Bell-state generation for spin qubits via dissipative coupling

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We theoretically investigate the dynamics of two spin qubits interacting with a magnetic medium. A systematic formal framework for this qubit-magnet hybrid system is developed in terms of the steady-state properties of the magnetic medium. Our particular focus is on the induced dissipative coupling between the spin qubits. In contrast to the conventional wisdom that dissipation is detrimental to quantum effects, here we show that a sizable long-lifetime entanglement can be established via a dissipative environment, in the absence of any coherent coupling. Moreover, we demonstrate that maximally-entangled two-qubit states (Bell states) can be achieved in this scheme when complemented by proper postselection. In this situation, the time evolution of the entanglement is governed by a non-Hermitian Hamiltonian, where dynamical phases are separated by an exceptional point. The resultant Bell state is robust against weak random perturbations and does not require the preparation of a particular initial state. Our study may find applications in quantum information science, quantum spintronics, and for sensing of nonlocal quantum correlations.

Introduction.—Entanglement between individually addressable qubits is the key to many quantum-information processes [1, 2]. The realization of qubits has been achieved in several systems, such as trapped atoms [3–6], quantum dots [7–9], superconducting circuits [10], and nitrogen-vacancy (NV) centers [11], etc. For example, the NV qubit has a long coherence time and a good performance in the initialization and readout of spin states [12–14]. However, because the direct dipolar interactions between NVs extend only up to tens of nanometers, the generation of entanglement between distant qubits has been one of the main adversities in building a scalable platform for practical applications. A potential solution to this problem is to exploit hybrid quantum devices [15], where qubits are interfaced with a solid-state system [16–21]. The latter, being long-range correlated, can act as a medium to induce an effective coherent coupling between the qubits, based on which certain two-qubit gates can be implemented [22, 23]. Meanwhile, the presence of a medium also enhances dissipation effects. To achieve a finite entanglement between qubits, the timescale set by the coherent coupling needs to be shorter than that of the local qubit relaxation. The competition between the two has thus been the focus of recent investigations [22–30].

Dissipation, however, is not always detrimental to quantum effects. Entanglement generation in an open quantum system by environment engineering was first discussed in the context of quantum optics [31, 32]. It was shown formally that two qubits can be entangled by undergoing Markovian dissipative dynamics [33]. Various proposals have been made to realize this, mainly in quantum optical and electronic systems [34–45]. In addition, dissipation is also investigated as a resource for quantum error correction [46–51] and other quantum information tasks [52–55]. Non-Hermitian Hamiltonians, frequently invoked to handle dissipative effects in the Hamiltonian form, can exhibit exceptional points [56] that have been shown to be sweet spots to enhance entanglement [57, 58].

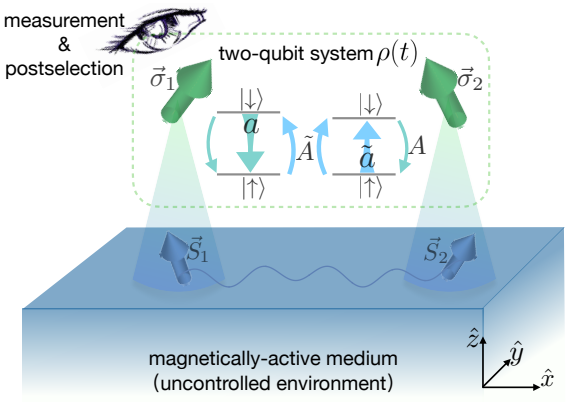

FIG. 1. A system composed of two spin qubits is coupled with a magnetic environment, which induces local relaxations a, $\tilde{a}$, mediates dissipative couplings A, $\tilde{A}$, as well as coherent couplings between two qubits. The two qubits may achieve a stable entangled state with large enough A and $\tilde{A}$, and even a Bell state with the help of measurement and postselection.

In this work, we discuss the dissipative coupling and entanglement generation induced by a generic noisy magnetic medium, in a hybrid quantum system sketched in Fig. 1. In particular, we demonstrate that, when complemented by proper postselections, a Bell state can be generated through an exceptional point in the time evolution governed by a non-Hermitian Hamiltonian. The qubits can be NVs or other isolated quantum defects and the medium is a generic solid-state system emitting magnetic field noise, which can arise from fluctuations of spin or pseudospin degrees of freedom [59]. Since many magnetic materials with different correlation properties are generally available, artificial design of the environment is not required as a first step, while spintronic engineering and tunability are promising for future studies.

To treat the induced coherent and dissipative couplings in a unified manner, we derive the full master equation [60, 61] that determines the time evolution of the

qubit entanglement. Specifically, two distinct types of dissipation are identified, bearing analogy to the local damping and the spin pumping-mediated viscosity in the classical spin dynamics [62]: One is the local relaxation, which originates in energy and information exchanges between a single qubit and the medium. The other is the dissipative coupling between the qubits induced by the correlated medium they both couple to. While the former is detrimental to quantum entanglement, we show the latter can help to establish a steady entanglement between qubits, even in a pessimistic scenario where the coherent coupling is absent. The long-time behavior of the qubits reflects a phase transition, as a function of system parameters. When the dissipative coupling is comparable to the local relaxation, the Lindbladian evolution induced by the medium can result in sizable robust entanglement between the qubits. This can be achieved for qubit separation on a lengthscale dictated by the relevant excitations responsible for dissipation (such as magnons for a magnetically ordered medium).

Model.—Let us consider an illustrative model consisting of two spin qubits weakly coupled to a magnet, with the following Hamiltonian:

$$H = H_{\rm S} + H_{\rm E} + H_{\rm SE}. \tag{1}$$

Here, $H_{\rm S} = -(\Delta_1\sigma_1^z + \Delta_2\sigma_2^z)/2$ is the Hamiltonian for the system with two qubits subjected to magnetic fields Δ_1 and Δ_2, respectively, along the z direction, $H_{\rm E}$ is an unspecified Hamiltonian of the medium as an environment for the system, and $H_{\rm SE} = \lambda(\vec{\sigma}_1\cdot\vec{S}_1 + \vec{\sigma}_2\cdot\vec{S}_2)$ describes the system-environment interaction with coupling strength λ, where $\vec{\sigma}_i$ stands for the Pauli matrice of the ith qubit, and $\vec{S}_i$ for local spin density operators it couples to within the medium. Without loss of generality, we assume $\Delta_1 \geq \Delta_2 \geq 0$. We will consider an axially-symmetric environment $H_{\rm E}$ in spin space, while a generalization would be straightforward. It would also be straightforward to generalize the treatment to the dipolar coupling between the qubit and the medium [25, 26].

The following Lindblad master equation of the density matrix of the two-qubit system can be derived microscopically based on the Born and Markov approximations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -i\big[H_{\rm S} + H_{\rm eff}, \rho\big] - \mathcal{L}[\rho]. \tag{2}$$

Leaving the derivation to the supplemental materials [63], we start with a phenomenological understanding of it on symmetry grounds. Here, $H_{\rm eff}$ is the medium-induced effective coherent coupling between qubits, participating in the unitary system evolution, while $\mathcal{L}[\rho]$ is the dissipative Lindbladian expanded in the usual form:

$$\mathcal{L}[\rho] = \sum_{nm} h_{nm}\big(\mathcal{O}_m^\dagger\mathcal{O}_n\rho + \rho\mathcal{O}_m^\dagger\mathcal{O}_n - 2\mathcal{O}_n\rho\mathcal{O}_m^\dagger\big), \tag{3}$$

where the coefficient matrix h is Hermitian and positive-semidefinite [60, 61], and $\mathcal{O} = (\sigma_1^-, \sigma_2^-, \sigma_1^+, \sigma_2^+, \sigma_1^z, \sigma_2^z)$ comprises qubit operators.

The most general form of $H_{\rm eff}$, allowed by the axial symmetry, is $H_{\rm eff} = J_z\sigma_1^z\sigma_2^z + J_\perp(\sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y) + D\hat{z}\cdot\vec{\sigma}_1\times\vec{\sigma}_2$, a summation of an XXZ model and a Dzyaloshinskii-Moriya (DM) interaction term [63]. The DM interaction must vanish if, for example, the structure is invariant under π z-rotation (see Fig. 1 for the coordinate frame). These coherent couplings induced by the magnetic medium can build up a finite entanglement within the timescale inversely proportional to the coupling strength [63], if it is shorter than the timescale set by dissipation. In the limiting case of a full isotropicity in spin space and $\Delta_i = 0$, $H_{\rm eff}$ is further reduced to a Heisenberg form $H_{\rm eff} = J\vec{\sigma}_1\cdot\vec{\sigma}_2$ resembling the RKKY coupling [64]. These effective coupling parameters are all real constants determined by the Green's functions of the medium [63], as is consistent with previous results from Schrieffer-Wolff transformation [22, 23, 25–30]. Direct dipolar interaction between qubits is typically negligible, except for very small spacings.

In the dissipative Lindblad part, h is block diagonal due to the axial symmetry. In general terms, we have

$$h = \begin{pmatrix}\tilde{a} & \tilde{A}\\ \tilde{A}^* & \tilde{a}\end{pmatrix} \oplus \begin{pmatrix}a & A^*\\ A & a\end{pmatrix} \oplus \begin{pmatrix}d & \mathfrak{D}\\ \mathfrak{D}^* & d\end{pmatrix}, \tag{4}$$

where $\tilde{a}, a, d$ and $\tilde{A}, A, \mathfrak{D}$ are real and complex phenomenological parameters, respectively. These parameters represent three types of dissipative effects: a and $\tilde{a}$ are associated with local decay and the reverse process. They govern local relaxation of individual qubits, giving rise to the relaxation time T_1 and contribute to the decoherence time T_2 of a single qubit [63]. In contrast, A and $\tilde{A}$ are related to cooperative decay and the reverse process involving both qubits, and are referred to as dissipative couplings, which depend on the distance between the two qubits. They are the focus of this work. d and $\mathfrak{D}$ are pure-dephasing parameters, originating from those terms in $H_{\rm SE}$ that commute with $H_{\rm S}$, namely $\lambda\sum_{i=1,2}\sigma_i^z S_i^z$. They only cause information but not energy exchange between the system and the medium, and in practice may be mitigated by dynamic decoupling [65–68]. We neglect pure-dephasing effects in the following discussion, though they may also lead to entanglement between multiple qubits as shown recently [69]. The Lindbladian (3) can then be brought into a diagonal form with four quantum-jump operators [70, 71]

$$J_1 = \sqrt{\frac{\tilde{a}+|\tilde{A}|}{2}}\left(\sigma_1^- + \sigma_2^-\right), \quad J_2 = \sqrt{\frac{\tilde{a}-|\tilde{A}|}{2}}\left(\sigma_1^- - \sigma_2^-\right),$$
$$J_3 = \sqrt{\frac{a+|A|}{2}}\left(\sigma_1^+ + \sigma_2^+\right), \quad J_4 = \sqrt{\frac{a-|A|}{2}}\left(\sigma_1^+ - \sigma_2^+\right), \tag{5}$$

yielding

$$\underline{\mathcal{L}}[\rho] = \sum_{i=1}^{4}\mathcal{D}_{J_i}[\rho], \tag{6}$$

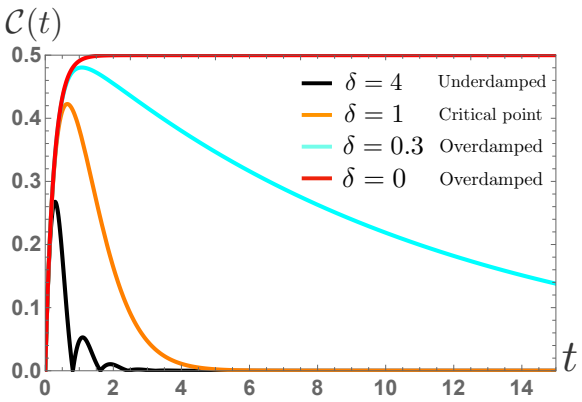

FIG. 2. Concurrence of two qubits as a function of time, with initial state $|\uparrow\downarrow\rangle$, where we set both local dissipation a and dissipative coupling $|A|$ to be 1. The black curve corresponds to the underdamped quantum regime. The orange curve is at the critical point $\delta = 1$, where entanglement decays as $\mathcal{C}(t) \propto te^{-2at}$. The cyan, $\delta = 0.3$, and the red, $\delta = 0$, curves are in overdamped quantum regime, where the lifetime of entanglement is extended dramatically.

where the dissipator is defined as $\mathcal{D}_J[\rho] \equiv J^\dagger J\rho + \rho J^\dagger J - 2J\rho J^\dagger$.

Microscopically, all parameters are given by the Green's functions of the medium in equilibrium [63], such that the fluctuation-dissipation theorem dictates that they are not independent: $\tilde{a} = e^{-\beta\Delta}a$ and $\tilde{A} = e^{-\beta\Delta}A$, where $\beta = 1/k_BT$ and $\Delta \equiv (\Delta_1 + \Delta_2)/2$. The zero temperature therefore corresponds to $\tilde{a} = \tilde{A} = 0$, where only the decay processes survive. Also, the thermodynamic stability of the magnetic medium imposes $a \geq |A|$ and $\tilde{a} \geq |\tilde{A}|$ [63], which ensures the matrix h is positive-semidefinite.

Dissipative coupling vs local relaxation.—Let us now explore the entanglement evolution of two qubits focusing on the dissipative effects, by setting ourselves in a pessimistic situation where the induced coherent dynamics is absent:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -i[H_\mathrm{S}, \rho] - \underline{\mathcal{L}}[\rho]. \tag{7}$$

Here, we treat the scenario of zero temperature $\tilde{a} = \tilde{A} = 0$ analytically to demonstrate the effects of local relaxation and dissipative couplings. Numerical results for finite temperature are presented in the Supplemental Material [63], which does not qualitatively change our conclusion below.

The qubits are initialized into a trivial product state, taking the example of $|\uparrow\downarrow\rangle$ for the sake of concreteness. We show the master equation (7) can be reduced to an equation for $x \equiv \mathrm{Re}\,\langle\uparrow\downarrow|\,\rho\,|\downarrow\uparrow\rangle$ [63]:

$$\ddot{x} + 4a\dot{x} + 4\left(\delta^2 + a^2 - |A|^2\right)x = 0, \tag{8}$$

where $\delta \equiv (\Delta_1 - \Delta_2)/2$ is the local field asymmetry. This equation resembles a damped oscillator with complex characteristic frequencies

$$\omega_\pm = \pm 2\omega_0 - i2a, \tag{9}$$

where $\omega_0 \equiv \sqrt{\delta^2 - |A|^2}$. The real part gives the coherent beating of the density matrix elements, while the imaginary part reflects decoherence. The contribution from local relaxation $-i2a$ leads to a decaying envelope factor e^{-2at} in the entanglement between two qubits (as detailed below), indicating its detrimental effect on quantum coherence as expected.

We identify three distinct parameter regimes for the quantum dynamics. In the underdamped regime, $\delta > |A|$, ω_0 is real valued. To quantify the time evolution of the entanglement between the two qubits, we calculate the *concurrence* [63, 72] as a function of time: $\mathcal{C}(t) = 2e^{-2at}|A\sin\omega_0 t|\sqrt{\delta^2 - |A\cos\omega_0 t|^2}/\omega_0^2$. See Fig. 2. The entanglement oscillates with frequency $2\omega_0$ as the system decays rapidly to the ground state $|\uparrow\uparrow\rangle$ on the time scale $\tau = 1/2a$. At the critical point $\delta = |A|$, $\omega_0 = 0$, there is no oscillation. The concurrence evolves as $\mathcal{C}(t) \propto te^{-2at}$, where the final steady state is also $|\uparrow\uparrow\rangle$. As shown in Fig. 2, we have a larger transient-peak entanglement and the decay process is slowed down moderately compared with the underdamped regime.

In the overdamped regime, $\delta < |A|$, ω_0 becomes purely imaginary and $\omega_\pm = -2i(a \pm \kappa_0)$, with $\kappa_0 = \sqrt{|A|^2 - \delta^2}$. The time-dependent concurrence is $\mathcal{C}(t) = 2e^{-2at}|A|\sinh\kappa_0 t\sqrt{|A\cosh\kappa_0 t|^2 - \delta^2}/\kappa_0^2$. In addition to a larger transient peak entanglement, the decay process has been slowed down dramatically. On a long time scale $t \gg 1/\kappa_0$, $\mathcal{C}(t) \propto e^{-2(a-\kappa_0)t}$. The entanglement can last for $\tau = 1/2(a - \kappa_0)$, which becomes $\tau = 1/2(a - |A|)$ when the two local fields are the same, $\delta = 0$. See Fig. 2. It is clear from this expression of the lifetime τ that the dissipative coupling A and the local relaxation a, though both originating from the qubits-magnet coupling, have opposite effects on the quantum entanglement in the nonunitary evolution. The local dissipation tends to destroy any quantum coherence whereas the dissipative coupling can be exploited to extend the lifetime of entanglement and even realize steady entangled states. With equal local fields $\delta = 0$, a finite entanglement can persist for a long time before eventually decaying to zero in the large dissipative coupling regime $|A| \lesssim a$. Based on their (greater) Green's function expressions [63, 73] $2a = i\lambda^2 G^>_{S_1^+S_1^-}(\Delta) = i\lambda^2 G^>_{S_2^+S_2^-}(\Delta)$, $2A = i\lambda^2 G^>_{S_1^+S_2^-}(\Delta)$, $|A| \lesssim a$ physically corresponds to the scenario with two qubits placed within a lengthscale dictated by the relevant excitations responsible for dissipation. For example, for qubits coupled to a magnetically ordered medium via processes of magnon absorption and emission, this lengthscale is set by the wavelength of the magnon at frequency Δ. Furthermore, the concurrence

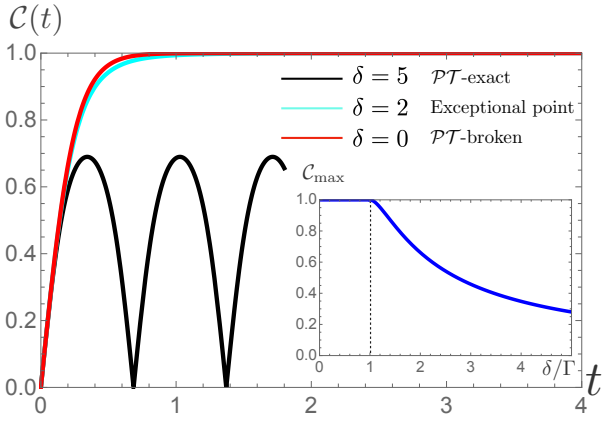

FIG. 3. Concurrence of qubits as a function of time with initial state $|\uparrow\downarrow\rangle$ under continuous measurements and postselections. We set $\Gamma = 2$. The black curve $\delta = 5$ is in the $\mathcal{PT}$-exact regime, where entanglement oscillates and its maximal value is less than 1. At the exceptional point (cyan curve), there is no oscillation and its maximal value is 1. In $\mathcal{PT}$-broken regime (red curve), entanglement is $\mathcal{C}(t) = \tanh 2\Gamma t$. The inset shows the maximal concurrence as a function of δ/Γ.

lifetime extends to infinity $\tau \to \infty$ when $|A|$ reaches its maximal allowed value $|A| = a$, and thus a steady entanglement is achieved, $\mathcal{C}(\infty) = 1/2$ for the final state

$$\rho_\infty = \left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |00\rangle\langle 00|\right)/2. \tag{10}$$

Noting that the singlet $|00\rangle$ cannot be evolved to a different state by the operative jump operators or the system Hamiltonian, it is a dark state—the system would stay in this pure state indefinitely. For this reason, a steady-state entanglement can also be reached at finite temperatures when $a = |A|$, $\tilde{a} = |\tilde{A}|$ [63], though with a smaller concurrence. We also remark that finite steady-state entanglement can be achieved in this optimal situation, irrespective of the initial two-qubit state as long as it is not a symmetric state [63].

We stress that the above critical point and the associated transition from underdamped to overdamped regime result from the dissipative couplings, which are the main focus of this article. We next show how to generate Bell states by exploiting this dissipative coupling, when combined with proper measurements and postselections.

Non-Hermitian Hamiltonian scheme.—Let us turn to the evolution of qubits under measurements (see Fig. 1), which is often invoked to perform feedback and conditional control as a valuable resource in controlling open quantum systems [74–78]. To this end, we rewrite the master equation (7) in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -i[\underline{H}_{\rm eff}, \rho] + 2\sum_{i=1}^{4} J_i \rho J_i^\dagger, \tag{11}$$

where

$$\underline{H}_{\rm eff} = H_S - i\sum_i J_i^\dagger J_i, \tag{12}$$

is a non-Hermitian Hamiltonian. Correspondingly, the commutator should now be understood as $[\underline{H}_{\rm eff}, \rho] \equiv \underline{H}_{\rm eff}\rho - \rho\underline{H}_{\rm eff}^\dagger = [H_S, \rho] - i\{\sum_i J_i^\dagger J_i, \rho\}$. By subjecting the two qubits to continuous measurements of the absolute value of their total spin z component $\underline{\sigma}^z = \sigma_1^z + \sigma_2^z$ and subsequently conditioning the postselection on zero outcomes, we can effectively forbid all quantum jump processes (5), as $[J_i, \underline{\sigma}^z] \neq 0$. This monitored dynamics of the two-qubit system formally eliminates the last term in Eq. (11) and reduces quantum dynamics to a non-Hermitian Hamiltonian form in the $\underline{\sigma}^z = 0$ subspace [56], $d\rho/dt = -i[\underline{H}_{\rm eff}, \rho]$, whose integration is appropriately normalized to give [79]:

$$\rho(t) = \frac{e^{-i\underline{H}_{\rm eff}t}\rho_0 e^{i\underline{H}_{\rm eff}^\dagger t}}{\mathrm{tr}\left(e^{-i\underline{H}_{\rm eff}t}\rho_0 e^{i\underline{H}_{\rm eff}^\dagger t}\right)}, \tag{13}$$

in terms of the initial qubits state ρ_0. Since the effective Hamiltonian (12) conserves the quantum number $\underline{\sigma}^z$ $[\underline{H}_{\rm eff}, \underline{\sigma}^z] = 0$, the subspace $\mathcal{H}$ spanned by $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$ is closed under time evolution. The eigenkets of $\underline{H}_{\rm eff}$ in $\mathcal{H}$ are

$$\begin{aligned} |s\rangle &= e^{i\varphi}\sin\theta\,|\uparrow\downarrow\rangle + \cos\theta\,|\downarrow\uparrow\rangle, \\ |a\rangle &= -\cos\theta\,|\uparrow\downarrow\rangle + e^{i\varphi}\sin\theta\,|\downarrow\uparrow\rangle, \end{aligned} \tag{14}$$

with associated eigenenergies $E_{s(a)} = \mp\omega - i(a+\tilde{a})$. Here, θ and φ are determined by the sum of dissipative couplings $\Gamma = |A| + |\tilde{A}|$ and the local field asymmetry δ: For $\delta \geq \Gamma$, $\omega = \sqrt{\delta^2 - \Gamma^2}$, $\cos\theta \equiv \Gamma/\sqrt{|\omega+\delta|^2 + \Gamma^2}$ and $e^{i\varphi}\sin\theta \equiv -i(\omega+\delta)/\sqrt{|\omega+\delta|^2+\Gamma^2}$; for $\delta < \Gamma$, the principal value is taken for $\omega = i\kappa = i\sqrt{\Gamma^2 - \delta^2}$.

Bell state generation.—We now show that steady Bell states can be generated based on the monitored dynamics governed by $\underline{H}_{\rm eff}$ (12), focusing on the two-qubit dynamics in the subspace $\mathcal{H}$, which applies to zero and finite temperatures. Similarly to the unmonitored scenario, we can identify three distinct parameter regimes: parity-time ($\mathcal{PT}$) symmetry broken regime, $\delta < \Gamma$, the exceptional point, $\delta = \Gamma$, and $\mathcal{PT}$-exact regime, $\delta > \Gamma$ [80–82].

In the $\mathcal{PT}$-broken regime, eigenvalues E_a and E_s are purely imaginary with $\mathrm{Im}\,E_a > \mathrm{Im}\,E_s$. Thus the probability in the eigenmode $|a\rangle$ ($|s\rangle$) grows (decays) in time, and all probability eventually flows into the eigenmode $|a\rangle$. For an arbitrary initial state $\rho_0 = \sum_{i,j=\{a,s\}} p_{ij}\,|i\rangle\langle j|$, one can analytically solve for $\rho(t)$ according to Eq. (13):

$$\rho(t) = \frac{p_{aa}e^{2\kappa t}|a\rangle\langle a| + p_{as}\,|a\rangle\langle s| + p_{sa}\,|s\rangle\langle a| + p_{ss}e^{-2\kappa t}\,|s\rangle\langle s|}{p_{aa}e^{2\kappa t} + p_{as}\,\langle s|a\rangle + p_{sa}\,\langle a|s\rangle + p_{ss}e^{-2\kappa t}}, \tag{15}$$

which ultimately evolves into the maximally entangled state $|a\rangle = \left(|\uparrow\downarrow\rangle + e^{i\tilde{\varphi}}\,|\downarrow\uparrow\rangle\right)/\sqrt{2}$ with $e^{i\tilde{\varphi}} = (-\kappa + i\delta)/\Gamma$,

when $t \gg 1/2\kappa$. Thus, the two qubits eventually reach the maximal concurrence in the $\mathcal{PT}$-broken regime, irrespective of the initial state as long as $p_{aa} \neq 0$. As an illustrative example, we evaluate the time-dependent concurrence, with initial state $|\uparrow\downarrow\rangle$ and equal local fields $\delta = 0$, $\mathcal{C}(t) = \tanh 2\Gamma t$, as shown in Fig. 3. In practice, the entanglement-growth rate 2κ needs to be larger than the postselection rate ($\sim$ the rate of leaking out of the subspace $\mathcal{H}$ of our interest) $2(a+\tilde{a}-\kappa)$ for the system to settle into the Bell state. The optimal scenario is when $\delta = 0$, $|A| = a$, and $|\tilde{A}| = \tilde{a}$, the same as that in the overdamped quantum regime without postselection.

At the exceptional point, $\underline{H}_{\text{eff}}$ is nondiagonalizable, since the eigenstates $|a\rangle$ and $|s\rangle$ coalesce into $(|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle)/\sqrt{2}$. The two qubits will gradually evolve into this sole state where they are maximally entangled. For example, starting with a trivial state $|\uparrow\downarrow\rangle$, the concurrence $\mathcal{C}(t) = 2\Gamma t\sqrt{1+\Gamma^2 t^2}/(1+2\Gamma^2 t^2)$, algebraically approaching 1. See Fig. 3.

In the $\mathcal{PT}$-exact regime, the eigenenergies E_s and E_a have nonzero real parts. The amplitudes of eigenmodes $|s\rangle$ and $|a\rangle$ keep oscillating without reaching a steady state, hence no steady entanglement. The frequency of entanglement oscillation is 2ω, as shown in Fig. 3. The maximal entanglement one can achieve is $\mathcal{C}_{\max}(\eta) = \sqrt{2-1/\eta^2}/\eta$, with $\eta = \delta/\Gamma$, which is less than 1. Notably, the second derivative of $\mathcal{C}_{\max}$ is discontinuous across the exceptional point ($\eta = 1$), reflecting a phase transition (see Fig. 3).

It is clear that we can achieve a Bell state by decreasing the local field asymmetry δ for a fixed dissipative coupling Γ to reach the $\mathcal{PT}$-broken regime. We remark that Γ is also potentially tunable by engineering the magnetic medium spintronically. This discussion, again, highlights the role of the nonlocal dissipative couplings in realizing an exceptional point in the dynamics, further triggering an entanglement transition in the long-time steady state. In the large nonlocal dissipative coupling regime, we achieve steady Bell states.

Discussion.—We remark that the non-Hermitian Hamiltonian scheme is precise when the rate of measurements is infinite. As this rate approaches zero, we recover the full Lindblad dynamics. It could be intriguing to explore, within our framework, possible phase transitions or crossovers induced by finite-rate measurements.

In our case, the possible forms of induced coherent interactions and quantum jump operators are determined by the axial symmetry of the media. This may render a general guidance in studying the dynamics of hybrid quantum systems with other classes of symmetries, especially their long-time entanglement behavior.

The theoretical framework developed here provides a good starting point for further studies on the relationship between the entanglement dynamics and thermodynamic properties of the medium. One may be able to manipulate entanglement by engineering the medium [41, 83–86], enabled by recent progress in the field of spintronics [87–89]. It is especially interesting to look into media with anisotropies, which have been shown to be good entanglement reservoirs [90–92]. By extending our equilibrium Green's function treatment to allow for a quasi-equilibrium spin chemical potential, we may study the scenario with a spintronically pumped medium, where local relaxations and dissipative couplings are tunable.

For practical consideration in a NV-magnet hybrid setup, one challenge is to bring down the detuning δ of the magnetic fields at the NV sites to be much smaller than the dissipative parameters, which are typically on the scale of MHz or less. It would also be necessary to put the induced effective Hamiltonian back into the picture, as the interplay between the coherent and dissipative evolution can be nontrivial in steady-state entanglement generation. Though the Markovian nature of the intrinsic dynamics can be justified when the NV frequency Δ is sufficiently above the magnon band gap Ω, such that the decay time of relevant correlations within the medium, $\sim (\Delta - \Omega)^{-1}$, is smaller than the timescale associated with the medium-induced qubit dynamics, disorder and defects may lead to low-energy excitations that contribute to non-Markovian evolution (which can also be interesting to look into). As a possible low-temperature implementation of the proposed post-selection scheme, we may post-select on the absence of any emitted magnons.

Lastly, NV centers have been demonstrated as good quantum probes of local fields and noise [93]. Here, we propose to extend this to nonlocal characteristics. For example, one may use it to detect quantum phase transitions and steady exotic phases that are characterized by nonlocal quantum correlations.

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. (Cambridge University Press, January 31, 2011).
[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[3] B. B. Blinov, D. L. Moehring, L. M. Duan, and C. Monroe, Nature **428**, 153 (2004).
[4] J. Volz, M. Weber, D. Schlenk, W. Rosenfeld, J. Vrana, K. Saucke, C. Kurtsiefer, and H. Weinfurter, Phys. Rev. Lett. **96**, 030404 (2006).
[5] R. Blatt and D. Wineland, Nature **453**, 1008 (2008).
[6] C. D. Bruzewicz, J. Chiaverini, R. McConnell, and J. M. Sage, *Applied Physics Reviews*, Applied Physics Reviews **6**, 021314 (2019).
[7] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
[8] F. Basso Basset, M. B. Rota, C. Schimpf, D. Tedeschi, K. D. Zeuner, S. F. Covre da Silva, M. Reindl, V. Zwiller, K. D. Jöns, A. Rastelli, and R. Trotta, Phys. Rev. Lett.

123, 160501 (2019).
[9] H. Qiao, Y. P. Kandel, S. K. Manikandan, A. N. Jordan, S. Fallahi, G. C. Gardner, M. J. Manfra, and J. M. Nichol, Nature Communications **11**, 3022 (2020).
[10] G. Wendin, Reports on Progress in Physics **80**, 106001 (2017).
[11] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. Hollenberg, Physics Reports **528**, 1 (2013).
[12] Y. Chu, M. Markham, D. J. Twitchen, and M. D. Lukin, Phys. Rev. A **91**, 021801 (2015).
[13] N. Bar-Gill, L. M. Pham, A. Jarmola, D. Budker, and R. L. Walsworth, Nature Communications **4**, 1743 (2013).
[14] G. Balasubramanian, P. Neumann, D. Twitchen, M. Markham, R. Kolesov, N. Mizuochi, J. Isoya, J. Achard, J. Beck, J. Tissler, V. Jacques, P. R. Hemmer, F. Jelezko, and J. Wrachtrup, Nature Materials **8**, 383 (2009).
[15] M. Wallquist, K. Hammerer, P. Rabl, M. Lukin, and P. Zoller, Physica Scripta **T137**, 014001 (2009).
[16] H. Wang, S. Zhang, N. J. McLaughlin, B. Flebus, M. Huang, Y. Xiao, E. E. Fullerton, Y. Tserkovnyak, and C. R. Du, arXiv e-prints , arXiv:2011.03905 (2020).
[17] C. Du, T. van der Sar, T. X. Zhou, P. Upadhyaya, F. Casola, H. Zhang, M. C. Onbasli, C. A. Ross, R. L. Walsworth, Y. Tserkovnyak, and A. Yacoby, Science **357**, 195 (2017).
[18] P. Andrich, C. F. de las Casas, X. Liu, H. L. Bretscher, J. R. Berman, F. J. Heremans, P. F. Nealey, and D. D. Awschalom, npj Quantum Information **3**, 28 (2017).
[19] T. van der Sar, F. Casola, R. Walsworth, and A. Yacoby, Nature Communications **6**, 7886 (2015).
[20] C. S. Wolfe, V. P. Bhallamudi, H. L. Wang, C. H. Du, S. Manuilov, R. M. Teeling-Smith, A. J. Berger, R. Adur, F. Y. Yang, and P. C. Hammel, Phys. Rev. B **89**, 180406 (2014).
[21] C. S. Wolfe, S. A. Manuilov, C. M. Purser, R. Teeling-Smith, C. Dubs, P. C. Hammel, and V. P. Bhallamudi, Applied Physics Letters **108**, 232409 (2016).
[22] L. Trifunovic, O. Dial, M. Trif, J. R. Wootton, R. Abebe, A. Yacoby, and D. Loss, Phys. Rev. X **2**, 011006 (2012).
[23] L. Trifunovic, F. L. Pedrocchi, and D. Loss, Phys. Rev. X **3**, 041023 (2013).
[24] L. D. Contreras-Pulido and R. Aguado, Phys. Rev. B **77**, 155420 (2008).
[25] B. Flebus and Y. Tserkovnyak, Phys. Rev. Lett. **121**, 187204 (2018).
[26] B. Flebus and Y. Tserkovnyak, Phys. Rev. B **99**, 140403 (2019).
[27] C. Mühlherr, V. O. Shkolnikov, and G. Burkard, Phys. Rev. B **99**, 195413 (2019).
[28] D. R. Candido, G. D. Fuchs, E. Johnston-Halperin, and M. E. Flatté, Materials for Quantum Technology **1**, 011001 (2020).
[29] T. Neuman, D. S. Wang, and P. Narang, Phys. Rev. Lett. **125**, 247702 (2020).
[30] M. Fukami, D. R. Candido, D. D. Awschalom, and M. E. Flatté, arXiv e-prints , arXiv:2101.09220 (2021).
[31] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 4728 (1996).
[32] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A **59**, 2468 (1999).
[33] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. **91**, 070402 (2003).
[34] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Büchler, and P. Zoller, Nature Physics **4**, 878 (2008).
[35] Y. Lin, J. P. Gaebler, F. Reiter, T. R. Tan, R. Bowler, A. S. Sørensen, D. Leibfried, and D. J. Wineland, Nature **504**, 415 (2013).
[36] M. E. Kimchi-Schwartz, L. Martin, E. Flurin, C. Aron, M. Kulkarni, H. E. Tureci, and I. Siddiqi, Phys. Rev. Lett. **116**, 240503 (2016).
[37] G. Kordas, S. Wimberger, and D. Witthaut, EPL (Europhysics Letters) **100**, 30007 (2012).
[38] S. Nicolosi, A. Napoli, A. Messina, and F. Petruccione, Phys. Rev. A **70**, 022511 (2004).
[39] N. Yokoshi and H. Ishihara, Journal of the Physical Society of Japan **86**, 083401 (2017).
[40] D. Kienzler, H.-Y. Lo, B. Keitch, L. de Clercq, F. Leupold, F. Lindenfelser, M. Marinelli, V. Negnevitsky, and J. P. Home, Science **347**, 53 (2015).
[41] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Phys. Rev. Lett. **107**, 080503 (2011).
[42] M. Benito, M. J. A. Schuetz, J. I. Cirac, G. Platero, and G. Giedke, Phys. Rev. B **94**, 115404 (2016).
[43] P.-B. Li, S.-Y. Gao, H.-R. Li, S.-L. Ma, and F.-L. Li, Phys. Rev. A **85**, 042306 (2012).
[44] X. Wang, H. Zhang, W. Zhang, X. Ouyang, X. Huang, Y. Yu, Y. Liu, X. Chang, D.-l. Deng, and L. Duan, Phys. Rev. A **102**, 032615 (2020).
[45] K. Ullah, E. Köse, R. Yagan, M. C. Onbaşl ı, and O. E. Müstecaplıoğlu, Phys. Rev. Research **4**, 023221 (2022).
[46] F. Reiter, A. S. Sørensen, P. Zoller, and C. A. Muschik, Nature Communications **8**, 1822 (2017).
[47] E. Kapit, Phys. Rev. Lett. **116**, 150501 (2016).
[48] J. Cohen and M. Mirrahimi, Phys. Rev. A **90**, 062344 (2014).
[49] C. D. Freeman, C. M. Herdman, and K. B. Whaley, Phys. Rev. A **96**, 012311 (2017).
[50] Z. Leghtas, S. Touzard, I. M. Pop, A. Kou, B. Vlastakis, A. Petrenko, K. M. Sliwa, A. Narla, S. Shankar, M. J. Hatridge, M. Reagor, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. H. Devoret, Science **347**, 853 (2015).
[51] S. Shankar, M. Hatridge, Z. Leghtas, K. M. Sliwa, A. Narla, U. Vool, S. M. Girvin, L. Frunzio, M. Mirrahimi, and M. H. Devoret, Nature **504**, 419 (2013).
[52] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, New Journal of Physics **16**, 045014 (2014).
[53] K. Modi, H. Cable, M. Williamson, and V. Vedral, Phys. Rev. X **1**, 021022 (2011).
[54] A. Górecka, F. A. Pollock, P. Liuzzo-Scorpo, R. Nichols, G. Adesso, and K. Modi, New Journal of Physics **20**, 083008 (2018).
[55] T. Botzung, S. Diehl, and M. Müller, Phys. Rev. B **104**, 184422 (2021).
[56] C. M. Bender, Reports on Progress in Physics **70**, 947 (2007).
[57] S. Lin, X. Z. Zhang, C. Li, and Z. Song, Phys. Rev. A **94**, 042133 (2016).
[58] H. Y. Yuan, P. Yan, S. Zheng, Q. Y. He, K. Xia, and M.-H. Yung, Phys. Rev. Lett. **124**, 053602 (2020).
[59] S. Zhang and Y. Tserkovnyak, arXiv e-prints , arXiv:2108.07305 (2021).
[60] H.-P. Breuer and F. Petruccione, *The Theory of Open*

Quantum Systems (Oxford University Press, 2007).
[61] D. A. Lidar, arXiv e-prints , arXiv:1902.00967 (2019).
[62] Y. Tserkovnyak, Phys. Rev. Research **2**, 013031 (2020).
[63] In this Supplemental Material, we present (i) General formalism for open quantum systems, (ii) Single spin dynamics, (iii) Derivation of the master equation for the two-qubit scenario, (iv) Symmetry-dictated possible forms of $H_{\rm eff}$, (v) Thermodynamic stability of the magnetic medium, (vi) Concurrence as a measure of entanglement, and (vii) Full entanglement dynamics.
[64] Here the microscopic expression of J is $2J = G^R_{S_1^x S_2^x}(0) + G^R_{S_2^x S_1^x}(0) = G^R_{S_1^y S_2^y}(0) + G^R_{S_2^y S_1^y}(0) = G^R_{S_1^z S_2^z}(0) + G^R_{S_2^z S_1^z}(0)$ when the spin space is isotropic.
[65] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
[66] K. Khodjasteh and D. A. Lidar, Phys. Rev. Lett. **95**, 180501 (2005).
[67] R. Lo Franco, A. D'Arrigo, G. Falci, G. Compagno, and E. Paladino, Phys. Rev. B **90**, 054304 (2014).
[68] G. A. Paz-Silva, S.-W. Lee, T. J. Green, and L. Viola, New Journal of Physics **18**, 073020 (2016).
[69] A. Seif, Y.-X. Wang, and A. A. Clerk, Phys. Rev. Lett. **128**, 070402 (2022).
[70] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
[71] We have gauged out the phases of $A, \tilde{A}$ by absorbing them into the definition of $\sigma^{\pm}$.
[72] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[73] We use common convention for the greater Green's function: $G^{>}_{AB}(\omega) = -i\int_{-\infty}^{\infty}\langle A(t)B(0)\rangle e^{i\omega t}dt$.
[74] H. M. Wiseman, *Quantum Measurement and Control* (Cambridge University Press, 2009).
[75] D. B. Hume, T. Rosenband, and D. J. Wineland, Phys. Rev. Lett. **99**, 120502 (2007).
[76] V. Negnevitsky, M. Marinelli, K. K. Mehta, H. Y. Lo, C. Flühmann, and J. P. Home, Nature **563**, 527 (2018).
[77] L. Jiang, J. S. Hodges, J. R. Maze, P. Maurer, J. M. Taylor, D. G. Cory, P. R. Hemmer, R. L. Walsworth, A. Yacoby, A. S. Zibrov, and M. D. Lukin, Science **326**, 267 (2009).
[78] X. H. H. Zhang and H. U. Baranger, Phys. Rev. Lett. **122**, 140502 (2019).
[79] D. C. Brody and E.-M. Graefe, Phys. Rev. Lett. **109**, 230405 (2012).
[80] W. D. Heiss, Journal of Physics A: Mathematical and Theoretical **45**, 444016 (2012).
[81] A. Galda and V. M. Vinokur, Phys. Rev. B **97**, 201411 (2018).
[82] A. Galda and V. M. Vinokur, Phys. Rev. B **94**, 020408 (2016).
[83] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 4728 (1996).
[84] B. Kraus and J. I. Cirac, Phys. Rev. Lett. **92**, 013602 (2004).
[85] C. A. Muschik, H. Krauter, K. Jensen, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Journal of Physics B: Atomic, Molecular and Optical Physics **45**, 124021 (2012).
[86] S. McEndoo, P. Haikka, G. D. Chiara, G. M. Palma, and S. Maniscalco, EPL (Europhysics Letters) **101**, 60005 (2013).
[87] S. Maekawa, S. O. Valenzuela, E. Saitoh, and T. Kimura, eds., *Spin Current*, Series on Semiconductor Science and Technology (Oxford University Press, 2015).
[88] Y. Tserkovnyak, Journal of Applied Physics **124**, 190901 (2018).
[89] A. Avsar, H. Ochoa, F. Guinea, B. Özyilmaz, B. J. van Wees, and I. J. Vera-Marun, Rev. Mod. Phys. **92**, 021003 (2020).
[90] J. Zou, S. K. Kim, and Y. Tserkovnyak, Phys. Rev. B **101**, 014416 (2020).
[91] A. Kamra, E. Thingstad, G. Rastelli, R. A. Duine, A. Brataas, W. Belzig, and A. Sudbø, Phys. Rev. B **100**, 174407 (2019).
[92] A. Kamra, W. Belzig, and A. Brataas, Applied Physics Letters **117**, 090501 (2020).
[93] F. Casola, T. van der Sar, and A. Yacoby, Nature Reviews Materials **3**, 17088 (2018).