + \cos(k_x) + \cos(k_y)\right]\Gamma^0, \end{aligned} \tag{1}$$

where, $\Gamma^1 = \sigma^3 \otimes \sigma^2$, $\Gamma^2 = -\sigma^3 \otimes \sigma^1$, $\Gamma^3 = \sigma^1 \otimes \sigma^0$, $\Gamma^4 = \sigma^2 \otimes \sigma^0$, $\Gamma^0 = \sigma^3 \otimes \sigma^3$, $\sigma^{1,2,3}$ are Pauli matrices, and $\sigma^0$ is the $2\times 2$ identity. This Hamiltonian is related to the original model presented in Ref. 1, by a unitary transformation (see supplemental material[30]). Before considering the continuum limit, we first recall some known features of this model (see Ref. 3 for more details). The energy bands derived from Eq. 1 are doubly degenerate, and there is a gap between the upper and lower pairs of bands which closes when $(\Delta_1, \Delta_2, \Delta_3) = (0, 0, \pm 2)$, or $(0, \pm 2, 0)$. The Bloch Hamiltonian has time-reversal symmetry $\mathcal{T} = \Gamma^2\Gamma^4\mathcal{K}$ ($\mathcal{K}$ is complex conjugation), and when $\Delta_1 = 0$, the lattice model has chiral symmetry, $\Pi = \Gamma^3$. When $\Delta_1 = \Delta_2 = 0$ there is a $C_4$ rotation symmetry:

$$\begin{aligned} \hat{U}_4 h^q(\vec{k})\hat{U}_4^{-1} &= h^q(R_4\vec{k}) \\ \hat{U}_4 &= \mathrm{diag}(e^{i3\pi/4}, e^{i\pi/4}, e^{-i\pi/4}, e^{-i3\pi/4}), \end{aligned} \tag{2}$$

where $R_4$ rotates $\vec{k}$ by $\pi/2$. In the presence of $C_4$ symmetry, there are two topologically distinct phases: the QI, which occurs when $|\Delta_3| < 2$, and a trivial insulator, which occurs when $|\Delta_3| > 2$.

To pass to the continuum, we consider the system close to a transition between the two $C_4$ symmetric phases. Without loss of generality, let us restrict our attention to the band crossing at $(\Delta_1, \Delta_2, \Delta_3) = (0, 0, +2)$. At this point, a pair of Dirac cones form at lattice moment $\vec{k} = (\pi, \pi)$. The low energy degrees of freedom near this critical point can be written in terms of Dirac fermions, with the continuum Lagrangian

$$\mathcal{L}_{\text{quad}} = \bar{\mathbf{\Psi}}[\gamma^0 i\partial_{\text{t}} + \gamma^1 i\partial_{\text{x}} + \gamma^2 i\partial_{\text{y}} + \boldsymbol{m}\cdot\boldsymbol{\tau}]\mathbf{\Psi} \tag{3}$$

where $\mathbf{\Psi}$ and $\bar{\mathbf{\Psi}} = \mathbf{\Psi}^\dagger\gamma^0$ are 4 component spinors, and $\boldsymbol{m}\cdot\boldsymbol{\tau} = m_1\tau^1 + m_2\tau^2 + m_3\tau^3$, with $m_1 \propto \Delta_1$, $m_2 \propto \frac{1}{\sqrt{2}}\Delta_2$,

$m_3 \propto \frac{1}{\sqrt{2}}(\Delta_3 - 2)$. The $\gamma$ and $\tau$ matrices are defined as $\gamma^0 = \sigma^0 \otimes \sigma^3$, $\gamma^1 = i\sigma^3 \otimes \sigma^1$, $\gamma^2 = i\sigma^3 \otimes \sigma^2$, $\tau^1 = -\sigma^2 \otimes \sigma^3$, $\tau^2 = -\sigma^1 \otimes \sigma^3$, $\tau^3 = -\sigma^3 \otimes \sigma^0$. The $\gamma$ and $\tau$ matrices each generate a representation of SU(2). We will refer SU(2) generated by the $\gamma$ matrices as the spin of the Dirac fermions[32], and the SU(2) generated by the $\tau$ matrices as the isospin. Eq. 3 therefore describes a pair of two-component 2D Dirac fermions with isospin coupled to a mass vector $\boldsymbol{m}$.

When $\boldsymbol{m}$ is constant (which occurs in the bulk of an insulator), the mass terms $m_1$ and $m_2$ break $C_4$ rotation symmetry (see Eq. 2). Based on the lattice model, we can identify the Lagrangian where $\boldsymbol{m} = (0, 0, m_3)$ as the QI for $m_3 < 0$, and the trivial insulator for $m_3 > 0$. Both of these phases consist of a pair of 2 component Dirac fermions with opposite masses. Intuitively we can think of this as a bilayer system where the layers, which are indexed by $\tau^3$, have opposite (integer) Hall conductance. In addition to the "light" fermions in Eq. 3, this theory also includes a pair of two component "heavy" regulator fermions, which will be left implicit for brevity.[33] Since we have specified $\Delta_3 \sim 2$ for our continuum limit we avoid the parameter regime of the model where the heavy fermions become massless.

To analyze the corner physics of the QI, we will consider the responses of the continuum Lagrangian Eq. 3 in the presence of a background gauge field $A$ and a non-constant $\boldsymbol{m}$, about which much is already known[34–37]. In the context of the QI, the spatial variation of $\boldsymbol{m}$ encodes domain walls and corners. For non-constant $\boldsymbol{m}$, the space of $C_4$-symmetric mass vectors $\boldsymbol{m}(x) \equiv (m_1(x), m_2(x), m_3(x))$ satisfies

$$\boldsymbol{m}(x) = (-m_1(R_4 x), -m_2(R_4 x), m_3(R_4 x)), \qquad (4)$$

where $x = (\mathrm{t}, \mathrm{x}, \mathrm{y})$ is the space-time coordinates and $R_4$ rotates the spatial component of $x$ by $\pi/2$. Assuming that $\boldsymbol{m}$ varies slowly over length scales $\propto |\boldsymbol{m}|^{-1}$, the fermions can be integrated out at one loop order, which leads to the topological response term

$$\mathcal{L}_{\text{top}} = \frac{\epsilon^{\mu\rho\kappa}}{8\pi} \boldsymbol{n} \cdot (\partial_\mu \boldsymbol{n} \times \partial_\rho \boldsymbol{n}) A_\kappa, \qquad (5)$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ is defined such that $\boldsymbol{m} \equiv m\boldsymbol{n}$ and $|\boldsymbol{n}|^2 = 1$. The value of $m$ does not affect the topological responses we are interested in, and will be taken to be a constant. Here, $j^\mu_{\text{sky}} = \frac{1}{4}\epsilon^{\mu\rho\kappa}\boldsymbol{n} \cdot (\partial_\rho \boldsymbol{n} \times \partial_\kappa \boldsymbol{n})$ is the skyrmion density of $\boldsymbol{n}$, and the response indicates that charge is bound to skyrmions, $j^\mu = \frac{1}{2\pi} j^\mu_{\text{sky}}$.

For our purposes, it will be useful to switch to a gauge field description of the skyrmions of $\boldsymbol{n}$[38] by defining a local SU(2) transformation $\Omega$, which rotates the unit vector $\boldsymbol{n}$ to an arbitrary constant unit vector $\boldsymbol{N}$ at each point in space:

$$\Omega^{-1}(x)\boldsymbol{n}(x) \cdot \boldsymbol{\tau}\Omega(x) = \boldsymbol{N} \cdot \boldsymbol{\tau}. \qquad (6)$$

The choice of $\boldsymbol{N}$ is inconsequential, and can be changed via a global isospin rotation. The transformation in Eq. 6 modifies the Lagrangian by rotating the mass vector $\boldsymbol{m} = m\boldsymbol{n} \to m\boldsymbol{N}$, and generating a covariant derivative, $D_\mu = \partial_\mu - iA_\mu - i\boldsymbol{b}_\mu \cdot \boldsymbol{\tau}$, where $b^j_\mu = \frac{i}{4}\operatorname{Tr}\left[\tau^j \Omega^{-1}\partial_\mu \Omega\right]$. Despite the fact that there are 3 gauge fields, $b^i_\mu$ (one per generator $\tau^i$ of SU(2)) there is only a U(1) gauge symmetry, which corresponds to the local U(1) $\subset$SU(2) isospin rotations that leave the mass vector $m\boldsymbol{N}$ invariant. In terms of the gauge fields, $\boldsymbol{b}_\mu$ we can rewrite Eq. 5 as

$$\mathcal{L}_{\text{top}} = \frac{1}{2\pi}\epsilon^{\mu\rho\kappa} b^N_\mu \partial_\rho A_\kappa, \qquad b^N_\mu \equiv \boldsymbol{N} \cdot \boldsymbol{b}_\mu \qquad (7)$$

where $b^N$ is the U(1) gauge field, which corresponds to the aforementioned U(1) $\subset$SU(2) gauge symmetry, and should be regarded as a background field that encodes the skyrmions of $\boldsymbol{n}$. Here, $j^\mu = -\frac{1}{2\pi}\epsilon^{\mu\rho\kappa}\partial_\rho b^N_\kappa$, from which we see that charge is bound to the vortices of $b^N$. The coefficient of $\frac{1}{2\pi}$ in Eq. 7 is quantized due to the U(1) charge and the U(1) isospin gauge symmetries[39].

Now we are ready to consider the corner charges of the $C_4$ symmetric QI. As we shall show, domain corners between the QI and trivial insulator correspond to $\pi$-vortices of $b^N$, and these $\pi$-vortices bind a half-integer charge. To see this, we consider a square sample of the QI embedded in a trivial insulator in a $C_4$ symmetric fashion. In the bulk of the QI $\boldsymbol{n} = (0, 0, -1)$, while deep in the trivial region $\boldsymbol{n} = (0, 0, 1)$. The gapped domain walls between the QI and the trivial insulator correspond to regions where $\boldsymbol{n}$ spatially interpolates between $(0, 0, -1)$ and $(0, 0, 1)$. For simplicity, we will consider the situation where $n_1 = 0$ near the domain walls. The model has additional chiral symmetry for this configuration of domain walls, although only $C_4$ symmetry is necessary to protect the corner charges. Since $n_1 = 0$ everywhere, it is convenient to switch to polar coordinates, $\boldsymbol{n} = (0, n_2, n_3) = (0, \sin(\varphi), \cos(\varphi))$, such that in the bulk of the QI $\varphi = \pi$, in the trivial region $\varphi = 0$, and at the domain walls $\varphi$ smoothly winds between $\pi$ and 0. Based on Eq. 4, $C_4$ symmetry requires that, $\varphi(x) \to -\varphi(R_4 x)$. So, $C_4$ symmetry requires that if domain walls normal to the $\pm$y-direction have $\varphi$ winding by $+\pi + 2q\pi$ ($q \in \mathbb{Z}$), then domain walls normal to the $\pm$x-direction must have $\varphi$ winding by $-\pi - 2q\pi$. If we rotate $\boldsymbol{n}$ according to Eq. 6, $b^N_\mu = \frac{1}{2}\partial_\mu\varphi$, and, based on our discussion above, the charge located at the domain corner is $Q_{\text{corner}} = -\frac{1}{2\pi}\oint b^N_i \cdot dl_i = -\frac{1}{4\pi}\oint d\varphi = \frac{1}{2} \mod (1)$ where the integral is over a loop that encircles the corner and is much larger than the width of the domain wall. We therefore find that the response theory correctly predicts the characteristic half-integer corner charge of the QI.

For the aforementioned domain corner configuration where $\boldsymbol{n} = (0, n_2, n_3)$, the Dirac equation has a localized zero-energy corner[34]. This mode is exactly the zero-energy corner mode of the QI with additional chiral symmetry[1], and acquires a gap if we break chiral symmetry and set $n_1 \neq 0$ at the corner. The corner mode can shift the corner charge by an integer. Indeed, it is generically possible to shift any bound charge by an integer via local perturbations, and so we are only concerned

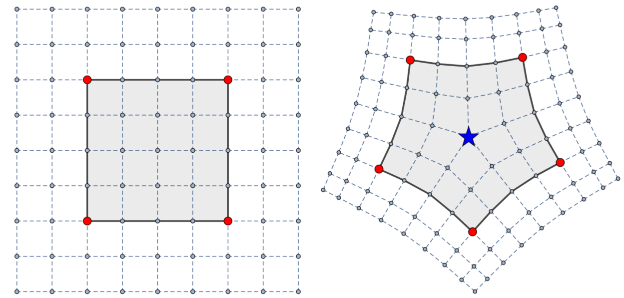

**Figure 1:** Left: A sample (gray) without a disclination, which has 4 corners (●). Right: A sample with a $\pi/2$ disclination ($\star$), which has 5 corners.

with the fractional part of the charge throughout this manuscript.

Let us now move on to another notable feature of the QI: half-integer charge bound to $\pi/2$ disclinations of a $C_4$ symmetric lattice. It is useful to think of this effect as the electromagnetic response to singular sources of curvature, which are fluxes of $C_4$ symmetry. This can be interpreted as the bulk response of the QI, similar to how the charge bound to magnetic vortices is a bulk response of quantum Hall insulators. As we shall explicitly show, this response is described by a Wen-Zee-like term[8,40,41]. Before deriving the response term, it will be useful to first discuss the connection between the corner and disclination responses in rotation-invariant HOTIs[2]. To demonstrate this connection, we will again consider a square sample of the QI embedded in a trivial insulator. The boundary of the QI traces out an angle of $2\pi$ with respect to the center of the sample, indicating that there are 4 corners at which the orientation of the boundary changes by $\pi/2$. If a single disclination with Frank angle $\pi/2$ is added to the bulk of the QI, the boundary will instead trace out an angle of $5\pi/2$, and the sample must have 5 corners (see Fig. 1). Thus, adding a disclination in the bulk requires the addition of an extra corner on the boundary. Since the QI has half-integer corner charges, there will be an extra, anomalous, half-integer of charge at the boundary of the disclinated 5-corner sample. In order to have a total integer charge, there must also be a half-integer charge bound to the disclination in the bulk. This argument indicates that charge conservation at the boundary is anomalous with respect to $C_4$ symmetry, since inserting a flux of $C_4$ symmetry into the bulk of the QI increases the charge at the boundary. This anomaly is canceled by the topologically non-trivial bulk of the QI. Here, we have only considered the $C_4$ symmetric QI, but it is straightforward to generalize this argument to other $C_n$ symmetric HOTIs.

We now consider coupling the continuum theory in Eq. 3 to curvature. Here we will consider a translationally invariant $C_4$ symmetric system ($\boldsymbol{m} = (0, 0, m_3)$). The first observation we make is that in the continuum theory of the $C_4$ symmetric phases, the discrete $C_4$ rotation symmetry is enlarged to a continuous SO(2) $\simeq$U(1) rotation symmetry. Under this symmetry, $\boldsymbol{\Psi} \to \hat{U}(\theta)\boldsymbol{\Psi}$, where $\hat{U}(\theta) = \exp\big(i\theta[\frac{1}{2}\gamma^0 + \tau^3]\big)$. This U(1) symmetry is explicitly broken down to the $C_4$ rotation symmetry of the lattice by subleading terms that are not included in the continuum description. To include curvature/disclinations in our description, we gauge the rotation symmetry and introduce the background gauge field (spin connection) $\omega$[42]. Here, the gauge symmetry is $C_4$, although the continuum Lagrangian has an enlarged U(1) symmetry. Because of this, the fluxes of $\omega$ are quantized in multiples of $\pi/2$ (i.e. the possible Frank angles of a square lattice).

To physically motivate the introduction of the spin connection $\omega$, we can consider two orthonormal vectors (frame-fields AKA vielbeins), $e^i_\mu$ $(i = x, y)$ and their inverses, $E^\mu_i$ and identify the latter with the primitive vectors of the underlying lattice[43] (in units of the lattice constant). The spin connection is then $\omega_\mu \equiv e^x_\rho \partial_\mu E^\rho_y$. Therefore, $\omega_\mu$ measures the local rotation of the lattice in the $\mu$-direction. The field strength of the spin connection, $\partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, encodes the curvature of the lattice that arises from discliantion defects, and should *not* be confused with the curvature of external spacetime, which we are not considering in this work.

The fermions in Eq. 3 couple to $\omega$ via a term proportional to the generator of spatial rotations in the covariant derivative: $D_\mu = \partial_\mu - iA_\mu - i\frac{1}{2}\omega_\mu\gamma^0 - i\omega_\mu\tau^3$. The response theory for the minimally coupled fermions is given by the Wen-Zee term

$$\mathcal{L}_{\rm geo} = \frac{\mathrm{sgn}(m_3) - 1}{2\pi}\epsilon^{\mu\rho\kappa}\omega_\mu\partial_\rho A_\kappa, \tag{8}$$

where the addition of the $-1/2\pi$ in the coefficient comes from the heavy fermions, which also couple to curvature. For the QI ($m_3 < 0$), $j^\mu = \frac{1}{\pi}\epsilon^{\mu\rho\kappa}\partial_\rho\omega_\kappa$, and $C_4$ disclinations of the lattice ($\pi/2$ vortices of $\omega$), bind a half-integer charge, while for the trivial phase ($m_3 > 0$) the Wen-Zee term vanishes. For $C_4$ symmetric systems, the coefficient of the Wen-Zee term is quantized in units of $1/2\pi$, and defined modulo $2/\pi$[2,40,41]. When boundaries are present, the Wen-Zee term is not gauge invariant. This indicates that the boundary of the QI is anomalous, with a mixed anomaly between spatial rotations and the U(1) charge symmetry. The anomalous conservation law is $\partial_\mu j^\mu|_{\rm boundary} = -\frac{1}{\pi}\epsilon^{\mu\rho}\partial_\mu\omega_\rho|_{\rm boundary}$, where both sides of the equation are evaluated at the 1D boundary of the QI. According to the integrated form of this equation, if we add $N_{\rm dis}$ disclinations to the bulk of the QI, the charge localized at the boundary of the QI changes by $\Delta Q_{\rm boundary} = -\frac{1}{2}N_{\rm dis}$. Hence, the charge localized at the boundary is not conserved if $C_4$ symmetry is gauged, in agreement with our earlier argument.

Having separately discussed the continuum interpretations of the corner responses and the disclination responses, we are now in a position to present a unified description of both of these features. To do this, we first note that the spatial variation of a vector can be interpreted as the parallel transport of a constant vector in the presence of an effective curvature. Indeed, if we consider a spatially varying mass vector $m(x)$, then using Eq. 6, we find that

$$m_i(x+dx) = m_i(x) + 2b^j_\mu \epsilon^{ijk} m_k(x) dx_\mu. \quad (9)$$

Eq. 9 can be interpreted as the parallel transport of $\boldsymbol{m}$ with respect to a new effective affine connection $2b^j_\mu \epsilon^{ijk}$ that encodes a new effective curvature. Let us now consider the effects of background curvature on $\boldsymbol{m}$. For a local patch of curved space, $\boldsymbol{m}$ satisfies

$$m_i(x+dx) = m_i(x) + 2[b^j_\mu + \omega_\mu \delta^{j,3}]\epsilon^{ijk} m_k(x) dx_\mu, \quad (10)$$

where delta function, $\delta^{j,3}$, originates from the fact that the spin connection only couples to the isospin $\tau^3$. Therefore, the variation of $\boldsymbol{m}$ in curved space receives contributions from a combination of the physical background geometry $\omega$ and the effective geometry, $\boldsymbol{b}$. Importantly, the effective geometry is induced by the domain walls and corners of the QI, and is independent of the curvature of the underlying lattice.

We will now consider coupling the fermions in Eq. 3 to background curvature, and allowing the mass vector $\boldsymbol{m} = m\boldsymbol{n}$ to vary. As before, we consider rotating the Lagrangian using $\Omega$, such that the theory has a constant mass vector $m\boldsymbol{N}$. After performing this rotation, the covariant derivative becomes $D_\mu = [\partial_\mu - iA_\mu - i\omega_\mu \frac{1}{2}\gamma^0 - i\boldsymbol{b}_\mu \cdot \boldsymbol{\tau} - i\omega_\mu \boldsymbol{s}\cdot\boldsymbol{\tau}]$, where $s^j = \frac{1}{4}\operatorname{Tr}\left[\tau^j \Omega^{-1}\tau^3\Omega\right]$ is the $j$-th projection of the isospin $\tau^3$ after the local rotation by $\Omega$. The theory therefore describes a system of fermions coupled to both physical lattice curvature and the effective curvature (the latter induced by the variation of $\boldsymbol{m}$). After integrating out the fermions, the response theory is

$$\mathcal{L}_{\text{full}} = \frac{\epsilon^{\mu\rho\kappa}}{2\pi}\left[b^N_\mu + \jmath\omega_\mu\right]\partial_\rho A_\kappa. \quad (11)$$

where $\jmath = (\boldsymbol{N}\cdot\boldsymbol{s} - 1)$ (the $-1$ is from the heavy regulator fermions, which are only coupled to the physical curvature). This response equation is a central result of this paper and indicates that charge is bound to fluxes of the combination $b^N + \jmath\omega$. The fluxes of $b^N$ correspond to domain corners of the QI, which are sources of the effective curvature. The fluxes of $\omega$ correspond to lattice disclinations, which are sources of the physical lattice curvature.

For locally $C_4$ symmetric systems, $\jmath = 0$ (trivial) or 2 (QI). However, if $C_4$ symmetry is locally broken, such as near domain walls, $\jmath$ is not quantized and can fluctuate. Because of this, it is possible for $\jmath\omega$ to have a non-vanishing flux even when the flux of $\omega$ vanishes. Indeed, fluxes of $\jmath\omega$ can naturally appear on boundaries, far from any bulk disclinations, as we shall show below.

Previously, we pointed out that adding a disclination to the bulk of a QI with boundaries leads to an additional corner and its corresponding half integer corner charge. This behavior can be explicitly examined using Eq. 11. Let us start with a curvature free system ($\omega = 0$) where a domain wall located at $\mathrm{x} = \mathrm{x_{dw}}$ separates a QI

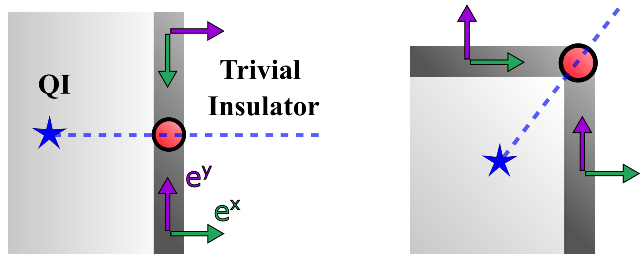


**Figure 2:** Left: A domain wall separating a trivial insulator and a QI which hosts a $\pi/2$ disclination ($\star$). The disclination induces a half-integer charge ($\bullet$) on the domain wall, and rotates the vielbeins ($e^x$ and $e^y$). Right: The orientation of the domain wall *relative* to the vielbeins.

($\mathrm{x} < \mathrm{x_{dw}}$) and a trivial insulator ($\mathrm{x} > \mathrm{x_{dw}}$)–we will implicitly assume that there are other domain walls located significantly far away such that the system has a global $C_4$ symmetry. As before, we shall consider boundary conditions such that $\boldsymbol{n} = (0, \sin(\varphi), \cos(\varphi))$ where $\varphi$ winds by $\pi + 2q\pi$ at the domain wall. In terms of the polar variables, $b^N_\mu = \frac{1}{2}\partial_\mu\varphi$, and $\jmath = \cos(\varphi) - 1$. In the absence of curvature, there are no localized charges in this region of the domain wall, since $Q = \frac{1}{2\pi}\oint b^N_i dl_i = \oint \frac{1}{4\pi}\partial_i\varphi dl_i = 0$ for a loop far away from any other domain walls or corners.

We will now add a $\pi/2$ disclination at $(\mathrm{x_0}, \mathrm{y_0})$ which is deep in the bulk of the QI ($\mathrm{x_0} \ll \mathrm{x_{dw}}$). To model this, we consider a local patch of space and choose $\omega$ as

$$\omega_x = 0, \quad \omega_y = \frac{\pi}{2}\Theta(\mathrm{x}-\mathrm{x_0})\delta(\mathrm{y}-\mathrm{y_0}), \quad \omega_t = 0, \quad (12)$$

where $\Theta$ is the step function. This configuration yields a point disclination at $(\mathrm{x_0}, \mathrm{y_0})$, around which there is a localized charge $Q_{\text{dis}} = 1/2$. Let us now examine how this disclination affects the domain wall at $\mathrm{x} = \mathrm{x_{dw}}$. From Eq. 12 we see the curl of $\omega$ vanishes near the domain wall. However, $\jmath$ varies in space near the domain wall, and at $(\mathrm{x_{dw}}, \mathrm{y_0})$ there is a localized charge $Q = \frac{1}{2\pi}\oint \jmath\omega_i dl_i = -1/2 \mod(1)$, where the integral is over a loop that encircles $(\mathrm{x_{dw}}, \mathrm{y_0})$ and does not approach the disclination or other domain walls. Additionally, the disclination adds a new domain corner at $(\mathrm{x_{dw}}, \mathrm{y_0})$. To see this, we recall that $\omega_\mu$ measures the rotation of the underlying lattice along the $\mu$-direction. The spin connection defined in Eq. 12 therefore indicates that the lattice rotates by $\pi/2$ at $(\mathrm{x} > \mathrm{x_0}, \mathrm{y} = \mathrm{y_0})$. Hence, the orientation of the domain wall rotates by $-\pi/2$ *relative* to the lattice at $(\mathrm{x_{dw}}, \mathrm{y_0})$ (see Fig. 2). We interpret this rotation of the domain wall as a domain corner. Eq. 11 therefore correctly describes the charge bound to the disclination of the QI, as well as the additional corner and corresponding corner charge of the disclinated system.

We can also consider a "fractional" version of the response in Eq. 11,

$$\mathcal{L}_{\text{frac}} = \frac{\epsilon^{\mu\rho\kappa}}{2\pi}\nu\left[b^N_\mu + \jmath\omega_\mu\right]\partial_\rho A_\kappa, \quad (13)$$

where $\nu$ is a rational number[44]. Eq. 13 characterizes a *fractional quadrupole insulator* (FQI), which has charge $\nu/2$ localized at corners and $\pi/2$-disclinations. Due to the fractional prefactor, the FQI is a symmetry enriched topologically ordered phase[45]. Based on our earlier observation that the bulk of the QI can be treated as a bilayer system where the layers have opposite (integer) Hall conductances, we expect that the bulk of the FQI can be realized in a fractional Chern insulator (FCI) bilayer where the FCIs have opposite (fractional) Hall conductances. In the supplemental material[30] (see also Ref. 46 and 47 therein), we consider a model for such a system with $\nu = 1/(2k+1)$ and show that it realizes the response in Eq. 13. We also show that a fractional quadrupole insulator with magnetic point group symmetry can be constructed by the breaking time reversal symmetry of a fractional quantum spin Hall insulator[48,49].

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*Note added.* During the preparation of this manuscript, an independent work appeared with some overlapping results in Ref. 50. These works were carried out independently.

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