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Observing Quasiparticles through the Entanglement Lens

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The low energy physics of interacting quantum systems is typically understood through the identification of the relevant quasiparticles or low energy excitations and their quantum numbers. We present a quantum information framework that goes beyond this to examine the nature of the entanglement in the corresponding quantum states. We argue that the salient features of the quasiparticles, including their quantum numbers, locality and fractionalization are reflected in the entanglement spectrum and in the mutual information. We illustrate these ideas in the specific context of the $d = 1$ transverse field Ising model with an integrability breaking perturbation.

Introduction— Our understanding of quantum many body theory has been deeply affected by the quantum information “revolution” which has, among other ideas, led to the systematic study of the entanglement present in quantum states [1]. The most prominent product of this study is the formulation of the area law for the entanglement entropy (EE), i.e., the von Neumann entropy $S = -\text{Tr} \rho_A \log \rho_A$ of the reduced density matrix ρ_A of a subsystem A , in ground states of local quantum systems [2, 3]. That in turn is the basis for an entire family of tensor network methods for representing and computing with ground states which has revolutionized the computational study of quantum many body systems [4–6]. Moreover, the area law for the EE has led to the identification of a universal subleading correction, the topological EE, which is a manifestation of topological order [7, 8]. Moving beyond the entropy, the entanglement spectrum (ES) has been widely explored [9–16]. Altogether the study of entanglement has provided a new route to exploring quantum phases and phase transitions, and has been very successful in describing a wide variety of exotic states and related phase transitions [17–19].

In this paper we wish to use the entanglement lens to look at very low energy dynamics. In this limit, as is well known, the dynamics in phases can be typically be understood in terms of the elementary excitations of the system. Specifically we wish to look at quasiparticles (QP) which we will take to be gapped particle-like excitations as opposed to collective excitations like Goldstone bosons [20]. There is however no expectation, as in traditional many-body physics, that quasiparticles necessarily have an adiabatic continuation to particle-like excitations of a free system. In particular, we also consider QPs that are fractionalized and/or cannot be created by local operators (e.g., spinons or solitons). In doing so, we also move beyond a body of results on excitations in integrable models [21–27] which prefigures some of our work although even where our results overlaps theirs, we are interested in their properties in generic, non-integrable, systems.

In what follows we study the entanglement structure of quasiparticle states. The questions of interest are: How is the entanglement present in the ground state modified by the presence of the quasiparticle? Are the quantum numbers of the quasiparticle reflected in the entanglement spectrum of the quasiparticle states? Can we distinguish quasiparticles created by local operators from those created by non-local operators via the quantum information content of the quasiparticle states? To this end we will present results on the $d = 1$ transverse field Ising model with an integrability breaking term added. In the symmetric/paramagnetic phase the quasiparticle is local while in the broken symmetry/ferromagnetic phase the quasiparticle is a domain wall/soliton and thus is created by a non-local operator. In the symmetric phase we find the EE exhibits a term above the ground state contribution that arises from the uncertainty in the position of the quasiparticle. More finely, we find an exact doublet structure in the entanglement spectrum for a symmetric bipartition that we trace to a combination of the symmetry of the partition and the existence of a Z_2 charge carried by the quasiparticle. In the broken symmetry phase the EE of the domain wall excitation exhibits an additional universal contribution that reflects the underlying broken symmetry. As the Z_2 charge is no longer well defined, the ES no longer has an exact spectral degeneracy. Finally, we show that the non-locality of the domain wall excitation can be detected by computing the long-range mutual information (MI) between spatially separated quibits in the corresponding quantum state. We briefly discuss generalizations of these ideas to symmetry fractionalization and topological QP in higher dimensions which will be presented in future publications.

Entanglement in a local QP – Our exemplar of a local QP will arise in the paramagnetic/symmetric phase of the $d = 1$ transverse field Ising model:

$$H = \sum_i J_z \sigma_i^z \sigma_{i+1}^z + h_x \sigma_i^x + J_x \sigma_i^x \sigma_{i+1}^x \quad (1)$$

where we have added a weak integrability breaking inter-

action $J_x \sigma_i^x \sigma_{i+1}^x$ for genericity. When $h_x > J_z + O(J_x)$, the system is in the paramagnetic phase. In the extreme limit $h_x \gg J_z$ the ground state is fully polarized in the x direction and the QP—which we will refer to as a Z_2 magnon—is generated by local spin flip operator $Q_i = \sigma_i^z$ which creates a Z_2 charge measured by the parity operator $P = \prod_i \sigma_i^x$. Away from the extreme paramagnetic limit Q_i remains a Z_2 odd operator but now spreads out over a correlation length ξ . With periodic boundary conditions, the system is translationally invariant and the low energy magnon states are momentum eigenstates of the form [21],

$$|\psi^{1M}\rangle_k = \sum_i e^{ikr_i} Q_i |\text{GS}\rangle \quad (2)$$

where we have ignored the subleading k dependence of the operator Q_i itself. With open boundary conditions we will have standing wave analogs thereof. In particular, since we are interested in the entanglement entropy difference between the GS and the excited state with $k = 0$, we will take $k = 0$ here and after.

We will now consider a spatial bipartition into regions of length pL and $(1-p)L$ that are much larger than the correlation length such that the contributions to the QP EE are given by,

$$S_p^{1M} = S_p^{\text{GS}} - p \ln(p) - (1-p) \ln(1-p) + O(1/L), \quad (3)$$

irrespective of k and irrespective of the exact ratio h_x/J_z of the Hamiltonian parameters. The additional EE due to the presence of the magnon over the ground state value originates from the uncertainty in its location relative to the cuts. For a finite correlation length, there is a finite probability of the magnon reaching across the cut, yielding $O(1/L)$ corrections. For a symmetric cut in the thermodynamic limit, the additional EE created by QP is $\ln(2)$ —one bit—as illustrated in Fig. 1.

We now consider the magnon entanglement spectrum, defined as $\epsilon_\gamma = -\log \lambda_\gamma^2$, which is obtained from a Schmidt decomposition

$$|\psi^{1M}\rangle = \sum_\gamma \lambda_\gamma |\gamma\rangle_A |\gamma\rangle_B. \quad (4)$$

Note that λ_γ^2 are the eigenvalues of the reduced density matrices ρ_A and ρ_B for the two subsystems. For a reflection symmetric cut (see inset in Fig. 1), the entanglement spectrum exhibits an *exact* two-fold degeneracy. We now show that this exact degeneracy of the Schmidt values arises from a combination of the reflection symmetry $R|\psi^{1M}\rangle = |\psi^{1M}\rangle$ and the non-trivial Z_2 charge of the magnon $P|\psi^{1M}\rangle = -|\psi^{1M}\rangle$: To this let us assume instead that the entanglement spectrum contains a non-degenerate eigenvalue λ_γ . The state $|\psi^{1M}\rangle$ is symmetric under R and the Schmidt decomposition is unique up to

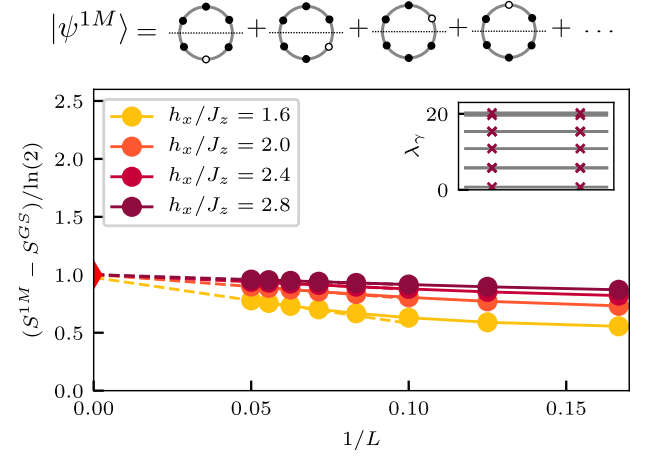


Figure 1. Difference of the EE between the ground state and the single magnon excitation with $k = 0$ in the paramagnetic phase of the transverse field Ising model with $J_x/J_z = 0.1$ for a reflection symmetric bipartition. The red marker on the y-axis shows the expected $\ln(2)$ contribution. Inset: The entanglement spectrum exhibits an inversion symmetry protected two-fold degeneracy.

unitary rotations among degenerate eigenspaces. Thus it must hold that $|\gamma\rangle_A = R|\gamma\rangle_B$ up to a $U(1)$ phase. Moreover, for a parity symmetric state, the Schmidt states are eigenstates with $P|\gamma\rangle_{A/B} = \pm|\gamma\rangle_{A/B}$. Since $P|\psi^{1M}\rangle = -|\psi^{1M}\rangle$, $|\gamma\rangle_A$ and $|\gamma\rangle_B$ must have opposite parity eigenvalues. This leads to a contradiction with the requirement $|\gamma\rangle_A = R|\gamma\rangle_B$ as R does not change the charge. Consequently, all Schmidt values have to be degenerate [28].

Our discussion above can be readily generalized to Z_M paramagnetic phases. For symmetric bipartitions, the quasiparticle carrying unit charge (modulo M) in the Z_M transverse Ising model again contributes an additional EE $\ln(2)$ in the large system limit due to its position uncertainty. However, the entanglement spectrum exhibits an exact two-fold degeneracy for finite system sizes only if M is even as we discuss in the Supplemental Material[29].

Entanglement in non-local QPs— We now move to the FM phase of the transverse field Ising model whose Z_2 symmetry breaking ground states are two-fold degenerate in the infinite system limit. The elementary excitations now are domain walls which are topological defects between $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ patterns. Creating a domain wall is clearly a non-local operation involving flipping all the spins on the left (or right) side of the intended domain wall location. As we would like to study finite systems, boundary conditions now become important. With periodic boundary conditions, the ground states take the form of macroscopic superpositions of the $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ states (“cat states”) which are eigenstates with $P = \pm 1$.

Separated from these via a gap we will find states with pairs of domain walls. As this is a somewhat complicated situation to think about, let us begin the simpler case of the FM Ising model with anti-periodic boundary conditions whose ground state contains a single domain wall. As the eigenstate preserves the Z_2 symmetry, for each configuration with fixed domain wall locations, there are two globally distinct patterns related by the Z_2 symmetry as illustrated in Fig. 2. Consequently, its EE with respect to the partial chain with length pL takes the form,

$$S_p^{1DW} = S_p^{GS} - p \ln(p) - (1-p) \ln(1-p) + \ln(2) + O(1/L) \quad (5)$$

where we assume that pL is much larger than the correlation length. The EE of a single domain wall excitation has three contributions: S_p^{GS} is the local EE of the symmetry broken Ising FM ground state with PBC which is non-universal and dependent on the local entanglement pattern near the cut. The p dependent contribution arises from the domain wall location uncertainty. The final $\ln(2)$ piece is the universal EE due to the macroscopic superposition for each fixed domain wall location. Provided the domain wall excitation is invariant under Z_2 , the cat-state with globally distinct patterns persists so the $\ln(2)$ contribution is robust against any perturbation. For a spatial symmetric bipartition, the additional EE generated by domain wall is then $2\ln(2)$ as one sees in the data in Fig. 2. We note that the entanglement spectrum of the symmetric bipartition does *not* exhibit an exact two-fold degeneracy. This lack of degeneracy arises from the fact that the domain wall does not carry a well defined Z_2 quantum number in the phase where the symmetry is broken.

Returning to the case of PBCs we might expect that the two domain wall states will exhibit an additional EE

$$S_p^{2DW} - S_p^{GS} = f^2(p) + \ln(2) \quad (6)$$

where $f^2(p)$ is the contribution from the joint location uncertainty of the two excitations. In the zero-correlation length limit, we expect $f^2(p) = -[p^2 \ln(p^2) + (1-p)^2 \ln((1-p)^2) + 2p(1-p) \ln(2p(1-p))]$. The second term in Eq. (6) is the universal $\ln(2)$ EE discussed above. Such a result would again indicate that non-local quasiparticles in symmetry breaking phases carry two types of information entropy, one from their positional uncertainty and the other universal piece from their non-locality. Unfortunately, away from this limit the explicit form of $f^2(p)$ is non-universal and varies between different eigenstates while depending on microscopic details such as the QP interaction and correlation length. To extract the universal $\ln(2)$ pieces, we consider the reduced density matrix of a small region with $p \ll 1$ so the positional uncertainty part almost vanishes and the EE converges to the universal $\ln(2)$. The spirit of the form in Eq. (6) reveals that the additional EE $S_p^{DW} - S_p^{GS}$ car-

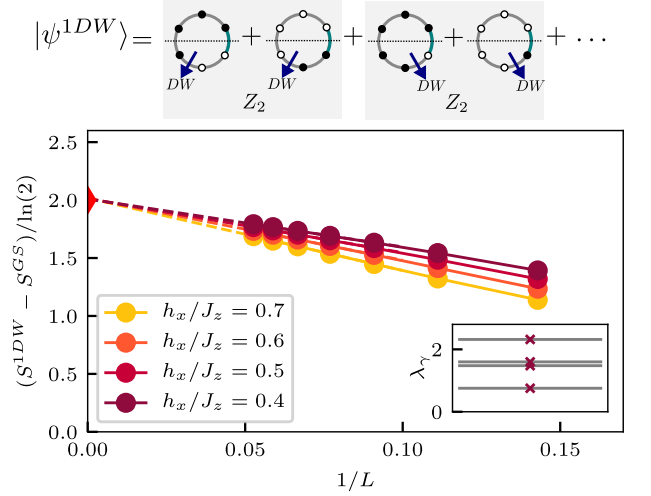


Figure 2. Difference of the EE between the symmetry broken ground state and the single domain wall excitation in the ferromagnetic phase of the transverse field Ising model with $J_x/J_z = 0.1$. We use anti-periodic boundary condition and choose a reflection symmetric bipartition. The red marker on the y-axis shows the expected $2\ln(2)$ contribution. Inset: The entanglement spectrum does not exhibit any symmetry protected degeneracies.

ried by the QP states is composed of two additive contributions from the positional uncertainty and the macroscopic superposition respectively. The second, universal, piece will generalize [30] and depends on the dimension of the ground state manifold in the symmetry breaking phase, e.g. for the Z_M FM Ising model it becomes $\ln(M)$.

LRMI in non-local QPs— Following the identification of this universal entanglement contribution, we propose another intimately related entanglement property, the long-range mutual information (LRMI) [31] between spatially widely separated qubits stemming from the presence of one or more non-local QPs. This LRMI provides a more accessible protocol than the universal contributions to the EE to distinguish between local and non-local QPs. We again begin with the state containing single domain wall induced by anti-periodic boundary conditions. As noted above, the EE of a single spin B is,

$$S_B^{1DW} = \ln(2) . \quad (7)$$

Next we consider the EE of two isolated spins A, C separated by $L/2$ as in Fig. 3(a) and (b), and find

$$S_{AUC}^{1DW} = 2\ln(2) - O(1/L) . \quad (8)$$

Due to the uncertainty in the location of the domain wall, the two spins can be parallel or anti-parallel which produces $\ln(2)$ EE in addition to the universal EE of $\ln(2)$ from the global superposition in a cat state.

Now we place B in the middle of A, C and calculate

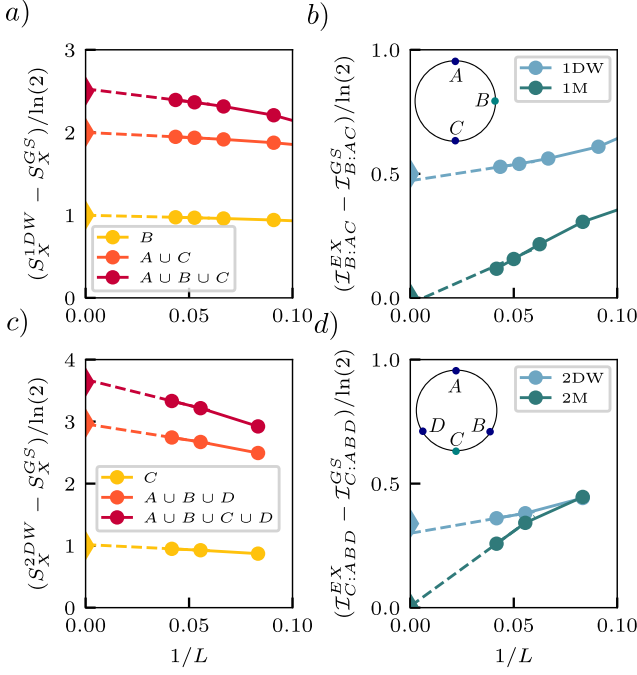


Figure 3. Mutual information for different partitions of the chain, differentiating the cases of magnon and domain-wall excitations, in the para- ($h_x/J_z = 1.6$) and ferromagnetic ($h_x/J_z = 0.2$) phase, respectively. Panel a) shows the entanglement entropy of several disconnected regions in the single domain wall excited state obtained by taking the anti-periodic boundary conditions. Panel c) Shows the entanglement entropy of several disconnected regions in the two domain wall excited state obtained by taking the periodic boundary conditions. Panel b) show the tripartite mutual information and its contributions for single domain wall and single magnon state. Panel d) show the quadripartite mutual information and its contributions for two domain wall and two magnon excitation. The markers on the y-axes show the expected contribution in the thermodynamic limit.

the EE for the three isolated spins,

$$S_{AUCUB}^{1DW} = 5/2 \ln(2) - O(1/L). \quad (9)$$

Finally, the mutual information between B and $A \cup C$ is

$$I_{B:AC} = -S_{AUCUB}^{1DW} + S_{AUC}^{1DW} + S_B^{1DW} = 1/2 \ln(2). \quad (10)$$

Such a non-vanishing MI implies that a measurement of $A \cup C$ would reduce the information carried by B . This is obvious in the sense that once A, C are parallel to each other, the domain wall is outside so the spin B configuration is fixed to be parallel to A, C . In general, the mutual information between the set of three spins B and A, C is always non-vanishing provided each pair of them is far apart.

For QP states containing a pair of domain wall excitation, we still find a non-vanishing mutual information to

appear between three isolated spins and the spin in the middle shown in Fig. 3(c) and (d).

$$I_{C:ABD} = S_{AUBUD}^{2DW} + S_C^{2DW} - S_{AUBUCUD}^{2DW} = \ln(2) - g \quad (11)$$

Here g is a non-universal term which depends on the spatial distribution of the two domain walls and their interaction with an upper limit $g < \frac{7}{9} \ln(2)$. Subsequently, the mutual information is bounded from below. This non-vanishing LRMI can be understood as follows: Due to the non-locality of the DW excitations, the entropy of a single spin can be strongly reduced when performing a measurement or projection to the spins infinite far away, *e.g.*, if we project (A, B, D) to the $(1, 0, 0)$ pattern, the region between B, D are nearly polarized so C is pinned into the 0 state.

The LRMI allows without prior knowledge of the phase to distinguish between ‘local’ and ‘non-local’ QPs, and is a generically applicable measure. In the Supplemental Material we show via large-scale DMRG simulations that we can indeed probe the phase diagram via entanglement of QPs with our MI measures. The LRMI between distant regions generates an experimentally accessible protocol to probe different types of QPs via partial measurement of conditional EE [32]. Namely, a measurement of subsystem A leads to a dramatic reduction of the bipartite EE of region B even though they are spatially separated apart. In particular, the non-vanishing mutual information we explored in Eqs. (10) and (11) can be extended to Renyi-entropies which are directly experimentally accessible [33–37].

Generalizations and Outlook— While we introduced our framework by considering eigenstates of the transverse field Ising chain as concrete examples, the entanglement lens can be applied to characterize QPs in a much broader sense [38]: For example, QPs can carry fractionalized quantum numbers, as it is for example the case in the Majumdar-Aghosh model [39], where domain walls between different dimerization patterns are $S = 1/2$ spinons. In this case, the symmetry fractionalization leads to additional characteristic pattern in the entanglement structure of the QP excitations. Moreover, we expect that QP excitations in symmetry protected topological (SPT) phases [40–42] still exhibit conditional mutual information despite the fact that the characteristic string order [43] is only present in the ground state and disappears in QP states. Going beyond the one-dimensional case, it is an exciting question how to characterize anyonic QPs that emerge in topologically ordered systems [44]. Here the non-local nature of the QP is expected to lead to long range mutual information that allow to characterize the underlying entanglement structure. Another exciting direction is to investigate the entanglement dynamics of individual QP excitations. Imagine we begin with an initial state by adding a static QP at

site- i to the ground state. The QP will delocalize due to the unitary time evolution and the entanglement growth after the quench turns out to differ qualitatively between local and non-local QP.

Conclusions— We proposed a unified way to characterize QPs using the EE and mutual information. We identified universal properties that provide a powerful framework to characterize one-dimensional symmetric and symmetry broken phases of matter via their local and non-local QP excitations, respectively. Thus the low energy QP state inherits the underlying symmetry structure of the ground state. Broadly speaking, our approach provides a new route to probe QP and collective excitations in complex quantum materials.

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