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Gravitational anomaly of $3+1$ dimensional $\mathbb{Z}_2$ toric code with fermionic charges and fermionic loop self-statistics

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Quasiparticle excitations in $3+1$ dimensions can be either bosons or fermions. In this work, we introduce the notion of fermionic *loop* excitations in $3+1$ dimensional topological phases. Specifically, we construct a new many-body lattice invariant of gapped Hamiltonians, the loop self-statistics $\mu = \pm 1$, that distinguishes two bosonic topological orders that both superficially resemble $3+1$d $\mathbb{Z}_2$ gauge theory coupled to fermionic charged matter. The first has fermionic charges and bosonic $\mathbb{Z}_2$ gauge flux loops (FcBl) and is just the ordinary fermionic toric code. The second has fermionic charges and fermionic loops (FcFl) and, as we argue, can only exist at the boundary of a non-trivial $4+1$d invertible phase, stable without any symmetries *i.e.*, it possesses a gravitational anomaly. We substantiate these claims by constructing an explicit exactly solvable $4+1$d model using a method that bootstraps a boundary theory into a bulk Hamiltonian, analogous to that of Walker and Wang, and computing the loop self-statistics in the fermionic $\mathbb{Z}_2$ gauge theory hosted at its boundary. We also show that the FcFl phase has the same gravitational anomaly as all-fermion quantum electrodynamics. Our results are in agreement with the recent classification of nondegenerate braided fusion 2-categories by Johnson-Freyd, and with the cobordism prediction of a non-trivial $\mathbb{Z}_2$-classified $4+1$d invertible phase with action $S = \frac{1}{2}\int w_2 w_3$.

CONTENTS

I. INTRODUCTION

In the past few decades it has been realized that sophisticated mathematical tools and structures can be applied to understand the classification of gapped many-body quantum phases. This

includes, for example, unitary modular tensor categories, which, having already had close connections to quantum field theory [1–6], found an application in the classification of $2+1$-dimensional topological orders [7–9], as well as cobordism groups, which turn out to be useful in classifying invertible and symmetry protected topological (SPT) phases in arbitrary dimensions [10–12]. More recently, braided fusion 2-categories have been introduced in order to classify topological orders in $3+1$-dimensions [13–15]. Validating any such mathematical classification scheme requires constructing a physical observable — *i.e.*, a quantized invariant of many-body lattice Hamiltonians — that distinguishes among the proposed phases. While in some cases this is (in principle) straightforward — *e.g.,* using interferometry to measure the mutual braiding statistics of anyons — in other cases it is more complicated. One example of the latter situation is the recent prediction [16], based on a classification of nondegenerate braided fusion 2-categories, of two distinct variants of $3+1$d $\mathbb{Z}_2$ gauge theory coupled to fermions: the ordinary fermionic toric code, and an anomalous variant that can only exist at the boundary of a non-trivial $4+1$d invertible phase. Assuming these phases do indeed exist, one can ask: what physical observable, defined in the context of gapped lattice spin Hamiltonians, distinguishes between them? Furthermore, are there exactly solved models that realize these two phases?

We answer these questions by defining a new many-body lattice invariant of bosonic $3+1$d gapped Hamiltonians, the loop self statistics $\mu = \pm 1$, and constructing models that realize both of these values. Importantly, the loop self statistics is well defined for $\mathbb{Z}_2$ gauge theory topological orders if and only if the gauge charge is a fermion. In this case there are two possibilities: (i) $\mu = 1$, the usual fermionic toric code with fermionic charges and bosonic loops (FcBl) and (ii) $\mu = -1$, the anomalous variant with fermionic charges and fermionic loops (FcFl). The definition of μ is reminiscent of the T-junction process (reviewed in Section II A) used to measure exchange statistics of identical anyons by applying a product of string operators that exchanges the anyons in such a way as to carefully cancel all non-universal phases (see [17] and Sec 8.3 and figure 10 in [9]). In the present case of loops in $3+1$d, the T-junction is replaced by a more complicated geometry, illustrated in Fig. II.2, and the string operators are replaced with 2d membrane operators. The process ends up effectively rotating the initial loop configuration in such a way as to reverse the orientation along the loop, as illustrated in Fig. II.4.

As already alluded to, the loop self statistics are also an anomaly indicator: the FcFl phase cannot exist in a standalone $3+1$d lattice model, but only at the boundary of a non-trivial $4+1$d invertible phase, and hence possesses a gravitational anomaly. At an intuitive level this is because the fermionic nature of both the charges and the loops in the FcFl phase prevents either one from

being condensed, making it impossible to drive a phase transition to a trivial phase, and hence leading to it being anomalous. More formally, if the FcFl phase could be realized strictly in $3+1$d, then, by "un-gauging" fermion parity, we would obtain a non-trivial $3+1$d fermionic invertible phase, which is absent in current classification schemes and believed not to exist.

The anomalous nature of the FcFl phase is also related to its connection to all-fermion quantum electrodynamics (QED), *i.e.*, $3+1$d QED with fermionic charges, monopoles, and dyons [18, 19]. To elucidate this connection, let us imagine condensing pairs of fermionic charges in all-fermion QED. The resulting Meissner effect confines the magnetic field in $\pm\pi$ flux tubes, with the domain wall between π and $-\pi$ flux trapping a neutral fermionic monopole. This intuitive picture motivates the construction of an exactly solved $4+1$d model using a method similar to that of Walker and Wang [20], where a boundary theory is bootstrapped into a bulk Hamiltonian in one dimension higher. Specifically, at a continuum level we view a spacetime trajectory of loops and particles in the anomalous $3+1$d theory as a configuration in 4-dimensional space, and assign to it an amplitude equal to its exponentiated action.

The fermionic nature of the charge worldlines leads us to pick a "blackboard" framing, as is typical in Walker–Wang constructions [21]. This same blackboard framing can be used to put a local orientation on the (possibly not globally orientable) 2d loop trajectory worldsheet. The key feature which then makes our exactly solved model non-trivial is that the 1d domain wall where this local orientation reverses (referred to as the "w_1 line" below) is decorated with with an additional *gauge neutral* fermion — the remnant of the fermionic monopole.[1] The choice of blackboard framing in our model should be viewed as a technical tool used to obtain a lattice model whose boundary excitations have the appropriate statistics, verified through computing the appropriate commutation relations of string or world-sheet operators. In particular, the notion of framing never needs to be explicitly referred to subsequently, when studying the lattice models.

Our model is specifically designed so that, when truncated, its boundary hosts the FcFl fermionic toric code topological order, as we explicitly verify. Furthermore, we explicitly show that the $4+1$d bulk is invertible — in this case, it means that the bulk can be disentangled by a shallow depth circuit for two stacked copies of this phase. In particular, this implies that there is no topological order in the bulk. In view of the generalized Walker–Wang prescription guiding the construction of our model, this is a reflection of the nondegenerate nature of the braided fusion 2-categories

[1] Note that binding an additional physical gauge charged fermion to this w_1 line results in effectively a gauge charged boson being bound to it. This gives another, equivalent, Walker–Wang model, where the w_1 line decoration is by a gauge charged boson. The key point is that it is impossible to get rid of both the gauge charge and the statistics of the w_1 line decoration when the gauge charge is a fermion. When the gauge charge is a boson, on the other hand, the decoration can be screened out, showing, at a heuristic level, why the loop self-statistics is not well defined in that case.

describing our boundary. In fact, our model can be interpreted as a gauge theory involving 2-form and 3-form gauge fields whose action, valued in a certain cohomology group, contains the data of the braided fusion 2-category. Further discussion of this connection and the work of [16] in particular is given in Section V below. The boundary of our model can also be driven into the all-fermion QED phase (after the addition of some ancilla boundary degrees of freedom), confirming that all-fermion QED and the FcFl phase indeed possess the same gravitational anomaly.

There have been several previous works related to ours. The anomalous nature of all-fermion QED has been studied in a continuum field theory context in [19] and [22], where the anomaly is diagnosed by putting the theory on a $\mathbb{CP}^2$ spacetime topology. The anomalous nature of all-fermion QED was also studied in [18], where a proof by contradiction exploited the edge-ability of any standalone $3+1$d model, and relied on an assumption of the existence of a gapped surface topological order for any invertible phase of fermions. In [16] the existence of two distinct variants of fermionic $\mathbb{Z}_2$ gauge theory was posited based on the classification of nondegenerate braided fusion 2-categories. The fact that our loop self-statistics μ are well defined only in the case of fermionic gauge charges turns out to be a reflection of the trivialness of a certain automorphism of the corresponding braided fusion 2-category in the work of [16]. The notion of a fermionic loop excitation was also introduced in a field theory context in [23], although the relation between this and our loop self-statistics is not completely clear.

The rest of this paper is structured as follows. In Section II we construct the loop exchange statistics μ in fermionic gauge theories, and show that it is independent of the various arbitrary choices made in the construction, assuming that the gauge charge is a fermion. In Section III, motivated by a continuum intuition coming from a decorated domain wall picture, we construct a $4+1$d lattice model and verify in Section IV that it hosts the FcFl phase on its $3+1$d boundary. We conclude with some remarks about the connection of our work to the classification of braided fusion 2-categories and the work of [16], as well as some future directions in Section V.

Note added: Near the completion of this work, we learned of another paper [24] in preparation that also constructs an exactly solvable lattice model for the nontrivial invertible bosonic phase in $4+1$ dimensions. Also, after the completion of the initial draft of this work we learned about mathematical work of Johnson-Freyd and Reutter [25] where a so-called 'Klein' invariant is defined. We believe that this Klein invariant should correspond to our loop self-statistics, modulo the fact that one is defined in the continuum field theory and the other for lattice many-body quantum systems.

II. FERMIONIC LOOP SELF-STATISTICS

A. Review of exchange statistics of identical point particles

Our loop self-statistics will be defined in analogy with the process that measures exchange statistics of identical (and for simplicity abelian) quasiparticle excitations (see [17] and Sec 8.3 and figure 10 in [9]), so let us first review this process in a way that will naturally generalize. We have a T-junction geometry, as in Fig. II.1, with all distances much longer than the correlation length. We choose 6 different states $|\mathfrak{c}_i\rangle$, $i = 1, \ldots, 6$, corresponding to the $\binom{4}{2} = 6$ configurations of two identical quasiparticles illustrated in Fig. II.1. These states have the property that if $|\mathfrak{c}_i\rangle$ and $|\mathfrak{c}_j\rangle$ both have a given location occupied by a quasiparticle, or both have it unoccupied, then the reduced density matrices of $|\mathfrak{c}_i\rangle$ and $|\mathfrak{c}_j\rangle$ in the neighborhood of that location are identical (in the latter case, being just the ground state reduced density matrix). We then choose string operators M_i, $i = 1, 2, 3$, which move a quasiparticle from the center out to one of the three outer endpoints. We require that the M_i be shallow circuits supported in thin neighborhoods of the intervals connecting the center to these outer endpoints, with Lieb–Robinson length much smaller than the lengths of the intervals (but possibly larger than the correlation length). We then compute

$$M_2 M_3^{-1} M_1 M_2^{-1} M_3 M_1^{-1} \, |\mathfrak{c}_1\rangle = \theta \, |\mathfrak{c}_1\rangle \tag{1}$$

where θ encodes the exchange statistics of the quasiparticles.

To argue that θ is well defined, one must show that (i) for a given choice of the states $\{|\mathfrak{c}_i\rangle\}$, the phase factor θ is independent of the choice of the M_i, and (ii) θ is independent of the choice of $\{|\mathfrak{c}_i\rangle\}$. By "choice of $\{|\mathfrak{c}_i\rangle\}$" we mean a potentially different set of states $\{|\mathfrak{c}'_i\rangle\}$ where all of the $|\mathfrak{c}'_i\rangle$ have the same topological charges as the $|\mathfrak{c}_i\rangle$ locally, but may differ by some topologically trivial

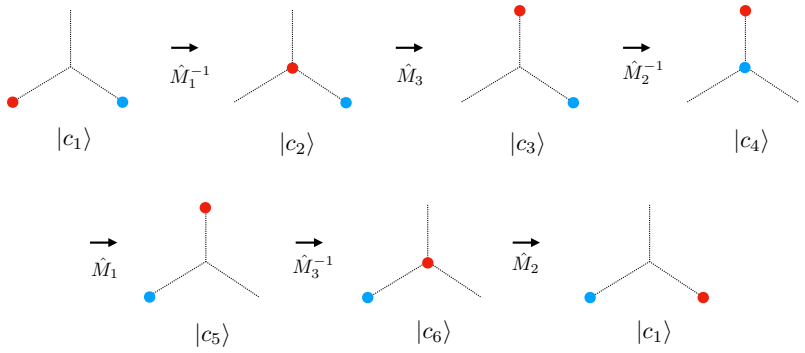

FIG. II.1. T-junction process used to measure statistics of identical particles

excitations; for example, the locations of the quasiparticles may have moved slightly.

To prove the first statement, let us take, for a fixed set of $\{|\mathfrak{c}_i\rangle\}$, a different choice of string operators M_i'. Then we must have $M_i' = F_i M_i$, where F_i is a shallow circuit. F_1 has the property that $F_1|\mathfrak{c}_1\rangle = \alpha|\mathfrak{c}_1\rangle$, $F_1|\mathfrak{c}_5\rangle = \alpha'|\mathfrak{c}_5\rangle$. Furthermore, $|\mathfrak{c}_5\rangle = U|\mathfrak{c}_1\rangle$ where U is a shallow circuit supported away from the support of F_1. Being supported on disjoint spatial regions, U and F_1 commute, so that $\alpha = \alpha'$. A similar argument applies to F_2 and F_3, so that:

$$\begin{aligned} F_1|\mathfrak{c}_1\rangle &= \alpha|\mathfrak{c}_1\rangle, \quad F_1|\mathfrak{c}_5\rangle = \alpha|\mathfrak{c}_5\rangle \\ F_2|\mathfrak{c}_3\rangle &= \beta|\mathfrak{c}_3\rangle, \quad F_2|\mathfrak{c}_1\rangle = \beta|\mathfrak{c}_1\rangle \\ F_3|\mathfrak{c}_5\rangle &= \gamma|\mathfrak{c}_5\rangle, \quad F_3|\mathfrak{c}_3\rangle = \gamma|\mathfrak{c}_3\rangle \end{aligned}$$

Thus,

$$\begin{aligned} &M_2' \left(M'_3\right)^{-1} M_1' \left(M'_2\right)^{-1} M_3' \left(M'_1\right)^{-1} |\mathfrak{c}_1\rangle \\ &\quad = F_2 M_2 M_3^{-1} F_3^{-1} F_1 M_1 M_2^{-1} F_2^{-1} F_3 M_3 M_1^{-1} F_1^{-1} |\mathfrak{c}_1\rangle \\ &\quad = \beta M_2 M_3^{-1} \gamma^{-1} \alpha M_1 M_2^{-1} \beta^{-1} \gamma M_3 M_1^{-1} \alpha^{-1} |\mathfrak{c}_1\rangle \\ &\quad = M_2 M_3^{-1} M_1 M_2^{-1} M_3 M_1^{-1} |\mathfrak{c}_1\rangle \end{aligned}$$

so we get the same value of θ.

To prove the second statement, suppose we have a different set of configuration states $\{|\mathfrak{c}_i'\rangle\}$. Then clearly there exists a shallow circuit V such that $|\mathfrak{c}_i'\rangle = V|\mathfrak{c}_i\rangle$ (just take one that moves the quasiparticles from their old positions to their new positions); conjugating M_i by V gives a set of string operators for $\{|\mathfrak{c}_i'\rangle\}$, and the composition of these new string operators used in computing θ is simply the V conjugate of the old composition (note that conjugation commutes with taking inverses). Thus the exchange phase stays the same.

We will now use a similar procedure to define self-exchange statistics of loop excitations in 3 spatial dimensions, when the gauge charge is a fermion.

B. Data used to define the loop self statistics in $3+1$d

Before delving into the details of the process used to define the loop self statistics, let us make some general remarks. The process will move a loop excitation in such a way that its final position is the same as its initial position, but the orientation along the loop is reversed. This means that the spacetime history of this process, with periodic boundary conditions identifying the initial and final state of the loop, is a Klein bottle. We believe that the resulting invariant is the same as the

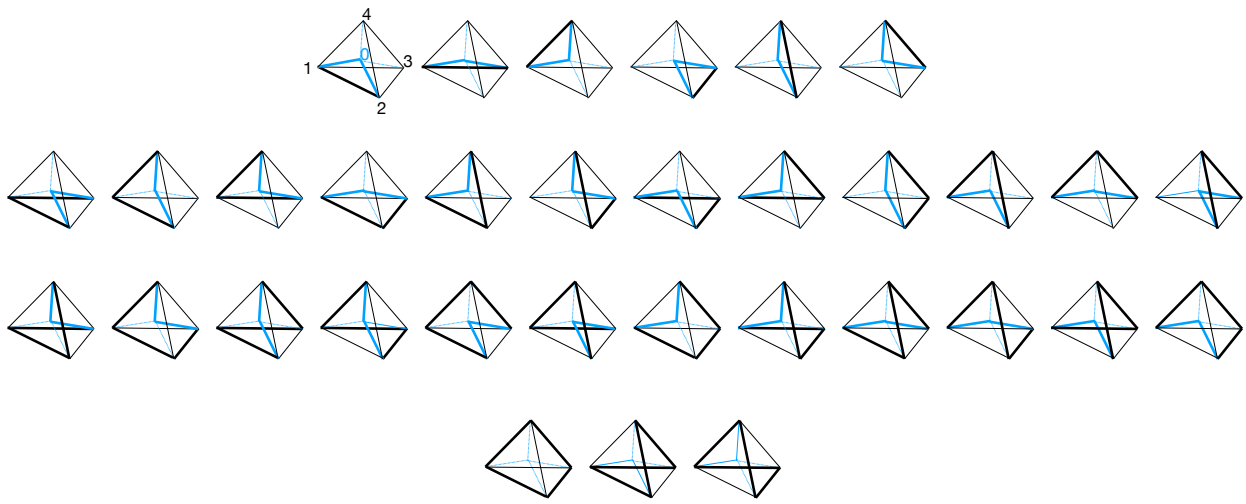

FIG. II.2. All 33 loop configurations. The length scale is much longer than the correlation length. Whenever two of these configurations look identical in some local region, we require that their reduced density matrices in this local region be identical. This in particular means that there are $6 = \binom{4}{2}$ different reduced density matrices in the neighborhood of each vertex. We have colored the interior edges blue and the outside edges black for clarity. The bottom three configurations appear twice in the sequence defining our invariant; all other configurations appear exactly once.

Klein invariant of Freyd and Reutter, defined in section 3.3 or [25]. We emphasize though that the key feature of the process defined below is that it defines a manifestly universal quantity in the lattice quantum many-body system, independent of arbitrary choices of the string movement operators. This is the reason for the large number of seemingly un-motivated steps in the process.

Our geometry is now a tetrahedron with vertices $1, 2, 3, 4$, together with a central vertex 0. Again, all length scales are much longer than the correlation length. Consider 33 loop configurations $\mathfrak{c}$ illustrated in Fig. II.2. The first piece of data we will need for defining our loop self-statistics is a corresponding set of 33 states $|\mathfrak{c}\rangle$, where occupied edges of $\mathfrak{c}$ form a $\mathbb{Z}_2$ gauge flux loop. We demand the following property of the $\{|\mathfrak{c}\rangle\}$: if two configurations $\mathfrak{c}$ and $\mathfrak{c}'$ look the same locally, then the reduced density matrices of $|\mathfrak{c}\rangle$ and $|\mathfrak{c}'\rangle$ in that local region are identical. More precisely, if $\mathfrak{c}$ and $\mathfrak{c}'$ both have the same edge occupied, or both have it unoccupied, then the reduced density matrices of $|\mathfrak{c}\rangle$ and $|\mathfrak{c}'\rangle$ in a neighborhood of the interior of that edge (not including its endpoints) are identical. Also, if $\mathfrak{c}$ and $\mathfrak{c}'$ have the same two edges adjoining a given vertex occupied, or both have all edges adjoining that vertex unoccupied, then the reduced density matrices of $|\mathfrak{c}\rangle$ and $|\mathfrak{c}'\rangle$ in a neighborhood of that vertex are identical (in the latter case, being just the ground state density matrix).

To produce such states $|\mathfrak{c}\rangle$, we can act with membrane operators on various plaquettes $(ij0)$ to produce the desired configuration of $\mathbb{Z}_2$ gauge flux loops, then act locally in the neighborhoods of the various edges $(i0)$ to remove local excitations, and then finally do the same in the neighborhoods of the various vertices. There is a possible obstruction that may potentially arise in this last step: there may be extra $\mathbb{Z}_2$ gauge charges stuck at the vertices, relative to the desired configuration. We will assume that these can always be removed by acting with gauge charge string operators along the various edges. It is possible that such an obstruction never arises, but we do not prove this here. We note that this obstruction certainly does not arise in the two universality classes we are interested in in this paper, namely that of the trivial $3+1$d fermionic toric code (FcBl), and the anomalous one (FcFl), as can be explicitly seen from the exactly solved model presented below.

The other piece of data we need for defining our loop self-statistics are the membrane operators M_{ij} $(i<j)$, which nucleate a $\mathbb{Z}_2$ gauge flux loop around the plaquette $(ij0)$. We impose the following conditions on M_{ij}. First, we demand that M_{ij} is a shallow circuit of local unitaries, with Lieb–Robinson length possibly longer than the correlation length but much shorter than the length scales associated with our tetrahedron. It could also be a circuit with tails, *i.e.*, a short time evolution of a quasi-local time dependent Hamiltonian. Second, suppose that $\mathfrak{c}$ is one of our 33 configurations, with edge (ij) unoccoupied, and that $\mathfrak{c}'$ differs from $\mathfrak{c}$ precisely in the occupation numbers of the three edges (ij),$(i0)$,$(j0)$. Then, if $\mathfrak{c}'$ is also one of our 33 allowed configurations, we demand that

$$|\mathfrak{c}'\rangle\langle\mathfrak{c}'| = M_{ij}|\mathfrak{c}\rangle\langle\mathfrak{c}|M_{ij}^{-1} \tag{2}$$

A construction of a particular set of M_{ij} satisfying these conditions is illustrated in Fig. II.3.

C. Definition of the loop self statistics μ

The loop statistics are defined using the process illustrated in Fig. II.4. There is some arbitrariness in the choice of initial configuration, but, for concreteness, we begin with a starting configuration $\mathfrak{c}_1$, consisting of the edges $(14),(10),(40)$ being occupied (the top left configuration in Fig. II.4). Then, we apply to $|\mathfrak{c}_1\rangle$ a sequence of 36 membrane operators or their inverses. Specifically, in each step, one of two possibilities occurs: 1) an edge (ij) changes from being unoccupied to being occupied or 2) an edge (ij) changes from being occupied to being unoccupied. We act with M_{ij} and M_{ij}^{-1} in these two cases respectively. In this way we generate a sequence of states $|\mathfrak{c}_j\rangle$,

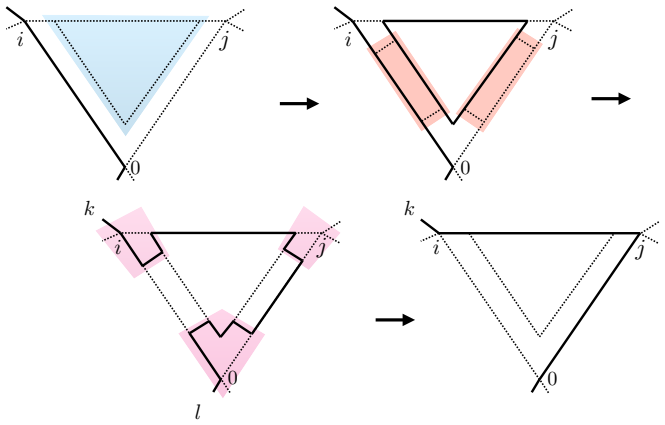

FIG. II.3. A construction of an operator M_{ij} satisfying the requisite conditions, described in Section II B. In this construction, M_{ij} is obtained as the composition of a face operator (blue), which nucleates a loop of gauge flux in the interior of the plaquette, with controlled shallow circuits, which splice this loop into the perimeter of the plaquette. The edge operators (red) are controlled by the occupation numbers of the corresponding edges, and the vertex operators (purple) are controlled by the local configurations near the corresponding vertices. All distances in the figure are much greater than the correlation length. This process is designed in such a way that Eq. (2) is satisfied, as can be checked by comparing the local reduced density matrices on both sides of this Eq. (2).

$j = 1, 2, \ldots, 37$, with $\mathfrak{c}_{37} = \mathfrak{c}_1$, *i.e.*, the configuration comes back to itself at the end. However, the orientation along the loop reverses at the end of the process, as illustrated in Fig. II.4. Note that since for each step in the process there is always an occupied edge e which is not touched by the membrane operator acting at that step, we can consistently define what it means for the orientation to not change during a step by requiring that the orientation along e be fixed. Note also that this orientation is not in any way a physical observable, *i.e.*, $|\mathfrak{c}_1\rangle = |\mathfrak{c}_{37}\rangle$.

Concretely, the process in Fig. II.4 is implemented by the following operator:

$$\begin{aligned}\mathbf{M} =& M_{34}^{-1} M_{12}^{-1} M_{23}^{-1} M_{12} M_{23} M_{14} M_{13}^{-1} M_{24}^{-1} M_{12}^{-1} M_{24} M_{34} M_{12} \\ & M_{23}^{-1} M_{14}^{-1} M_{24}^{-1} M_{14} M_{13} M_{24} M_{34}^{-1} M_{12}^{-1} M_{14}^{-1} M_{34} M_{23} M_{14} \\ & M_{13}^{-1} M_{24}^{-1} M_{34}^{-1} M_{13} M_{34} M_{12} M_{23}^{-1} M_{14}^{-1} M_{13}^{-1} M_{23} M_{13} M_{24}\end{aligned} \tag{3}$$

We define the loop self-statistics μ by

$$\mathbf{M}|\mathfrak{c}_1\rangle = \mu|\mathfrak{c}_1\rangle. \tag{4}$$

D. μ is well defined when the gauge charge is a fermion.

We have to check that μ is well defined, *i.e.,* μ does not depend on the various arbitrary choices made above. First, we will check that, for a given fixed set of $\{|\mathfrak{c}\rangle\}$, μ is independent of the particular choices of M_{ij}. Second, we will check that, when the gauge charge is a fermion, μ is independent of the choice of the $\{|\mathfrak{c}\rangle\}$. From these two facts, it will follow that μ is an invariant of the phase, rather than just of a particular Hamiltonian. The ground states of two Hamiltonians in the same phase can be related by a shallow circuit (or, more generally, a short time evolution of a quasi-local pseudo-Hamiltonian evolution), and conjugating by this circuit allows us to turn membrane operators M_{ij} associated with one Hamiltonian into those associated with the other, and the latter are a valid choice. Since the operator $\mathbf{M}$ also ends up being conjugated, its eigenvalue μ does not change.

1. μ against choices of M_{ij}

Consider two different choices of membrane operators, $\{M_{ij}\}$ and $\{M'_{ij}\}$, and suppose that $|\mathfrak{c}\rangle$ is a configuration state that is acted on by M_{ij}, or is the result of acting with M_{ij}^{-1}, in the expression

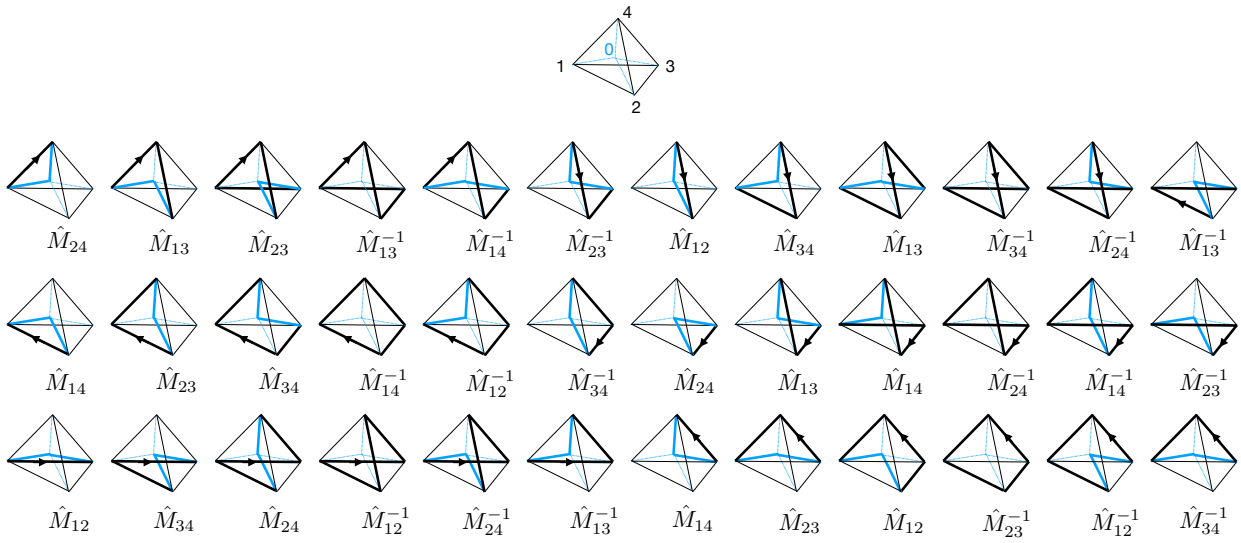

FIG. II.4. Sequence of moves used in defining the loop self-statistics. We start with the configuration $|\mathfrak{c}_1\rangle$ in the upper left, and, at each step, apply the operator below the configuration to obtain the next configuration (reading left to right and up to down). This sequence of operators is written as a product in Eq. (3). Note that the loop comes back to itself at the end, but the orientation along it reverses. At the top we illustrate our labeling scheme for the vertices of the tetrahedron.

for $\mathbf{M}$. Then we claim that

$$M'_{ij}|\mathfrak{c}\rangle = u^i_{ij}(\mathfrak{c})u^j_{ij}(\mathfrak{c})u^0_{ij}(\mathfrak{c})M_{ij}|\mathfrak{c}\rangle \tag{5}$$

where $u^i_{ij}(\mathfrak{c})$, $u^j_{ij}(\mathfrak{c})$, and $u^0_{ij}(\mathfrak{c})$ are $U(1)$ phases that depend only on the local occupation numbers, in the configuration $\mathfrak{c}$, of the edges that end at vertices i, j, and 0, respectively. We prove Eq. (5) in Appendix A 1. Intuitively, it just states that the phase ambiguity associated to M_{ij} depends locally on the configuration $|\mathfrak{c}\rangle$ that M_{ij} is acting on.

Now, let us compute the loop self-statistics μ associated with the new set of membrane operators M'_{ij}, *i.e.*, insert M'_{ij} in place of M_{ij} in the definition of $\mathbf{M}$ in Eq. (3) and act on $|\mathfrak{c}_1\rangle$, as in Eq. (4). This expression is unaltered by inserting projectors $|\mathfrak{c}_i\rangle\langle\mathfrak{c}_i|$, for an appropriate i, between any two of the operators appearing in the expression for $\mathbf{M}$, since, by Eq. (2), the operators M'_{ij} just map between the configurations $|\mathfrak{c}_i\rangle$, up to an overall $U(1)$ phase. We then use Eq. (5) or its conjugate

$$\langle\mathfrak{c}|\left(M'_{ij}\right)^{-1} = \overline{u^i_{ij}}(\mathfrak{c})\overline{u^j_{ij}}(\mathfrak{c})\overline{u^0_{ij}}(\mathfrak{c})\langle\mathfrak{c}|M^{-1}_{ij} \tag{6}$$

on every one of the 36 operators that appears in the expression. We thus see that the new value of μ, computed using the M'_{ij}, is equal to the old value, computed using the M_{ij}, times a product of various $U(1)$ phases $u^i_{ij}(\mathfrak{c})$, $u^j_{ij}(\mathfrak{c})$, $u^0_{ij}(\mathfrak{c})$, and their complex conjugates. We show in Appendix A 2 that these phases cancel, *i.e.*, each phase appears the same number of times as its complex conjugate. The core reason for this is that the expression in Eq. (3) for $\mathbf{M}$ was engineered in such a way that, locally near each vertex, each change in the configuration appears exactly the same number of times as its inverse.

2. *μ against choices of $\{|\mathfrak{c}\rangle\}$ when the gauge charge is a fermion*

Now let us examine the dependence of μ on the choice of $\{|\mathfrak{c}\rangle\}$. First, note that if U is a shallow circuit of local unitaries, then replacing $\{|\mathfrak{c}\rangle\}$ with $\{U|\mathfrak{c}\rangle\}$ does not change μ. Indeed, we can simply conjugate the operators M_{ij} by U to obtain valid membrane operators for $\{U|\mathfrak{c}\rangle\}$. Now suppose that $\{|\mathfrak{c}'\rangle\}$ is some other arbitrary set of configuration states, satisfying all of the properties in Section II B, and let us try to deform $\{|\mathfrak{c}\rangle\}$ into $\{|\mathfrak{c}'\rangle\}$. By possibly repeatedly applying shallow circuits, we can bring $\{|\mathfrak{c}\rangle\}$ near $\{|\mathfrak{c}'\rangle\}$, and by applying shallow circuits on neighborhoods of the edges we can ensure that the reduced density matrices of $\{|\mathfrak{c}\rangle\}$ and $\{|\mathfrak{c}'\rangle\}$ on occupied edges are identical. However, there is a potential obstruction to the most natural way of completing this

deformation on the vertices, since it may be the case that the disagreement near the vertices is by a gauge charge. In this case, there is no local unitary, acting near the vertices, that connects the two states, since they lie in different topological superselection sectors.[2] Of course, by modifying the unitary on the edges by gauge charge string operators it may be possible to get rid of these extra gauge charges and connect the two sets of configurations.

To investigate this question, let us therefore consider the topological (shallow-circuit) equivalence classes of states $\{|\tilde{\mathfrak{c}}\rangle\}$ which agree with $\{|\mathfrak{c}\rangle\}$ on the interiors of the edges but may disagree on the vertices. We may imagine that the states $\{|\tilde{\mathfrak{c}}\rangle\}$ are just the states $\{|\mathfrak{c}\rangle\}$, but with additional gauge charge decoration $x_v(\mathfrak{c}) = 0, 1$ at the various vertices $v = 0, 1, 2, 3, 4$, which depends on the configuration $\mathfrak{c}$ only through the local portion of $\mathfrak{c}$ near v. This means that x_v must be a function of the occupancy numbers on incident edges. Such a function is a polynomial in binary variables y_j where j ranges over all vertices other than v, *e.g.*, $x_{v=0} = y_1 + y_2 + y_1 y_2 \in \mathbb{Z}_2[y_1, y_2, \ldots]/(y_j^2 + y_j)$. Since the number of occupied incident edges is always 0 or 2 in our set of configurations, the occupancy variables obey the condition that $y_j y_k y_\ell = 0$ for any distinct j, k, ℓ and $\sum_j y_j = 0$. In addition, if $y_j = 0$ for all j, then x_v must be zero; with no incident flux tube near v, we can detect a gauge charge at v. So, x_v has no constant term. Hence, the most general function $x_v(y_j)$ is a $\mathbb{Z}_2$-linear combination of quadratic functions; a linear function $x_v = y_j$ is equal to $x_v = y_j^2 = y_j(\sum_{k\neq j} y_k)$. Therefore, we introduce coefficients $[avb] = [bva] \in \mathbb{Z}_2$ associated with each corner at v in a triangle avb so that

$$x_v(\mathfrak{c}) = \sum_{\text{triangle } avb} [avb] \cdot \delta_{avb}(\mathfrak{c}) \tag{7}$$

$$\text{where } \delta_{avb}(\mathfrak{c}) = (\text{quadratic function } y_a y_b) = \begin{cases} 1 & \text{if both edges } av \text{ and } vb \text{ are occupied in } \mathfrak{c}, \\ 0 & \text{otherwise.} \end{cases}$$

Such a decoration is subject to the consistency condition that every loop contain an even number of $\mathbb{Z}_2$ gauge charges. This requirement translates to conditions on the coefficients $[avb]$:

$$[jk0] + [k0j] + [0jk] = 0 \tag{8}$$

$$[ijk] + [jkl] + [kli] + [lij] = 0 \tag{9}$$

where $i, j, k, l \in \{1, 2, 3, 4\}$ are distinct. Equation (8) comes from loops around triangles and Eq. (9) from loops over four edges in Fig. II.2. We have some loop configurations that occupy 5 edges,

[2] We exclude here the possibility of Cheshire charge [26], since our interest will be mostly in the case of fermionic gauge charges, which cannot condense on loops.

but we will see that we do not need to impose another set of constraints from these configurations. Assuming this system of equations, we prove in Appendix B that

$$t = [avb] + [bvc] + [cva] \in \mathbb{Z}_2 \tag{10}$$

is a constant independent of any distinct a, b, c, v.

Next, we show that a gauge charge decoration x_v that is determined by a consistent set of coefficients $[abc]$, is realized by modified membrane operators. For clarity, we may imagine that these gauge charges are offset from the vertices by a common spatial vector which is long compared to the correlation length but short compared to the size of the tetrahedron. Then, the modified membrane operators are

$$\tilde{M}_{ij} = M_{ij} S_{0i}^{[0ij]+t\mathfrak{c}(0i)} S_{0j}^{[0ji]+t\mathfrak{c}(0j)} S_{ij}^{t\mathfrak{c}(ij)} \tag{11}$$

where S_{ab} is a string operator inserting a gauge charge at a and another at b. The appearance of the edge occupation number $\mathfrak{c}(ab) = \mathfrak{c}(ba) = 0, 1$ in the exponents is a shorthand for S_{ab} being controlled on whether edge ab is occupied. Let us explain why this modification realizes the charge decoration. We have to check that the changes in charge decoration numbers are in accordance with those given by x_v. Upon the action by $\tilde{M}_{ij}$, the charge decoration number changes by

$$\begin{aligned}
\Delta x_0 &= [0ij] + t\mathfrak{c}(0i) + [0ji] + t\mathfrak{c}(0j) = [i0j] + t\mathfrak{c}(0i) + t\mathfrak{c}(0j) && \text{by Eq. (8)}, \\
\Delta x_i &= [0ij] + t\mathfrak{c}(0i) + t\mathfrak{c}(ij), \\
\Delta x_j &= [0ji] + t\mathfrak{c}(0j) + t\mathfrak{c}(ij).
\end{aligned} \tag{12}$$

This may be summarized as

$$\Delta x_v = [avb] + t\mathfrak{c}(av) + t\mathfrak{c}(vb). \tag{13}$$

On the other hand, if we toggle the edge occupation numbers on the three edges of a triangle avb,

then Eq. (7) implies that

$$\begin{aligned}
\Delta x_v &= [avb]\Delta\delta_{avb} + \sum_{k\neq a,v,b} ([avk]\Delta\delta_{avk} + [kvb]\Delta\delta_{kvb}) \qquad (14)\\
&= [avb]\big((\mathfrak{c}(av)+1)(\mathfrak{c}(vb)+1) - \mathfrak{c}(av)\mathfrak{c}(vb)\big) + \sum_{k\neq a,v,b} ([avk]\mathfrak{c}(vk) + [kvb]\mathfrak{c}(kv))\\
&= [avb](\mathfrak{c}(av)+\mathfrak{c}(vb)+1) + \sum_{k\neq a,v,b} ([avk]+[kvb])\mathfrak{c}(kv)\\
&= [avb](\mathfrak{c}(av)+\mathfrak{c}(vb)+1) + \sum_{k\neq a,v,b} (t+[avb])\mathfrak{c}(kv) \qquad \text{by Eq. (10)}\\
&= [avb](\mathfrak{c}(av)+\mathfrak{c}(vb)+1) + (t+[avb])(\mathfrak{c}(av)+\mathfrak{c}(bv)) \qquad \text{for } \sum_{l\neq v}\mathfrak{c}(lv)=0\\
&= [avb] + t\mathfrak{c}(av) + t\mathfrak{c}(bv).
\end{aligned}$$

This completes the proof that our modified membrane operator realizes a given charge decoration. Since charge decoration is realized by some string operators whose end points are at vertices where flux loop passes through, the requirement that any loop configurations occupying 5 edges must have an even number of gauge charge decorations, which we did not impose when we solved Eqs. (8) and (9), is automatically satisfied.

In Appendix B we find that there are exactly two classes of solutions of Eqs. (8) and (9), distinguished by the constant $t = 0, 1$ of Eq. (10). When $t = 0$, there is a new set of coefficients $[ab] = [ba] \in \mathbb{Z}_2$, where each $[ab]$ is associated with an edge ab, such that $[abc] = [ab] + [bc]$. Then, the modified membrane operators are[3]

$$\begin{aligned}
\tilde{M}_{\text{triangle } abc} &= M_{\text{triangle } abc}\, S_{ab}^{[ab]+[bc]} S_{ac}^{[ac]+[cb]} \qquad (15)\\
&\cong M_{\text{triangle } abc}\, S_{ab}^{[ab]} S_{ac}^{[ac]} S_{bc}^{[bc]}\\
&= U\, M_{\text{triangle } abc}\, U^\dagger\\
\text{where } U &= \prod_{\text{edge } yz} S_{yz}^{[yz]\mathfrak{c}(yz)}. \qquad (16)
\end{aligned}$$

Here, $\cong$ in the second line denotes modification of the membrane operator by $(S_{ab}S_{ac}S_{bc})^{[bc]}$; this modification does not affect μ due to Eq. (5). The appearance of $\mathfrak{c}(yz)$ in the exponent of U means that the operator $S_{yz}^{[yz]}$ is controlled on the edge occupation on yz. Since S_{yz} is a string-like shallow quantum circuit, U is also a shallow quantum circuit. Therefore, any charge decoration with $t = 0$ does not affect μ.

[3] When applied to our calculation of μ, the vertex $a = 0$.

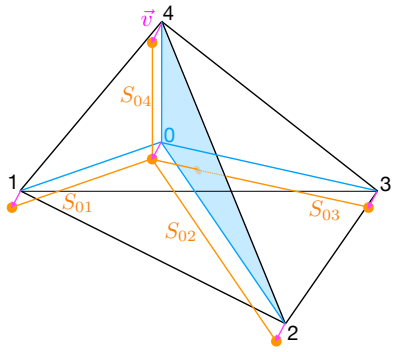

FIG. II.5. There is a unique triple of distinct j, i, k, such that the only non-trivial commutation relation is S_{0j} with M_{ik}. In the case shown in the figure, this non-trivial commutation relation is between S_{03} and M_{24}.

Since any two solutions with $t = 1$ differ by a $t = 0$ solution, it remains to confirm that μ is unaffected under any one particular $t = 1$ solution. One such $t = 1$ solution is given by $[012] = 1$, $[034] = 1$, $[0jk] = 0$ for all other distinct $j, k \in \{1, 2, 3, 4\}$, and $[ijk] = 1 + [0jk] + [0ji]$ for all distinct $i, j, k \in \{1, 2, 3, 4\}$. It is straightforward to see that, for this charge decoration, the new value of μ, calculated from the decorated $\tilde{M}_{ij}$ membrane operators, will differ by at most a sign from the old value. Indeed, this sign difference has two contributions: (i) the anticommutation of the gauge string operators with the membrane operators which they intersect, and (ii) in the case of the gauge charges being fermions, the anticommutation of the gauge string operators S_{i0} with each other since all of them share the vertex 0 in common.

Consider the first contribution. Note that for a generic displacement $\vec{v}$, there will be a unique triple of distinct j, i, k, such that the only nontrivial commutation relation is S_{j0} with M_{ik}, as illustrated in Fig. II.5 with $(ik) = (24)$ and $j = 3$. As we check in Fig. II.6, these commutation relations contribute a factor of -1 to the loop statistics.

The second contribution, due to the statistics of the gauge charges, is nontrivial only in the case of fermionic gauge charges, in which case the string operators S_{i0} anticommute for certain pairs. We find in Fig. II.6 that the product of all of the fermionic string operators appearing in our decorated 36-step process yields a factor of -1. Hence, the second contribution is -1 in the case of fermionic gauge charges, and $+1$ in the case of bosonic gauge charges.

Thus, in the case of bosonic gauge charges μ is not well defined, since its value changes by an overall factor of -1 when the loop configurations are nontrivially decorated by gauge charges. However, in the case of fermionic gauge charges the two contributions to the sign always cancel

and μ is well defined. Furthermore, for the usual $3+1$-dimensional fermionic toric code, *i.e.*, the FcBl model, $\mu = 1$, as can be checked by an explicit calculation in the $3+1$d Walker–Wang model based on the premodular category $\{1, f\}$, where f is a fermion.

We show in Appendix D that, if μ is well defined, it must always be equal to ± 1, and in the discussion we give an argument, based on some physical assumptions, that any stand-alone $3+1$d realization of the fermionic toric code must have $\mu = 1$. Futhermore, in the next sections, we construct a $4+1$d invertible exactly solved Hamiltonian (the "FcFl model") which realizes a $3+1$d

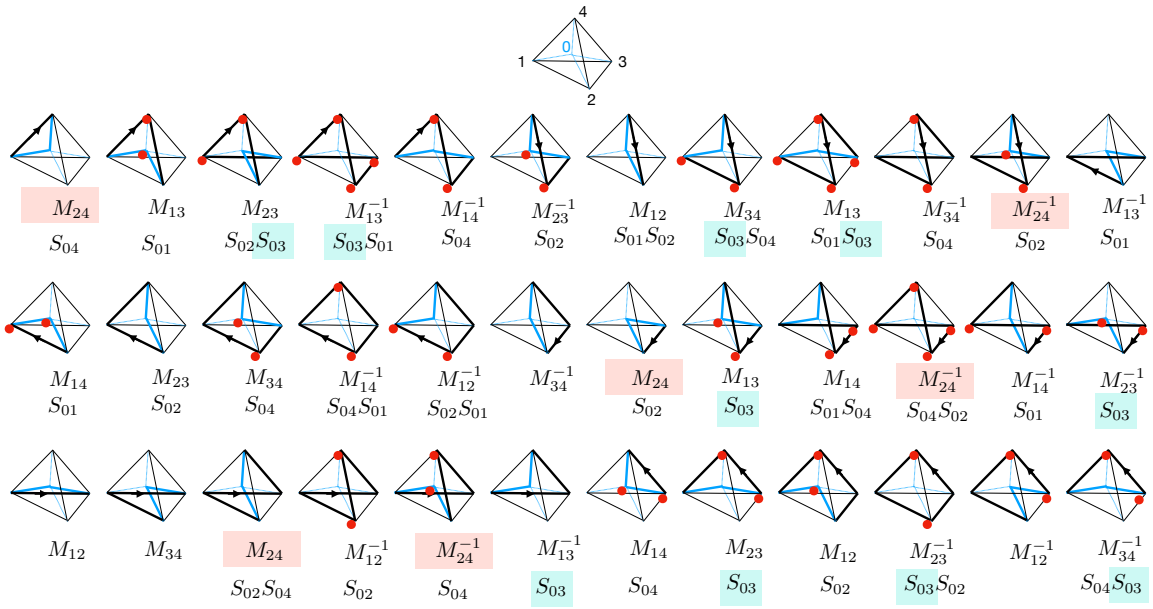

FIG. II.6. The red dots indicate the gauge charges decorating the various vertices, for the non-trivial $t = 1$ decoration described in the text. The red rectangles highlight the steps at which the membrane operator M_{24} or its inverse is applied. The blue rectangles highlight an application of the string operator S_{03}. Commuting such a membrane operator past such a string operator gives rise to a minus sign in a $\mathbb{Z}_2$ gauge theory. As is apparent from the figure, it takes an odd number of anticommutations to cancel off all of the membrane operators against each other and all of the string operators against each other, so the overall sign contribution from such anticommutations is -1. Note that the string operators were chosen to square to $+1$. Furthermore, we can assume that, in the case of fermionic gauge charges, the only pairs of anti-commuting string operators are (S_{01}, S_{02}) and (S_{03}, S_{04}) (one can always get this from any other choice by multiplying some of the string operators by gauge charge detection operators at 0). We can check from the figure that the sign from anti-commuting S_{01}, S_{02} past each other, in such a way as to cancel all of these operators, is 1, whereas the sign from anti-commuting S_{03}, S_{04} past each other is -1, leading to an overall sign of -1 due to the fermionic statistics of the gauge charges.

fermionic toric code with $\mu = -1$. Taken together, these facts imply that our FcFl model is not shallow circuit equivalent to a product state. On the other hand, we will see that two copies of it are shallow circuit equivalent to a product state, so that the FcFl model is a $\mathbb{Z}_2$-classified invertible phase of matter.

III. EXACTLY SOLVED MODEL

In this section we construct a $4+1$d exactly solved Hamiltonian which realizes an anomalous $\mu = -1$ fermionic toric code on its boundary. Our construction is analogous to that of Walker and Wang [20], in the sense that it bootstraps a boundary topological order into a bulk Hamiltonian in one higher dimension. However, our dimensions are shifted up by one from the case discussed in [20], and the input data is a braided fusion 2-category rather than a premodular category. Because such generalizations of Walker–Wang models have not been studied before, we first warm up by constructing simpler exactly solved models which realize nonanomalous bosonic and fermionic toric codes on their respective boundaries. As we will explicitly check, these simpler models will be short-range entangled, *i.e.*, their ground states on geometries with no boundary will be small-depth (shallow) circuit disentanglable. The model realizing the anomalous fermionic toric code, on the other hand, will be invertible but not shallow circuit disentanglable.

A guiding principle to define explicit Hamiltonians below is as follows. We would like the ground state to be a superposition of vacuum-to-vacuum processes involving topological excitations of an input theory, where the amplitude is the corresponding transition amplitude. If the input theory is in $3+1$-dimensional spacetime, then each component of our ground state wavefunction is a picture of a dynamical process drawn in a 4-dimensional canvas. The input theory has point-like and loop-like excitations (generally in every dimension up to space codimension 2), and a picture of ours consists of worldlines and worldsheets. Fluctuation operators of these worldmembranes will be generators of our picture, and the sum of these picture generators will be our $4+1$d Hamiltonian. Whenever the fluctuation changes the topology of worldmembranes, the topological interaction of the input theory gives nontrivial change in the transition amplitude, which must be encoded in the fluctuation operators. The point of our construction below is that desired fluctuation operators can be constructed such that they all commute and the ground state is nothing else but what we want.

We assume that we have a cellulation of a 4-dimensional space which refines to a triangulation. We will use the Poincaré dual cellulation that refines to a triangulation as well, where a

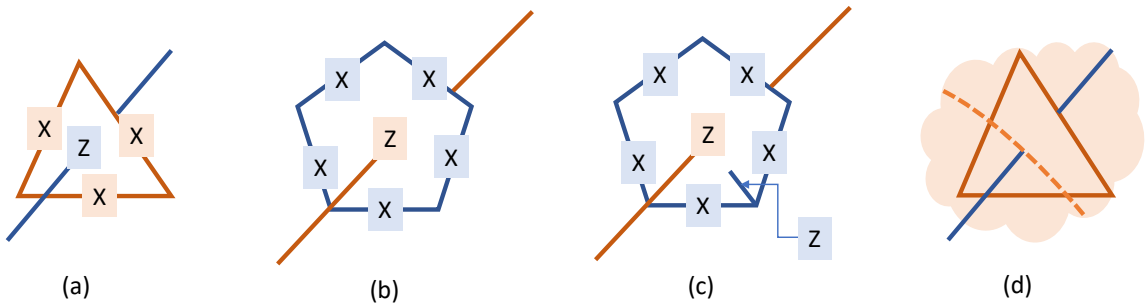

FIG. III.1. Red segments in the figure denote a primary 2-cell; they are drawn as if they were one-dimensional, but they are two-dimensional and the fourth direction is not shown. Blue segments are secondary 1-cells. (a) A worldsheet fluctuation operator T_{Bl}. The Z factor on the blue secondary 1-cell, Poincaré dual to the primary 3-cell bounded by the depicted primary 2-cycle, captures the mutual braiding between a point-like excitation and a line-like excitation of the input theory, the $3+1$-dimensional toric code. The X tensor factors implement the fluctuation of worldsheet. (b) A worldline fluctuation operator L_{Bc}. A secondary 2-cell is not necessarily a pentagon. (c) A worldline fluctuation operator L_{Fc}. The extra Z factor on the secondary 1-cell whose projection under ϕ intersects the interior of the secondary 2-cell that bounds the secondary 1-cycle with X factors, captures the fact that the worldline is a trajectory of a fermion. (d) This is meant to illustrate the worldsheet fluctuation operator T_{Fl}, which consists of the same operator content as T_{Bl}, and, in addition, extra nonPauli operators that captures the change in twisting of the orientation domain wall (orange dashed curve) of the primary 2-cycle.

k-dimensional cell of the original cellulation corresponds to a $(4-k)$-dimensional dual cell of the Poincaré dual. The simplest choice would be the 4-dimensional cubic lattice whose 0-cells are identified with $\mathbb{Z}^4 \subset \mathbb{R}^4$. To distinguish the two cellulations, we call the former the *primary* cellulation consisting of primary cells, and the latter the *secondary*. As usual, the sizes of all primary and secondary cells are uniformly bounded from below and above. The sizes of cells define our lattice spacing, the smallest length scale in our construction.

Suppose there is one qubit ($\mathbb{C}^2$) on each primary 2-cell and on each secondary 1-cell. These degrees of freedom can be interpreted as 3-form and 2-form gauge fields, respectively; a short discussion of this gauge theory interpretation is given in Section V. In view of the guiding principle above, the primary 2-cells and secondary 1-cells will support worldsheets and worldlines of line-like and point-like excitations of an input theory, respectively. Figure III.1 might be helpful to understand the construction of Hamiltonian terms.

A. Bosonic charge and bosonic loop excitations (BcBl)

a. Hamiltonian. We define a *worldsheet* fluctuation operator T_{Bl} for each primary 3-cell e^3, which is Poincaré dual to a secondary 1-cell e_1, by the tensor product of Pauli X over all the primary 2-cells $f^2 \in \partial e^3$ at the boundary of the 3-cell e^3 and Pauli Z on e_1. If the primary cellulation is the cubic lattice, each worldsheet fluctuation operator is a product of 7 Pauli operators. In addition, we define a *worldline* fluctuation operator L_{Bc} for each secondary 2-cell f_2, which is Poincaré dual to a primary 2-cell f^2, by the tensor product of Pauli X over all the secondary 1-cells $\ell_1 \in \partial f_2$ at the boundary of f_2 and Pauli Z on f^2. If the primary cellulation is the cubic lattice, each worldline fluctuation operator is a product of 5 Pauli operators. We define a Hamiltonian H^{WW}_{BcBl} to be the negative sum of all the worldsheet and worldline fluctuation operators.

$$H^{WW}_{BcBl} = -\sum_{f_2:\text{cells}} L_{Bc}(f_2) - \sum_{e^3:\text{cells}} T_{Bl}(e^3) \tag{17}$$

It is natural to include terms that enforce closedness of worldlines and worldsheets, but they turn out to be redundant. A worldline closedness is enforced by demanding that the product $\pi(v_0)$ of Z along the boundary of every secondary 0-cell v_0 should take eigenvalue $+1$. But this product of Z around v_0 is equal to $\pi(v_0) = \prod_{e^3:v_0\in\partial e_1} T_{Bl}(e^3)$ where e_1 is Poincaré dual to e^3. Likewise, the worldsheet closedness is enforced by demanding that $\pi(a^1) = \prod_{f_2:a^1\in\partial f^2} L_{Bc}(f_2)$ should take eigenvalue $+1$ where f_2 is Poincaré dual to f^2.

b. Commutativity. It is obvious that any pair of worldsheet fluctuation operators T_{Bl} commute since the tensor factors on the primary qubits are all X and those on the secondary are all Z. The same is true for any pair of worldline fluctuation operators L_{Bc}. The less obvious case is when a worldsheet fluctuation operator T_{Bl} on e^3 meets a worldline fluctuation operator L on f_2 where the Poincaré dual f^2 of f_2 is on the boundary of e^3. But $f^2 \in \partial e^3$ happens precisely when the Poincaré dual e_1 of e^3 is on the boundary of f_2. Hence, when the X factor of T_{Bl} on f^2 anticommutes with the Z factor of L, the X factor of L on e_1 anticommutes with Z factor of T_{Bl}. Therefore, H^{WW}_{BcBl} consists of commuting terms.

c. Disentangling circuit. Let U be a two-qubit unitary:

$$U = \begin{Bmatrix} |{+}{+}\rangle \\ |{+}{-}\rangle \\ |{-}{+}\rangle \\ |{-}{-}\rangle \end{Bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{Bmatrix} \langle{+}{+}| \\ \langle{+}{-}| \\ \langle{-}{+}| \\ \langle{-}{-}| \end{Bmatrix}, \qquad \begin{cases} U(X\otimes I)U^\dagger = X\otimes I \\ U(I\otimes X)U^\dagger = I\otimes X \\ U(Z\otimes I)U^\dagger = Z\otimes X \\ U(I\otimes Z)U^\dagger = X\otimes Z \end{cases} \tag{18}$$

where the braced column matrices denote the basis in terms of $|\pm\rangle = \pm X|\pm\rangle$. Note that U is invariant under exchange of the two qubits in its support. Consider a shallow quantum circuit

$$V = \prod_{e_1 \sim f^2} U_{e_1, f^2} \tag{19}$$

where $e_1 \sim f^2$ ranges over all pairs of a secondary 1-cell e_1 and a primary 2-cell f^2 such that the Poincaré dual of e_1 has f^2 in its boundary. This product is well defined because U is diagonal in the X basis. Under conjugation by V, every worldsheet fluctuation operator becomes a single-qubit operator Z on the associated secondary 1-cell, and every worldline fluctuation operator becomes Z on the associated primary 2-cell. Hence, $VH^{WW}_{BcBl}V^\dagger$ is the negative sum of single-qubit operator Z over all qubits in the system. This implies in particular that the ground state of H^{WW}_{BcBl} is the unique common eigenstate of all the worldsheet and worldline fluctuation operators with eigenvalue $+1$ and is disentangled by V.

In fact, the disentangling circuit is supplied by a general fact as follows.

Lemma III.1. *Suppose there are n qubits and let $\mathcal{S} = \{P_1, P_2, \ldots, P_n\}$ be a set of Pauli operators of form*

$$P_j = Z(j) \otimes \bigotimes_{i \in \Gamma(j)} X(i) \tag{20}$$

where the arguments denote the qubit the operator acts on and $\Gamma(j)$ is a subset of $\{1, 2, \ldots, n\} \setminus \{j\}$. If P_j commutes with P_k for all j, k, then some product of U defined in Eq. (18) maps P_j to a single-qubit operator $Z(j)$ for every j.

This can be thought of as a characterization of so-called graph states [27].

Proof. If $i \in \Gamma(j)$, then P_i has the factor $Z(i)$ anticommuting with $X(i)$ of P_j. The commutativity demands that this anticommutation must be canceled by another anticommutation which can only be given by a factor $X(j)$ of P_i. This means that $j \in \Gamma(i)$. Hence, we have an undirected graph with n nodes where there is an edge between i and j if and only if $i \in \Gamma(j)$ or equivalently $j \in \Gamma(i)$. If we apply U over every edge of this graph, then $\mathcal{S}$ is disentangled. □

d. Amplitudes in the ground state. Given a primary 1-cell e^1, consider its Poincaré dual e_3 and consider the product $\prod L$ of all the worldline fluctuation operators on the boundary ∂e_3. The X factors of these operators are on the boundary of $b_2 \in \partial e_3$ and hence all cancel since $\partial\partial e_3 = 0 \bmod 2$. The remaining factors are Z on the primary 2-cells whose boundary includes e^1.

The product $\prod L$ taking an eigenvalue $+1$ means that the ground state consists of configurations of primary 2-cells that must define a 2-cycle with $\mathbb{Z}_2$ coefficients.

Likewise, the product $\prod T_{Bl}$ of all the worldsheet fluctuation operator on the boundary ∂h^4 of a primary 4-cell h^4 has surviving factors Z on all secondary 1-cells whose boundary contain the Poincaré dual of h^4. The product $\prod T_{Bl}$ taking an eigenvalue $+1$ means that the ground state consists of configurations of secondary 1-cells that must define a 1-cocycle with $\mathbb{Z}_2$ coefficients.

The Hamiltonian terms drive fluctuations in the worldsheet and worldline configurations. In the basis where Z is diagonal, the qubit states $|1\rangle$ and $|0\rangle$ on a primary 2-cell indicate that the worldsheet of a loop excitation has and has not swept that location in spacetime, respectively. Likewise, the qubit state $|1\rangle$ and $|0\rangle$ on a secondary 1-cell represent the occupancy of the worldline of a charge on that spacetime location. Hence, the Hamiltonian terms correctly capture the property that if the worldline of a charge links with the worldsheet of a loop, there must be an amplitude factor of -1 relative to the configuration where they are not linked. Therefore H^{WW}_{BcBl} is a Walker–Wang Hamiltonian in $4+1$d with respect to the $3+1$d toric code topological order.

e. Linking number. Given any exact secondary 1-chain a_1, let $\tilde{a}_2$ be any secondary 2-chain whose boundary is a_1. Likewise, given any exact primary 2-chain b^2, let $\tilde{b}^3$ be any primary 3-chain whose boundary is b^2. Since a secondary chain is canonically a cochain, we can consider mod 2 intersection numbers $\mathrm{Int}_2(\tilde{a}_2, b^2)$ and $\mathrm{Int}_2(a_1, \tilde{b}^3)$ by the evaluation of the cochain on the chain.

The amplitude ± 1 for the configuration of a_1 and b^2 can be computed starting from the vacuum by applying various fluctuation operators, and a choice of set of fluctuation operators determines $\tilde{a}_2$ and $\tilde{b}^3$. The amplitude is precisely the mod 2 intersection numbers, which must be the same: if we first build b^2 by worldsheet fluctuation operators and then insert a_1 by worldline fluctuation operators, then we will be computing $\mathrm{Int}_2(\tilde{a}_2, b^2)$; if we first build a_1 and then insert b^2, then we will be computing $\mathrm{Int}_2(a_1, \tilde{b}^3)$. Hence, the existence of the ground state of H^{WW}_{BcBl} shows that the intersection number is well defined independent of $\tilde{a}_2$ and $\tilde{b}^3$. We conclude that we may *define* the mutual linking number modulo 2 as a *function* of the exact cycles:

$$\mathrm{Lnk}(a_1, b^2) = \mathrm{Int}_2(\tilde{a}_2, b^2) = \mathrm{Int}_2(a_1, \tilde{b}^3) \tag{21}$$

Note that this conclusion only uses Poincaré duality with $\mathbb{Z}_2$ coefficients, which is valid even for nonorientable manifolds.

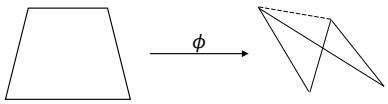

FIG. III.2. A 2-cell that has four or more sides may be mapped to a folded figure under ϕ.

B. Fermionic charge and bosonic loop excitations (FcBl)

We keep using the worldsheet fluctuation operator of H^{WW}_{BcBl}. The worldline fluctuation operator, on the other hand, must be modified since we want the worldline to be one of a fermion, not of a boson. The canonical way to describe a fermion via worldlines is to consider *framed* worldlines. If a worldline closes, the holonomy of the frame along the line is valued in $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and the closed worldline has quantum amplitude ± 1.

In this section we take a simplified approach, motivated by the idea of a "blackboard" framing determined by a projection to 2 dimensions. Although we could in principle just work with a 4-dimensional hypercubic lattice with a generic linear projection to 2d — thereby directly generalizing [21] — we instead find it useful to work in the more general setting of an arbitrary cellulation of a general 4-dimensional spatial manifold.

Consider therefore a piecewise linear map

$$\phi : \mathcal{K} \to \mathbb{R}^2 \tag{22}$$

called a *projection*, from the 2-skeleton $\mathcal{K}$ of the secondary cellulation down to $\mathbb{R}^2$. We require that ϕ should map every secondary 1-cell to a straight line segment of nonzero length, and the images of 1-cells are transverse to one another. These conditions are generically satisfied: we can simply project all the 0-cells to points in generic position on $\mathbb{R}^2$, and connect two projected points if the pair is the boundary of some 1-cell. We also require that the image of a 2-cell f_2 is defined as in Fig. III.2. That is, if $v_0(1), v_0(2), \ldots, v_0(n)$ are the vertices of f_2, then we subdivide f_2 into $n-2$ triangles t_j, each formed by $v_0(1), v_0(j), v_0(j+1)$ for $j = 2, \ldots, n-1$, and map each t_j injectively to the triangle in $\mathbb{R}^2$ formed by $\phi(v_0(1)), \phi(v_0(j)), \phi(v_0(j+1))$.[4] So, every 2-cell is projected to a polygon, which may be folded.

a. Hamiltonian. We have to define a worldline fluctuation operator for each secondary 2-cell f_2. First, assume that ϕ is injective on f_2 (*e.g.*, f_2 is a triangle). Take the worldline fluctuation operator L_{Bc} of H^{WW}_{BcBl}, which consists of Pauli X along the boundary of f_2 and Pauli Z on the

[4] Previously in [28, §II] we introduced a similar projection, but required that every 2-cell is mapped injectively. We no longer require this injectivity.

Poincaré dual f^2 of f_2. We multiply this operator by Pauli Z tensor factors on the secondary 1-cells e_1 such that there is a factor Z on e_1 if and only if $\phi(e_1)$ intersects the interior of $\phi(f_2)$. This defines a term $L_{Fc}(f_2)$. Given two such 2-cells f_2 and f_2', $L_{Fc}(f_2)$ and $L_{Fc}(f_2')$ then commute for a nontrivial reason. We will not give the argument here because it is verbatim the same as in [28, §II.C.a].

Generally, even if $\phi|_{f_2}$ is not injective, the projected polygon $\phi(f_2)$ defines a subdivision $(f_2)_1, \ldots, (f_2)_n$ of f_2 such that each subcell $(f_2)_i$ is injectively mapped under ϕ. For example, in Fig. III.2 a quadrilateral decomposes into two triangles. Except for the Pauli Z on f^2, we follow the prescription for each $(f_2)_i$ as if it were a genuine 2-cell and the folding lines (the dashed line in Fig. III.2) were occupied by qubits. Then, we multiply all these to obtain $\hat{L}(f_2) = \prod_i \left[L_{Fc}((f_2)_i) \setminus Z(f^2)\right]$. Since the factors are commuting, the multiplication order does not matter and $\hat{L}(f_2)$ is unambiguous. The product $\hat{L}(f_2)$ does not contain any X factor on the folding lines because a folding line is shared by two subcells. However, there could be Z factors on folding lines in $\hat{L}(f_2)$. We define

$$L_{Fc}(f_2) = Z(f^2) \left\langle 0_{\text{folding lines}} \right| \hat{L}(f_2) \left| 0_{\text{folding lines}} \right\rangle . \tag{23}$$

where $|0_{\text{folding lines}}\rangle$ is the $+1$ eigenvalue eigenstate of all of the Z operators on the folding lines. That is, we remove any Z factor on the folding line but keep any signs that may have been accumulated while forming $\hat{L}(f_2)$.

The Hamiltonian H^{WW}_{FcBl} is the negative sum of all the worldsheet fluctuation operators, which are the same as in H^{WW}_{BcBl}, and all the worldline fluctuation operators, which is the product of the worldline fluctuation operator of H^{WW}_{FcBl} and the extra Z factors determined by ϕ.

$$H^{WW}_{FcBl} = -\sum_{f_2:\text{cells}} L_{Fc}(f_2) - \sum_{e^3:\text{cells}} T_{Bl}(e^3) \tag{24}$$

As in H^{WW}_{BcBl}, we do not have to include the worldline closedness terms $\pi(v_0)$. Also, it turns out that we do not have to include worldsheet closedness terms; this is not immediately obvious, but will be handled by a disentangling circuit below.

b. Commutativity. The worldsheet fluctuation operators are the same as those of the BcBl model, so we already know that they commute with each other. Any worldsheet fluctuation operator commutes with any worldline fluctuation operator because of the same reason as in H^{WW}_{BcBl}; the extra Z factors in the FcBl worldline fluctuation operator are on the secondary cellulation, on which the worldsheet fluctuation operator has Z factors only. Any two worldline fluctuation operators commute because each is a product of commuting operators associated with sub-2-cells. Therefore, H^{WW}_{FcBl} consists of commuting terms.

c. Disentangling circuit. The unitary V of Eq. (19) transforms the worldsheet fluctuation operator to the single-qubit operator Z. Hence, $H' = VH^{WW}_{FcBl}V^\dagger$ has a completely disentangled ground state on the secondary cellulation. Next, we focus on the worldline fluctuation operator $L = L_{Fc}(b_2)$ after V. Letting b^2 denote the primary 2-cell that is Poincaré dual to b_2, Eq. (18) implies that $VLV^\dagger$ consists of three kinds of Pauli tensor factors: Z on b^2, Z on some secondary 1-cells e_1 near b^2 determined by ϕ (denoted as $e_1 \propto b^2$), and X on primary 2-cells p^2 if they are on the boundary of (Poincaré dual of) an odd number of such e_1's (denoted as $p^2 \in \delta e_1$). On the ground state of H', the Z on the secondary 1-cells e_1 have eigenvalue $+1$, so we may ignore them. Hence,

$$VLV^\dagger \bmod Z \text{ on secondary qubits} = Z(b^2) \prod_{e_1 \propto b^2} \prod_{p^2 \in \delta e_1} X(p^2). \tag{25}$$

Now, there is exactly one $VLV^\dagger$ term for each primary qubit on which $VLV^\dagger$ act by Z. The other factors of $VLV^\dagger$ is X (ignoring Z on secondary qubits). Any $VLV^\dagger$ term commutes with another, and Lemma III.1 guarantees that $VH^{WW}_{FcBl}V^\dagger$ can be disentangled. We have completed the disentangling circuit.

A consequence of the disentangling circuit is that the ground state of H^{WW}_{FcBl} is unique. Namely, it is the superposition of all cycles in the trivial 2-homology and 1-cohomology class with an appropriate amplitude ± 1. This implies that the sign is a *function* of cycles. Since the mutual linking in Eq. (21) is a function of cycles, we conclude there is a well-defined amplitude $(-1)^{\mathrm{Frm}}$ assigned to each worldline a_1 (in the trivial homology class), regardless of which 2-chain bounding this worldline is used to compute the amplitude. This assignment is defined whenever we have a projection ϕ.

In Appendix E we elaborate on Frm to construct a commuting Pauli Hamiltonian H_{Fc}, which is a topologically ordered model with a fermionic point-like excitation on any combinatorial manifold of dimension two or higher.

C. Fermionic charge and fermionic loop excitations (FcFl)

In this subsection we finally construct a Hamiltonian for a non-trivial invertible $4+1$d bosonic phase. Again we find it useful to do this for a general cellulation of a general 4-manifold, though we can easily specialize to a hypercubic lattice. We refer to our model as the FcFl model, because there is a sense in which the loop is fermionic. Specifically, there is an extra factor (± 1) in the amplitude for the worldsheet of fermionic loop excitations, in addition to the framing of fermionic

worldline and the mutual linking between worldlines and worldsheets. This extra ± 1 factor can be read off as follows. On a 2-cycle representing the worldsheet, there is an orientation domain wall, which is a 1-cycle W^1. We regard W^1 as a worldline of a (fictitious) fermionic charge and read off the associated amplitude of W^1. The cycle W^1 may or may not link with the worldsheet, but there is no contribution to the quantum amplitude from this linking; the imagined fermionic charge along W^1 is gauge-neutral.

The orientation domain wall of a surface is only defined as a homology class; however, we always choose a preferred cycle W^1. Our prescription to this end is determined by globally defined (local) orientations for each primary 2-cell. Given any primary 2-chain b^2 with $\mathbb{Z}_2$ coefficients we promote it to a 2-chain $\xi(b^2)$ with $\mathbb{Z}_4$ coefficients,

$$\xi : \mathbb{Z}_2[\text{2-chains}] \to \mathbb{Z}_4[\text{2-chains}], \tag{26}$$

by the rule $\mathbb{Z}_2 \ni 0 \mapsto 0 \in \mathbb{Z}_4$ and $\mathbb{Z}_2 \ni 1 \mapsto 1 \in \mathbb{Z}_4$, where every cell in a chain with $\mathbb{Z}_4$ coefficients carries its orientation. Then, for a closed 2-chain b^2 we define

$$W^1(b^2) = \tfrac{1}{2}\partial\xi(b^2) \tag{27}$$

where ∂ is the mod 4 boundary operator and $\frac{1}{2}$ is well defined because $\partial\xi(b^2)$ vanishes mod 2.

a. Hamiltonian. There is one qubit on each primary 2-cell and one qubit on each secondary 1-cell. We continue to use the worldline fluctuation operators L_{Fc} of H^{WW}_{FcBl}, for which we need to fix a projection ϕ_2 from the 2-skeleton of the secondary cellulation. Also, we fix another projection ϕ^2 from the 2-skeleton of the primary cellulation; ϕ is piecewise linear, mapping 1-cells to transverse straight line segments. See Section III B. Such a projection exists for the standard hypercubic cellulation of $\mathbb{R}^4$. The worldsheet fluctuation operator T_{Fl} on a primary 3-cell e^3 (*i.e.*, a secondary 1-cell e_1) is then defined as follows.

$$T_{Fl} = D \cdot \underbrace{Z(e_1) \cdot \prod_{f^2 \in \partial e^3} X(f^2)}_{T_{Bl}} \tag{28}$$

The operator D is a diagonal operator in the Z basis, acting on qubits on primary 2-cells, and calculates the amplitude change due to the orientation domain wall change. We define D on the subspace of closed chains below; if there is open boundary detected on the support of D, then we let D have eigenvalue 0 on that state. The change ΔW^1 of the orientation domain wall can be computed using that fact that for all 2-chains x^2 and y^2,

$$\xi(x^2 + y^2) = \xi(x^2) + \xi(y^2) \quad \text{if} \quad x^2 \cap y^2 = \emptyset, \tag{29}$$

and is given by, with $s^2 \equiv b^2 \cap \partial e^3$,

$$\begin{aligned}\Delta W^1 &= W^1(b^2+\partial e^3) - W^1(b^2)\\ &= \tfrac{1}{2}\partial\left(\xi\big((b^2+s^2)+(\partial e^3+s^2)\big) - \xi\big((b^2+s^2)+s^2\big)\right)\\ &= \tfrac{1}{2}\partial\left(\xi(\partial e^3+s^2) - \xi(s^2)\right).\end{aligned} \tag{30}$$

Hence, ΔW^1 is local. Reading the existing orientation domain wall, and gluing ΔW^1 in with interpretation that they are fermionic worldlines, materialized by the projection ϕ^2, we compute the change in the amplitude locally. This is D. Therefore, T_{Fl} is local.

In addition to the worldline and worldsheet fluctuation operators, we also include

$$\pi(a^1) = \prod_{f^2:a^1\in\partial f^2} Z(f^2), \qquad \pi(v_0) = \prod_{e_1:v_0\in\partial e_1} Z(e_1) \tag{31}$$

in H^{WW}_{FcFl} which enforce closed primary 2-chains and closed secondary 1-chains. So, the full Hamiltonian reads

$$H^{WW}_{FcFl} = -\sum_{a^1:\text{cells}} \pi(a^1) - \sum_{b_2:\text{cells}} L_{Fc}(b_2) - \sum_{e^3:\text{cells}} T_{Fl}(e^3) - \sum_{v_0:\text{cells}} \pi(v_0). \tag{32}$$

The term T_{Fl} is not a tensor product of Pauli matrices.

b. Commutativity. The π terms commute as they are diagonal, and they also commute with L_{Fc} and T_{Fl} as the fluctuation operators preserve the subspace of closed chains. The worldline fluctuation operators L_{Fc} commute with each other as we remarked before, which relies on [28, §II.C.a]. A worldline fluctuation operator L_{Fc} and a worldsheet fluctuation operator T_{Fl} commute for the same reason as in H^{WW}_{FcBl} since the diagonal factor D of T_{Fl} does not overlap with X factors of L_{Fc}. It remains to check the commutation of the operators T_{Fl} with themselves. It suffices to check this on the subspace of closed chains since T_{Fl} annihilates other states. But for a state with closed chains, the action of T_{Fl} is to change the cycle configuration and insert a sign that is a *function* of cycle configurations. Hence, T_{Fl} commute among themselves, and H^{WW}_{FcFl} is a commuting Hamiltonian.

c. Unique ground state of H^{WW}_{FcFl}. We have defined H^{WW}_{FcFl} such that there is a ground state where all terms (without the overall minus sign) assume +1 on that ground state. This does not immediately imply that there is only one ground state. The uniqueness of the ground state is proven for H^{WW}_{BcBl} and H^{WW}_{FcBl} since we have the circuits that disentangle entire spectra of these Hamiltonians, but we do not yet know the uniqueness of the ground state of H^{WW}_{FcFl}. However, we can still show the uniqueness, independent of disentangling transformations.

Any ground state must be in the subspace of closed chains; otherwise, the π terms will not assume $+1$ and the energy will be higher. Then, the worldline and worldsheet fluctuation operators hybridize all configurations in a (co)homology class $[h] \in H_2(\mathcal{K}^4;\mathbb{Z}_2) \oplus H^1(\mathcal{K}^4;\mathbb{Z}_2)$. Suppose that

($\star$) if a product of L_{Fc} and T_{Fl} is diagonal in Z basis, *i.e.*, the product does not change the configuration, then it acts as $+1$ on the space of all cycles in $[h]$.

We have proved this condition for products over null-homologous cycles; the existence of one ground state is a proof. Then, we can consider a basis of closed chains with the amplitudes in place implied by the fluctuation operators. In this basis L_{Fc} and T_{Fl} have off-diagonal elements that are either $+1$ or 0. The Perron–Frobenius theorem implies that there is a unique ground state in $[h]$ under the assumption ($\star$). If ($\star$) is violated for $[h]$, then there is no ground state in $[h]$.

It remains to show that ($\star$) is violated for all but the zero homology class. Suppose a primary 2-cycle s^2 (worldsheet) is in a nontrivial homology class. Then, there is a secondary 2-cycle s_2 that intersects this 2-cycle at odd number of positions by Poincaré duality. Take the product $\prod L$ of all worldline fluctuation operators L_{Fc} on s_2. Since L_{Fc} on any secondary 2-cell b_2 has X factors on ∂b_2, the product $\prod L$ has no X factors, and $\prod L$ is a diagonal operator with Z factors on every primary 2-cell dual to secondary 2-cells on s_2. These Z factors probe the parity of the intersection number between s^2 and s_2, which is odd. But this is contradictory to the fact that each L_{Fc} has eigenvalue $+1$ on a hypothetical ground state with a nontrivial primary 2-cycle s^2. A similar argument rules out any nontrivial homology class for secondary 1-cycles; the product of worldsheet fluctuation operators T_{Fl} over nontrivial primary 3-cycle detects any nontrivial secondary 1-cycles. Therefore, any ground state is in the trivial homology sector, and H^{WW}_{FcFl} has only one ground state.

D. Disentangling two FcFl states

We are going to construct a shallow quantum circuit that maps the stacked tensor product of two FcFl states to the stacked tensor product of two FcBl states. Since we have a shallow disentangling circuit for an FcBl state, the two FcFl states will be disentangled. We put one FcFl state on our usual system with qubits on primary 2-cells and secondary 1-cells. The other FcFl state is put on the dual system of the same cellulation, *i.e.*, on a system with qubits on secondary 2-cells and primary 1-cells. We do not try to find a shallow circuit between the two FcFl states, but we note that on $\mathbb{R}^4$ with the standard hypercubic cellulation the two states are equal up to a small "diagonal" translation, and hence are considered two identical states laid side by side.

Let c denote any configuration of a primary 2-cycle and a secondary 1-cycle in the trivial homology and cohomology class. Similarly, let c' denote any configuration of a secondary 2-cycle and a primary 1-cycle. We write $c = a \oplus b$ and $c' = a' \oplus b'$ where a is the secondary (or primary if primed) 1-cycle and b is the primary (or secondary if primed) 2-cycle. We have

$$|\text{2FcFl}\rangle = \sum_{c,c'} \eta^f(c,c')\,|c,c'\rangle \tag{33}$$

where the sign η^f is determined by three contributions:

- linking of a with b and that of a' and b';
- framing of a and a';
- framing of $W^1(b)$ and $W_1(b')$.

If the last contribution was absent, the wavefunction becomes that of the two FcBl states $|\text{2FcBl}\rangle$. Now, we introduce a nominally different basis for the closed chain subspace. Define an involution ω by

$$\begin{aligned} (c,c') &= a' \oplus b \oplus b' \oplus a \\ \mapsto \omega(c,c') &= (a' + W^1(b)) \oplus b \oplus b' \oplus (a + W_1(b')) \\ \mapsto (\omega \circ \omega)(c,c') &= (c,c'). \end{aligned} \tag{34}$$

This involution ω is simply a permutation on the set $\{(c,c')\}$ of all closed chains. We use the ω-permuted basis to write

$$|\text{2FcBl}\rangle = \sum_{c,c'} \eta^b(c,c')\,|\omega(c,c')\rangle \tag{35}$$

where the sign η^b is determined by two contributions:

- linking of $a + W_1(b')$ with b and linking of $a' + W^1(b)$ with b';
- framing of $a + W_1(b')$ and framing of $a' + W^1(b)$.

Therefore, the difference $\eta^f \eta^b = (-1)^{q+r_1+r^1}$ for a given pair c, c' is given by the linking number difference q and the framing difference $r_1 + r^1$:

$$\begin{aligned} q(c,c') &= \text{Lnk}(W_1(b'), b) + \text{Lnk}(W^1(b), b') \\ r^1(c,c') &= \text{Frm}(a' + W^1(b)) + \text{Frm}(a') + \text{Frm}(W^1(b)) \\ r_1(c,c') &= \text{Frm}(a + W_1(b')) + \text{Frm}(a) + \text{Frm}(W_1(b')) \end{aligned} \tag{36}$$

We will show below that each of q, r^1, r_1 is computed by a shallow quantum circuit. The circuit for q will require global orientation of the 4-space. That is, we construct a shallow quantum circuit Q such that

$$Q\eta^f(c,c')\,|c,c'\rangle = \eta^b(c,c')\,|c,c'\rangle \tag{37}$$

for all c, c'. Then,

$$|\text{2FcBl}\rangle = \underbrace{\sum_{c,c'} |\omega(c,c')\rangle\,\langle c,c'|\,Q}_{\Omega}\,|\text{2FcFl}\rangle\,. \tag{38}$$

Here, Ω can be realized by a shallow quantum circuit. One can apply control-$\sqrt{X}$ or control-$\sqrt{X}^\dagger$ from each 2-cell to each of its boundary 1-cells, depending on the local orientation differences. The displayed expression for Ω does not determine the circuit; we are only specifying the action of Ω on 2-cycles of trivial (co)homology.

After showing that each of q, r^1, r_1 is computed by a shallow quantum circuit in the subsections below, we will have completed the construction of a disentangling circuit for two FcFl states on any *orientable* 4-space with a cellulation which refines to a triangulation.

1. *Circuit for $q(c,c')$*

We claim that, *if the ambient 4-space is oriented*,

$$\tfrac{1}{2}\,\text{Int}_4(b,b') = \text{Lnk}(W_1(b'),b) + \text{Lnk}(W^1(b),b') \tag{39}$$

where Int_4 is the mod 4 intersection number which we will define shortly. It is well defined to take the half because $\text{Int}_4(b,b') = 0 \bmod 2$ if b is null-homologous. $\text{Int}_4(b,b')$ is defined by regarding b' as a 2-cochain over $\mathbb{Z}_4$ and evaluating it at b as a 2-chain over $\mathbb{Z}_4$. Geometrically, this can be described as follows. Each 2-cell carries a preferred local orientation, so at an intersection point we have two mutually transverse 2-dimensional ordered bases. Concatenating these, starting with the primary cell's basis, gives an ordered local basis of the full 4-dimensional space, which can be compared to the ambient orientation. This gives a sign and the mod 4 intersection number is the mod 4 sum of all the signs.

Consider the "error" $\Delta(b,b') = \frac{1}{2}\,\text{Int}_4(b,b') + \text{Lnk}(W_1(b'),b) + \text{Lnk}(W^1(b),b')$ in our claim. We observe that $\frac{1}{2}\,\text{Int}_4(b,b') + \text{Lnk}(W^1(b),b')$ is independent of how we choose preferred local orientations for each 2-cell of b. Indeed, if we change the local orientation for a cell nonintersecting

with b', then the intersection number is unaffected and there is no new linking. If we change the local orientation for an intersecting cell, then Int_4 changes by $\pm 2 = 2 \bmod 4$ and the newly created piece of W^1 links with b'. Thus both $\frac{1}{2}\mathrm{Int}_4(b, b')$ and $\mathrm{Lnk}(W^1(b), b')$ change by 1, whereas $\mathrm{Lnk}(W_1(b'), b)$ is unaffected, so $\Delta(b, b')$ does not change. Therefore, $\Delta(b, b')$ is independent of local orientations of 2-cells.

We further claim that $\Delta(b, b')$ is constant even if deform b or b' by the boundary of a 3-cell. Suppose we deform b by ∂e^3. Any deformation on b' will be handled by symmetry. Since our Δ is independent of local orientations of 2-cells, we may choose them so that $W^1(\partial e^3) = \frac{1}{2}\partial\xi\partial e^3 = 0$. Recall that ξ is a lifting from 2-chains over $\mathbb{Z}_2$ to 2-chains over $\mathbb{Z}_4$ determined by local orientations. This means that e^3 can be given an orientation, *i.e.*, a choice of a mod 4 chain denoted by ξe^3, such that

$$\partial\xi e^3 = \xi\partial e^3. \tag{40}$$

Here, the boundary operator ∂ on the left-hand side is mod 4 and that on the right-hand side is mod 2. Then, by Eq. (30) we have, with a mod 2 chain $u = b \cap \partial e^3$,

$$\begin{aligned} W^1(b + \partial e^3) - W^1(b) &= \tfrac{1}{2}\partial\left(\xi(\partial e^3 + u) - \xi u\right) \\ &= \tfrac{1}{2}\partial\left(\xi\partial e^3 - \xi u - \xi u\right) \\ &= \tfrac{1}{2}\partial(-2\xi u) = \partial u, \end{aligned} \tag{41}$$

where in the second line we used

$$\xi(f^2 + g^2) = \xi f^2 - \xi g^2 \qquad \text{for any } f^2 \supseteq g^2. \tag{42}$$

By standard equivalence between linking and intersection, the changes in the linking numbers are

$$\begin{aligned} \mathrm{Lnk}(W_1(b'), b + \partial e^3) - \mathrm{Lnk}(W_1(b'), b) &= \mathrm{Lnk}(W_1(b'), \partial e^3) = \mathrm{Int}_2(W_1(b'), e^3), \\ \mathrm{Lnk}(W^1(b + \partial e^3), b') - \mathrm{Lnk}(W^1(b), b') &= \mathrm{Lnk}(\partial u, b') = \mathrm{Int}_2(u, b'). \end{aligned} \tag{43}$$

To calculate the change in Int_4, let us be more explicit about the lifting of coefficients from $\mathbb{Z}_2$

to $\mathbb{Z}_4$ by writing $\mathrm{Int}_4(b,b') = \mathrm{Int}_4[\xi b, \xi b']$. Then, with the mod 2 chain $u = b \cap \partial e^3$,

$$
\begin{aligned}
&\mathrm{Int}_4[\xi(b+\partial e^3), \xi b'] - \mathrm{Int}_4[\xi b, \xi b'] \mod 4 \\
&= \mathrm{Int}_4[\xi((b+u)+(\partial e^3+u)), \xi b') - \mathrm{Int}_4[\xi b, \xi b'] && \text{decomposition of } b+\partial e^3 \\
&= \mathrm{Int}_4[\xi(b+u)+\xi(\partial e^3+u), \xi b'] - \mathrm{Int}_4[\xi b, \xi b'] && \text{Eq. (29)} \\
&= \mathrm{Int}_4[\xi b - \xi u, \xi b'] + \mathrm{Int}_4[\xi \partial e^3 - \xi u, \xi b'] - \mathrm{Int}_4[\xi b, \xi b'] && \text{Eq. (42)} \\
&= -2\,\mathrm{Int}_4[\xi u, \xi b'] + \mathrm{Int}_4[\partial \xi e^3, \xi b'] && \text{Eq. (40)} \\
&= -2\,\mathrm{Int}_4[\xi u, \xi b'] + \mathrm{Int}_4[\xi e^3, \partial \xi b'] && \mathrm{Int} = \text{cochain evaluation} \qquad (44) \\
&= -2\,\mathrm{Int}_4[\xi u, \xi b'] + \mathrm{Int}_4[\xi e^3, 2W_1(b')] && \text{by def. of } W_1 \\
&= 2\,\mathrm{Int}_2(u, b') + 2\,\mathrm{Int}_2(e^3, W_1(b')) && (2x) \bmod 4 = 2(x \bmod 2)
\end{aligned}
$$

where the reader might want to recall that we write W_1 as it is a chain of secondary 1-cells, which are dual of primary 3-cells, so it can be regarded as a 3-cocycle. Therefore, Δ is constant under changes of the 2-cycles by the boundary of any 3-chain.

Finally, since the cycles that enter the FcFl ground state are all (co)homologically trivial, it suffices to evaluate $\Delta(b,b')$ for zero cycles b, b', for which it is obvious that $\Delta(b,b') = 0$. We have proved Eq. (39) and

$$q(c,c') = \tfrac{1}{2}\,\mathrm{Int}_4(b,b'). \tag{45}$$

The mod 4 intersection number is computable by a depth 1 quantum circuit. Indeed, the local preferred orientations of 2-cells f^2 and its dual f_2 determines the contribution ± 1 to the intersection number when they are both occupied by 2-cycles b, b'. This information is fixed once and for all, and the quantum gate $\mathrm{diag}(1,1,1,\pm i)$ reads this off.

2. *Circuit for r_1 and r^1*

The construction will be the same for primary and secondary cellulations, so we only discuss r^1. By Eq. (E5) and Eq. (E6) we know that $r^1(c,c') = \mathrm{BiFrm}(W^1(b), a')$, which we show can be evaluated by a shallow circuit. In fact, the extension $\overline{\mathrm{BiFrm}}$ can be evaluated by a shallow circuit:

Lemma III.2. *Under the assumption of Proposition E.8, there exists a shallow circuit R such that $R\,|a^1, b^2\rangle = (-1)^{\overline{\mathrm{BiFrm}}(a^1, W^1(b^2))}\,|a^1, b^2\rangle$ for all cycles a^1 and b^2.*

Proof. Remark E.9 says that $\overline{\mathrm{BiFrm}}$ may be evaluated by arbitrarily extending the 2-skeleton $\mathcal{K}$ of our 4-space. We introduce an auxiliary 0-cell ∞ and project it to a generic position "infinitely"

away (which can be specified finitely but we will not) from the image of any other cell. We build a cone over the 2-skeleton $\mathcal{K}$ with an apex ∞. We can obviously extend ϕ over the 2-skeleton of the cone to have ϕ^e, and all the line segments from a 0-cell of $\mathcal{K}$ to ∞ on the projection plane $\mathbb{R}^2$ are almost parallel to each other since ∞ is far away. Since $\mathcal{K}$ is locally finite, the topology of line segments in the vicinity of $\phi(\mathcal{K}) \subset \mathbb{R}^2$ stabilizes as ∞ is pushed to "infinity." This extension fulfills the assumption of Proposition E.8, since all 1-cycles become null-homologous.

Now, the cycle a is the boundary of the cone $\tilde{a}$, a 2-chain, over a itself. For any 2-cell $f \in \tilde{a}$ in the cone, the Pauli factors of the operator $L_{Fc}(f)$ on the 1-cells of $\mathcal{K}$ (those that do not touch ∞) are locally determined. In particular the thorns of $S(a)$ are locally determined. Since every coefficient of cells in $W^1(b^2)$ is locally determined, we find a desired shallow circuit R. □

IV. BOUNDARY TOPOLOGICAL ORDER

For any of our models BcBl, FcBl, and FcFl, we have shown that the bulk does not admit any deconfined topological excitation since the stack of two copies of a model can be completely disentangled. We have also shown that the unique ground state on a closed 4-manifold is a superposition of null-homologous cycles. However, in the presence of boundaries we will show that the boundary hosts nontrivial topological excitations and that the ground state becomes degenerate whenever the 3-dimensional boundary has nontrivial first, and hence second, homology.

Before proceeding with the detailed construction, let us make some comments. First, the boundary Hamiltonians discussed below are not unique, and in general we expect the various models to admit many different boundary topological orders. However, we will show that, in particular, the FcFl model admits an anomalous boundary topological order, *i.e.*, one which is not possible strictly in $3+1$ dimensions (modulo an assumption that there are no nontrivial $3+1$-dimensional fermionic invertible phases). This fact by itself is sufficient to show that the FcFl model defines a nontrivial invertible phase of matter. For if it did not, we could find a shallow depth disentangler for it, and by truncating this disentangler appropriately we could then disentangle the $3+1$-dimensional surface from the $4+1$-dimensional bulk and obtain a standalone $3+1$-dimensional model of the anomalous topological order, a contradiction.

Imagine we have a closed 4-space which we will cut out to open up a 3-dimensional boundary. We choose a connected set of *primary* 4-cells, and declare that the qubits contained in (the closure of) the chosen 4-cells are our system qubits. This choice of a connected subset of qubits leaves a complementary set of *external* qubits. By construction, the system boundary is a closed 3-manifold

and inherits the cell structure of the 4-space. Namely, the system boundary is filled with primary 3-cells, each of which is the intersection of a system primary 4-cell and an external primary 4-cell.

Given our Hamiltonian, be it BcBl, FcBl, or FcFl, on a closed 4-space, we define a new Hamiltonian as

$$H = -\sum_j h_j \quad \longrightarrow \quad H_{\mathrm{w/bd}} = -\sum_j \langle 0^{\otimes \mathrm{external}}|\, h_j \,|0^{\otimes \mathrm{external}}\rangle\,. \tag{46}$$

In other words, if a term h_j contains a Z factor at an external qubit, we replace it with a number 1, but if a term h_j contains an X factor at an external qubit, we drop the term. One may regard the prescription as introducing (infinitely) strong single-qubit terms $-Z$ on external qubits. The Hamiltonian with boundary is still commuting. Any surviving pair of two terms that are modified by the qubitwise sandwiching must have had diagonal factors on external qubits, and the commutativity of the modified terms is inherited from that of the unmodified terms.

The original Hamiltonian H contains (in the algebra generated by its terms) cycle-enforcing terms $\pi(a^1) = \prod_{f^2:\partial f^2 \ni a^1} Z(f^2)$ and $\pi(v_0) = \prod_{e_1:\partial e_1 \ni v_0} Z(e_1)$. These terms survive under the qubitwise sandwiching to become cycle-enforcing terms on the nonexternal, system qubits. It is possible that all but one factor of a closed-chain enforcing term may be sandwiched out to produce a single-qubit Z, in which case, effectively, the set of external qubits is enlarged. Indeed, for any secondary 1-cell e_1 (a primary 3-cell) at the system boundary, there exists an external primary 4-cell v_0 such that $e_1 \in \delta v_0$. If 4-cell v_0 had no other system qubit on its boundary, the cycle-enforcing term $\pi(v_0)$ becomes a single-qubit Z under the sandwiching, and the qubit at e_1 is disentangled out. If we consider a hypercubic lattice with a flat cubic lattice boundary, then every secondary qubit within the boundary is disentangled out.

A. Gapped boundary

Our Hamiltonian with boundary is gapped in a strong sense: if O is an operator on a not-too-big ball-like subsystem (excluding external qubits) that commutes with every term of $H_{\mathrm{w/bd}}$, then O belongs to the complex algebra generated by terms of $H_{\mathrm{w/bd}}$ near the support of O. Colloquially, this is to say that any commuting term that can be added to $H_{\mathrm{w/bd}}$ is already in $H_{\mathrm{w/bd}}$. Suppose that K is any operator supported on a "small" ball-like region, not necessarily commuting with any Hamiltonian term. We multiply K by t_j, the projector onto $+1$-eigenspace of h_j of $H_{\mathrm{w/bd}}$, on the left and right for all j such that the support of t_j overlaps with that of K, to obtain an

operator

$$K' = \underbrace{(\prod_j t_j)}_{\Pi(K)} K \underbrace{(\prod_j t_j)}_{\Pi(K)}. \tag{47}$$

The support of K' is only slightly larger than that of K, and thus still supported on a ball-like region, but K' now commutes with all Hamiltonian terms. Our claim implies that K' is a $\mathbb{C}$-linear combination of products of Hamiltonian terms near the support of K. This is the local topological order condition, that is used to prove the gap stability against perturbations [29]. It demands that if K is any observable on a small box, then the expectation value of K on any true, global ground state be the same as that on any state of form $\Pi(K)\,|\psi\rangle$ where $|\psi\rangle$ is arbitrary, bearing no reference to $H_{\mathrm{w/bd}}$. Indeed, since K' belongs to the algebra generated by operators, each of which takes a definite eigenvalue on $\Pi(K)$, we see that

$$\Pi(K) \cdot K \cdot \Pi(K) = \Pi(K) \cdot K' \cdot \Pi(K) = c(K)\Pi(K) \tag{48}$$

for some scalar $c(K)$.

We have not specified how "small" or "not-too-big" subsystems have to be where O or K is supported. In fact, we only need that the ℓ-neighborhood of a region we consider contains only null-homologous (co)cycles, where ℓ is the maximum diameter of the support of a Hamiltonian term. Since Eq. (48) is the Knill-Laflamme criterion for error correction [30], the ground state subspace of $H_{\mathrm{w/bd}}$ obeys the "homogeneous" topological order condition [31], a geometry-free form of an error correction property.

Let us prove the claim. Given an operator O that commutes with every Hamiltonian term, we expand O in the Pauli operator basis,

$$\begin{aligned} O &= \sum_j P_j \\ 0 = [O, \pi] &= \sum_j [P_j, \pi] \end{aligned} \tag{49}$$

where P_j are distinct tensor products of Pauli matrices, and π is either $\pi(v_0)$ or $\pi(a^1)$. Each commutator, if nonvanishing, is a Pauli operator $2P_j\pi$, and is thus different from any other appearing in the sum. Since Pauli operators are an orthonormal operator basis (under Hilbert–Schmidt inner product), we see that every commutator must vanish. We can write each P_j as a product of some Pauli Z operators and some Pauli X operators up to a phase. The X part of P_j must form a cycle because P_j commutes with $\pi(a^1)$ and $\pi(v_0)$ for all a^1 and v_0. Being supported on a ball-like region, the cycle must be null-homologous. But the fluctuation operators of $H_{\mathrm{w/bd}}$ explore

all null-homologous cycles within the ball-like region, and hence, the X part can be canceled by multiplying P_j by fluctuation operators. This multiplication results in P_j', a diagonal operator in Z basis, which may have a larger support than P_j, but not more than the size of the fluctuation operators.

Our next goal is to show that if D is any diagonal operator on a ball-like region which commutes with every Hamiltonian term, then it belongs to the algebra generated by Hamiltonian terms. A diagonal operator D commutes with a fluctuation operator if and only if D commutes with the X part of fluctuation operators. Now, because Z is diagonal, each Pauli summand D_k of $D = \sum_k D_k$ in the Pauli basis expansion is a product of Z's. By the same argument as before, each such D_k must individually commute with the X part of fluctuation operators. Thus, each such D_k may be viewed as a dual chain that is closed relative to the spatial boundary. Because the support of D_k is topologically trivial, this dual chain is exact. However, the terms $\pi(v_0)$ and $\pi(a^1)$ are precisely the generator of null (co)homology cycles, relative to the spatial boundary, which shows that each D_k, and hence D, belongs to the algebra generated by π's.

Therefore, each P_j' belongs to the algebra of π's, so P_j belongs to the algebra of π's and the fluctuation terms, and, finally, O belongs to the algebra of the Hamiltonian terms. Note that a similar prescription introduces a gapped boundary to the $2+1$d and any higher dimensional toric code.

B. Operators for topological excitations

1-dimensional operators

Consider a secondary 1-cycle a_1 near the system boundary which may be homologically nontrivial. Since we have a direction normal to the surface, there is a homologous shift a_1' of a_1 into the exterior of the system, also near the surface. So, a_1 is entirely in the interior of the system, and a_1' is entirely in the exterior.

a. For bosonic charges. The null-homologous cycle $a_1 + a_1'$ is the boundary of some 2-chain b_2, and the product $L_{Bc}(b_2)$ of L_{Bc} over b_2 is an X string along $a_1 + a_1'$ times Z factors on the primary 2-cells, each of which is Poincaré dual to cells of b_2. Let us strip off the X factors of $L_{Bc}(b_2)$ along a_1' and apply the qubitwise sandwiching on external qubits, to obtain a system operator $S_{Bc}(a_1)$. This string operator $S_{Bc}(a_1)$ commutes with every term of $H^{WW}_{BcBl\ \mathrm{w/bd}}$; the stripped off part from $L_{Bc}(b_2)$ is not seen by terms of $H^{WW}_{BcBl\ \mathrm{w/bd}}$. When a_1 is homologically nontrivial, $S_{Bc}(a_1)$ is not generated

by terms of $H^{WW}_{BcBl\ \mathrm{w/bd}}$.

If we truncate a long $S_{Bc}(a_1)$, then at the end point the cycle-enforcing term is violated. This excitation is topological and deconfined as witnessed by the following 3-dimensional operator. In the 4-space in which our system lives, there is a primary 3-sphere r^3 that encloses v_0 but not v_0'. The product $M(r^3)$ of secondary qubit operator Z over r^3 is a product of cycle-enforcing terms $\pi(v_0)$ over the 4-ball bounded by r^3, and $M(r^3)$ anticommutes with $S_{Bc}(a_1)$. The sphere r^3 is in fact a 3-dimensional hemisphere if we neglect the external qubits, where the equatorial boundary is a 2-sphere near the system boundary. Since there is no restriction on the geometry (size) of these operators, the excitation at v_0 is topological. We will later find a strictly 2-dimensional operator that does not penetrate into the 4-dimensional bulk.

Our construction of $S_{Bc}(a_1)$ depends on a_1' and b_2, but the resulting operator is unique up to $\pi(a^1)$'s within the system if the cellulation is so refined that the distance between a_1 and its shift a_1' is smaller than the injectivity radius r_{inj} of the system boundary manifold. Indeed, the difference ΔS_{Bc} resulting from these arbitrary choices, before we take the qubitwise sandwiching, is a product of primary-qubit Z which as a whole commutes with every term of H^{WW}_{BcBl}. We know that ΔS_{Bc} must be a product of terms of H^{WW}_{BcBl} because we have a Hamiltonian-disentangling circuit Eq. (19). The group of all primary-qubit diagonal Pauli operators in the group of terms of H^{WW}_{BcBl} consists of $L_{Bc}(c_2)$ for secondary 2-cycles c_2. Since $\pi(a^1)$'s viewed as secondary 2-chains generate all null-homologous secondary 2-cycles, it remains to show that if $\Delta S_{Bc} = L_{Bc}(c_2)$, then c_2 is null-homologous. To see this, just note that the cycle c_2 is contained in the r_{inj}-neighborhood of $\mathrm{Supp}(a_1)$, which deformation-retracts to $\mathrm{Supp}(a_1)$, that contains no nontrivial 2-cycle, so c_2 is null-homologous.

The quasiparticle at the end of truncated string operator is a boson as seen by a T-junction exchange process in Eq. (1); any segment of S_{Bc} commutes with any other.

b. For fermionic charges. For FcBl, a similar string operator S_{Fc} near the boundary exists. It is likewise a product of X along a secondary cycle a_1, but is additionally decorated with Z thorns. The construction is the same as that for string operators of $H^{WW}_{BcBl\ \mathrm{w/bd}}$; take a pair of a secondary 1-cycle within the system and its shift to the exterior, pick a secondary 2-chain bounded by the pair to multiply L_{Fc} over, and read off the system part. We have an extra property that the secondary part of S_{Fc} localizes along a_1 by Lemma E.6. The Z factors on primary 2-cells that were responsible for linking parities do not localize, but they are unique up to $\pi(a^1)$'s as we have shown in the construction of S_{Bc} above. The string operator S_{Fc} that is first localized by Lemma E.6 and then truncated by the system boundary, is equivalent to one that is first truncated and then

localized. These two procedures commute because the cycle-enforcing terms $\pi(v_0)$ that localize the string before the system boundary truncation can be first truncated and then localize the string operator. The same string operators are applicable for FcFl; we have used the worldline fluctuation operators only which are the same for both FcBl and FcFl.

The string operators of FcBl indeed transport fermions as one can easily check by a T-junction process with Eq. (1). A concrete expression for string operators on a hypercubic lattice is shown in Fig. E.1(c). Note that the fourth direction (into the 4-dimensional bulk) is irrelevant for the evaluation of exchange statistics of boundary quasiparticles, because any potential Z thorns in the fourth direction will commute with all other operators, as there are no X's in the fourth direction. Also, factors of Z on primary qubits are not relevant for this computation.

2-dimensional operators

Let b^2 be a primary 2-cycle that sits within the system boundary, which may represent a nontrivial homology of the system boundary. Similarly to the string operator construction above, we consider a translate $(b^2)'$ of b^2 along the direction normal to the system boundary towards the exterior. We know that $b^2 + (b^2)' = \partial c^3$ for some primary 3-chain c^3. Before the qubitwise sandwiching, the product of T_{Bl} over c^3 is a Pauli operator with X factors on $b^2 \sqcup (b^2)'$ and some Z factors on secondary 1-cells in between. The product of T_{Fl} over c^3 is more complicated, but only in the diagonal part. That is, for $T = T_{Bl}$ and $T = T_{Fl}$ we have

$$\prod_{a_1 \in c^3} T(a_1) = \left(\prod_{a_1 \in c^3} Z(a_1) \right) \cdot D \cdot \left(\prod_{f^2 \in b^2} X(f^2) \right) \cdot \left(\prod_{g^2 \in (b^2)'} X(g^2) \right) \tag{50}$$

where D is some operator on the primary qubits which is diagonal in Z basis. In the case of T_{Bl}, D is the identity. This follows from the fact that each term $T(a_1)$ maps a configuration of primary 2-cycle to another with some sign, where the change in the cycle is the boundary of primary 3-cell, Poincaré dual to a_1. The change in cycles is expressed by the X factors in Eq. (50). To define membrane operators, we strip off the X factors on $(b^2)'$ and apply the qubitwise sandwiching on the remaining operator:

$$M(b^2) = \langle 0^{\otimes \text{external}}| \left(\prod_{e_1 \in c^3} Z(e_1) \right) D \, |0^{\otimes \text{external}}\rangle \left(\prod_{f^2 \in b^2} X(f^2) \right). \tag{51}$$

Starting with T_{Bl}, we obtain $M_{Bl}(b^2)$ that consists of X factors on b^2 that is within the system boundary and some Z factors on secondary 1-cells, also within the system boundary. With T_{Fl},

the operator $M_{Fl}(b^2)$ differs from $M_{Bl}(b^2)$ only in the diagonal operator on primary qubits. For either case, Bl or Fl, on the hypercubic lattice 4-space and the cubic lattice boundary, no Z factors on secondary 1-cells survive after qubitwise sandwiching.

The membrane operator $M(b^2)$ is 2-dimensional and commutes with every Hamiltonian term of $H_{\mathrm{w/bd}}$. The commutativity is obvious for the worldsheet fluctuation operators in these Hamiltonians. The part of $M_{Bl}(b^2)$ seen by the worldline fluctuation operators is unchanged under stripping and qubitwise sandwiching, so the commutativity is retained. The constructed operator $M(b^2)$ complements the 1-dimensional string operators we constructed above; the anticommutation relations come from the primary-qubit X factors of $M(b^2)$ and Z factors of $S(a_1)$. In other words, the membrane operators over primary homology 2-cycles and the string operators over secondary homology 1-cycles are a generating set for the algebra acting on the ground state subspace. Here, the homologies are those of the boundary 3-manifold.

C. Loop self-statistics $\mu = -1$ on boundary of the FcFl model

Any truncation of M_{Bl} satisfies the requirements of Section II B to compute the loop self-statistics μ, since M_{Bl} is a tensor product of single-qubit unitary operators. It is obvious that $\mu_{Bl} = +1$, since X factors are on primary cells and Z factors are on the secondary cells, so any truncated versions commute with one another.

To compute the loop self-statistics μ_{Fl} with M_{Fl}, we resort to a representation of M_{Fl} inside the boundary theory of a model Fc×FcBl, which will be shown to be isomorphic by a shallow quantum circuit to Fc×FcFl. Here, Fc is a topologically ordered model in $4+1$d defined in Remark E.11 that has a fermionic charge. We will employ the circuits developed in Section III D.

a. $Fc\times FcBl \cong Fc\times FcFl$. Recall that the amplitude $\langle b^2, a_1|\mathrm{FcFl}\rangle$ is given by the product of three signs: the linking between a_1 and b^2, the frame parity of a_1, and the frame parity of $W^1(b^2)$. On the other hand, the amplitude $\langle b^2, a_1|\mathrm{FcBl}\rangle$ is given by the product of two signs: the linking between a_1 and b^2 and the frame parity of a_1. For a fixed primary homology 1-cycle h^1, the amplitude $\langle x^1|\mathrm{Fc}; [h^1]\rangle$ with $x^1 \in [h^1]$ is given by the frame parity $\overline{\mathrm{Frm}}(x^1)$ as constructed in Proposition E.10. Given a projection from our 2-skeleton down to $\mathbb{R}^2$, $\overline{\mathrm{Frm}}$ may not be unique, precisely when the degree-1 homology of the 2-skeleton is nonzero. We choose an arbitrary $\overline{\mathrm{Frm}}$

extending Frm. As in Section III D we consider

$$\langle x^1, b^2, a_1 \mid \mathrm{Fc}\times\mathrm{FcFl}\rangle = (-1)^r \langle x^1 + W^1(b^2), b^2, a_1 \mid \mathrm{Fc}\times\mathrm{FcBl}\rangle \qquad (52)$$
$$\text{where} \quad r = \overline{\mathrm{BiFrm}}(x^1, W^1(b^2)).$$

We know from Lemma III.2 that $r : (y^1, b^2) \mapsto \overline{\mathrm{BiFrm}}(y^1, W^1(b^2))$ where y^1 is any primary 1-cycle and b^2 is any primary 2-cycle, is computable by a shallow circuit R. We also know that

$$\Omega|_{\mathrm{cycles}} = \sum_{x^1\in[h^1], b^2\in[0], a_1\in[0]} |x^1 + W^1(b^2), b^2, a_1\rangle \langle x^1, b^2, a_1| \qquad (53)$$

is implemented by a shallow circuit, where Ω does not change the homology class of the primary 1-cycle since W^1 is always null-homologous. Therefore, we have a shallow-circuit equivalence

$$\Omega R\,|\mathrm{Fc}\times\mathrm{FcFl}\rangle = |\mathrm{Fc}\times\mathrm{FcBl}\rangle \qquad (54)$$

regardless of what superselection sector the ground state $|Fc\rangle$ is in.

In the presence of spatial boundary, the circuit R has gates that straddle system qubits and external qubits; there could be an external primary 2-cell that meets a system primary 2-cell along a system 1-cell, and some gate of R needs to access both 2-cells to infer W^1. However, being a diagonal circuit, R becomes a unitary circuit under the qubitwise sandwiching, which is nothing but restriction of domain of the unitary gates, and hence the sandwiched circuit locally computes $\overline{\mathrm{BiFrm}}$. The layer of Ω does not have any gates that straddle system qubits and external qubits, and Ω restricts to a shallow circuit under the qubitwise sandwiching. Hence, the shallow-circuit equivalence between Fc×FcFl and Fc×FcBl continues to hold in the presence of spatial boundary.

b. Representation of T_{Fl} in the bulk. We will find a representation of open membrane operators of FcFl via operators in Fc×FcBl. (The worldsheet fluctuation terms T_{Fl} of H^{WW}_{FcFl} were rather implicit, which is why it is easier to work with H^{WW}_{FcBl}.) Before we consider open membrane operators of FcFl, we first find a representation of the worldsheet fluctuation operator T_{Fl}. Then, we will be able to read off an open membrane operator representation.

Suppose there is a local orientation for each primary 2-cell. Consider an operator

$$T'(g^2) \equiv X(g^2)S(W^1(g^2)) \qquad (55)$$

associated with each primary 2-cycle g^2, where $X(g^2) \equiv \prod_{f^2\in g^2} X(f^2)$ is a product of Pauli X over all 2-cells of g^2. Such an operator gives

$$\begin{aligned} T'(g^2)\,|x^1, b^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1)} &= |x^1 + W^1(g^2), b^2 + g^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1)+\mathrm{Frm}(W^1(g^1))+\overline{\mathrm{BiFrm}}(x^1, W^1(g^2))} \\ &= |x^1 + W^1(g^2), b^2 + g^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(g^1))}. \end{aligned} \qquad (56)$$

Then, we will have

$$\begin{aligned}
&R\Omega T'(g^2)\Omega R\,|x^1,b^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1)+\mathrm{Frm}(W^1(b^2))}\\
=&R\Omega T'(g^2)\Omega\,|x^1,b^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2))}\\
=&R\Omega T'(g^2)\,|x^1+W^1(b^2),b^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2))}\\
=&R\Omega\,|x^1+W^1(b^2)+W^1(g^2),b^2+g^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2)+W^1(g^2))}\\
=&R\,|x^1+W^1(b^2)+W^1(g^2)+W^1(b^2+g^2),b^2+g^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2)+W^1(g^2))}\\
=&\,|x^1+W^1(b^2)+W^1(g^2)+W^1(b^2+g^2),b^2+g^2\rangle\\
&\qquad\cdot(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2)+W^1(g^2))+\overline{\mathrm{BiFrm}}(x^1+W^1(b^2)+W^1(g^2),W^1(b^2+g^2))}\\
=&\,|x^1+W^1(b^2)+W^1(g^2)+W^1(b^2+g^2),b^2+g^2\rangle\\
&\qquad\cdot(-1)^{\overline{\mathrm{Frm}}(x^1+W^1(b^2)+W^1(g^2)+W^1(b^2+g^2))+\overline{\mathrm{Frm}}(W^1(b^2+g^2))}
\end{aligned}\tag{57}$$

where in the second from the last equality we use $\overline{\mathrm{BiFrm}}(u^1,u^1)=0$ for any 1-cycle u^1. That is,

$$\begin{aligned}
&(\Omega R)^\dagger T'(g^2)(\Omega R)\,|x^1,b^2\rangle\,(-1)^{\overline{\mathrm{Frm}}(x^1)+\mathrm{Frm}(W^1(b^2))}\\
&\qquad=|(x^1)',b^2+g^2\rangle\,(-1)^{\overline{\mathrm{Frm}}((x^1)')+\mathrm{Frm}(W^1(b^2+g^2))}.
\end{aligned}\tag{58}$$

where $(x^1)'=x^1+W^1(b^2)+W^1(g^2)+W^1(b^2+g^2)$. Since $(x^1)'$ is always homologous to x^1, if we sum over all 1-cycles in the homology class $[x^1]$, we have

$$\begin{aligned}
&(\Omega R)^\dagger T'(\partial c^3)Z(c^3)(\Omega R)\,|\mathrm{Fc}\rangle\,|b^2,a_1\rangle\,(-1)^{\mathrm{Frm}(W^1(b^2))}\\
&\qquad=|\mathrm{Fc}\rangle\,|b^2+\partial c^3,a_1\rangle\,(-1)^{\mathrm{Frm}(W^1(b^2+\partial c^3))+\mathrm{Int}(c^3,a_1)}.
\end{aligned}\tag{59}$$

This is the correct relation of amplitudes in the ground state $|\mathrm{FcFl}\rangle$. Therefore, $T'(\partial c^3)Z(c^3)$ is a correct representation of the worldsheet fluctuation operator T_{Fl} acting on the state of form $|\mathrm{Fc}\rangle\,|b^2,a_1\rangle$ where b^2,a_1 are cycles. Note that it is important that the first factor $|\mathrm{Fc}\rangle$ is the ground state of H_{Fc}.

We wish to express $T'(g^2)$ as a product of operators O, each of which is associated with a 2-cell:

$$T'(g^2)\equiv X(g^2)S(W^1(g^2))=\pm\prod_{f^2\in g^2}O(f^2)\tag{60}$$

where we allow for the product to be ambiguous up to a sign. The challenge is to find operators $O(f^2)$ *and* a local orientation of all 2-cells f^2 such that the product of $O(f^2)$ over an arbitrary null-homologous 2-cycle g^2 gives $S(W^1(g^2))$ as in Eq. (60). To this end, we assume that the primary cellulation is a triangulation. We use a set of coefficients $[xyz]=[zyx]\in\mathbb{Z}_2$ where x,y,z are any

distinct primary 0-cells subject to conditions that

$$[xyz] + [yzx] + [zxy] = 0, \qquad [xwy] + [ywz] + [zwx] = 1 \tag{61}$$

for any distinct x, y, z, w. These coefficients are similar to those used in Section II D 2 with $t = 1$, but previously the coefficients $[xyz]$ were defined over five vertices. In Appendix B we show that there exists a solution to Eq. (61) over arbitrarily many vertices, and any two solutions differ by $\Delta[xyz] = [xy] + [yz]$ where $[xy] = [yx] \in \mathbb{Z}_2$ are free parameters. Ignoring secondary qubits for now, we define an open membrane operator $O(xyz)$ for any primary triangle xyz:

$$O(xyz) \equiv X(xyz)S(xy)^{[xyz]+\mathfrak{c}(xy)}S(xz)^{[xzy]+\mathfrak{c}(xz)}S(yz)^{\mathfrak{c}(yz)} \tag{62}$$

where $S(xy) = S(yx)$ are fermion string segments on primary 1-cell xy constructed in Remark E.14, and $\mathfrak{c}(xy) = 1$ if and only if the underlying configuration of primary 2-chain has $\mathbb{Z}_2$-boundary at xy. The form of $O(xyz)$ is virtually the same as $\tilde{M}$ of Section II D 2, but here a triangle xyz ranges over all primary 2-cells, as opposed to the case of $\tilde{M}$, where we only looked at a finite, small number of triangles. Every fermion string segment is a product of Pauli X on xy and some Zs around it and squares to $+1$. Due to $\mathfrak{c}(xy)$ in the exponents, the open membrane operator $O(xyz)$ is not a product of Pauli operators. Note that there is a distinguished vertex x in Eq. (62). We assume that a distinguished vertex is chosen arbitrarily for each triangle.

The open membrane operators commute up to a sign regardless of distinguished vertices of triangles:

$$O(xyz)O(yzw)O(xyz)^{-1}O(yzw)^{-1} = \pm 1. \tag{63}$$

So, the product of O over a primary 2-cycle is well defined up to a sign. Now, we claim that $O(f^2 = xyz) = O(xyz)$ satisfies Eq. (60). We will prove this claim below.

c. A local orientation for triangles. In the definition Eq. (62) of open membrane operators, we have not specified orientation for triangles; the choice of a distinguished vertex of a triangle has nothing to do with the orientation of the triangle. Hence, our goal is to define a local orientation for all triangles such that Eq. (60) is met. There is essentially a unique local orientation for triangles such that the operator $O(g^2)$ that we have defined (up to sign) takes the form of Eq. (60), *i.e.*, has the fermion string operators precisely on the $W^1(g^2)$ line determined by the orientation. If xyz and yzw are the only triangles of a 2-cycle b^2 which touch edge yz, then other open membrane operator factors are "too far" from yz. So, the product $O(xyz)O(yzw) = \pm O(yzw)O(xyz)$ has

to act by the fermion string operator on yz if and only if the orientations of xyz and yzw do not agree along yz. Put differently, each of the orientations of xyz and yzw induces an ordering on the edge yz, and the fermion string operator has to be produced upon multiplication of the two open membrane operators if and only if these induced orderings on bc are the same. Given such a local orientation, Eq. (60) immediately follows. Note that $O(xyz)^2 = \pm S(xy)S(yz)S(zx)$ is the closed fermion string operator along the boundary of triangle xyz.

To find the requisite local orientation, we define

$$\chi(xyz|yzw) = \begin{cases} 1 & \text{if } \langle 0(yz)| \, O(xyz)O(yzw) \, |0(yz)\rangle = 0, \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

where $|0(yz)\rangle$ is an eigenstate of single-qubit Pauli Z on edge yz. The product $O(xyz)O(yzw)$ may have $S(yz)$ factor, in which case the factor gives an X factor on edge yz and $\chi(xyz|yzw) = 1$. Obviously, χ is symmetric, *i.e.*, $\chi(yzw|xyz) = \chi(xyz|yzw)$. We find it convenient to define χ for arbitrary pairs of triples of vertices. Each string operator is some operator acting on an edge, and for the definition of χ we use neither the fact that the string operator transports a fermion, nor that it is a tensor product of Pauli operators. With this abstraction, although we have only defined the open membrane operators $O(xyz)$ for primary 2-cells xyz, we may use the same formula Eq. (62) to define O for an arbitrary triple of vertices, regardless of their distances. This can be regarded as an embedding of our primary 2-skeleton into a very high dimensional simplex. To make sense of Eq. (62) we have to introduce a qubit, unless it is already there, for each triple (triangle) of vertices and for each pair (edge) of vertices.

Then, for any distinct vertices x, y, z, u, v, w, we will show that

$$\begin{aligned} \chi(uvx|uvy) + \chi(uvy|uvz) + \chi(uvz|uvx) &= 1 \qquad \text{(edge)} \\ \chi(wxy|wyz) + \chi(wyz|wzx) + \chi(wzx|wxy) &= 0 \qquad \text{(vertex)} \end{aligned} \tag{65}$$

Note here that the vertices in the argument of χ may be arbitrarily far apart. Once we show this, Lemma C.2 will imply that there exists a local orientation for all triangles such that $S(yz)$ appears as a factor in the product $O(xyz)O(yzw)$ if and only if the orientations of xyz and yzw do not agree along yz. This is what we have sought.

Now, we prove Eq. (65) from the definition Eq. (62) of O using Eq. (61). We put a dot above a vertex symbol, *e.g.*, $\dot{x}yz$, to denote that the dotted vertex x is distinguished for Eq. (62). It is not difficult to enumerate all cases to check Eq. (65),[5] but the vertex condition follows without any

[5] The following enumerates all dotting possibilities for a vertex-sharing triple of triangles without an edge common to all three, up to re-labeling: $\dot{w}xy, \dot{w}yz, \dot{w}zx$ / $\dot{w}xy, \dot{w}yz, w\dot{z}x$ / $\dot{w}xy, w\dot{y}z, wz\dot{x}$ / $\dot{w}xy, w\dot{y}z, w\dot{z}x$ / $\dot{w}xy, wy\dot{z}, w\dot{z}x$ / $w\dot{x}y, w\dot{y}z, w\dot{z}x$ / $wx\dot{y}, w\dot{y}z, w\dot{z}x$.

further calculation. Since relevant edges wx, wy, wz in the vertex condition are distinct, we simply consider the product $O(wxy)O(wyz)O(wzx)O(xyz) = O(\partial(wxyz))$ where $wxyz$ is a 3-simplex. The calculation of Section II D 2 implies that if we multiply O one at a time to cover $\partial(wxyz)$, the loop ℓ along the boundary of any intermediate 2-chain contains an even number of charges, and as ℓ disappears eventually, all charges must disappear, too. This means that the remaining string operator in $O(\partial(wxyz))$ must be $\mathbb{Z}_2$-closed, which implies the vertex condition of Eq. (65). As for the edge condition, the following enumerates all dotting possibilities for edge-sharing triples of triangles:

$$\begin{aligned} &uv\dot{x}, uv\dot{y}, uv\dot{z}; \qquad uv\dot{x}, uv\dot{y}, \dot{u}vz; \qquad uv\dot{x}, \dot{u}vy, \dot{u}vz; \\ &uv\dot{x}, \dot{u}vy, u\dot{v}z; \qquad \dot{u}vx, \dot{u}vy, \dot{u}vz; \qquad \dot{u}vx, \dot{u}vy, u\dot{v}z. \end{aligned} \tag{66}$$

It is straightforward to prove the edge condition[6] by checking all of the above possibilities, using

$$\begin{aligned} &\chi(uv\dot{x}|uv\dot{y}) = 1, && \chi(uv\dot{x}|\dot{u}vy) = 1 + [uvy], \\ &\chi(\dot{u}vx|u\dot{v}y) = 1 + [uvx] + [vuy], && \chi(\dot{u}vx|\dot{u}vy) = 1 + [uvx] + [uvy] = [xvy]. \end{aligned} \tag{67}$$

d. $\mu_{Fl} = -1$. We have found a local orientation for all primary 2-cells and shown that the open membrane operators O multiply over a 2-cycle to leave a string operator along the W^1 of the 2-cycle, implying that $O(b^2)$ of Eq. (62) fulfills Eq. (60). Hence, we have found an explicit representation of M_{Fl} in the presence of an auxiliary ground state $|\mathrm{Fc}\rangle$. If we change the orientation of some triangle, then the open membrane operator at that triangle is multiplied by unconditional closed fermion string operator along its perimeter. Therefore, we have obtained a set of (big) open membrane operators M_{ij} satisfying all requirements of Section II B — simply multiply O over some large triangle.

Then, applying our indicator definition in Section II C, we see that nontrivial commutators among open membrane operators come from nontrivial commutators among decorating open fermion string operators. The overall effect of these nontrivial commutators of fermion string operators is calculated in Fig. II.6, and the result is that $\mu_{Fl} = -1$.

V. DISCUSSION

In this work we defined an invariant of 3+1d fermionic $\mathbb{Z}_2$ gauge theories, the loop self-statistics μ. We demonstrated that μ can be measured using a process that rotates the gauge

[6] This is the step where we really need the second condition ($t = 1$) of Eq. (61); if $[xwy] + [ywz] + [zwx] = 0$, then the edge condition would not hold.

flux loop by an angle of π around its diameter, in a way that is independent of the arbitrary choices made in the process. In other words, all of the nonuniversal geometric phases cancel out. Crucially, the gauge charge has to be a fermion in order for μ to be well defined; if the gauge charge is a boson, μ is only defined up to sign.

For the ordinary 3+1d fermionic toric code, we saw rather easily that $\mu = +1$. Less trivially, we also constructed a 4+1d model H^{WW}_{FcFl} (the FcFl model) which realizes a $\mu = -1$ fermionic $\mathbb{Z}_2$ gauge theory on its boundary. We demonstrated that this FcFl model is invertible by explicitly disentangling two stacked copies of it into a product state. Further, we argued that any stand-alone 3+1d fermionic $\mathbb{Z}_2$ gauge theory must have $\mu = +1$, for otherwise we would be able to construct a nontrivial invertible fermionic phase in 3+1d, which is believed not to exist. Hence, the FcFl model in 4+1d realizes a $\mathbb{Z}_2$ classified nontrivial bosonic phase of matter, stable without any symmetry. We argued that a $\mathbb{Z}_2$ fermionic gauge theory with $\mu = -1$ can be obtained by Higgsing all-fermion QED, so, by a standard argument, the FcFl model can realize all-fermion QED on its boundary.

Let us now discuss the connection of our model to the work of [16] in a little more detail. Our exactly solved models have two types of degrees of freedom: those living on secondary 1-cells, and those living on primary 2-cells. These can be interpreted as $4-1 = 3$-form and $4-2 = 2$-form gauge fields respectively. Ref. [16] refers to these as E and M respectively, and shows in [16, Eq. (7)] that a general $3+1$ dimensional action for these is given by $H^5(K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,3),\mathrm{U}(1))$, which by the Künneth formula is equal to a direct sum of the three tensor factors $H^5(K(\mathbb{Z}_2,3),\mathrm{U}(1)) \simeq \mathbb{Z}_2$, $H^3(K(\mathbb{Z}_2,3),\mathrm{U}(1)) \otimes H^2(K(\mathbb{Z}_2,2),\mathrm{U}(1)) \simeq \mathbb{Z}_2$ and $H^5(K(\mathbb{Z}_2,2),\mathrm{U}(1)) \simeq \mathbb{Z}_2$. These are generated by the classes $(-1)^{\mathrm{Sq}^2 E}, (-1)^{EM}, (-1)^{M\mathrm{Sq}^1 M}$ respectively. The first of these determines whether the charge is a boson (as in the BcBl model) or a fermion (as in the FcBl and FcFl models). The second class just states that the charge braids nontrivially with the magnetic flux loop. Finally, the last class, $(-1)^{M\mathrm{Sq}^1 M}$, is just equal to the loop self-statistics μ.

We would also like to make the following observation, noted by a referee of an earlier draft of this work. Our loop self-statistics invariant can be written in a somewhat more symmetric way by re-ordering certain terms in the process. The result can be checked to still be universal and equal to our invariant. The expression is:

$$\begin{aligned}\mathbf{M} =& M_{24}[M_{13},M_{23}]M_{14}^{-1}\cdot M_{12}[M_{34},M_{13}]M_{24}^{-1}\cdot M_{23}[M_{14},M_{34}]M_{12}^{-1}\\ & M_{13}[M_{24},M_{14}]M_{23}^{-1}\cdot M_{34}[M_{12},M_{24}]M_{13}^{-1}\cdot M_{14}[M_{23},M_{12}]M_{34}^{-1}\end{aligned}$$

There are several avenues for future exploration. First, it would be nice to have a direct

argument that $\mu = +1$ in any stand-alone 3+1d model, *i.e.*, one which does not rely on the classification of invertible fermionic phases in 3+1d. In fact, it is likely sufficient to make a weaker assumption: namely, that any putative invertible fermionic phase in 3+1d admits a gapped 2+1d boundary that is possibly topologically ordered. For, in that case, one can cut a ball out around the 0 vertex in Fig. II.4, large compared to the correlation length. This ensures that the only locations in which the circuit calculating μ is potentially nontrivial are along nonintersecting edges from the outer vertices to the surface of the ball, and by examining the process defining μ, one sees that all of these cancel. Thus, in a sense, the circuit calculating μ localizes onto the 0 vertex. Another potential approach to showing that $\mu = +1$ in any stand-alone 3+1d model is to argue that, if $\mu = -1$ say, then an open membrane operator on a disc has to square to an operator on the disc's boundary which is the worldline operator for a gauge neutral fermion. But, such a particle does not exist in the theory, leading to a contradiction. It remains to make any of these arguments fully rigorous.

Another avenue for further exploration is to construct continuum versions of these models, and in particular to find an action formulation. In fact, thinking about 2d world-sheets embedded in 4d and the various ways one can put Pin structures on these motivated much of our early thinking on this problem. We would expect any such model to be closely related to that discussed in [22]. It would also be nice to make an explicit connection between our FcFl model and the 2-form Chern-Simons field theory description of the 4+1d bulk that realizes all-fermion QED on its boundary [19].

ACKNOWLEDGMENTS

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Appendix A: Details in defining the loop self-statistics

1. Proof of Eq. (5)

Consider two different choices of membrane operators M_{ij} and M'_{ij}. Note that our convention is that $i < j$, which will be notationally important in the following discussion. Let us write $M_{ij} = M_{ij;\text{int}} M_{ij;\text{bdry}}$ where $M_{ij;\text{int}}$ acts in the interior of the plaquette $(ij0)$ and $M_{ij;\text{bdry}}$ acts near the

boundary. Similarly, $M'_{ij} = M'_{ij;\mathrm{int}} M'_{ij;\mathrm{bdry}}$. Then, $M'_{ij;\mathrm{int}} M^{-1}_{ij;\mathrm{int}}$ is a membrane operator which preserves the ground state reduced density matrix inside the plaquette $(ij0)$ and creates some topologically trivial excitations at its boundary. By virtue of being topologically trivial, these can be removed by a shallow circuit G^{-1}_{bdry} acting near this boundary, so the operator

$$F^{ij0}_{ij} \equiv G^{-1}_{\mathrm{bdry}} M'_{ij;\mathrm{int}} M^{-1}_{ij;\mathrm{int}} \tag{A1}$$

has support on the interior of the plaquette $(ij0)$ and maps the ground state to itself up to a $U(1)$ phase. In fact, since all of the configuration states $|\mathfrak{c}\rangle$ look like the ground state in the interior of the plaquette $(ij0)$, all of these $|\mathfrak{c}\rangle$ are eigenstates of F^{ij0}_{ij} with the same eigenvalue. Now,

$$\begin{aligned} F_{ij} &\equiv M'_{ij} M^{-1}_{ij} \\ &= M'_{ij;\mathrm{int}} M'_{ij;\mathrm{bdry}} M^{-1}_{ij;\mathrm{bdry}} M^{-1}_{ij;\mathrm{int}} \\ &= \left(M'_{ij;\mathrm{int}} M'_{ij;\mathrm{bdry}} M^{-1}_{ij;\mathrm{bdry}} (M'_{ij;\mathrm{int}})^{-1} \right) \left(M'_{ij;\mathrm{int}} M^{-1}_{ij;\mathrm{int}} \right) \\ &= \left(M'_{ij;\mathrm{int}} M'_{ij;\mathrm{bdry}} M^{-1}_{ij;\mathrm{bdry}} (M'_{ij;\mathrm{int}})^{-1} G_{\mathrm{bdry}} \right) F^{ij0}_{ij} \end{aligned} \tag{A2}$$

The operator in parentheses of the last line is a shallow circuit supported near the boundary of the plaquette $(ij0)$. Since every $|\mathfrak{c}\rangle$ is an eigenstate of F^{ij0}_{ij} and any state of form $M_{ij}|\mathfrak{c}'\rangle$ is an eigenstate of F_{ij}, we see that any state of form $M_{ij}|\mathfrak{c}'\rangle$ is an eigenstate of this circuit. We can partition this circuit as

$$\left(M'_{ij;\mathrm{int}} M'_{ij;\mathrm{bdry}} M^{-1}_{ij;\mathrm{bdry}} (M'_{ij;\mathrm{int}})^{-1} G_{\mathrm{bdry}} \right) = F^0_{ij} F^i_{ij} F^j_{ij} F^{i0}_{ij} F^{j0}_{ij} F^{ij}_{ij} \tag{A3}$$

where F^{i0}_{ij}, F^{j0}_{ij}, and F^{ij}_{ij} are supported on neighborhoods of the interiors of edges $(i0)$, $(j0)$ and (ij), respectively, and F^0_{ij}, F^i_{ij}, and F^j_{ij} are supported on neighborhoods of the vertices 0, i, and j, respectively, as illustrated in Fig. A.1. Furthermore, we can ensure that, first, the states $|\mathfrak{c}\rangle$ are eigenstates of F^{ij}_{ij} with a common eigenvalue f^{ij}_{ij}, and second, the states $|\mathfrak{c}\rangle$ are eigenstates of F^{i0}_{ij} with eigenvalues $f^{i0}_{ij}(\mathfrak{c})$ that depend on $\mathfrak{c}$ only through the occupation number of the edge $(i0)$, and similarly for F^{j0}_{ij}. Indeed, we can pick any partition we like, and then modify F^{i0}_{ij}, F^{j0}_{ij}, and F^{ij}_{ij} locally near the endpoints of the corresponding intervals as needed to remove local excitations. Note that if there are gauge charges left at any of the endpoints, they will be left behind at *both* endpoints of all of these operators; in this case we can multiply G_{bdry} by a closed gauge string operator surrounding the plaquette to get to a situation where there are no gauge charges left at the endpoints.

Thus, finally we have

$$F_{ij} = F^0_{ij} F^i_{ij} F^j_{ij} F^{i0}_{ij} F^{j0}_{ij} F^{ij}_{ij} F^{ij0}_{ij} \tag{A4}$$

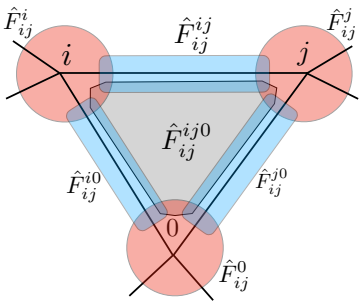


FIG. A.1.

Now, suppose that $|\mathfrak{c}\rangle$ is a configuration state that gets acted on by M_{ij}, or is the result of acting with M_{ij}^{-1}, in the expression for $\mathbf{M}$. Then the above facts taken together imply that $M_{ij}|\mathfrak{c}\rangle$, which is equal to some other configuration state $|\mathfrak{c}'\rangle$ up to a $U(1)$ phase, is an eigenstate of all seven of the operators appearing on the right-hand side of Eq. (A4). The corresponding eigenvalues have the following properties. The eigenvalues f_{ij}^{ij0} and f_{ij}^{ij} are independent of $\mathfrak{c}'$. The eigenvalues $f_{ij}^{i0}(\mathfrak{c}')$ and $f_{ij}^{j0}(\mathfrak{c}')$ depend on $\mathfrak{c}'$ only through the occupation numbers of the edges $(i0)$ and $(j0)$, respectively. Finally, the eigenvalues $f_{ij}^{0}(\mathfrak{c}')$, $f_{ij}^{i}(\mathfrak{c}')$, and $f_{ij}^{j}(\mathfrak{c}')$ depend on $\mathfrak{c}'$ only through how this configuration looks locally near the vertices 0, i, and j, respectively. To see these, we simply note that for any two configurations $\mathfrak{c}', \mathfrak{c}''$ which look locally the same in a ball-like region R (*e.g.*, the neighborhood of a vertex or of the interior of an edge), the corresponding states $|\mathfrak{c}'\rangle$ and $|\mathfrak{c}''\rangle$ can be transformed into each other by a unitary supported away from R; such a unitary will commute with any operator supported on R.

We thus have

$$M'_{ij}|\mathfrak{c}\rangle = F_{ij}^0(\mathfrak{c}')F_{ij}^i(\mathfrak{c}')F_{ij}^j(\mathfrak{c}')F_{ij}^{i0}(\mathfrak{c}')F_{ij}^{j0}(\mathfrak{c}')F_{ij}^{ij}F_{ij}^{ij0}M_{ij}|\mathfrak{c}\rangle \tag{A5}$$

Let us define $u_{ij}^i(\mathfrak{c}) = F_{ij}^i(\mathfrak{c}')F_{ij}^{i0}(\mathfrak{c}')$, $u_{ij}^j(\mathfrak{c}) = F_{ij}^j(\mathfrak{c}')F_{ij}^{j0}(\mathfrak{c}')$, and $u_{ij}^0(\mathfrak{c}) = F_{ij}^0(\mathfrak{c}')F_{ij}^{ij}F_{ij}^{ij0}$. Note that these still depend only on how $\mathfrak{c}$ looks locally near the vertices i, j, and 0, respectively. We finally have:

$$M'_{ij}|\mathfrak{c}\rangle = u_{ij}^i(\mathfrak{c})u_{ij}^j(\mathfrak{c})u_{ij}^0(\mathfrak{c})M_{ij}|\mathfrak{c}\rangle$$

which is Eq. (5).

2. μ is independent of the choice of M_{ij}

Now let us prove that the various phases in the expression for the loop self-statistics in terms of the M'_{ij} cancel in pairs with their complex conjugates. Fix a particular (ij). Then, as shown

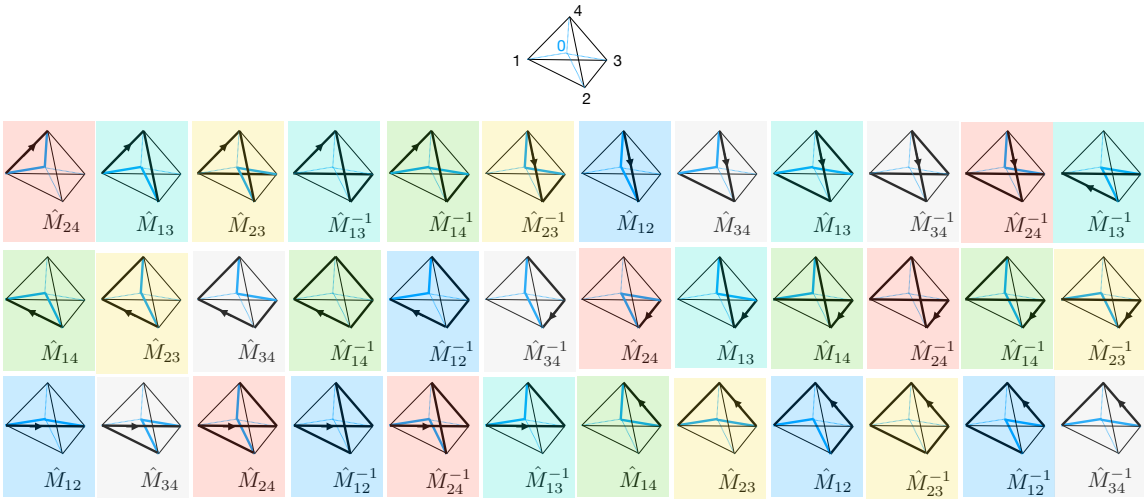


FIG. A.2. There are six different colors, corresponding to the six possible choices of (ij). Each corresponding M_{ij} occurs precisely three times, as does its inverse. We start with the configuration in the top left, and act on each configuration state with the operator directly below it.

in Fig. A.2, each M_{ij} occurs precisely three times in the expression for $\mathbf{M}$, as does its inverse. Then we have three instances for each of $u_{ij}^i, u_{ij}^j, u_{ij}^0, \overline{u_{ij}^i}, \overline{u_{ij}^j}, \overline{u_{ij}^0}$, evaluated on different configurations $\mathfrak{c}$. The fact that these all cancel is illustrated, for the various choices of (ij), in Figs. A.3 to A.5, which are color coded in the same way as Fig. A.2, and show only the portion of the configuration near the relevant $(ij0)$ plaquette. Specifically, consider first u_{ij}^0 and $\overline{u_{ij}^0}$. Then the factors of $u_{ij}^0(\mathfrak{c})$ generated on each of the three configurations at the top row cancel with the corresponding factors of $\overline{u_{ij}^0}$ generated on the configurations directly below. This is because, when one acts with M_{ij}^{-1} on each of the configurations in the bottom row, the local configuration near 0 becomes identical to that of the corresponding configuration directly above; recall that, according to Eq. (6), the $\mathfrak{c}$ that appears in the factor of $\overline{u_{ij}^0}$ generated by the action of $\left(M'_{ij}\right)^{-1}$ is the configuration obtained after $\left(M'_{ij}\right)^{-1}$ has acted. Similarly, the factors of u_{ij}^i and $\overline{u_{ij}^i}$ cancel between the configurations on the top and bottom row connected by a line labeled with i, and likewise for u_{ij}^j and $\overline{u_{ij}^j}$.

Appendix B: Decoration coefficients

Let $V = \{0, 1, 2, \ldots\}$ be a label set of 5 or more elements. Whenever a, b, c are distinct elements of the label set, we denote by $[abc] = [cba]$ a value in $\mathbb{Z}_2 = \{0, 1\}$. Suppose for any distinct

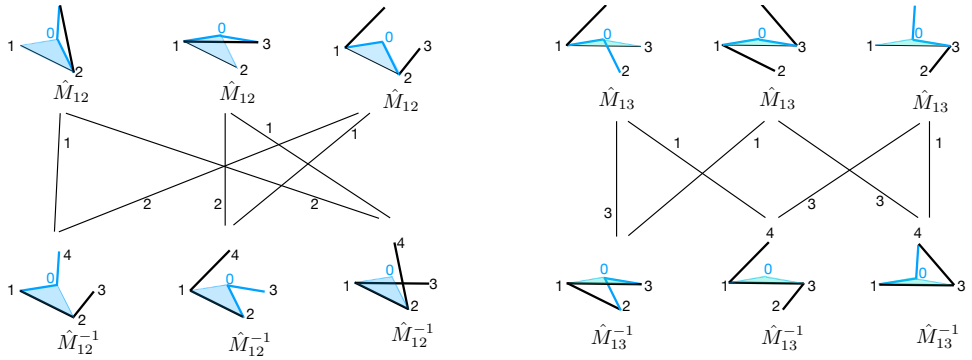


FIG. A.3. Cancellation of phases for $(ij) = (12)$ and (13)

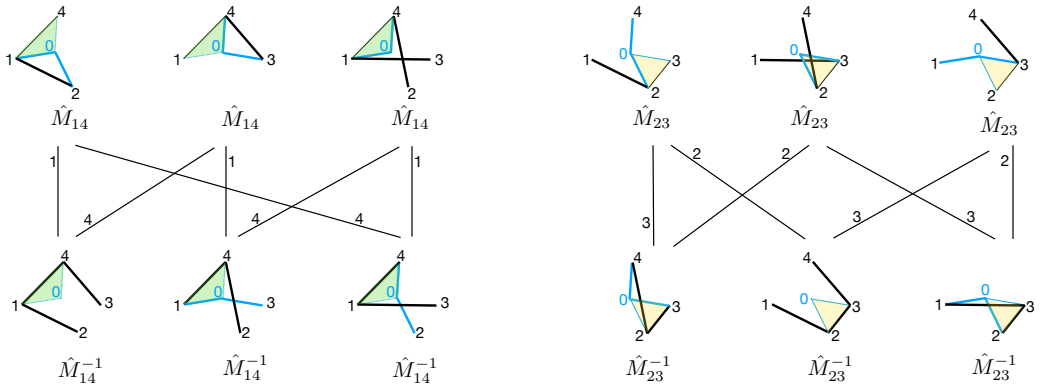


FIG. A.4. Cancellation of phases for $(ij) = (14)$ and (23)

$a, b, c, d \in V$

$$0 = (ab0) \equiv [ab0] + [b0a] + [0ab] \qquad \text{if } a, b \neq 0, \tag{B1}$$

$$0 = (abcd) \equiv [abc] + [bcd] + [cda] + [dab]. \tag{B2}$$

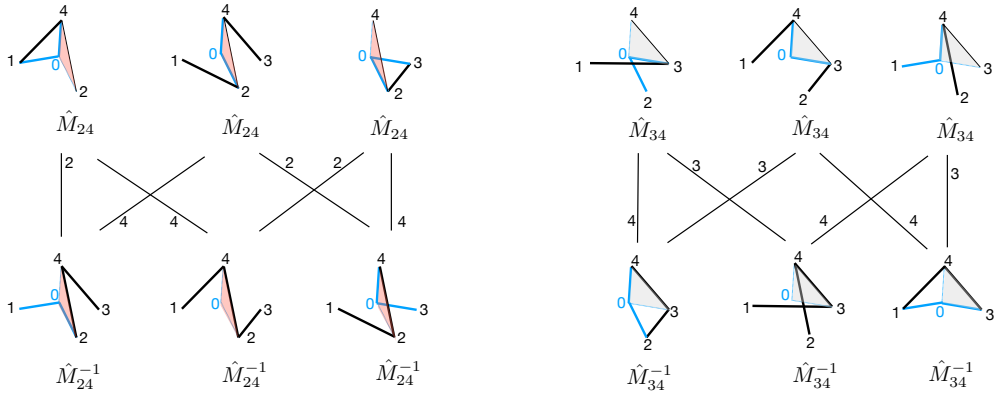


FIG. A.5. Cancellation of phases for $(ij) = (24)$ and (34)

Regard $[0ab] = [ba0]$ as primitive variables for distinct a, b; we have not checked whether they are indeed independent variables. Solve for $[a0b]$ and $[abc]$ in terms of these primitive variables:

$$\begin{aligned} [a0b] &= [0ba] + [0ab], \\ [abc] &= [bc0] + [c0a] + [0ab] = [0cb] + [0ac] + [0ca] + [0ab]. \end{aligned} \tag{B3}$$

Then, it readily follows that for all distinct $a, b, c \in V$

$$(abc) = [abc] + [bca] + [cab] = 0. \tag{B4}$$

The condition $(abcd) = 0$ puts a constraint on our primitive variables:

$$0 = (abcd) = [a0b] + [b0c] + [c0d] + [d0a] \tag{B5}$$

which further implies that for all distinct $x, a, b, c \in V$

$$\begin{aligned} (abcde) &\equiv [abc] + [bcd] + [cde] + [dea] + [eab], \\ (01234) &= (01234) + (0123) + (0234) + (023) = [401] + [103] + [302] + [204] = 0, \\ (abcde) &= [eab] + [bad] + [dac] + [cae] = 0 \end{aligned} \tag{B6}$$

where the last line is because our manipulation is invariant under label permutations. Equation (B6) implies that for all distinct $x, a, b, c \in V$

$$t(x; abc) \equiv [axb] + [bxc] + [cxa] = t(x; adc). \tag{B7}$$

Since any two unordered triples can be "connected" via a sequence of unordered triples where any neighboring pair in the sequence share two elements, this means that $t(x) = t(x; abc)$ *is independent of* abc. Then, (B4) implies that for all distinct $a, b, c, d \in V$

$$t(a) + t(b) + t(c) + t(d) = t(a; bcd) + t(b; acd) + t(c; abd) + t(d; abc) = 0. \tag{B8}$$

This means that $t(x) = t(x; 012) = t(0) + t(1) + t(2) = t$ is a constant for all $x \neq 0, 1, 2$. If we have only 5 labels, then the left-hand side of a formula $\sum_{a=0}^{4} t(a) = t(0)$ is invariant under label permutations, so we have $t = t(a)$ for all $a \in V$. If we have 6 or more labels, then $t(a) = t(5)+t(4)+t(3) = t$ for $a = 0, 1, 2$. Therefore, $t(a) = t$ *is always a constant for all* $a \in V$ *if* $|V| \geq 5$.

a. $t = 0$. We regard $[b0c]$ as a variable on "edge" bc, *i.e.*, the set $\{[b0c] : b, 0, c \text{ are distinct}\}$ is a 1-cochain on the abstract simplex (a complete graph) formed by $\{1, 2, \ldots\} = V \setminus \{0\}$. The equation $0 = t = [b0c]+[c0d]+[d0b]$ is a cocycle condition, but the degree 1 cohomology of a simplex

whose dimension is ≥ 3 is trivial, so we have $[b0c] = [b0] + [c0]$ for some choice of $\{[b0] \in \mathbb{Z}_2 : b \neq 0\}$; here, $[b0]$ is a new symbol. Defining $[bc] = [bc0] + [c0]$, we see that $[bc] + [cb] = [bc0] + [c0] + [cb0] + [b0] = [bc0] + [cb0] + [b0c] = 0$ by (B4). Then, (B2) implies that

$$\begin{aligned}[bcd] &= [cd0] + [d0b] + [0bc] = [cd0] + [d0b] + ([b0c] + [bc0]) \\ &= [cd0] + ([d0b] + [b0c]) + [bc0] = [cd0] + [c0d] + [bc0] \\ &= [cd0] + ([d0] + [c0]) + [bc0] = [cd] + [bc].\end{aligned} \tag{B9}$$

Conversely, if $[ab] = [ba] \in \mathbb{Z}_2$ are arbitrarily given and we set $[abc] = [ab] + [bc]$, which is equal to $[ba] + [cb] = [cba]$, then $t = 0$ and all Eqs. (B2) and (B4) are obviously satisfied. Therefore, *we conclude that $t = 0$ if and only if there is a choice of $\{[ab] = [ba] \in \mathbb{Z}_2 \ : \ a, b \in V, a \neq b\}$ such that $[abc] = [ab] + [bc]$.*

b. $t = 1$. Since the difference of any two $t = 1$ solutions is a $t = 0$ solution, it suffices to find one $t = 1$ solution. If $V = \{0, 1, 2, 3, 4\}$, a solution is obtained by setting for any distinct $a, b, c \in \{1, 2, 3, 4\}$,

$$[0ab] = 0 \text{ except } [012] = [034] = 1. \tag{B10}$$

This satisfies our sole nontrivial condition (B5); there are three cases to consider $(abcd) = (1234)$, (1324), and (1243).

For larger V, we use induction. Suppose we have a solution $\{[abc]\}$ on $V \supset \{0, 1\}$ and we seek to find a solution on $V \sqcup \{\star\}$. Define

$$[0 \star a] = 0 \text{ for all } a \in V, \quad \text{and} \quad [01\star] = 0. \tag{B11}$$

Since $[0\star a]$ and $[01\star]$ are new variables and no linear relation involving two primitive variables may be generated by (B5), we see that this is an allowed choice. It remains to find $[0y\star]$ for $y \in V \setminus \{0, 1\}$. Equation (B7) implies that for any $y \in V \setminus \{0, 1\}$

$$\begin{aligned}[\star 0y] + [y01] + [10\star] &= t = 1, \\ ([0 \star y] + [0y\star]) + [y01] + ([01\star] + [0 \star 1]) &= 1, \\ [0y\star] + [y01] &= 1.\end{aligned} \tag{B12}$$

The last line determines $[0y\star]$ because $[y01]$ is given by the induction hypothesis. This completes the induction for a $t = 1$ solution. (The same induction is applicable for $t = 0$ once we have a solution with $|V| = 5$.)

Appendix C: Local orientations of triangles

Let $V = \{0, 1, 2, \ldots\}$ be a set (of vertices). We call an unordered pair of distinct vertices an edge, and an unordered triple of distinct vertices a triangle. We may regard V as a simplicial complex of one simplex V; we will be interested in the 2-skeleton of this simplicial complex.

Definition C.1. An *alignment* χ is a symmetric $\mathbb{Z}_2$-valued function on all pairs of edge-sharing triangles. An alignment χ is *regular* if χ satisfies the following two conditions.

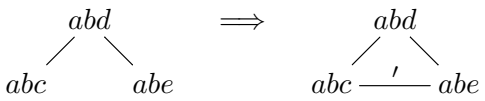
- (edge) for any triple abc, abd, abe of triangles that share a common edge ab, we have $\chi(abc|abd) + \chi(abd|abe) + \chi(abe|abc) = 1$.
- (vertex) for any triple vab, vbc, vca of triangles that share a common vertex v but without any edge common to all, we have $\chi(vab|vbc) + \chi(vbc|vca) + \chi(vca|vab) = 0$.

An orientation for a triangle is by definition a preferred cyclic ordering of its vertices. Let us define a regular alignment on V where each triangle is given an orientation. On each edge, the orientation of a triangle that contains the edge induces an ordering of vertices of the edge. For two triangles abc, bcd that share an edge, we define $\chi(abc|bcd)$ to be 1 if and only if the induced orderings on the shared edge agree, *i.e.*, the value of the alignment is nonzero if and only if the local orientation gives an orientation domain wall element on the shared edge. Let us show that this alignment is regular. For any triple of triangles that share a common edge, we have three induced orderings on the common edge. Either all three orderings agree or only two agree while the third does not. In any case, the sum of alignment values must be odd. This is the edge condition. For any triple of triangles that share a common vertex but without any edge common to the three triangles, either exactly two alignment values are 1 or none of them is 1. In any case, the sum of χ values around the vertex must be even. This is the vertex condition. Therefore, χ is regular. Let us say in this case that χ is *compatible* with the local orientation of triangles.

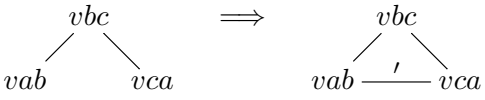
Lemma C.2. *For a regular alignment χ on V, there exists a local orientation for all triangles compatible with χ.*

Proof. If a triangle abc is oriented, and bcd is an edge-sharing triangle, then the orientation of bcd is determined by $\chi(abc|bcd)$ in order for χ to be compatible: the induced orderings on bc must be opposite if $\chi(abc|bcd) = 0$; otherwise they must be the same. We write $abc - bcd$ to denote that the two triangles abc, bcd are given orientations that are compatible with $\chi(abc|bcd)$. Then, the

edge and vertex conditions translate to the following diagrammatic inference rules.



(edge condition) (C1)



(vertex condition) (C2)

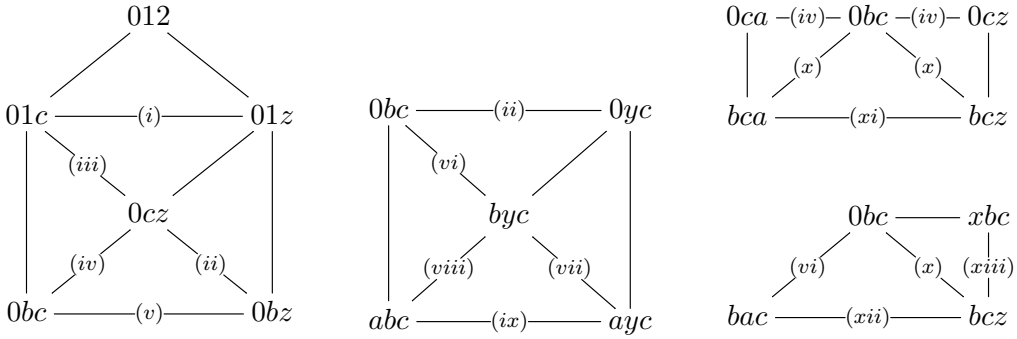
Here, distinct letters stand for distinct vertices, and primes denote inferred compatibility.

To define a local orientation, we order $V = \{0, 1, 2, \ldots\}$ by the integer ordering. Orient the first triangle 012 once and for all. Then, we orient all triangles by

$$012 \text{ —— } 01c \text{ —— } 0bc \text{ —— } abc \tag{C3}$$

for any a, b, c such that $1 \leq a < b < c$.

It remains to show that the orientation defined by (C3) is compatible with χ for any pair of edge-sharing triangles. To this end we start with an example chain of inference:

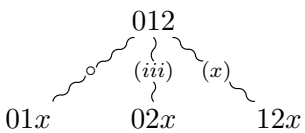


(C4)

Here, any unmarked line exists due to (C3). The letters denote arbitrary vertices that are distinct subject to the ordering implied by the notation. For example, in the lower right diagram it may be that $b = 1$. The lines marked by roman numeral is inferred in order by either the edge or vertex condition. As each line is a mathematical statement, equivalent statements are labeled the same.

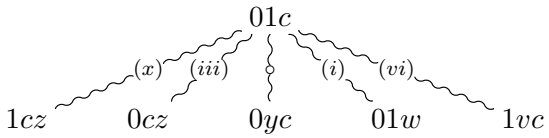
The diagrams in (C4) actually prove the compatibility for all cases. To see this, we enumerate all edge-sharing pairs of triangles in a systematic fashion below, where a class of pairs is given a wiggly line. A wiggly line with a circle means that it is part of the definition (C3). A wiggly line with a roman numeral means that it follows from the corresponding labeled edge in (C4).

For triangles sharing an edge with triangle 012, we have



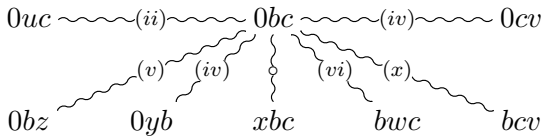
. (C5)

For triangles sharing an edge with triangle $01c$ where $c > 2$, we have



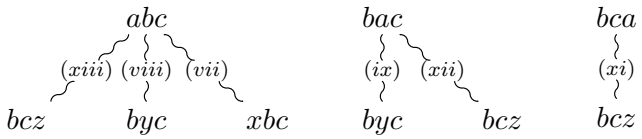
. (C6)

For triangles sharing an edge with triangle $0bc$ where $b, c > 1$, we have



. (C7)

For triangles sharing an edge with triangle abc where $a, b, c > 0$, we have



(C8)

where we do not have to consider cases where $c < b$ by symmetry $b \leftrightarrow c$. □

Appendix D: Proof that the loop self-statistics $\mu = \pm 1$

To prove that $\mu = \pm 1$, it is useful to first construct a certain 2-dimensional complex $\mathcal{L}$, as follows. The vertices of this complex are configurations of a gauge flux loop on a tetrahedron $\mathfrak{T}$ with vertices $1, 2, 3, 4$ and a central 0 vertex. Each vertex of $\mathcal{L}$ is denoted by a sequence of $\mathfrak{T}$-vertices, *e.g.,* (1203), up to cyclic and reverse reordering, *e.g.,* (1203) = (2031) = (1302). Specifically, they are the 33 configurations that appear in the definition of μ, together with four additional configurations (123), (124), (134), and (234), consisting of triangular loops on the surface of $\mathfrak{T}$. Two such configurations are connected by an edge if there exists a membrane operator M_{ij} that maps one to the other.

The faces of $\mathcal{L}$ are all quadrilaterals. By dividing up each quadrilateral into two triangles, we will have a simplicial complex, though we will not need to do this. The quadrilaterals are defined as follows. First, for each edge (ij) of $\mathfrak{T}$ with $i, j \in \{1, 2, 3, 4\}$ we define four quadrilaterals of $\mathcal{L}$ as follows. Given i, j, let $k, l \in \{1, 2, 3, 4\}$ be the two vertices of $\mathfrak{T}$, different from i, j. Then, the four quadrilaterals, defined by specifying the $\mathcal{L}$-vertices around their perimeters, are:

$$\{(ij0), (ijl0), (kijl0), (kij0)\}, \qquad \{(ij0), (ijk0), (lijk0), (lij0)\}, \tag{D1}$$
$$\{(ij0), (ijk0), (ijk), (kij0)\}, \qquad \{(ij0), (ijl0), (ijl), (lij0)\}.$$

Second, for each $(ijkl)$ of $(1234), (1324)$, and (1243), which are the three permutations of (1234) up to cyclic and reverse reordering, we define four quadrilaterals

$$\{(ijkl),(ijkl0),(ijk0),(ijk0l)\}, \qquad \{(ijkl),(ijk0l),(ij0l),(ij0kl)\}, \tag{D2}$$
$$\{(ijkl),(ij0kl),(i0kl),(i0jkl)\}, \qquad \{(ijkl),(i0jkl),(0jkl),(0ijkl)\}.$$

The resulting 2-complex $\mathcal{L}$ is in fact a 2-manifold $\mathbb{RP}^2$, the real 2-dimensional projective plane, as illustrated explicitly in Fig. D.1.

Each quadrilateral in $\mathcal{L}$ is associated with the eigenvalue of a certain group commutator of membrane operators acting on a certain configuration state. Up to permutations of $\mathfrak{T}$-vertices,

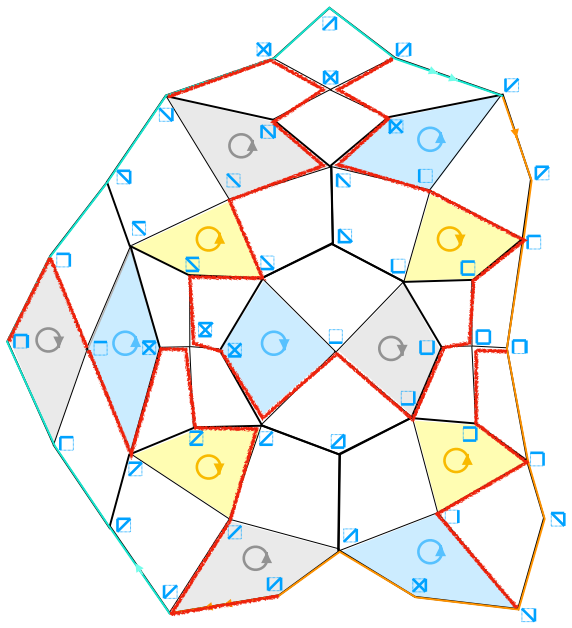


FIG. D.1. The complex defined in the text is explicitly seen to be the 2 dimensional projective plane. Note that the outer orange edges are identified with the outer blue ones as indicated by the arrows. The figures at the vertices represent the corresponding gauge flux loop configuration, with the 0 vertex, and the edges connecting to it, suppressed. The red path corresponds to the sequence of membrane operators and their inverses used to define μ, see Eq. (3). As we can explicitly see from the figure, this path is nontrivial in the fundamental group of $\mathbb{RP}^2$. The three sets of four quadrilaterals associated to the three group commutators of the form $M_{jk}^{-1}M_{il}^{-1}M_{jk}M_{il}$ are colored in yellow, grey, and blue.

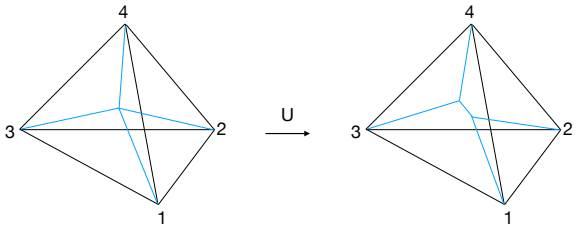


FIG. D.2. There exists a shallow quantum circuit U that transforms any gauge flux configuration on the skeleton on the left to the corresponding configuration on the right. The occupation of the extra middle link on the right is always uniquely determined. Note that the Lieb–Robinson of U may be longer than this extra middle link, which is assumed to be much shorter than the other links. Now, for the configurations on the right, it is clear that one may choose membrane operators such that those associated to (012) and (034) commute. Conjugating these by U, one obtains membrane operators for the configurations on the left, again with the property that M_{12} and M_{34} commute.

there are three types of such eigenvalues:

$$\begin{cases} M_{jk}^{-1}M_{il}^{-1}M_{jk}M_{il} \text{ acting on } |(ij0)\rangle\,, \\ M_{jk}^{-1}M_{ik}^{-1}M_{jk}M_{ik} \text{ acting on } |(ij0)\rangle\,, \\ M_{jk}^{-1}M_{ik}^{-1}M_{jk}M_{ik} \text{ acting on } |(ilj0)\rangle\,, \end{cases} \tag{D3}$$

where $i,j,k,l \in \{1,2,3,4\}$ are distinct. Here, the second and third lines have the same operators evaluated on different states. We claim that, after a possible re-definition of membrane operators M_{**}, these eigenvalues can all be set equal to $+1$.

To prove this, we first consider the first type of group commutator, $M_{jk}^{-1}M_{il}^{-1}M_{jk}M_{il}$. Note that there are exactly three such group commutators. Each of these three acts on four possible states, yielding three sets of four quadrilaterals in Fig. D.1, each set of four colored in a different color. Also, the three group commutators of this form involve disjoint sets of membrane operators, so we can look at them in turn. Let us thus consider $M_{34}^{-1}M_{12}^{-1}M_{34}M_{12}$ (colored yellow in Fig. D.1). This group commutator acts on the four states $(130), (240), (140), (230)$, which are eigenstates of it with eigenvalues $\alpha,\beta,\gamma,\delta \in U(1) \subset \mathbb{C}$. Also, as an operator, it is localized near the center 0 vertex. To see that $\alpha,\beta,\gamma,\delta$ can all be set equal to one, we simply note that, as illustrated in Fig. D.2, there exists a choice of membrane operators such that M_{12} and M_{34} commute identically. The same argument can be repeated for the other two group commutators of the form $M_{jk}^{-1}M_{il}^{-1}M_{jk}M_{il}$.

Now let us consider the second type of group commutator, $M_{jk}^{-1}M_{ik}^{-1}M_{jk}M_{ik}$. Note that as a shallow quantum circuit, it is supported in the vicinity of the $(k0)$ edge, so its eigenvalue is the same on $|(ij0)\rangle$ and $|(ilj0)\rangle$. Again, it is easy to see that one can modify the membrane operators

near the k vertex to set the eigenvalues of all group commutators of this form to 1. Explicitly, note that for each k, there are three commutators of this form — this is just the number of ways of choosing two out of the three edges that connect the vertex k of $\mathfrak{T}$ to noncentral vertices of $\mathfrak{T}$. For each such commutator $M_{jk}^{-1}M_{ik}^{-1}M_{jk}M_{ik}$, we now modify the membrane operator M_{ik} by multiplying it by a phase that depends on the occupation number of the (jk) edge, in such a way as to make the commutator equal to $+1$. Note that this modification does not affect the eigenvalues of the other two commutators associated to k, so by this argument they can all be set equal to $+1$. Furthermore, since all these modifications take place near the $1, 2, 3, 4$ vertices, they are far away from the central 0 vertex, and hence do not change the value of $M_{jk}^{-1}M_{il}^{-1}M_{jk}M_{il}$ (which has already been set equal to 1).

Thus, we can assume that, with an appropriate choice of membrane operators, all of the group commutator actions corresponding to the quadrilaterals in Fig. D.1 have eigenvalues $+1$. Now consider computing μ^2, which corresponds to tracing out the red path in Fig. D.1 twice. This doubled path is contractible — it can be deformed to a trivial path by sliding it over the various quadrilaterals (in fact, over each quadrilateral exactly once). Since the eigenvalue corresponding to each quadrilateral is $+1$, we see that such path deformations do not change the eigenvalue associated to the path. Since the eigenvalue associated to the trivial path is $+1$, we thus see that $\mu^2 = 1$, so that $\mu = \pm 1$, as desired.

Appendix E: Frame parity of fermion worldlines

In this appendix, we collect results about Frm and its extension $\overline{\text{Frm}}$, improving upon [28, Lem II.4]. The object $\overline{\text{Frm}}$ is a quadratic form on all 1-cycles over $\mathbb{Z}_2$. This function Frm resembles a spin structure in that Frm can be used to define a Hamiltonian with emergent fermions, and the set of all possible $\overline{\text{Frm}}$ is acted on transitively and freely by the first cohomology group of the underlying space. In essence, $\overline{\text{Frm}}$ measures in $\mathbb{Z}_2$ how every 1-cycle is twisted. However, it is different from spin structures since there is no obstruction to define $\overline{\text{Frm}}$ given an underlying space — our results will be applicable to any 2- or higher dimensional combinatorial manifold, regardless of the Stiefel–Whitney classes.

We will begin by introducing a continuous map from 2-skeleton down to $\mathbb{R}^2$ which encodes how each null-homologous worldline is twisted, and define a $\mathbb{Z}_2$-valued function Frm. We will note that Frm is essentially unique and independent of the continuous map down to $\mathbb{R}^2$, and construct fermion string operators by examining localizations of worldline fluctuation operators. The results

on fermion string operators are used to analyze a commuting Pauli Hamiltonian that is topologically ordered, is fixed under an entanglement renormalization group flow, and has a unique nontrivial topological charge that is a fermion whenever the underlying space has dimension 3 or higher. We will extend Frm to $\overline{\text{Frm}}$ on all 1-cycles and consider the action on $\overline{\text{Frm}}$ by first cohomology group of the underlying space.

We do not consider any primary chains or any primary qubits in the following discussion on Frm. For simplicity we only consider simplicial complexes; for a more general cell complex, one can use the prescription in Section III B, especially Eq. (23).

1. Projection and Frm

Definition E.1. For a 2-dimensional simplicial complex $\mathcal{K}$,[7] a *projection* is a continuous map $\phi : \mathcal{K} \to \mathbb{R}^2$ such that all 1-cells are mapped to transverse straight line segments and each 2-cell, a triangle, is mapped injectively to the triangle in $\mathbb{R}^2$ defined by its sides.

Unless states otherwise, $\mathcal{K}$ will always denote a simplicial 2-complex equipped with a projection ϕ.

For each 2-cell f_2, we define $L_{Fc}(f_2)$ to be the product of Pauli X along the boundary of f_2 and Pauli Z on every 1-cell e_1 such that $\phi(e_1)$ intersects the interior of $\phi(f_2)$. This definition is the same as in Section III B (except for Z on the primary qubit on the Poincaré dual of f_2).

By construction, the frame parity $\text{Frm}(a_1)$ of a null-homologous 1-cycle a_1 is computed by any 2-chain $\tilde{a}_2$ bounded by a_1:

$$L(\tilde{a}_2) = \prod_{f_2 \in \tilde{a}_2} L_{Fc}(f_2) = (-1)^{\text{Frm}(a_1)} \prod X \prod Z, \tag{E1}$$

$$(-1)^{\text{Frm}(a_1)} = \langle a_1 | \, L(\tilde{a}_2) \, |0\rangle \, . \tag{E2}$$

where $|0\rangle$ is the $+1$ eigenstate of all the relevant Z operators.

Proposition E.2. *The map* Frm *from any finite null-homologous cycle a_1 to a sign, is a well-defined function that can be evaluated by any 2-chain $\tilde{a}_2$ with $\partial \tilde{a}_2 = a_1$.*

Proof. Let $a_1 = \partial \tilde{a}_2$ be any null-homologous 1-cycle of $\mathcal{K}$, and let $\tilde{a}_2'$ be another 2-chain such that $\partial \tilde{a}_2' = a_1$. We have to show that

$$L(\tilde{a}_2) L(\tilde{a}_2') = L(\tilde{a}_2 + \tilde{a}_2') = + \prod Z, \tag{E3}$$

[7] a locally finite cell complex in which every cell is a simplex and the intersection of any two simplices is a simplex.

which cannot have any X factor because $\partial(\tilde{a}_2 + \tilde{a}'_2) = 0$. Consider a cone over the support N of $\tilde{a}_2 + \tilde{a}'_2$.[8] The subcomplex N is $\mathbb{Z}_2$-closed, so if c is the chain of all 3-cells of the cone, the boundary ∂c is $\tilde{a}_2 + \tilde{a}'_2$. Let us extend ϕ so that the extension ϕ^e is defined on the 2-skeleton of the cone. This is straightforward: place an additional generic point p on $\mathbb{R}^2$ for the apex of the cone, and connect the images of the 0-cells of N to p by straight lines. Every 2-cell of the cone consisting of some 1-cell of N and the apex, has an obvious image under ϕ^e. Define $L(\partial e_3) = \prod_{f_2 \in \partial e_3} L_{Fc}(f_2)$ for any 3-cell e_3 of the cone. Since $\partial c = \tilde{a}_2 + \tilde{a}'_2$, we have

$$\prod_{e_3 \in c} L(\partial e_3) = L(\tilde{a}_2 + \tilde{a}'_2). \tag{E4}$$

Each e_3 is a tetrahedron, and there are only two different projections of a tetrahedron, up to homotopy, depending on whether a vertex is inside a triangle. By direct calculation, we confirm that $\langle 0| L(\partial e_3) |0\rangle = +1$, and therefore the claim is proved. □

Remark E.3. If we extend $(\mathcal{K}, \phi)$ by adding more cells to obtain $(\mathcal{K}^e, \phi^e)$ where $\phi^e|_{\mathcal{K}} = \phi$, the induced function Frm^e on null-homologous 1-cycles of $\mathcal{K}^e$ agrees with Frm on those of $\mathcal{K}$. This is simply because $\mathrm{Frm}^e(a_1)$ for a null-homologous 1-cycle a_1 of $\mathcal{K}$ can be evaluated by some 2-chain of $\mathcal{K}$, over which L_{Fc} are the same for $\mathcal{K}$ and $\mathcal{K}^e$. Once we have an extension $(\mathcal{K}^e, \phi^e)$, the function $\mathrm{Frm}^e(a_1)$ where a_1 is a cycle of $\mathcal{K}$ may be evaluated by a 2-chain of $\mathcal{K}^e$. ◊

Proposition E.4. *Let ϕ' be another projection $\mathcal{K} \to \mathbb{R}^2$, which defines a different set of operators L'_{Fc}, and let* Frm' *be the resulting function by Eq. (E1). Assume that $\mathcal{K}$ is the 2-skeleton of an n-dimensional simplicial manifold with $n \geq 2$ such that every 2-cell of $\mathcal{K}$ has at most m 1-cells in its boundary and every 0-cell has at most m 1-cells in its coboundary. Then,* $\mathrm{Frm} + \mathrm{Frm}'$ *on the space of all null-homologous cycles is locally computable,* i.e., *there exists a quantum circuit of depth at most a constant depending only on m which implements $|a\rangle \mapsto (-1)^{\mathrm{Frm}(a)+\mathrm{Frm}'(a)} |a\rangle$ for all null-homologous cycles a.*

Thus, we may speak of *the* frame parity Frm without specifying a particular projection for any "uniform" cellulation of a combinatorial manifold, if we are interested in shallow-circuit-equivalence of many-body states. Note that Frm is *not* defined on 1-cycles of nonzero homology class. We will consider extensions of Frm to all 1-cycles in Proposition E.10.

Proof. Since $\mathcal{K} = \mathcal{K}_2$ is a simplicial 2-complex, the projection is, by definition, determined by the image of the 0-skeleton $\mathcal{K}_0$. Suppose that $\phi'(\mathcal{K}_0)$ and $\phi(\mathcal{K}_0)$ differ only by the image of one

[8] This argument is the same as in [28, §II.C.b].

point $v_0 \in \mathcal{K}_0$. Since $\mathcal{K}$ is the 2-skeleton of a manifold, any chain e that passes through v_0 is homologous to one $u(e)$ that does not. For any chain e, we may choose $u(e)$ such that the change $u(e) + e$ is a null-homologous cycle supported on the union $E(v_0)$ of the boundaries of 2-cells that touch v_0. Therefore, for any 1-cycle a, the change $\mathrm{Frm}(a) + \mathrm{Frm}(u(a))$ can be computed by a diagonal unitary $F(v_0)$ supported on $E(v_0)$, and so can be $\mathrm{Frm}'(a) + \mathrm{Frm}'(u(a))$ by $F'(v_0)$. (The unitary $F(v_0) : |a\rangle \mapsto (-1)^{\mathrm{Frm}(a)+\mathrm{Frm}(u(a))} |a\rangle$ does not change the underlying cycle a.) But, we know $\mathrm{Frm}'(u(a)) = \mathrm{Frm}(u(a))$ due to Remark E.3. Hence, $\mathrm{Frm}(a) + \mathrm{Frm}'(a)$ is computed by a local diagonal unitary $G(v_0) = F(v_0)F'(v_0)$ supported on $E(v_0)$.

Now, consider general cases where $\phi(\mathcal{K}_0)$ and $\phi'(\mathcal{K}_0)$ are arbitrarily different. Let $\phi^{(0)} = \phi$ and inductively define $\phi^{(t)}$ for $t \geq 1$ as follows. Choose a point $v \in \mathcal{K}_0$ such that $\phi^{(t-1)}(v) \neq \phi'(v)$; if there is no such a point, then $\phi^{(t-1)} = \phi'$. Choose any maximal subset $V(t)$ of $\mathcal{K}_0$ such that for any $v, v' \in V(t)$ we have $E(v) \cap E(v') = \emptyset$ whenever $v \neq v'$, and define $\phi^{(t)}(p) = \phi'(p)$ if $p \in V(t)$ and $\phi^{(t)}(p) = \phi^{(t-1)}(p)$ if $p \notin V(t)$. That is, as t increases, we gradually "move" projected points under ϕ to match those under ϕ'. At any given t, the gates $G(v)$ for $v \in V(t)$ do not overlap, so we may apply them in parallel. The assumption on neighbor numbers implies that for some $t_{\max}$ upper bounded only through m, we have $\phi^{(t_{\max})} = \phi'$, and hence the cumulative change $\mathrm{Frm} + \mathrm{Frm}'$ is computed by a quantum circuit of depth $t_{\max}$ that is bounded only through m. □

Remark E.5. The function Frm is not just a function but a $\mathbb{Z}_2$-valued quadratic form, *i.e.*, $\mathrm{Frm}(a_1 + a_1') - \mathrm{Frm}(a_1) - \mathrm{Frm}(a_1')$ is bilinear. To see this, for any pair of secondary 2- and 1-chain (b_2, a_1) we define $\mathrm{BiFrm}((b_2, a_1))$ to be the mod 2 number of times the *thorns* of $L(b_2)$, the $\prod Z$ part of Eq. (E1), intersect a_1. Since L_{Fc} are Pauli operators of which multiplication ignoring signs may be regarded as $\mathbb{Z}_2$-vector addition, we see that BiFrm is bilinear. Since $\mathrm{BiFrm}((\tilde{a}_2, \partial\tilde{a}_2'))$ is the sign upon rearranging Pauli factors in the product $L(\tilde{a}_2)L(\tilde{a}_2')$, we conclude that

$$\mathrm{Frm}(a_1 + a_1') = \mathrm{Frm}(a_1) + \mathrm{Frm}(a_1') + \mathrm{BiFrm}((\tilde{a}_2, a_1')) \tag{E5}$$

where $\tilde{a}_2$ is any secondary 2-chain such that $\partial\tilde{a}_2 = a_1$. Since Frm is a function of null-homologous 1-cycles, the association

$$(a_1, a_1') \mapsto \mathrm{BiFrm}(a_1, a_1') = \mathrm{BiFrm}((\tilde{a}_2, a_1')) = \mathrm{BiFrm}((\tilde{a}_2', a_1)) \tag{E6}$$

is a well-defined bilinear, symmetric function on the space of all null-homologous 1-cycles. We will see again, after Corollary E.7 below, that BiFrm is independent of 2-chains bounded by one of the argument 1-chains. ♢

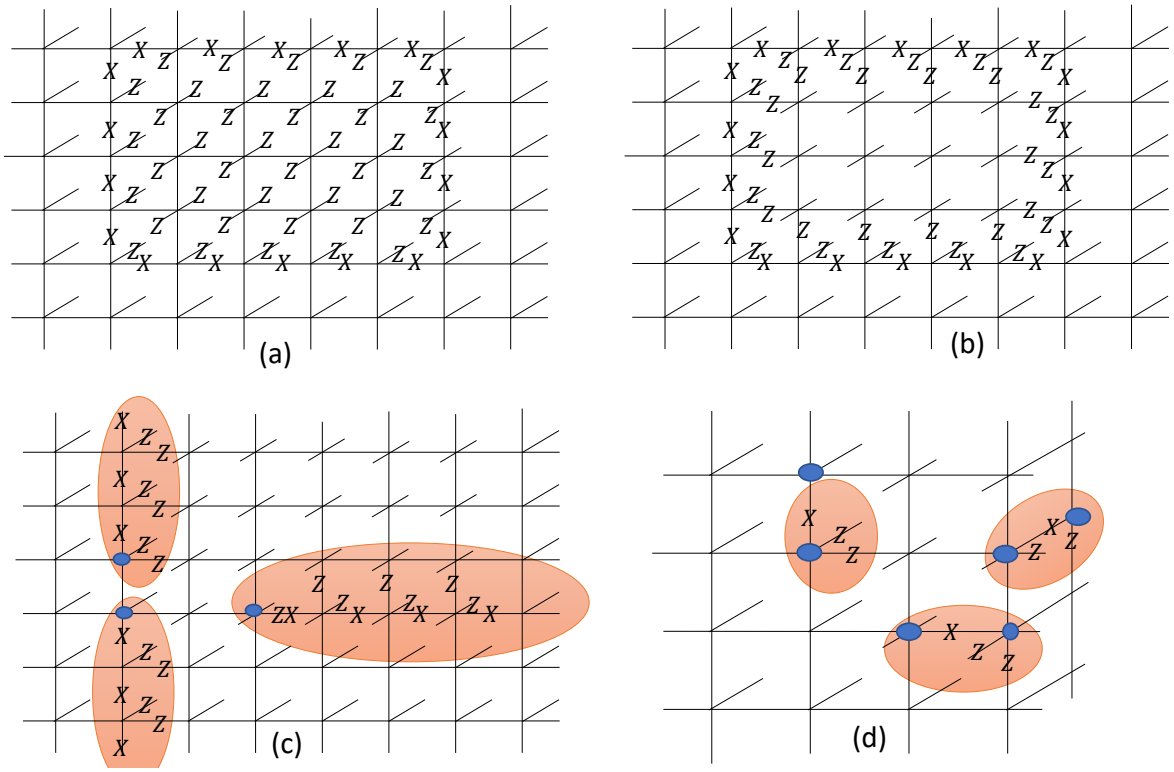


FIG. E.1. The secondary cellulation of the system boundary is shown. The drawing specifies a projection ϕ. The fourth direction into the 4-dimensional bulk is not shown. (a) A product of worldline fluctuation operators over a 2-chain b_2. The X factors are along ∂b_2, and Z factors are "thorns." (b) After multiplying by the cycle-enforcing operators in the interior where the winding parity of b_2 is odd, the operator over b_2 localizes along ∂b_2. (c) If we break up the closed string operator, there will be a quasiparticle at the end of the string. The three semi-infinite operators have end configurations such that when any pair of them are joined by overlaying the solid dot, the resulting operator commutes with every Hamiltonian term. Among the three, one pair is anticommuting, showing that the quasiparticle is a fermion. With primary qubits and worldsheets that link with worldlines, the fermion is confined, but in a system with worldlines only, the fermion is a deconfined topological excitation. (d) Any fermion string operator is built out of three types of segments.

2. Fermion string operators

Let us show that the product $L(b_2) = \prod_{f_2 \in b_2} L_{Fc}(b_2)$ of L_{Fc} over a 2-chain b_2 localizes around ∂b_2 by multiplying $L(b_2)$ by certain cycle-enforcing operators $\pi(v_0) = \prod_{e_1 \in \delta v_0} Z(e_1)$. For any chain x, let $\mathrm{Supp}(x)$ be (the topological closure of) the union of all cells in x.

Lemma E.6. *For any 2-chain b_2, there exists a subset V_0 of vertices in $\mathrm{Supp}(b_2) \setminus \mathrm{Supp}(\partial b_2)$ such that the operator $S = L(b_2) \prod_{v_0 \in V_0} \pi(v_0)$ is supported on the collection of 1-cells that meet*

$\mathrm{Supp}(\partial b_2)$.

Proof. We first define *winding parity* of b_2 at an *interior* vertex $v_0 \in \mathrm{Supp}(b_2) \setminus \mathrm{Supp}(\partial b_2)$. Let $N(v_0)$ be a chain of all 2-cells of b_2 which meet v_0. The projected boundary $\phi(\partial N(v_0))$ is a 1-cycle in the plane $\mathbb{R}^2$, one which does not pass through $\phi(v_0)$. Let γ be an integral piecewise linear 1-cycle of $\mathbb{R}^2$ that reduces to $\phi(\partial N(v_0))$ modulo 2. We may regard γ as a collection of (not necessarily disjoint) oriented immersed circles on $\mathbb{R}^2 = \mathbb{C}$. The winding parity is defined to be the complex contour integral

$$\left(\frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z - \phi(v_0)} \right) \bmod 2. \tag{E7}$$

The lifting γ is arbitrary but any lifting differs from γ by a cycle that is zero modulo 2. So, the difference in the contour integral is two times some contour integral over a cycle, which vanishes modulo 2, and therefore the winding parity is well defined.

A thorn of $L(b_2)$ may be on a 1-cell in $\mathrm{Supp}(b_2)$ (internal) or not (external). We claim that external thorns of $L(b_2)$ appear around an interior 0-cell $v_0 \in \mathrm{Supp}(b_2) \setminus \mathrm{Supp}(\partial b_2)$ if and only if the winding parity of b_2 at v_0 is odd. Indeed, an external thorn on e_1 with $\partial e_1 \ni v_0$ exists if and only if $\phi(e_1)$ intersects an odd number of $\phi(f_2)$ for $f_2 \in N(v_0)$. Consider a straight line segment $\ell(x)$ between $\phi(v_0)$ and any point x on $\phi(\partial N(v_0))$. As x moves along $\phi(\partial N(v_0))$, the line segment $\ell(x)$ may overlap with $\phi(e_1)$, in which case, tautologically, there is a 2-cell in $N(v_0)$ whose projection intersects $\phi(e_1)$. We can choose an integral lifting γ of $\phi(\partial N(v_0))$ such that every nonzero coefficient is ± 1. This is easy as follows. We follow a line segment of $\phi(\partial N(v_0))$ until we encounter a junction and we make an arbitrary choice what segment to continue on, and stop if we return to the starting position. Repeat by choosing a segment we have never visited. Since every junction joins an even number of segments, this procedure takes us through every segment exactly once. With such a lifting γ, we let x traverse $\phi(\partial N(v_0))$, and the signed number of times that $\ell(x)$ overlaps with $\phi(e_1)$, where $\phi(e_1)$ just fixes a reference axis, is the value of the contour integral above. Hence the mod 2 reduction of this count determines the external thorn on e_1.

An internal thorn on $s_1 \in \mathrm{Supp}(b_2)$ where both vertices of ∂s_1 are in the interior of $\mathrm{Supp}(b_2)$, is determined by $N(\partial s_1)$ that is the chain of all 2-cells of b_2 which meet either of the boundary vertices t_0, u_0 of s_1. We should count the 2-cells of $N(\partial s_1)$ that intersect s_1 with their interior. There are two classes of such 2-cells: those in $N(\partial s_1) \setminus N(t_0)$ or in $N(\partial s_1) \setminus N(u_0)$. There cannot be any contribution from $N(t_0) \cap N(u_0)$ because the 2-cells in this intersection has s_1 as a boundary 1-cell. But, even if we count the number of times that $\phi(s_1)$ overlaps with $\ell(x)$ where x runs over all $\partial N(t_0)$ and all $\partial N(u_0)$, any contribution from $N(t_0) \cap N(u_0)$ is counted twice. Hence, we conclude

that an internal thorn on $s_1 \in \mathrm{Supp}(b_2)$ exists if and only if the sum of the winding parities of b_2 at the two end points of s_1 is odd.

Therefore, if we multiply $L(b_2)$ by the cycle-enforcing operators $\pi(v_0)$ for all interior 0-cells v_0 at which the winding parity is odd, then we obtain an operator without any thorns around any interior 0-cells of $\mathrm{Supp}(b_2)$. See Fig. E.1(a,b). □

Corollary E.7. *Given a null-homologous* 1*-cycle* a_1*, the operator* $S(a_1)$ *obtained by canceling interior thorns of* $L(b_2)$ *where* $\partial b_2 = a_1$ *is independent of the choice of* b_2 *up to cycle-enforcing operators* π*. In addition,* $S(a_1)$ *commutes with any other* $S(a_1')$ *where* a_1' *is any null-homologous* 1*-cycle.*

We will refer to $S(a_1)$ as the fermion string operator along a_1.

Proof. If b_2' is another 2-chain such that $\partial b_2' = a_1$, then $L(b_2')L(b_2) = L(b_2' + b_2)$ is a product of Z only; the sign is $+1$ by Proposition E.2. By Lemma E.6, $L(b_2'+b_2)$ must localize around $\partial(b_2'+b_2) = 0$, implying that $L(b_2' + b_2)$ is in fact a product of cycle-enforcing operators π. If S and S' are two localized operators from $L(b_2)$ and $L(b_2')$, respectively, then $S'S^{-1}$ is an operator supported on the collection of 1-cells that meet $\mathrm{Supp}(a_1)$, and we have shown that $S'S^{-1} = (L(b_2')\prod \pi)(L(b_2)\prod \pi)$ is a product of π's. □

Proposition E.8. *Assume that for any two* 1*-cycles* x_1, x_1' *there is a cycle* x_1'' *homologous to* x_1' *such that no* $L_{Fc}(f_2)$ *with a* 2*-cell* f_2 *intersects both* $S(x_1)$ *and* $S(x_1'')$*. Then, for arbitrary* 1*-cycle* a *there is an operator* $S(a)$*, unique up to a sign, such that* $a \mapsto S(a)$ *modulo signs and cycle-enforcing operators* π *is a* $\mathbb{Z}_2$*-linear extension of* S *in Corollary E.7. Two operators* $S(a)$ *and* $S(a')$ *for any two* 1*-cycles* a, a' *commute.*

Proof. Take $x_1 = x_1' = a_1$ in the assumption. Then, by Corollary E.7 the string operator $S(a_1+x_1'')$ is supported on two clearly separated "tubes," one along a_1 and the other along x_1''. We *define* that the string operator $S(a_1)$ is the part along a_1:

$$S(a_1 + x_1'') = S(a_1)S(x_1''). \tag{E8}$$

This definition leaves the sign of $S(a_1)$ *undetermined* if a_1 is homologically nontrivial, but the operator content (Pauli factors) is unique up to cycle-enforcing terms. Clearly, this definition is a $\mathbb{Z}_2$-linear extension of S in Corollary E.7.

Being a product of worldline fluctuation and cycle-enforcing terms, the operator $S(a_1 + x_1'')$ commutes with every worldline fluctuation term L_{Fc}. But any L_{Fc} can meet only one of $S(a_1)$

and $S(a_1')$ by assumption, so every L_{Fc} commutes with $S(a_1)$. Therefore, for any a_1, the fermion string operator $S(a_1)$ commutes with every L_{Fc}.

If a_1' is another cycle, we have a cycle y_1' homologous to a_1' and separated from a_1. The product of two string operators $S(a_1'+y_1') = S(a_1')S(y_1')$, being a product of L_{Fc} and cycle-enforcing terms, commutes with $S(a_1)$. Since $S(y_1')$ is separated from $S(a_1)$, we know $S(y_1')$ commutes with $S(a_1)$. Therefore, $S(a_1)$ commutes with $(S(a_1')S(y_1')) \cdot S(y_1')^{-1} = S(a_1')$. □

3. $\overline{\text{Frm}}$ for all 1-cycles

Now, we consider how to extend Frm beyond the space of null-homologous 1-cycles. We begin with an observation that Corollary E.7 makes it possible to define BiFrm in Eq. (E6) more directly. Namely, for two null-homologous 1-cycles a_1 and a_1', the bilinear form $\text{BiFrm}(a_1, a_1')$ equals the mod 2 number of times that the thorns (the Z part) of the fermion string operator $S(a_1)$ intersect a_1'. The ambiguity in the string operator by cycle-enforcing terms does not affect the value $\text{BiFrm}(a_1, a_1')$ since the cycle-enforcing terms always overlap with a cycle an even number of times. Since two closed fermion string operators commute, BiFrm is symmetric.

In Proposition E.8 we have defined a fermion string operator for any a_1, null-homologous or not. Define an extension $\overline{\text{BiFrm}}(a_1, a_1')$ for *any* 1-cycles a_1, a_1' to be the mod 2 number of times that the Z thorns of $S(a_1)$ intersect a_1'. Clearly, $\overline{\text{BiFrm}}$ is linear in the second argument. Since $S(a_1)$ and $S(a_1')$ are products of Pauli X and Z and they commute, we must have $\overline{\text{BiFrm}}(a_1, a_1') = \overline{\text{BiFrm}}(a_1', a_1)$. We have shown that $\overline{\text{BiFrm}}$ is a symmetric bilinear form on the $\mathbb{Z}_2$ vector space of all 1-cycles of $\mathcal{K}$, extending BiFrm.

Remark E.9. Analogously to Remark E.3, $\overline{\text{BiFrm}}(a_1, a_1')$ for 1-cycles a_1, a_1' of $\mathcal{K}$, can be evaluated by any extension $(\mathcal{K}^e, \phi^e)$ obtained by adding more cells, as long as the assumption of Proposition E.8 is satisfied. Here, an extension means that $\mathcal{K} \subseteq \mathcal{K}^e$ and $\phi^e|_{\mathcal{K}} = \phi$. This is because the string operator $S(a_1)$ can still be obtained by some 2-chain of $\mathcal{K}$ whose boundary contains a_1. Additional 1-cells from the extension may give extra thorns to $S(a_1)$, but those extra thorns do not contribute to $\overline{\text{BiFrm}}$ since a_1' is on $\mathcal{K}$. ◊

Proposition E.10. *Under the assumption of Proposition E.8, there exists a quadratic form* $\overline{\text{Frm}}$*, an extension of* Frm *to all* 1*-cycles with the associated biliear form* $\overline{\text{BiFrm}}$*. Any other extension is given by* $\overline{\text{Frm}} + h^1$ *for some* 1*-cocycle* h^1 *of* $\mathcal{K}$*; The set of all possible extensions is acted on transitively and freely by the first cohomology group of* $\mathcal{K}$ *over* $\mathbb{Z}_2$*.*

Proof. Choose a $\mathbb{Z}_2$ basis $[h_1(1)], \ldots, [h_1(n)]$ of the first homology of $\mathcal{K}$. Define inductively for all $m = 1, 2, \ldots, n$

$$\overline{\text{Frm}}\left(z_1 + \sum_{j=1}^{m} c(j)h_1(j)\right) = \overline{\text{Frm}}\left(z_1 + \sum_{j=1}^{m-1} c(j)h_1(j)\right) + c(m)\overline{\text{BiFrm}}\left(z_1 + \sum_{j=1}^{m-1} c(j)h_1(j), h_1(m)\right) \tag{E9}$$

where $z_1 \in [0]$ and $c(j) \in \mathbb{Z}_2$. This is the unique way to define a quadratic form given that $\overline{\text{Frm}}(h_1(j)) = 0$ for all j. This shows the existence of an extension.

Any two extensions that share an associated bilinear form differ by a linear form Δ; over $\mathbb{Z}_2$, the bilinear form measures the deviation of a quadratic form from being a linear form. This difference Δ must be coclosed because on an exact 1-cycle, we already have $\overline{\text{Frm}} = \text{Frm}$. If Δ is coexact, then Δ vanishes on all 1-cycles. Therefore, the first cohomology acts transitively and freely on the set of all possible extensions with $\overline{\text{BiFrm}}$ fixed. □

4. Fermionic charge only (Fc)

Remark E.11. Consider a wavefunction (on a system given a projection ϕ for secondary qubits but without any primary qubits)

$$|\text{Fc}; [h_1]\rangle = \sum_{a_1 \in [h_1]} (-1)^{\overline{\text{Frm}}(a_1)} |a_1\rangle \tag{E10}$$

where a_1 range over a homology class $[h_1]$ of 1-cycles. There is a commuting Pauli Hamiltonian

$$H_{Fc} = -\sum_{f_2 \colon \text{cells}} L_{Fc}(f_2) - \sum_{v_0} \underbrace{\prod_{e_1 \in \delta v_0} Z(e_1)}_{\pi(v_0)} \tag{E11}$$

whose ground state subspace is spanned by $\{|\text{Fc}; [h_1]\rangle : h_1 \text{ is any 1-cycle.}\}$. Here, L_{Fc} lacks the factors of $Z(f^2)$ on primary qubits. Since $[h]$ is any homology class, the basis of the ground subspace of this Hamiltonian can be identified with the first homology group of $\mathcal{K}$. ◊

Remark E.12. The state $|\text{Fc}\rangle_{\mathcal{K}}$ on a combinatorial manifold embeds naturally into a system of a refined triangulation. By refinement we mean an injective linear chain map $\iota : \mathcal{K} \to \mathcal{K}'$ such that 0-cells are mapped to 0-cells and $\iota \circ \partial = \partial \circ \iota$. When the refined triangulation is uniform in the sense of Proposition E.4, we can choose any projection $\phi' : \mathcal{K}' \to \mathbb{R}^2$, and a handy choice of ϕ' is one that has the same images for 0-cells of $\mathcal{K}$. Consider two string operators $S(\iota\partial b_2)$ on $\mathcal{K}'$ and $S(\partial b_2)$ on $\mathcal{K}$, acting on the span of $\{\iota\partial b_2 \ : \ b_2 \text{ is a 2-chain of } \mathcal{K}\}$. Lemma E.6 says that the only difference

between these two string operators is in the thorns around $\iota\partial b_2$. Indeed, some 1-cell e_1 of $\mathcal{K}$ may have been subdivided in $\mathcal{K}'$ so that $\iota(e_1)$ consists of two or more 1-cells of $\mathcal{K}'$, in which case there can be some thorns attached to $e_1 \setminus \partial e_1$. But there is no worldline segment from the embedded cycles on these thorns. Hence, any null-homologous 1-cycles of $\mathcal{K}$ embeds into $\mathcal{K}'$ with frame parity unchanged.

The embedding of $|\mathrm{Fc}\rangle$ is realized by a shallow quantum circuit. To this end, we aim to disentangle a qubit on a 1-cell e_1' of $\mathcal{K}'$ that is not in the image of ι. Choose one f_2' of the 2-cells in the coboundary of e_1'. There is a local unitary that conjugates $L_{Fc}(f_2')$ to $X(e_1')$. We choose a composition U of Clifford gates, control-X and control-Z, where the control is on e_1' and the "target" ranges over all other 1-cells in the support of $L_{Fc}(f_2')$. This local unitary U commutes with any product $L_{Fc}(f_2')L_{Fc}(p_2')$ for any p_2' that intersects f_2' along e_1'. Also, U conjugates the cycle-enforcing terms at the end points of e_1' to strip off $Z(e_1')$. Hence, the unitary U disentangles one qubit at e_1' and maps $|\mathrm{Fc}\rangle_{\mathcal{K}'}$ to a state of form $|\mathrm{Fc}\rangle$ on a nonsimplicial complex that lacks e_1' but admits the prescription of L_{Fc} in Section III B. After disentangling all qubits on 1-cells outside the image of ι, we recover the state $|\mathrm{Fc}\rangle_{\mathcal{K}}$.

Therefore, we can say that if $\mathcal{K}$ is the 2-skeleton of a combinatorial n-manifold where $n \geq 3$, the Hamiltonian H_{Fc} is a $\mathbb{Z}_2$-gauge theory with *fermionic* charges. We only have to check this for some cellulation, *e.g.*, the hypercubic lattice, which we do explicitly in Fig. E.1. This justifies our naming, fermion string operators. Furthermore, we may speak of *the fermion* string operator $S(a_1)$ for any null-homological 1-cycle a_1.

One might wonder why we need $n \geq 3$. Even if $n = 2$, the Hamiltonian H_{Fc} is well defined and all the lemmas and propositions above remain true; however, there are "too few" thorns in the product $L(b_2)$ of L_{Fc} over a 2-chain b_2, and the string operators transports a boson. In fact, H_{Fc} on a 2-dimensional manifold is shallow-circuit-equivalent to the toric code Hamiltonian on that manifold. $\Diamond$

Remark E.13. On a combinatorial 3- or higher dimensional manifold with sufficiently refined triangulation, there is only one deconfined topological charge with respect to H_{Fc}. We can see this as follows. Since H_{Fc} consists of commuting Pauli terms, we may identify an excitation with a set of flipped terms. Consider any excited state where only L_{Fc} are flipped. We know from Proposition E.2 that the product of L_{Fc} over the boundary of a 3-ball is $+\prod Z$ which must assume $+1$ on $|\mathrm{Fc}\rangle$. Hence, the excitation consisting of flipped L_{Fc}'s corresponds to a 2-cocycle b, *i.e.*, a flux. Disallowing extensive energy for an excitation, we see that b has to be exact, and the

excitation is caused by some Zs that form a 1-cochain whose coboundary is b. In particular, if the excitation is localized around a region in which no nontrivial second cohomology class may be supported, then the excitation can be annihilated by Zs near the region. A small neighborhood of a 1-chain qualifies as such a region. Now, more general excitation contains flips of the cycle-enforcing terms π. An even number of violated π terms can be canceled by a 1-chain of X operators whose boundary 0-cells correspond the flipped π's. Therefore, any excitation localized around a point can be annihilated locally if and only if there are an even number of flips of π. Since a truncated fermion string operator creates an excitation with one flipped π, the fermion at the string end is the only deconfined topological excitation of H_{Fc}. $\Diamond$

Remark E.14. In the setting of Remark E.13, we can assign to each 1-cell a specific fermion string segment that has only one X factor on the 1-cell and some Z factors. Any long fermion string operator will be a product of the operator segments. An example set of such segments is depicted in Fig. E.1(d). A complete set of segments can always be found by (i) assigning a specific excitation that is a fermion to each 0-cell and (ii) letting segments transport the excitation from an end to the other end of the segment. The latter (ii) is possible because we have the unique deconfined topological charge in H_{Fc}. The former (i) is achieved by choosing a base point and some fermion string operator on a closed path from the base point to a given 0-cell v and back. If the path bounds a 2-chain b_2 that has no interior points, *i.e.*, every 0-cell of $\mathrm{Supp}(b_2)$ belongs to $\mathrm{Supp}(\partial b_2)$, which may happen if the path almost self-intersects, then it may be tricky to truncate the path so that at the end point there is a localized nontrivial excitation. To avoid such degenerate paths, we require that the path is sufficiently non-self-intersecting that *if a plaquette term L_{Fc} overlaps with the path, then it may do so only along one 1-cell.* This requirement is fulfilled for a sufficiently refined triangulation. For such a non-self-intersecting path, a truncation of string operator at v gives a fermion there. The excitation at the base point is specified after all other 0-cells are assigned with a fermion, by letting one of other 0-cells be a new base point. $\Diamond$

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