Spontaneous symmetry breaking, spectral statistics, and the ramp

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Spontaneous Symmetry Breaking, Spectral Statistics, and the Ramp

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Ensembles of quantum chaotic systems are expected to exhibit energy eigenvalues with random-matrix-like level repulsion between pairs of energies separated by less than the inverse Thouless time. Recent research has shown that exact and approximate global symmetries of a system have clear signatures in these spectral statistics, enhancing the spectral form factor or correspondingly weakening level repulsion. This paper extends those results to the case of spontaneous symmetry breaking, and shows that, surprisingly, spontaneously breaking a symmetry further enhances the spectral form factor. For both RMT-inspired toy models and models where the symmetry breaking has a description in terms of fluctuating hydrodynamics, we obtain formulas for this enhancement for arbitrary symmetry breaking patterns, including broken Abelian symmetries Z_n and $U(1)$, and partially or fully broken non-Abelian symmetries.

I. INTRODUCTION

This paper is concerned with the statistical properties of energy levels of chaotic quantum systems exhibiting spontaneous symmetry breaking (SSB). The phenomenon of SSB can occur whenever a system possesses a symmetry and a suitable thermodynamic limit. This limit can be achieved either with a system extended in space or, in the cases examined in this paper, a zero-dimensional system with a large number of degrees of freedom. SSB is said to occur at a given energy density (energy per degree of freedom) if, after first taking the thermodynamic limit, a vanishingly small symmetry breaking perturbation in the Hamiltonian leads to a non-symmetric equilibrium state (reviews include [1, 2]). Because the system's symmetry constrains the structure of Hamiltonian, SSB also manifests as a reorganization of the energy levels as a function of energy density. The purpose of this paper is to understand how this reorganization manifests in the correlations between energy levels and how the symmetry unbroken case is recovered in a finite size system.

More precisely, the theory developed here computes the so-called spectral form factor (SFF), the simplest version of which is the Fourier transform of the 2-point correlation between pairs of energy levels. Because it is a Fourier transform, the SFF naturally lives in the time domain. For a quantum chaotic system, the expectation is that, after a time known as the Thouless time, the SFF will approach a random matrix form determined by the symmetry of the Hamiltonian. This paper develops a theory of the SFF in zero-dimensional system exhibiting SSB. In particular, the theory explains how the SFF deviates from the appropriate symmetric random matrix form at early time and how random matrix behavior is recovered at late time.

Although the SFF is understood in a wide variety of regimes [3–5], including systems with symmetries [6],

there is yet to be a systematic study of the SFF in systems with spontaneously broken systems. This paper fills this gap in the zero-dimensional case, with the case of spatially extended systems left to future work. There have also been a few other studies of the interplay of quantum chaos, eigenstate thermalization, and spontaneous symmetry breaking including [7–9]; see [10] for a review of notions of quantum chaos.

We analyze the spectral correlations in these systems as a function of energy density. At high energy density, one typically finds symmetric random-matrix-like energy levels, a hallmark of quantum chaos [11–14]. SSB then sometimes occurs as the energy density is lowered, in which case it corresponds to a breaking of ergodicity in the thermodynamic limit. Our theory quantitatively explains how ergodicity and symmetry are restored at finite system size from the point of the view of the energy spectrum.

Because symmetry restoration is a long-time process, the theory must deal with special slow dynamical processes associated with the order parameter of the broken symmetry for which a hydrodynamic-like effective theory is the right description [15–20] (see [21] for an accessible introduction to the Schwinger-Keldysh technique underlying these effective theories). In previous work [22], we showed how the field theory formulation of fluctuating hydrodynamics could be adapted to predict a random-matrix-like spectral form factor at late times and to compute finite time corrections to random matrix theory (RMT). The present paper can be viewed as an extension of the earlier theory to systems with the additional physics of spontaneous symmetry breaking [23, 24].

A. Setup and observables

We now describe the setup and observables in more detail. A quantum system has symmetry group G if (1) G acts on the Hilbert space of the system by some faith-

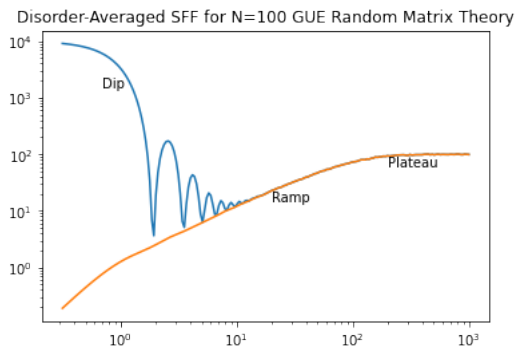Disorder-Averaged SFF for N=100 GUE Random Matrix Theory

FIG. 1. SFF (blue) of a random matrix exhibiting the dip-ramp-plateau structure. The connected SFF only exhibits the ramp and plateau.

ful (but typically reducible) representation $\mathcal{U}$ and (2) the Hamiltonian of the system commutes with every representation, $\mathcal{U}(g)H = H\mathcal{U}(g)$. We focus on systems where the representation of G acting on the full Hilbert space is unitary and linear. For disordered systems corresponding to an ensemble of Hamiltonians, we require that every element of the ensemble commute with the same $\mathcal{U}$s. Because it commutes with the Hamiltonian, such a symmetry constrains the energy spectrum by forbidding certain matrix elements in the Hamiltonian. More precisely, the Hamiltonian breaks into decoupled blocks labelled by the irreducible representations (irreps) of G.

In the simplest quantum chaotic case, each irrep block will consist of a number of copies (equal to the dimension of the irrep) of a system-specific matrix with random-matrix-like level correlations [11–14]. Furthermore, the matrices for different irreps will be independent of each other. We quantify the correlations in these spectra using a filtered form of the spectral form factor (SFF) [25] which zooms in on the particular energy density where SSB occurs. The spectral form factor with filter function f is

$$\text{SFF}(T, f) = \overline{|\text{Tr}[U(T)f(H)]|^2}, \quad (1)$$

where $U(T) = e^{-iHT}$ is the time evolution operator and overline denotes a disorder average over an ensemble of Hamiltonians where each representative is symmetric. We will say more about this averaging shortly; it is necessary in order to render the SFF a smoothly varying function of time T.

It is also useful to define the connected and disconnected contributions to the SFF. The disconnected part,

$$\text{SFF}_{\text{dis}}(T, f) = |\overline{\text{Tr}[U(T)f(H)]}|^2, \quad (2)$$

is distinguished by the full SFF because the squaring happens after the averaging. The connected part is the difference

$$\text{SFF}_{\text{con}}(T, f) = \text{SFF}(T, f) - \text{SFF}_{\text{dis}}(T, f) \quad (3)$$

It can be shown that both the connected and disconnected parts are positive definite. In the standard dip-ramp-plateau picture, the dip comes from the disconnected component, and the ramp and plateau come from the autocorrelation of $\text{tr}\, U(T)f(H)$ found in the connected component of the SFF.

For a quantum chaotic system with no symmetry, the expectation is that, after a Thouless time the SFF will agree with the random matrix result [11–14]. Up to the Heisenberg time, which is proportional to the level density, the RMT result is [26] the linear ramp,

$$\text{SFF}(T, f) = \int dE f^2(E) \frac{T}{\pi \mathfrak{b}}, \quad (4)$$

where $\mathfrak{b} = 1, 2, 4$ is the Dyson index from random matrix theory determined by the presence or absence of antiunitary symmetries. If we choose f to be a Gaussian filter of the form $f = \exp\left(-\frac{(E-E_0)^2}{4\sigma^2}\right)$, then the result is

$$\text{SFF}(T, f) = \frac{\sqrt{2\pi}\sigma T}{\pi \mathfrak{b}} \quad (5)$$

provided E_0 sits within the spectrum of H.

If instead we have a system with G symmetry which is unbroken at energy E_0, then the Gaussian filtered SFF will be

$$\text{SFF}(T, f) = \sum_R |R|^2 \frac{\sqrt{2\pi}\sigma T}{\pi \mathfrak{b}}, \quad (6)$$

where the sum is over irreps R appearing in the spectrum and $|R|$ is the dimension of R. As discussed above, each irrep block is composed of $|R|$ identical copies of an independent random matrix (one for each state in R), so the $|R|^2$ factor arises because all the subblocks are perfectly correlated. See appendix for more details.

The new feature associated with SSB at energy E_0 is that the different irrep blocks will no longer be effectively independent. At large but finite system size, the energy eigenstates will be 'cat states' which transform in different representations of the symmetry group but have nearly-identical energies. This means that the spectra of different blocks will be strongly correlated, and this additional correlation will cause the SFF to take a larger value at early to intermediate times. Then at fixed system size (and for typical forms of SSB), the system will crossover to the unbroken behavior at very long time provided the order parameter fluctuates rapidly compared to the Heisenberg time of each block.

The remainder of the paper is organized as follows. In section we consider the case of spontaneously broken Z_n symmetry in various toy models, obtaining analytic and numerical results in excellent agreement. Next, in section , we do the same for more general finite groups G. Next, we consider a hydrodynamic calculation of the symmetry-breaking SFF in sections and , focusing first on Abelian Lie groups and then on general Lie Groups.

II. Z_n SSB WITH ZERO SPATIAL DIMENSIONS

We will start with the simplest possible case of discrete SSB: the case of a Z_n symmetry and a charge-1 order parameter. Since any discrete Abelian group is a product of Z_n factors (with possibly different ns), this case captures most of the interesting physics of discrete Abelian SSB. We also restrict attention to the case of zero spatial dimensions, so the reader should have in mind a cluster or similar sort of system where a large number of degrees of freedom can interact without geometric restrictions.

The charge-1 order parameter is described by a basis $|\phi\rangle \in \Phi$ with ϕ an integer from 0 to $n-1$. Although in this chapter the symmetry and the corresponding order parameter are discrete, we will occasionally call ϕ a Goldstone mode when analogy with the continuous case would be helpful.

In addition to ϕ, we take the other degrees of freedom to be described by a state ψ in an N-dimensional Hilbert space Ψ, for some large L. In a physical system, L would be exponential in the system size. A state in the total Hilbert space will be an nL-dimensional superposition of states of the form $|\phi\rangle \otimes |\psi\rangle$. In this section, we will require that $|\psi\rangle$ transform trivially under G. We briefly consider more general behavior in appendix .

The Hamiltonian is built from a collection of operators H_k acting on Ψ that are associated with (possibly trivial) transitions of the order parameter from sector ϕ to sector $\phi + k$. Using the shift operator M_k defined as $M_k |\phi\rangle = |\phi + k\rangle$, we write this decomposition as

$$H = \sum_k M_k \otimes H_k$$
$$(M_k)_{ij} = \delta_{i,j+k}, \tag{7}$$

with the arguments of the delta function in M_k all taken mod n. Hermiticity of H requires that $H_k^\dagger = H_{n-k}$ since $M_k^\dagger = M_{n-k}$. For instance, the $n = 4$ Hamiltonian written out in block matrix form is

$$H = \begin{pmatrix} H_0 & H_3 & H_2 & H_1 \\ H_1 & H_0 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_3 \\ H_3 & H_2 & H_1 & H_0 \end{pmatrix} \tag{8}$$
with $H_0 = H_0^\dagger, H_1 = H_3^\dagger, H_2 = H_2^\dagger$.

At this point, we must ask what sorts of matrices H_0 and H_k make good models of the sorts of systems we see in real life. As a simple model, consider the case where each H_k is chosen independently consistent with the constraints imposed by Hermiticity. This should be a reasonable description of the spectral properties of generic chaotic systems after all other modes have decayed. Each block has matrix elements with variance J_k^2/L, and the physics of SSB is modeled by the condition $J_0 \gg J_{k\neq 0}$. This leads to a diffusive motion in the order parameter space.

The H_k could also have more structure. For example, one could have $H_k = g_k(H_0)$ for some simple slowly-varying matrix function g. A salient case is where g_k is roughly constant over the energy range of H_0, and is 0 unless $k = \pm 1$. In this world, the Hamiltonian can be roughly written as $I \otimes H_0 + K \otimes g(H_0)$, where $K_{ij} = \delta_{i,j+1} + \delta_{i,j-1}$. We can simultaneously diagonalize H_0 and K to diagonalize this matrix. Given an eigenstate $|\psi\rangle$ of H_0 with $H_0 |\psi\rangle = E_\psi |\psi\rangle$ and an eigenstate $|q\rangle$ of K with $K |q\rangle = \lambda_q |q\rangle$ the energy of $|q\rangle \otimes |\psi\rangle$ is $E_\psi + g(E_\psi)\lambda_q$. Especially when n is large, it makes sense to talk about a dispersion relation depending on q. We can think of the state as a particle with internal degrees of freedom propagating ballistically in order parameter space. In a real life system, the most realistic choice of g would simply be an identity matrix, corresponding to a temperature/energy-independent kinetic term for the Goldstone mode. More general gs allow more complicated dependence. And most realistic systems add an ordered matrix such as the identity in this paragraph to a disordered random collection as in the previous paragraph. For instance, in the continuous case one can consider a bound state of interacting atoms in a physical Mexican hat potential. The system has Goldstone mode θ, the angle to the center of mass of the atoms. The transition matrix is $\frac{1}{2I}\partial_\theta^2$, where I is the moment of inertia of the system. By the eigenstate thermalization hypothesis, this moment of inertia has some rough slow dependence of E, but also some RMT contributions.

A.   The SFF With a Purely Random Kinetic Term

Consider first the case of purely random H_k. To calculate the SFF, it is convenient to work with two copies of the system with total Hamiltonian. If the Hamiltonian of a single copy of the system is H_{sys},

$$H_{\text{tot}} = H_{\text{sys}} \otimes I - I \otimes H_{\text{sys}}^*. \tag{9}$$

The SFF of a single copy of the system is then

$$\text{SFF}(T, 1) = \overline{\text{tr} \exp(-iH_{\text{tot}}T)}. \tag{10}$$

Note that the complex conjugation in the definition of H_{tot} is added for convenience.

As a reminder, if all the $H_{k\neq0}$ are set to zero, then the order parameter is frozen and the SFF is controlled by the diagonal H_0 blocks. H_0 is a random matrix, so its SFF is given by the random matrix result, but since each block is identical and there are n blocks, the total SFF of the system is n^2 times the random matrix result. Nonzero $H_{k\neq0}$ blocks cause the order parameter to fluctuate in time. We want to calculate the modifications to the SFF by summing over quantum trajectories of the order parameter. This calculation is viewed as a kind of path integral calculation on the doubled system with Hamiltonian H_{tot}. The averaging operation discussed in the introduction corresponds here to averaging over the elements of the H_ks.
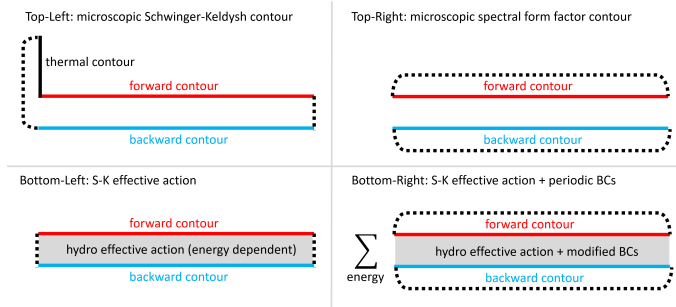
FIG. 2. A pictorial representation of how one goes from an exact expression for the SFF (top right) to an effective action on periodic boundary conditions (bottom right). The process causes an effective coupling between same-time variables on opposite contours, analogous to how performing this procedure on on a Schwinger-Keldysh contour also introduces coupling between the two legs. In fact, the two procedures will produce the same action, up to exponentially small corrections.
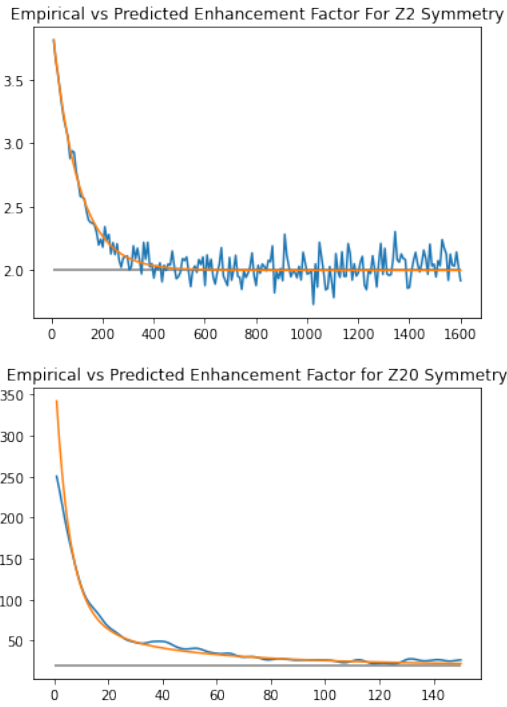

FIG. 3. The predicted (orange) vs displayed (blue) enhancements for two Z_n-symmetric Hamiltonians. The pure RMT prediction for a matrix with that symmetry group is in gray.

Let's take a moment to consider what sort of formula we should expect for this sum. As with other ramp-related quantities, it is helpful to juxtapose it with Schwinger-Keldysh/CTP contour $e^{-\beta H} e^{iTH} e^{-iTH}$. We can generally factor the Hilbert space into slow degrees of freedom Φ and fast degrees Ψ. In this paper Φ is the order parameter of a symmetry and the various Φ sectors will be related by that symmetry. But Φ could
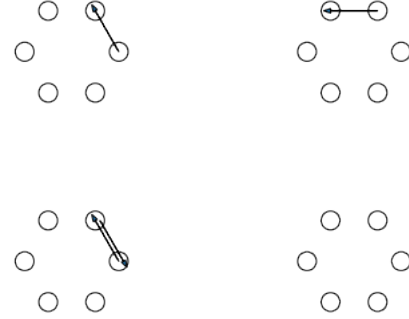

FIG. 4. The two types of processes. In the top figure, the first and second copies of the system both have identical jumps. In the bottom figure, the first system jumps and jumps back.

just as well include conserved quantities, gauge bosons, or fermion degrees of freedom with a chiral symmetry.

At any point along the contour we have some density matrix $\rho_{\Phi\Psi} \propto e^{-\beta H}$ indicating the density for both order parameter ϕ and the microscopic degrees of freedom ψ (equivalently, this can be viewed as a state of the two-replica system). If we trace out the Ψ modes, ρ_Φ evolves by multiplication by a time-dependent superoperator, $e^{-\mathrm{Trans}(E)T}$, which is generated from a sort of transfer matrix $\mathrm{Trans}(E) : \Phi^2 \to \Phi^2$ that acts on two replicas of the order parameter space. More precisely,

$$\rho_\Phi(T) = \int dE\, e^{-\mathrm{Trans}(E)T} P_E \rho_\Phi(0) \qquad (11)$$

where P_E projects down to an energy window around E.

If it weren't for the symmetry relating the different ϕs, the superoperator $\mathrm{Trans}(E)$ would approximately annihilate doubled-system states $|\phi_1\rangle |\phi_1'\rangle$ with $\phi_1 \neq \phi_1'$. One intuitive reason for this is that with no similarities between the different ϕ sectors we would be free to add a phase to one and not the others. The only terms that would be invariant under this are $\phi_1\phi_2 \to \phi_1\phi_2$ or $\phi_1\phi_1 \to \phi_2\phi_2$. Averaging over disorder, the first of these has to have zero amplitude in $\exp(-\mathrm{Trans}(E)T)$ because we can add some small amount of energy to one sector but not another and rotate the term by a relative phase. As such, without symmetry between sectors we could essentially replace

$$\mathrm{Trans}(E)_{\phi_1\phi_1'\phi_2\phi_2'} \to \delta_{\phi_1\phi_1'}\delta_{\phi_2\phi_2'}\mathrm{Trans}(E)_{\phi_1\phi_2}. \qquad (12)$$

This is the case considered in [22]. But here the symmetry allows for constructive interference, and we need to trace over $\mathrm{Trans}(E)$ in its full glory. Equation 12 is no longer valid. If the matrix elements of H_0 and $H_{k\neq 0}$ are independent random numbers, then there are only two perturbative processes that contribute to the path integral and thus to $\mathrm{Trans}(E)$, shown in figure 5. The first is when both systems go from (possibly distinct) ψ_is to

(possibly distinct) ψ_fs using the same H_k. The second is when one of the two systems goes from ψ_i to ψ_f to ψ_i, using H_k and $H_k^\dagger$. The both-sides-jump perturbation is parameterized by the two times t_1, t_2 of the two jumps, and by the energies E_{final} after the jump. The amplitude for any one of these processes for any given ψ_f depends on the square of the magnitude of the matrix element. The double integral over time gives one "center of mass" time and an energy delta function,

$$\int dt_1 dt_2 |\langle \psi_f| H_k |\psi_i\rangle|^2 e^{i(E_{\text{final}} - E_{\text{init}})(t_2 - t_1)} \propto T\delta(E_{\text{final}} - E_{\text{init}}). \tag{13}$$

Disorder-averaging over the matrix elements and converting the sum over final states into an integral, the total amplitude for this process to happen once after time T is a quantity depending on the root-mean-square (rms) matrix elements of H_k, which can be simplified using Fermi's golden rule.

$$\sum_{E_{\text{final}}} \int dt_1 dt_2 |H_k \text{ rms matrix element}|^2 e^{i(E_{\text{final}} - E_{\text{init}})(t_2 - t_1)}$$
$$= r_k(E_{\text{init}})T \tag{14}$$

where the rate r_k is

$$r_k(E) = 2\pi |H_k \text{ rms matrix element}|^2 \rho(E) \tag{15}$$

with $\rho(E)$ the density of states; this is just the Fermi's golden rule. This is thus an amplitude $r_k(E)$ per unit time for a process that multiplies the states of both replicas by M_k.

For the $\psi_1 \to \psi_2 \to \psi_1$ processes, we have the condition $t_2 > t_1$, which produces a factor of $\frac{1}{2}$ relative to when the two jumps happen on separate contours. However there is a factor of 2 because the two jumps can happen in either replica of the system. The net result of these two jumps is that both subsystems are in the same sector, so the total amplitude is $r_k(E)$ for a process that multiplies the states of both replicas by I.

Summing all processes we get

$$\text{Trans}(E) = \sum_{k \neq 0} r_k(E) \left(I \otimes I - M_k \otimes M_k \right) =$$
$$\frac{1}{2}\sum_{k \neq 0} r_k(E) \left(2I \otimes I - M_k \otimes M_k - M_{-k} \otimes M_{-k} \right) \tag{16}$$

The second equality follows from $r_k = r_{n-k}$ and $M_{-k} = M_{n-k} = M_k^\dagger$. Note that this is a tensor product of different spaces, two factors of Φ, rather than a factor of Φ and a factor of Ψ. This means that the transfer matrix has four Φ indices, i.e. it is a superoperator. The enhancement factor to the SFF is just $\int dE f^2(E) \, \text{tr} \, e^{-\text{Trans}(E)T}$. We can see this prediction borne out in Figure 3.

Note that the late time enhancement of the SFF follows from the number of zero modes of $\text{Trans}(E)$. The general
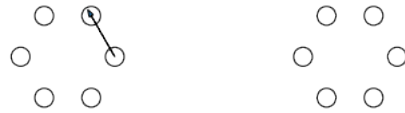

FIG. 5. An additional figure, where one system jumps from one sector to another, with a coherent amplitude $g_k(E)$

spectrum of the transfer matrix is obtained from states of the form

$$\sum_{\phi\phi'} e^{iq\phi + iq'\phi'} |\phi\rangle |\phi'\rangle, \tag{17}$$

where q, q' are $\frac{2\pi}{n}$ times an integer $\in \{0, \cdots, n-1\}$. The q, q' state has eigenvalue

$$\sum_{k \neq 0} r_k(1 - \cos k(q + q')). \tag{18}$$

The set of zero modes is given by $q = -q'$, hence, there are n zero modes of $\text{Trans}(E)$. This implies that the late time SFF enhancement is n, as expected in a situation where the symmetry has been restored.

B. The SFF With A Mixed Kinetic Term

The effect of adding in a slowly varying kinetic term like $H_k = g_k(H_0)$ is to add in additional processes which can contribute to transfer matrix. More precisely, let's break our Hamiltonian into

$$H_{\text{sys}} = \sum_k M_k \otimes H_k + \sum_k M_k \otimes g_k(H_0), \tag{19}$$

with the H_ks fully independent of H_0 and the g_ks reasonably slowly varying analytic functions of small enough value to be treated by perturbation theory. We will get an additional contribution to the transfer matrix of

$$\text{Trans}(E)_{\text{kinetic}} = -i \sum_k g_k(E) \left(M_k \otimes I - I \otimes M_k \right), \tag{20}$$

which is just the amplitude of a single jump caused by the perturbation f_j. Here again the tensor product refers to two factors of Φ. We see this prediction confirmed in Figure 6.

III. GENERAL DISCRETE SSB

In this section we will generalize from the simple Abelian case from last section and consider an arbitrary (finite) non-Abelian symmetry group G. Apart from its intrinsic interest, one motivation for considering Z_n SSB
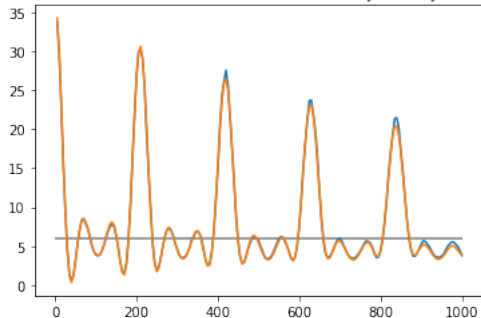
FIG. 6. Each H_0 is a 500 by 500 matrix, with consecutive blocks connected by $c_1 R + c_2 I$, where R is a random complex matrix, $c_1=0.02$, and $c_2=0.03$. The numerical results are in blue, the analytic predictions are red. The pure RMT prediction for a matrix with that symmetry group is in gray.

is that we can approach $U(1)$ SSB by taking the limit $n \rightarrow \infty$. Unfortunately, there is no way to make arbitrarily good approximations of continuous non-Abelian groups with finite groups. Nonetheless, such discrete non-Abelian groups do appear in nature and can be spontaneously broken. A standard example is a valence bond solid, which preserves the $SU(2)$ spin symmetry but breaks the crystal point group symmetry.

We will start with some discrete symmetry group G, and some order parameter $\phi \in \Phi$. It will be helpful to think of ϕ not necessarily as a number, but as a thing which transforms under G. For instance in a ferromagnet, ϕ is a unit vector on the unit sphere S^2 indicating the direction of polarization. It transforms under the rotation group $G = SO(3)$. In effect, each element $g \in G$ induces a function $f_g(S^2) \rightarrow S^2$, where $f_{g_1} \circ f_{g_2} = f_{g_2 g_1}$. Mathematicians call this a group action [27]. We say that Φ is the orbit of ϕ under G. There is some (possibly trivial) subgroup G' which maps ϕ to itself. This is called the stabilizer of ϕ, and we have $|G'||\Phi| = |G|$.

In addition to the order parameter space Φ, we will also have an 'internal' state $|\psi\rangle \in \Psi$, which transforms trivially under G. Once again we will set Φ to be N-dimensional. There is again some Hamiltonian random H_0 (variance J_0^2/N) within each of those ϕ subspaces, and a number of different matrices $\{H_i\}$ connecting different sectors. Let's again choose each element independently, with variance J_i^2/N, and where $J_i \ll J_0$. Because of the symmetry, each of the H_is will show up in more than one place.

Consider the example of the group D_4, which can be realized as the symmetry group of a square. This is a discrete group with 8 elements generated by $90°$ counter-clockwise rotations (a) and reflections about one diagonal (b). These obey the relations $a^4 = b^2 = e$ (e is the identity) and $ab = ba^{-1}$. The center is $\{e, a^2\}$. The group has four one-dimensional irreps and one two-dimensional irrep.

The order parameter corresponds to a choice of one

corner of the square, so it takes 4 values. Suppose that the only allowed jump of the order parameter corresponds to the rotation a and the inverse rotation $a^{-1} = a^3$. Let the corresponding operators acting on Ψ be H_a and H_{a^3}. Invariance under the reflection b requires $H_a = H_{a^3}$. The block matrix form of the system Hamiltonian is thus

$$H = \begin{pmatrix} H_0 & H_a & 0 & H_a \\ H_a & H_0 & H_a & 0 \\ 0 & H_a & H_0 & H_a \\ H_a & 0 & H_a & H_0 \end{pmatrix} \quad (21)$$

More generally, the system Hamiltonian is given by

$$H_{\text{sys}} = I \otimes H_0 + \sum_i M_i \otimes H_i + M_i^\dagger \otimes H_i^\dagger \quad (22)$$

where the M_is are matrices which are mostly 0s with a few 1s connecting pairs of ϕs related by the same group action. More precisely, the M_is are indexed by Φ^2/G. That is, two ordered pairs of ϕs share the same M_i if their pairs are connected by an element of G.

In the non-Abelian case, we only consider the situation where the H_i matrices are completely random and uncorrelated with H_0 and each other. In this case, the walk over the order parameter space is a random walk. This is a good model for, say, a particle stuck in one of k identical wells (with enough internal structure to be chaotic), which jumps between wells using diffusive Poisson-frequency instantons. If the wells are placed on the vertices of a k-gon, the relevant group would be dihedral group D_k. If they were the vertices of a $k-1$-simplex, the relevant group would be the symmetric group S_k. This paradigm can be mathematically modeled.

A. The SFF With a Purely Random Kinetic Term

As in the Abelian case, to calculate the SFF, we want to have two copies of the system with total Hamiltonian

$$H_{\text{tot}} = H_{\text{sys}} \otimes I - I \otimes H_{\text{sys}}^* \quad (23)$$

The SFF of a single copy of the system is still

$$\text{SFF}(T, 1) = \overline{\text{tr} \exp(-iH_{\text{tot}}T)}. \quad (24)$$

The analysis proceeds as in the Abelian case, in particular, we still have a path integral picture and treat the order parameter jumps in perturbation theory.

For each jump type M_i, there is a corresponding rate r_i. The transfer matrix is

$$\text{Trans}(E) =$$

$$\frac{1}{2} \sum r_i(E) \Big(\{M_i, M_i^T\} \otimes I + I \otimes \{M_i, M_i^T\}$$

$$-2M_i \otimes M_i - 2M_i^T \otimes M_i^T \Big)$$

$$= \frac{1}{2} \sum r_i(E) \Big((M_i \otimes I - I \otimes M_i^T)(M_i^T \otimes I - I \otimes M_i)$$

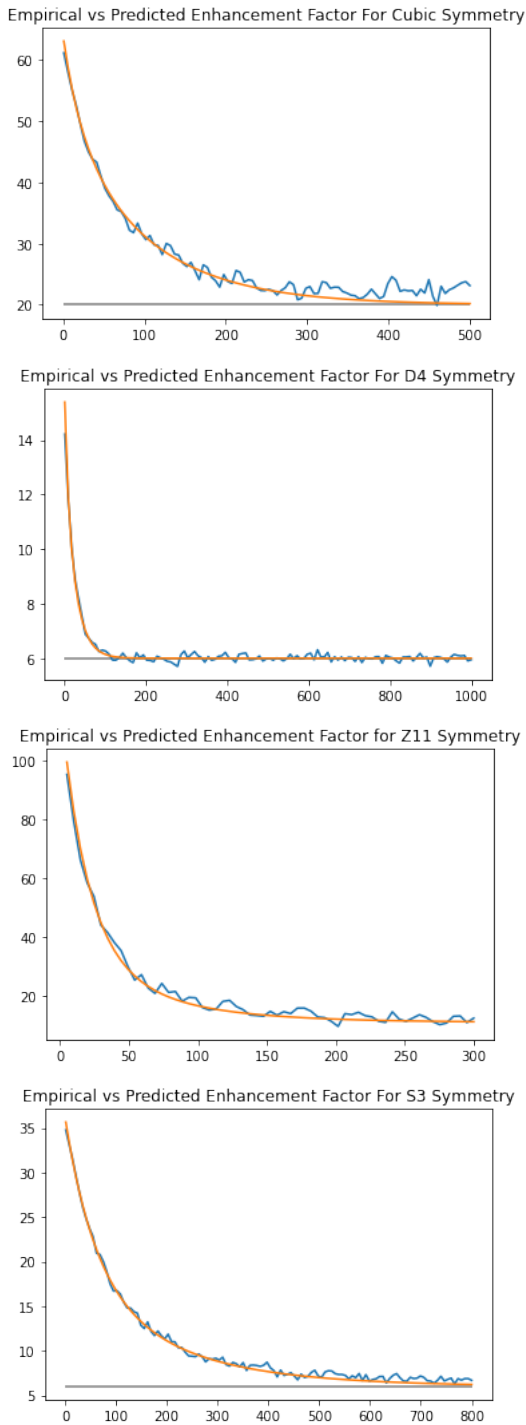$$+ (M_i^T \otimes I - I \otimes M_i)(M_i \otimes I - I \otimes M_i^T) \Big), \quad (25)$$

FIG. 7. The predicted (orange) vs displayed (blue) enhancements for a variety of group actions: Cubic Symmetry acting on the vertices of a cube, D_4 acting on the corners of a square, Z_{11} acting on 11 points, S_3 acting by multiplication on elements of S_3.

where we used the fact that the M_i have real matrix elements. The enhancement factor to the SFF is $\mathrm{tr}\, e^{-\mathrm{Trans}(E)T}$. Figure 7 shows the predicted and realized enhancements for various choices of G and Φ. In appendix we show that at long times this gives the RMT result consistent with symmetry G.

IV. ABELIAN SSB HYDRO AND THE SFF

We now turn to the description of SFFs in systems with spontaneously broken continuous symmetries. We begin with the Abelian case, focusing again on the simplest case of $U(1)$ symmetry and a charge 1 order parameter. Here the full power of the corresponding Schwinger-Keldysh hydro effective theory is revealed, so we first review that theory and then describe its modification to treat the SFF as in [22].

A. Quick Review of Hydrodynamics

At the broadest level, hydrodynamics is the program of creating effective field theories (EFTs) for systems based on the principle that macro-physics should be driven primarily by conservation laws. We will utilize the technology of the CTP formalism. For an accessible introduction, see [28], and for more details see [15, 16]. Additional information about fluctuating hydrodyamics can be found in [17–20]. We first discussed the application of hydro EFTs for non-SSB spectral statistics in [22].

The CTP formalism starts with the following partition function of a Schwinger-Keldysh contour:

$$Z[A_1^\mu(t,x), A_2^\mu(t,x)] = \mathrm{tr}\left(e^{-\beta H}\mathcal{P}e^{i\int dt d^d x A_1^\mu j_{1\mu}}\mathcal{P}e^{-i\int dt d^d x A_2^\mu j_{2\mu}}\right),$$

(26)

where $\mathcal{P}$ is a path ordering on the Schwinger-Keldish contour.

For $A_1 = A_2 = 0$, this is just the thermal partition function. Differentiating with respect to the As generates insertions of the conserved current density j_μ along either leg of the Schwinger-Keldysh contour. Thus Z is a generator of all possible contour ordered correlation functions of current operators.

One way to enforce the conservation law $\partial^\mu j_{i\mu} = 0$ is to require

$$Z[A_1^\mu, A_2^\mu] = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\left(i\int dt dx W[B_{1\mu}, B_{2\mu}]\right),$$
$$B_{i\mu}(t,x) = \partial_\mu \phi_i(t,x) + A_{i\mu}(t,x).$$

(27)

Here the fields ϕ_i have been "integrated in" and represent the slow fluctuating modes of the system. Insertions of the currents are obtained by differentiating Z with respect to the background gauge fields $A_{i\mu}$. A single such

functional derivative gives a single insertion of the current, and so one presentation of current conservation is the identity $\partial_\mu \frac{\delta Z}{\delta A_{i\mu}} = 0$.

This is derived as follows. Since Z only depends on $A_{i\mu}$ via the combined field $B_{i\mu}$, the functional derivative of Z reduces to a functional derivative of the action,

$$\frac{\delta Z}{\delta A_{i\mu}} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \frac{\delta}{\delta(\partial_\mu \phi_i)} \exp\left(i \int dt dx W\right). \quad (28)$$

Acting with ∂_μ and suppressing integration variables, we get

$$\int \partial_\mu \frac{i\delta W}{\delta(\partial_\mu \phi_i)} e^{i \int W}. \quad (29)$$

Because W does not depend explicitly on ϕ_i (only on its derivatives), this is the function integral of a functional total derivative and hence vanishes.

The functional W is not arbitrary. The key assumption of hydrodynamics is that the action W is local. Moreover, when expressed in terms of

$$\begin{aligned} B_a &= B_1 - B_2, \\ B_r &= \frac{B_1 + B_2}{2}, \end{aligned} \quad (30)$$

there are several constraints which follow from unitarity:

- W terms all have at least one power of B_a, that is $W = 0$ when $B_a = 0$,

- Terms odd (even) in B_a make a real (imaginary) contribution to the action,

- A KMS constraint imposing fluctuation-dissipation relations.

When calculating SFFs, one typically sets the external sources A to zero, so the action can be written purely in terms of the derivatives of the ϕs.

The ϕs have a physical interpretation as phases transforming under the $U(1)$ symmetry. Performing a symmetry operation corresponds to adding a constant to ϕ. If the symmetry is compact, this requires that adding $2\pi R$ to ϕ is actually a gauge transformation that doesn't change the state at all.

B. The Hydrodynamic SFF

As discussed in [22] and illustrated in figure 2, in order to calculate a spectral form factor, one performs the hydrodynamic integral with periodic boundary conditions in time. For instance, in a system with only energy conservation, the simplest hydro Lagrangian is

$$L = C\partial_t \phi_a \partial_t \phi_r. \quad (31)$$

One can show by taking derivatives with respect to the As that the energy is $E = \int d^d x C \partial_t \phi_r$. So the SFF becomes

$$\mathrm{SFF}(T, f) = \int \frac{\mathcal{D}E\mathcal{D}\phi_a}{2\pi} f^2(E) \exp\left(-i \int dt \phi_a \partial_t E\right) \quad (32)$$

This is a purely Gaussian integral. One can show that (subsection) that integrating out the nonzero frequencies with proper regularization leaves us with a prefactor in front of the measure of $(2\pi)^{-1}$. This leaves us with just an integral over the zero modes,

$$\mathrm{SFF}(T, f) = \int \frac{dE d\phi_a}{2\pi} f^2(E) = \frac{T}{2\pi} \int dE f^2(E), \quad (33)$$

the well-known result for GUE systems (for GOE there is an extra factor of two because one can reverse time on one contour relative to the other).

C. Hydro and the Symmetry-Broken Spectral Form Factor

Before we get into details, let's ask the most basic question: why should we expect spontaneous symmetry breaking to have any effect whatsoever on the spectral hydrodynamic path integral? For systems with spatial extent, SSB allows novel terms in the hydro action[16]. But it is impossible to write such terms in 0+1d. However, we know from previous sections that in the case of discrete symmetry, SSB has a clear signature in the connected SFF at short times.

The resolution lies in the details, and in particular, the boundary conditions. In conventional fluctuating hydrodynamics, when we break our field into ϕ_a and ϕ_r, it is standard practice to consider an overall constant addition to ϕ_r as a gauge symmetry. This is because phases aren't observable, only differences in phases like ϕ_a or $\phi_r(t_1) - \phi_r(t_2)$ are. But once a symmetry is spontaneously broken, phases do become observable. Water is invariant under a total rotation, but for ice you get a distinct quantum state.

What does this mean for boundary conditions? When calculating the SFF using conventional fluctuating hydrodynamics, it is sufficient to ensure that ϕ_a and $\partial_t \phi_r$ are periodic in time. But for SSB SFF, ϕ_r itself needs to be periodic. We can think of restoring the symmetry as gauging away ϕ_r.

Of course, it is itself a phase defined on a circular manifold, and the path integral can wrap around that manifold any integer number of times. This introduces a summation into our calculation. We will handle the details below.

D. The SSB Hydro Action

We can write the Lagrangian as

$$L = M\partial_t\phi_a(\partial_t\phi_r + b\partial_t^2\phi_r) + iM\frac{b}{2\beta}(\partial_t\phi_a)^2 \quad (34)$$

plus terms with more derivatives and/or fields. We show in appendix that this Lagrangian can be derived as a continuous limit of the discrete Z_n model in the previous section. The nature of spontaneous symmetry breaking means that in order to study it in 0+1d, we need some sort of large-N limit. Thus, in our case, we are justified in dropping nonlinearities and interactions.

E. Hamiltonian Approach

We have formulated the SFF as a path integral involving a hydro-like effective action. In this subsection, we evaluate this path integral by converting to a corresponding Hamiltonian description. In the next two subsections, we give a direct Lagrangian calculation. Again, the contribution of charge to the SFF is

$$Z = \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left(i\int dt L\right)f(q_1)f(q_2),$$
$$L = M\partial_t\phi_a(\partial_t\phi_r + b\partial_t^2\phi_r) + iM\frac{b}{2\beta}(\partial_t\phi_a)^2, \quad (35)$$

where f is some function (perhaps a Gaussian) to regularize over the infinite sum over charges/momenta.

We make use of the fact that a path integral in one dimension is equivalent to a quantum mechanics problem. The Hilbert space is just a suitable space of functions on the configuration space of the path integral, in this case a basis is given by $U(1)^2$.

Viewed as an effective field theory, this path integral arrives from integrating out a number of microscopic modes. But like any Lagrangian path integral it is also equivalent to a Schrodinger-like evolution on the square of the Goldstone manifold (the imaginary ϕ_a^2 part of the action corresponds to a non-unitarity for the fictitious Schrodinger dynamics).

To go from a Lagrangian quadratic in velocities to a Hamiltonian quadratic in momenta essentially involves inverting the Lagrangian. It is convenient to convert back to ϕ_1 and ϕ_2. Dropping the dissipative term in the equations of motion, the Lagrangian is

$$L = \frac{M}{2}(\partial_t\phi_1)^2 - \frac{M}{2}(\partial_t\phi_2)^2 + i\frac{Mb}{2\beta}(\partial_t\phi_1 - \partial_t\phi_2)^2. \quad (36)$$

The corresponding canonical momenta are

$$\pi_1 = \frac{\partial L}{\partial(\partial_t\phi_1)} = M\partial_t\phi_1 + i\frac{Mb}{\beta}(\partial_t\phi_1 - \partial_t\phi_2) \quad (37)$$

and

$$\pi_2 = \frac{\partial L}{\partial(\partial_t\phi_1)} = -M\partial_t\phi_2 - i\frac{Mb}{\beta}(\partial_t\phi_1 - \partial_t\phi_2). \quad (38)$$

The Hamiltonian for this fictitious particle on the doubled system is then

$$H_{eff} = \pi_i\partial_t\phi_i - L = \frac{\pi_i^2}{2M} - \frac{\pi_2^2}{2M} - i\frac{b}{2\beta M}(\pi_1+\pi_2)^2. \quad (39)$$

H_{eff} is not the Hamiltonian of any underlying system, rather it is a mathematical trick that comes from evaluating the Lagrangian of an effective field theory.

Since each ϕ_i is periodic with period $2\pi R$, the eigenvalues of π_i are quantized in terms of integer charges q_i to be $\frac{q_i}{R}$. Hence, the eigenstates of the Hamiltonian are

$$H_{eff}\psi_{q_1,q_2} = E_{q_1,q_2}\psi_{q_1,q_2},$$
$$E_{q_1,q_2} = (q_1^2 - q_2^2)\frac{1}{2MR^2} - i\frac{b}{2\beta MR^2}(q_1 + q_2)^2, \quad (40)$$
$$\psi_{q_1,q_2}(\phi_1,\phi_2) = e^{iq_1\phi_1/R}e^{iq_2\phi_2/R}.$$

The non-decaying solutions correspond to $q_1 = -q_2$, $E_{q_1,q_2} = 0$. At long times, these are the only remaining contributions, and the long-time SFF is just equal to the number of charge sectors allowed in the sum. At general times, the overall sum is

$$Z = \text{tr}\left[f(q_1)f(q_2)e^{-iH_{eff}T}\right] = \sum_{q_1,q_2} f(q_1)f(q_2)\exp(-iE_{q_1,q_2}T). \quad (41)$$

It is worth taking a moment to point out that in the large-N limit necessary for 0d spontaneous symmetry breaking, the fictitious Es in expressions (40) and (41) are $1/N$ quantities, meaning the slowest decay time is actually extensive in the system size.

F. Lagrangian Approach

In this section, we will start be developing technology for arbitrary Gaussian SFFs. We have our path integral of $\exp\left(i\int \phi_a D\phi_r + \phi_a D_{aa}\phi_a\right)$, where $D = a\prod_{k=1}^K(\partial_t - \lambda_k)$, with $\text{Re}(\lambda_k) < 0$ for all k, is some differential operator. The form of the action means that the determinant depends only on D, and we will set D_{aa} to zero throughout this section.

We proceed via spacetime discretization, and say that $\mathcal{D}\phi_a\mathcal{D}\phi_r = \prod_{j=0}^{T/\Delta t-1}\frac{c}{2\pi}d\phi_{aj}d\phi_{rj}$, the factor c in the measure is to be determined. We compute the path integral by going to the Fourier basis

$$\tilde{\phi}_i(\omega) = \sum_j \frac{e^{i\omega t_j}}{\sqrt{T/\Delta t}}\phi_i(t_j), \quad (42)$$

where $t_j = j\Delta t$ and $\omega = 2\pi n/T$ for $n = 0,\cdots,T/\Delta t-1$. The action becomes

$$\sum_j \Delta t\phi_a(t_j)[D\phi_r](t_j) =$$
$$\sum_\omega \Delta t\tilde{\phi}_a(-\omega)a\prod_{k=1}^K\left(\frac{e^{-i\omega\Delta t}-1}{\Delta t} - \lambda_k\right)\tilde{\phi}_r(\omega). \quad (43)$$

With only the ra cross terms, the $\tilde{\phi}(-\omega)$ integral just gives 2π times a delta function of $\Delta t D\tilde{\phi}_r(\omega)$.

Hence, the path integral evaluates to

$$Z = \prod_\omega \frac{c}{a\Delta t} \prod_{k=1}^K \left(\frac{e^{-i\omega\Delta t} - 1}{\Delta t} - \lambda_k \right)^{-1}. \quad (44)$$

What constant c in the measure keeps the integral from blowing up? This is pretty clearly a UV question. We want some c such that

$$Z = \prod_{n=0}^{T/\Delta t - 1} \frac{c}{a\Delta t} \prod_{k=1}^K \left(\frac{e^{-2\pi i n\Delta t/T} - 1}{\Delta t} - \lambda_k \right)^{-1} = \\ \prod_{n=0}^{T/\Delta t - 1} \frac{c}{a\Delta t^{1-K}} \prod_{k=1}^K \left(e^{-2\pi i n\Delta t/T} - 1 - \lambda_k\Delta t \right)^{-1} \quad (45)$$

doesn't blow as $\Delta t \to 0$. If we switch the order of the two products we get

$$Z = \left(\frac{c}{a\Delta t^{1-K}} \right)^{T/\Delta t} \prod_{k=1}^K \left(1 - e^{\lambda_k T} \right)^{-1} \quad (46)$$

For this not to blow up, we need to regularize with $c = a\Delta t^{1-K}$.

Now that we have a general technology for regularizing CTP integrals, let's apply it to the specific case of the hydro Lagrangian $L = M\partial_t\phi_a(\partial_t\phi_r + b\partial_t^2\phi_r) + iM\frac{b}{2\beta}(\partial_t\phi_a)^2$. The corresponding differential operator is

$$D = -Mb\partial_t^2(\partial_t + b^{-1}). \quad (47)$$

In this case, $a = -Mb$ (although only the magnitude of a is relevant), $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = -b^{-1}$. However, there are two things keeping the hydro path integral from being a vanilla Gaussian integral: the zero-frequency modes where one integrates over the hydro manifold, and the topological aspect of trajectories that wind around the manifold. For now, we will focus on the saddle point with no winding.

In the non-winding sector, the non-zero-frequency modes are treated as Gaussian variables, while the integral zero frequency modes are integrated over exactly. The zero modes $\tilde{\phi}_i(0)$ range over a period of length $2\pi R\sqrt{T/\Delta t}$, and there are two of them. The path integral in the order parameter sector is thus

$$Z_{\text{no-winding}} = \\ \frac{c}{2\pi}(2\pi R)^2 \frac{T}{\Delta t} \prod_{n=1}^{T/\Delta t - 1} \frac{c}{a\Delta t^{1-K}} \prod_k \left(e^{-2\pi i n\Delta t/T} - 1 - \lambda_k\Delta t \right)^{-1}. \quad (48)$$

The $\lambda_1 = \lambda_2 = 0$ terms in the product over k give $\left(\frac{\Delta t}{T} \right)^2$, and the $\lambda_3 = -1/b$ term gives $\frac{\Delta t}{b(1-e^{-T/b})}$. These pieces combine to give

$$Z_{\text{no-winding}} = \frac{c}{2\pi} \frac{\Delta t}{T} (2\pi R)^2 \frac{\Delta t}{b(1-e^{-T/b})}. \quad (49)$$

After inserting in our expression for $c = Mb\Delta t^{-2}$ we have

$$Z_{\text{no-winding}} = \frac{1}{2\pi} \frac{M}{T} (2\pi R)^2 \frac{1}{(1-e^{-T/b})}. \quad (50)$$

If we wanted, we could add in higher derivatives to our original action. This would just result in more decaying terms corrections like the $1 - e^{-T/b}$ in the denominator.

But there is a complication to equation (50)! The periodicity of ϕ actually means that there is an infinitude of saddle-points. In particular, the saddle-points of the action are parameterized by n_1, n_2, winding numbers of ϕ_1, ϕ_2 around the circle. As a function of $n_r = \frac{n_1+n_2}{2}, n_a = n_1 - n_2$, the winding contribution to the action is

$$\Delta S_{\text{winding}}(n_r, n_a) = M\frac{(2\pi R)^2}{T} \left(n_a n_r + \frac{ib}{2\beta} n_a^2 \right) \quad (51)$$

The full path integral is obtained by summing over these modes, and since each winding sector has the same Gaussian part, we get

$$Z = \\ \frac{1}{2\pi} \frac{M(2\pi R)^2}{T(1-e^{-T/b})} \sum_{n_r, n_a} \exp\left(iM\frac{(2\pi R)^2}{T} n_a n_r - M\frac{(2\pi R)^2}{T} \frac{b}{2\beta} n_a^2 \right). \quad (52)$$

G. Dealing With the Discrete Sum

Equation (52) is a divergent sum that needs to be regulated. In particular, there are an infinite number of $n_a = 0$ solutions that add up. To regularize them, we need to remember that n_r is proportional to the momentum/charge of the system, and so to get a finite enhancement we should only be summing over a finite set of charges, rather than all charges from $-\infty$ to ∞ (or the allowed set of microscopic charges). Then we can evaluate the path integral with an $f(Q)$ insertion, where f is a regulating function. (For now we will assume it can be naively inserted and removed at will, and will analyze it more carefully in the next subsection.) We get

$$Z = \frac{1}{2\pi} \frac{M(2\pi R)^2}{T(1-e^{-T/b})} \sum_{n_r, n_a} f(Q_1)f(Q_2)\times \\ \exp\left(iM\frac{(2\pi R)^2}{T} n_a n_r - M\frac{(2\pi R)^2}{T} \frac{b}{2\beta} n_a^2 \right). \quad (53)$$

Here charge Q_i is related to n_i by

$$Q_i = 2\pi MR^2 n_i/T. \quad (54)$$

Moreover, in writing Eq. (52) we implicitly assumed that f is slowly varying so that the Gaussian part of the path integral is not appreciably modified.

Note that this expression for Q in terms of winding numbers is such that, if we use it inside the path integral

with an $f(Q)$ insertion, then we recover the correct sum over discrete charges of $f(Q)$ in the final answer. However, the quantization of Q in the path integral arising from the winding sectors is different than the microscopic quantization. Either way of doing the sum, direct microscopic calculation or path integral calculation, will give the same answer, and they are related, roughly speaking, by a Poisson resummation. The situation is similar to the comparison between the Hamiltonian and Lagrangian descriptions of a particle on a ring.

For long times, we can rewrite n_a in terms of Q_a to see that the $\propto -Q_a^2 T$ term in the exponent suppresses $Q_a \neq 0$ contributions. The path integral is thus

$$\frac{1}{2\pi} \frac{M(2\pi R)^2}{T(1 - e^{-T/b})} \sum_{n_r} f^2(2\pi M R^2 n_r/T) = \int dQ f^2(Q),$$

(55)

exactly what we want. Again, this integral over charge Q approximates the discrete sum on the left hand side and is not a sum over the microscopic charges, but in the limit of slowly varying f where Eq. (52) is valid, we obtain the same answer.

Note that the functional substitution $Q = M\partial_t\phi_r$ would give us the exact same action as in the non-SSB case. The reason we get a more complicated short-time behavior is that we are evaluating the path integral in a different way. One way to look at the change is as a change in boundary conditions. Before breaking the symmetry, we treated an overall addition to ϕ_r as a gauge symmetry, and allowed arbitrary additions of it over the period. Whereas now we require that ϕ_r change by a quantized amount equal to $2\pi R n_r$.

H. A More Careful Accounting of f

We can express the filter function as

$$f(q) = \int_0^{2\pi R} dx \tilde{f}(x) e^{iqx}.$$

(56)

Inserting any operator $f(Q)$ into the path integral is thus equivalent to a double integral over basic insertions of the form $e^{iQ_1 x_1} \otimes e^{iQ_2 x_2}$. Since Q_i is conjugate to ϕ_i, this insertion when acting on a state of definite ϕ_i shifts the value by $-x_i$. When placed after the final resolution of the identity in the path integral, one gets a delta function $\delta(\phi_i(0) - \phi_i(T) + x_i)$ that effectively shifts the boundary condition to $\phi_i(T) = \phi_i(0) + x_i + 2\pi R n_i$. The path integral (35) can thus be written as

$$Z = \int dx_1 dx_2 \int \mathcal{D}_{x_1}\phi_1 \mathcal{D}_{x_2}\phi_2 \exp\left(i \int dt L\right) \tilde{f}(x_1)\tilde{f}(x_2),$$

$$L = M\partial_t\phi_a(\partial_t\phi_r + b\partial_t^2\phi_r) + iM\frac{b}{2\beta}(\partial_t\phi_a)^2,$$

(57)

where $\mathcal{D}_x$ is the path integral measure with twisted periodic boundary conditions $\phi_i(T) = \phi_i(0) + x_i + 2\pi R n_i$.

If we repeat the analysis in the last two sections, we find that these twisted boundary conditions don't change the Gaussian part of the path integral at all. Defining $x_r = \frac{x_1+x_2}{2}$, $x_a = x_1 - x_2$, equation (52) becomes

$$Z = \frac{1}{2\pi} \frac{M(2\pi R)^2}{T(1 - e^{-T/b})} \int_0^{2\pi R} dx_1 \int_0^{2\pi R} dx_2 \sum_{n_r,n_a} \tilde{f}(x_1)\tilde{f}(x_2) \exp(\Delta S)$$

(58)

with

$$\Delta S = iM \frac{((2\pi R)n_r + x_r)((2\pi R)n_a + x_a)}{T} -$$
$$M\frac{((2\pi R)n_a + x_a)^2}{T} \frac{b}{2\beta},$$

$$Z = \frac{1}{2\pi} \frac{M(2\pi R)^2}{T(1 - e^{-T/b})} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \tilde{f}(x_1)\tilde{f}(x_2)$$
$$\times \exp\left(iM\frac{x_r x_a}{T} - M\frac{x_a^2}{T}\frac{b}{2\beta}\right),$$

(59)

where in the last equation we extend $\tilde{f}$ to be a period function with period $2\pi R$. This formula is fully general for any regulating function f no matter how rapidly it varies.

Let's investigate its behavior at short times. At short times we can consider the $\tilde{f}$s roughly constant compared to the Gaussian integral. Performing this integral and factoring in the determinant $2\pi\frac{T}{M}$ from the delta function, we have the short-time formula

$$Z_{\text{short time}} = \frac{1}{(1 - e^{-T/b})}(2\pi R)^2 \tilde{f}(0)^2 = \left(\sum_{q=-\infty}^{\infty} f(q)\right)^2.$$

(60)

More generally, we can replace $\tilde{f}(x_1) = \frac{1}{2\pi R}\sum_{q_1'} e^{-iq_1' x_1}$. If we do the same for x_2, and define $q_r' = \frac{q_1'+q_2'}{2}$, $q_a' = q_1' - q_2'$, we have a Gaussian integral that works out to

$$Z = \frac{1}{(1 - e^{-T/b})} \sum_{q_1',q_2'} f(q_1')f(q_2') \exp\left(i\frac{T}{M}q_a'q_r' + \frac{2b}{\beta}\frac{Tq_a'^2}{M}\right).$$

(61)

If we make the $T \gg b$ assumption of subsection , this is the exact same formula as (41). For long T the $q_a' \neq 0$ terms become zero and we have

$$Z_{\text{long time}} = \sum_{q'} f^2(q').$$

(62)

V. NON-ABELIAN HYDRO AND THE SFF

In this section, we treat the case of spontaneous symmetry breaking of continuous non-Abelian symmetries. As in the continuous Abelian case, a modified version of the hydro theory of non-Abelian SSB provides a useful way to formulate the SFF.

A. Overview of Non-Abelian Hydro

In Abelian $U(1)$ hydrodynamics, the phase field ϕ can be regarded as an element of the group $U(1)$ by exponentiating it. In non-Abelian hydrodynamics [29], we replace the exponentiated phase field with a field g that takes values in the (non-Abelian) global symmetry group G. The hydro generating function is expressed as

$$Z_{\text{hydro}}[A_1^\mu(t,x), A_2^\mu(t,x)] =$$
$$\int \mathcal{D}g_1 \mathcal{D}g_2 \exp\left(i \int dt d^d x W[B_{1\mu}, B_{2\mu}]\right), \quad (63)$$
$$B_\mu = A_\mu + i g^{-1} \partial_\mu g,$$

a non-Abelian generalization of Eq. (27). This action and generating function will always have a G premultiplication symmetry. Note that in the case of $G = U(1)$, we recover the original Abelian hydro formalism. For the purposes of this paper we will focus on the unsourced $A = 0$ case, and also restrict to zero spatial dimensions.

Note that any action constructed out of Bs will have a G premultiplication symmetry for both the right and left replica, for a total symmetry group of $G_L \times G_R$. In conventional hydro, this symmetry is broken to the diagonal by the future and past boundary conditions, but these boundary conditions are not present in the SFF case. A representative Lagrangian for 0d non-Abelian hydro would be

$$L = B_{ta}(M + Mb\partial_t)B_{tr} + iM\frac{b}{2\beta}B_{ta}^2 \quad (64)$$

where the Bs are in the adjoint representation of the Lie algebra of G, and there is an implied summation over the representation indices.

Action (64) has two time derivatives. Since we are working in 0+1d QFT, this means there is a quantum mechanics interpretation, just as in the Abelian case there was an interpretation in terms of non-unitary evolution on $U(1)^2$.

B. Hamiltonian Approach for Full SSB

We can use a Legendre transform to go from the Lagrangian (64) to a Hamiltonian,

$$H_{eff} = \frac{1}{2M}\sum L_{i1}^2 - \frac{1}{2M}\sum L_{i2}^2 - i\frac{b}{2\beta M}\left(\sum L_{i1} + L_{i2}\right)^2, \quad (65)$$

where we sum over group generators L_i, which are canonically conjugate to the velocity components of $ig^{-1}\partial_t g$. The sum $\sum_i L_i^2$ is called the Casimir operator L^2, and has a number of important properties. Intuitively, it plays the same role as a Laplacian, but on a group manifold. Just as the Laplacian operator commutes with any momentum operator, L^2 commutes with all elements of the group G. As such it can be shown to be constant within

any irreducible representation of G. For the Abelian case $G = U(1)$, where irreducible representations R are parameterized by integer charges q, $L^2(R) = q^2$.

Let's evaluate

$$Z = \text{tr}\left[e^{-iH_{eff}T}f(L_1^2)f(L_2^2)\right], \quad (66)$$

where we promoted the filter function $f(Q)$ from the Abelian case to $f(L^2)$ in the non-Abelian case. Wavefunctions transform in the square of the regular representation of R_{reg} of G. The regular representation has the contains $|R_i|$ copies of representation R_i. $R_{reg} \times R_{reg}$ can be decomposed as

$$R_{reg} \times R_{reg} = \bigoplus_{\bar{R}} \bar{R}^{\oplus n(\bar{R})_{R_{reg}R_{reg}}}$$
$$n(\bar{R})_{R_{reg}R_{reg}} = \sum_{R_1 R_2} |R_1||R_2|n(\bar{R})_{R_1 R_2}, \quad (67)$$

where $n(\bar{R})_{R_1 R_2}$ is the number of times $\bar{R}$ appears in $R_1 \times R_2$. For instance if R_1 and R_2 are the spin $1/2$ and spin 1 representations of $SU(2)$ and $\bar{R}$ is the spin $1/2$ representation, then $n(\bar{R})_{R_1 R_2} = 1$.

The Hamiltonian depends on the Casimir of each R_i and on the composite system Casimir $(L_1 + L_2)^2$. The enhancement factor is thus

$$Z = \sum_{R_1, R_2, \bar{R}} f(L^2(R_1))f(L^2(R_2))|R_1||R_2||\bar{R}| \times$$
$$\exp\left(-i\frac{[L^2(R_1) - L^2(R_1)]T}{2M} - \frac{bL^2(\bar{R})T}{2\beta M}\right)n(\bar{R})_{R_1 R_2}, \quad (68)$$

Here R_1 and R_2 are representations of G on the first and second replica of the system, whereas $\bar{R}$ is a representation that lives in the Hilbert space of H_{eff} on the full doubled system.

The longtime behavior is given by the $\bar{R}$ trivial case where $L^2(\bar{R}) = 0$. This requires $\bar{R}$ to be the trivial 1D representation. If $\bar{R}$ is trivial, $n(\bar{R})_{R_1 R_2}$ is zero unless R_1 is the complex conjugate of R_2, in which case it is one. So the long time value is

$$Z(T \to \infty) = \sum_{R_1} f^2(L^2(R_1))|R_1|^2. \quad (69)$$

This is exactly what one would predict from random matrix theory. We can also make use of $|R_1||R_2| = \sum_{\bar{R}} n(\bar{R})_{R_1 R_2}|\bar{R}|$ to show that in the short time limit this is

$$Z(T \to 0) = \sum_{R_1}\sum_{R_2} f(L^2(R_1))f(L^2(R_2))|R_1|^2|R_2|^2. \quad (70)$$

Since the regular representation has $|R_i|$ copies of $|R_i|$, this is essentially saying all states of all charges constructively interfere.

C. Partial SBB

If a subgroup G' of G is unbroken, this means overall G' transformations that affect both replicas are unphysical. Thus we should gauge them out. In general, one gauges out a group by inserting a projector

$$P = \frac{1}{\text{Vol } G'} \int_{G'} dg' g' \qquad (71)$$

into the path integral. If we break representations $\bar{R}$ of G into representations of $\bar{R} = \bigoplus \bar{R}' \, G'$, this projects out all nontrivial representations of G'. For instance if $G' = G$, this will remove all the nontrivial $\bar{R}'$ leaving us with the result for RMT with symmetry group G. If G' is trivial, nothing is projected out and equation (68) still holds. In general, different symmetry-breaking patterns can be thought of as different degrees of freedom being observable, which means different projection operators P are needed as boundary conditions.

For example, let's imagine a system with $SO(3)$ symmetry, which is broken down to $SO(2)$ by an order parameter. We are interested in some term in equation (68), for example $R_1 = \mathbf{5}, R_2 = \mathbf{3}, \bar{R} = \mathbf{3}$. We can verify that $n(\bar{R})_{R_1 R_2} = 1$, meaning that when we multiply our representations we get one copy of $\bar{R} = \mathbf{3}$. What happens to this representation under the projection in equation (71)? To answer this, let's break of the representation of G into representations of G'. The vector representation of $SO(3)$ breaks into a scalar and vector representation of $SO(2)$. Integrating g' over the vector representation (as over any nontrivial representation of any group) we get zero. Integrating over the trivial representation we of course get one. So the projection operator projects down from three dimensions to one, and the $|\bar{R}|$ in equation (68) becomes 1 instead of 3.

VI. DISCUSSION

In this paper we extended the understanding of quantum chaotic level repulsion to include systems with spontaneous symmetry breaking. We started with toy models with discrete symmetries, solved them, and confirmed our solutions with exact diagonalization. Next we used hydrodynamics, extending known techniques for unbroken symmetry to the case of spontaneous breaking. The technique is powerful enough to prove the correct long-time behavior, and flexible enough to handle any possible symmetry breaking pattern. Interestingly, we found that SSB typically enhances the SFF beyond that of a system with unbroken symmetry. In terms of the spectral form factor we have schematically

No Symmetry < Symmetry < Spontaneously Broken Symmetry. (72)

One is left wondering how gauge symmetry might fit into that hierarchy.

In terms of SSB, the next step would be to handle higher dimensional systems. At least two interesting phenomena would reveal themselves in this case. First, the presence of sound poles associated with Goldstone modes. In higher dimension spontaneous symmetry breaking allows new terms in the hydrodynamic Lagrangian consistent with unitarity, such as $\phi_a \partial_x^2 \phi_r$. This would allow the hydrodynamic variables to have sound poles, leading to a potentially rich new phenomenology in the SFF.

Higher dimensions also allows the possibility of topological effects. For instance, in a periodic system the Goldstone mode could wrap around the manifold several times. This new topological charge would lead to an expansion in the number of sectors and an additional enhancement to the SFF that could last for exponentially long times until the system tunnels into a topologically uncharged state. More exotic Goldstone manifolds and spatial manifolds would result in even more exciting topological concerns.

Finally, there is the issue of the plateau structure, entirely ignored in this paper. The lack of the hydrodynamic description of plateau behavior is made all the more striking by the fact that certain systems with some sort of resonant behavior (the peaks in figure 6) can have 'ramp' values of the SFF exceeding the final plateau value.

Appendix A: SFF For Unbroken Non-Abelian Symmetries

For systems with non-Abelian symmetry group G, the Hilbert space can be decomposed into a sum of irreducible representations R of the Hamiltonian with dimension $|R|$. These are the non-Abelian analogs of the charge sectors in systems with $U(1)$ symmetries. Just as no term in an $U(1)$-invariant Hamiltonian can mix states with different charges, no term in a G-invariant Hamiltonian can mix states in different irreducible representation. Written in the $|R|$ subspace, the Hamiltonian should decompose into a tensor product of a chaotic random matrix and an $|R|$ by $|R|$ identity. In equations we have

$$H = \bigoplus_R I_{|R|} \otimes H_R, \qquad (A1)$$

where each H_R is an independent matrix with independent eigenvalues. If we write $Z_R(T, f) =$

$\mathrm{tr}\, e^{-iH_R T} f(H_R)$, then $Z(T,f) = \sum_R |R| Z_R(T,f)$. Squaring this, and setting the cross-terms to 0, we recover equation 6. For systems with spontaneous symmetry breaking, the Z_Rs are no longer independent and it no longer makes sense to set the cross-terms to 0. Resolving this complication is the key goal of this paper.

Appendix B: Appendix: Internal Charge

For many realistic systems, there is charge/momentum contained within the state ψ, not just encoded in the pattern of ϕs. A nice example of this is a superconductor. There is some charge in the condensate, but plenty of other charge in the system, including in uncondensed electrons and atomic nuclei. Likewise, if we look at a mechanical example of rotational SSB like a planet with a cloud of moons orbiting a star, there is some orbital angular momentum of the overall system, but the system of planets and moons has its own intrinsic angular momentum. What sort of Hamiltonian captures this situation?

Let's say the states are characterized by an order parameter ϕ, an internal angular momentum L, and an internal state $|\psi\rangle$. The simplest G-invariant Hamiltonian we can write for this is given by equation (22) for with internal-G-invariant H_0 and H_is. This candidate H actually has two copies of the symmetry, an internal one and an external one.

To make the model more realistic, we need to have some coupling which breaks us down to just one copy of the symmetry. First let's do it for Z_n. We can parameterize the sectors by order parameter $0 \le \phi < n$ and q. Then we have a coupling connecting the ϕ, q and ϕ, q' sectors proportional to $e^{2\pi i/n \phi(q-q')}$. This obviously doesn't commute with bland translation or bland conservation of q. But it does retain an overall Z_n symmetry. Written in block matrix form, such a Hamiltonian for a Z_4 symmetry might take the form

$$H = \begin{pmatrix} H_0^0 & I & 0 & I \\ I & H_0^1 & I & 0 \\ 0 & I & H_0^2 & I \\ I & 0 & I & H_0^3 \end{pmatrix} \qquad (B1)$$

Where H_0^ϕ is, itself, a block matrix of the form

$$H_0^\phi = \begin{pmatrix} Q_{00} & e^{\frac{2\pi i}{4}\phi(0-1)}Q_{01} & e^{\frac{2\pi i}{4}\phi(0-2)}Q_{02} & e^{\frac{2\pi i}{4}\phi(0-3)}Q_{03} \\ e^{\frac{2\pi i}{4}\phi(1-0)}Q_{10} & Q_{11} & e^{\frac{2\pi i}{4}\phi(1-2)}Q_{12} & e^{\frac{2\pi i}{4}\phi(1-3)}Q_{13} \\ e^{\frac{2\pi i}{4}\phi(2-0)}Q_{20} & e^{\frac{2\pi i}{4}\phi(2-1)}Q_{21} & Q_{22} & e^{\frac{2\pi i}{4}\phi(2-3)}Q_{23} \\ e^{\frac{2\pi i}{4}\phi(3-0)}Q_{30} & e^{\frac{2\pi i}{4}\phi(3-1)}Q_{31} & e^{\frac{2\pi i}{4}\phi(3-2)}Q_{32} & Q_{33} \end{pmatrix}$$
$$(B2)$$

While the Q_{ii} matrices all have to be square, Hermitian matrices and $Q_{ij} = Q_{ji}^\dagger$, in general the internal-charge-i subspace and the internal-charge-j subspace can be of totally difference sizes.

If we have a more general group, the elements are parameterized by order parameter ϕ, irrep R and vector k within that irrep. We connect ϕ, R and ϕ, R' by

$R_{G(\phi)} M R'_{G^{-1}(\phi)}$ where $G(\phi)$ is a group element that gets us to ϕ, and M is invariant under the unbroken part of the symmetry.

Appendix C: Appendix: Long-Time Behavior with Non-Abelian Discrete Groups

The enhancement factor for non-Abelian discrete symmetry groups is $\mathrm{tr}\left[e^{-\mathrm{Trans}(E)T}\right]$ with transfer matrix

$$\mathrm{Trans} = \frac{1}{2} \sum r_i (M_i \otimes I - I \otimes M_i^T)(M_i^T \otimes I - I \otimes M_i) + (M_i^T \otimes I - I \otimes M_i)(M_i \otimes I - I \otimes M_i^T). \tag{C1}$$

In this section, we count the zero modes of the transfer matrix. Such a zero mode must be annihilated by $M_i \otimes I - I \otimes M_i^T$ for all Ms, which is a heavy constraint.

For our analysis, we will need to decompose a vector space C^Φ into irreps. This space is a representation of G, with the G matrices forming permutation matrices which permute the elements of Φ according to the group action. The matrices M_i are matrices acting on this space which commute with every element of G. We can decompose the representation C^Φ into irreducible representations of R as

$$C^\Phi = \bigoplus_R R^{\oplus K_R}, \tag{C2}$$

where each representation R appears K_R times. A vector v in C^Φ can be written as $v_{R,k,\mu}$, where R denotes the irreducible representation it transforms in, $0 \le k < K_R$ indicates which copy of R, and μ is the index within the representation R. In this case, the requirement that the M_is commute with elements of G mean that they can act only on the k index, in a way not depending on the μ index. If we sum over a large enough collection of Ms (in particular, enough Ms so that their action on the k indices don't all commute) then the only vectors on $C^{\Phi \otimes \Phi}$ which are annihilated by the transfer matrix are ones in which the left and right k indices are maximally entangled. We are free to choose the representation R and the indices μ_1 and μ_2 for the right and left replicas. So we have $\sum_R |R|^2$ zero modes. This is also the random matrix theory prediction for the long-time ramp enhancement.

Appendix D: Appendix: Deriving Hydro from the Discrete Case

Let's talk about how a term like equation (20) gives rise to a Goldstone-like theory. It can be thought of as decoupled diffusion on both the right and left copies of the system, with positive and negative imaginary diffusivities. Of course, this is just QFT with a canonical

kinetic term

$$L = \frac{1}{2}\phi_1\partial_t^2\phi_1 - \frac{1}{2}\phi_2\partial_t^2\phi_2 = \phi_a\partial_t^2\phi_r. \qquad \text{(D1)}$$

This is already similar to the Goldstone theory, but it needs an aa term. This comes from the contribution of the matrix in equation (16). This matrix can be thought of as generating an un-intuitive sort of random walk on Φ^2. The most un-intuitive part is that the transfer probabilities aren't all positive. We can see from the formula that there is a morass of positive and negative signs. In order to build intuition, let's consider the case of Z_n symmetry acting on n elements. We know that the matrices M_i are indexed by Φ^2/G, which in this case is just the collection of jump sizes, ranging from the trivial jump to jumping $n-1$ to the right. For simplicity, let's just look at $(M_1)_{ij} = \delta_{i,j+1}$, the nearest-neighbor jump.

What sort of transfer matrix does this give rise to? Remembering that the transfer matrix is a linear map from the vector space R^{Φ^2} to itself and thus has a total of 4 Φ indices, it is unilluminating to try to write the whole thing. But we can write that it is

$$T_{ij,i'j'} \propto 2\delta_{i,i'}\delta_{j,j'} - \delta_{i,i'+1}\delta_{j,j'+1} - \delta_{i,i'-1}\delta_{j,j'-1} \quad \text{(D2)}$$

If we go from a discrete Z_n to a continuous $U(1)$, this transfer matrix corresponds to ϕ_r undergoing diffusion while ϕ_a doesn't change at all. When we combine this with the transfer matrix in equation (20) we add the covariances/correlators. Equation (78) only has ar correlators. Adding in an rr correlator results in a Lagrangian

$$L = \phi_a\partial_t^2\phi_r + iC\phi_a\partial_t^2\phi_a. \qquad \text{(D3)}$$

This is a generic hydrodynamic action for a superfluid, obtained entirely through taking the continuous limit of the transfer matrices.

Appendix E: Appendix: Time-Reversal Symmetry

Let's consider spontaneous time-reversal symmetry breaking. First, what is a good RMT-like toy model of the phenomenon? Consider a block Hamiltonian of the form

$$H = \begin{pmatrix} H_0 & H_1 \\ H_1 & H_0^* \end{pmatrix}, \qquad \text{(E1)}$$

where H_0 is an $N \times N$ GUE matrix and whose real and imaginary parts have variance J^2/N, and H_1 is a randomly selected GOE matrix whose elements are all independent and have variance J_1^2/N. This matrix has

an anti-unitary time reversal symmetry which conjugates the elements and switches the two blocks (which we will label the $+$ and $-$ blocks). But within each block, there is no time-reversal symmetry. Since the system will choose a block and only slowly tunnel back and forth, we say that the system spontaneously breaks the time-reversal symmetry.
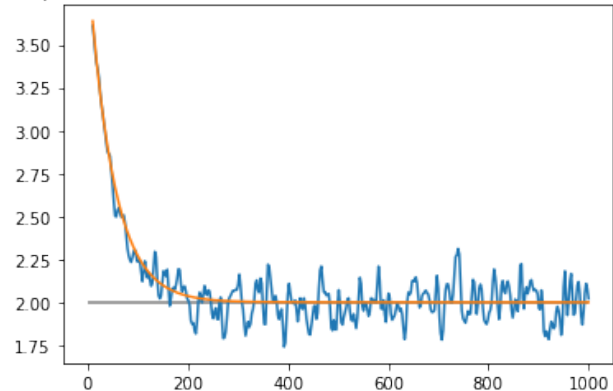

FIG. 8. Predicted (orange) and observed (blue) enhancement relative to GOE expectation for the model in equation (81).

We can study this using a similar instanton-like approach as with Z_2 SSB. We start with a doubled system. Since one copy of the system has two sectors, a pair of copies has $2 \times 2 = 4$ sectors. If the system is in the $++$ or $--$ sectors, it has exactly the same transfer matrix as a GUE matrix with two sectors. The processes contributing are the $++ \to --$ / $-- \to ++$ process with amplitude given by Fermi's golden rule, and the negative-amplitude $++ \to ++$ / $-- \to --$ processes. This gives an enhancement of $\text{tr}\, e^{MT} = 1 + e^{-rT}$, (where r is the transition rate) over the GUE result. However, we can also time-reverse the left contour with respect to the right. Now we have a system starting in either the $+-$ or $-+$ states, and the left replica is performing a time-reversed version of the right-replica's evolution. This gives another $1 + e^{-rT}$ contribution. The overall enhancement factor is thus $2(1 + e^{-rT})$ times the GUE ramp, which is of course $1 + e^{-rT}$ times the GOE ramp.

One thing which bears discussion and which is not fully understood is what happens when the Thouless time becomes comparable to the Heisenberg time for individual subsystems, in large part because we don't have a path integral-like picture for how the ramp gives way to the plateau. For GOE-like systems, plateau-like behavior sets in gradually even before the Heisenberg time. One guess then to assume that the ramp behavior for time reversal SSB is thus $(1 + e^{-rT})\text{SFF}_{\text{GOE}}(T)$ even when $t_{\text{Thouless}} \lesssim t_{\text{Heis}}$.

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