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We provide a unified semiclassical theory for the conserved current of nonconserved quantities, and manifest it in two physical contexts: the spin current of Bloch electrons and the charge current of mean-field Bogoliubov quasiparticles. We reveal that the previously overlooked torque quadrupole density and Berry phase correction to the torque dipole density are essential to assure a circulating spin current with null net flow at equilibrium. The band geometric origin of spin transport is ascertained to be the momentum space Berry curvature instead of the spin Berry curvature, paving the way for material related studies. In superconductors the attained conserved charge current corresponds to the quasiparticle charge current renormalized by the condensate backflow. Its circulation at equilibrium gives an orbital magnetization, which involves the characteristics of superconductivity, such as the Berry curvature arising from unconventional pairing and an orbital magnetic moment induced by the charge dipole of moving quasiparticles.

Introduction.—In condensed matter physics the current of a nonconserved quantity is intriguing, such as the spin current of spin-orbit coupled Bloch electrons [1] and the charge current of mean-field superconducting quasiparticles [2]. The conventionally defined current as the anticommutator of that quantity and velocity is in general not circulating and leads to nonvanishing net current flow even at equilibrium. There has not been a unified recipe showing a conserved current whose circulation characterizes the corresponding orbital magnetization and whose net flow vanishes at equilibrium.

In spintronics research, conserved spin currents have been investigated [3–15]. A natural way towards a bulk conserved current is to include the current density due to the source term of the continuity equation, and a conserved spin current was attempted along this line [6–8]. However, that proposal cannot address equilibrium currents of the pivotal role, hence the spin orbital magnetization characterizing the circulating conserved spin current has not been touched on and whether or not the net flow vanishes at equilibrium is unclear. Without a knowledge about the spin orbital magnetization, it is unknown what the transport component of the conserved current is in the presence of statistical forces, i.e., gradients of chemical potential and temperature, as the circulating current should be discounted to obtain the transport one [16–18]. Even in the case of electrically induced transport, the band geometric origin of the conserved current remains to be unveiled, and a conductivity formula amenable to a momentum space electronic structure code is absent [19]. These unknowns severely limit the utility of the conserved current in spintronics studies [20].

In the context of superconductivity, although charge is ultimately conserved, that of bare mean-field Bogoliubov quasiparticles is not, as these quasiparticles are not eigenstates of charge [21]. The conventional charge current of such quasiparticles is not conserved [2], with the source term arising from the charge transfer between quasiparticles and condensate [21–24]. How to understand the current due to this source term in a semiclassical description of quasiparticles is quite elusive, and the orbital magnetization as a circulating conserved charge current was not addressed yet. Moreover, the connection and possible unification of this subject and the conserved spin current of Bloch electrons, in which the nonconservation related symmetry breaking are respectively spontaneous and explicit, have not been studied.

In this Letter we present a unified semiclassical theory at steady states for the conserved current of nonconserved internal degrees of freedom (denoted by operator $\hat{\boldsymbol{s}}$). We uncover that the attained current has a vanishing net flow at equilibrium and acquire the orbital magnetization of nonconserved quantities. In the context of superconductivity, we recognize that this conserved current corresponds to a semiclassical description of the charge current renormalized by the condensate backflow due to the coupling between quasiparticles and condensate [21, 23]. A Berry phase formula for the orbital magnetization in superconductors is found, which consists of both the local and global charge circuits of quasiparticles with distinct geometric origins. The global circuit is due to the momentum space Berry curvature derivable from not only the parent Bloch states but also the unconventional superconducting pairing, whereas the local one includes a nontrivial orbital magnetic moment induced by the charge dipole moment of a moving quasiparticle, which has never been seen for charge conserved particles.

The aforementioned problems on the conserved spin current of Bloch electrons are solved. In particular, the torque quadrupole density and a Berry phase correction to the torque dipole density are unveiled to be vital to ensure a circulating equilibrium spin current with vanishing net flow. Moreover, we figure out the long desired but absent connection of the transport conserved current to band geometry, paving the way for material related studies of the conserved current in spin transport. The key geometric quantity is the momentum space Berry curvature, instead of the so called spin Berry curvature for the conventional spin current [25–27]. To get to this result, we also identify a nonconservation induced variant of side jump physics [28, 29] due to the intra-scattering change of spin dipole moment.

Equilibrium current.—For the convenience of presentation we describe the theory in terms of Bloch electrons. In view of the continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{J}^{\boldsymbol{s}}(\boldsymbol{r})=\boldsymbol{\tau}(\boldsymbol{r})$ at steady states, in order to construct a conserved current of $\hat{\boldsymbol{s}}$ we need study the local density of the conventional $\boldsymbol{s}$-current, represented by the operator $\hat{\boldsymbol{J}}^{\boldsymbol{s}}=\frac{1}{2}\{\hat{\boldsymbol{v}},\hat{\boldsymbol{s}}\}$ with $\hat{\boldsymbol{v}}$ being the velocity, and that of the $\boldsymbol{s}$-generation rate, represented by $\hat{\boldsymbol{\tau}}=d\hat{\boldsymbol{s}}/dt$. As we concern current densities up to the first order of spatial gradients in order to capture the magnetization current, pursuing $\boldsymbol{\tau}(\boldsymbol{r})$ up to the second order is indispensable. To this end we develop a second order semiclassical theory in combination with the field variational approach [30]. The derivation of this theory is involved hence delegated to the Supplemental Material [31], whereas the results take physically transparent forms and are set forth below.

The conventional $\boldsymbol{s}$-current density at equilibrium up to the first order of spatial gradients reads

$$J^{i\boldsymbol{s}}=\int f_nJ_n^{i\boldsymbol{s}}-\partial_lD^{li\boldsymbol{s}},\ D^{li\boldsymbol{s}}=\int(f_nd_n^{li\boldsymbol{s}}+g_n\Omega_n^{li\boldsymbol{s}}).\quad(1)$$

The first term in the zeroth order of gradients is in general nonzero [1], and the second term in the first order may not be circulating, in contrast to the orbital magnetization current which is always circulating [16]. Therefore, the net flow of this conventional $\boldsymbol{s}$-current through a cross section of a sample may not vanish even at equilibrium. Thus, this current alone cannot afford a physical description of the flow of nonconserved quantities. The non-circulating nature of the gradient term in Eq. (1) originates from the nonconservation of $\boldsymbol{s}$ and follows directly from the fact that the tensor $D^{li\boldsymbol{s}}$, which is the equilibrium dipole density of the conventional $\boldsymbol{s}$-current [30], is not antisymmetric with respect to Cartesian indices i and l for such $\boldsymbol{s}$. Here $d_n^{li\boldsymbol{s}}$ $=\mathrm{Re}\langle(\hat{r}-r_c)^l\,\hat{J}^{i\boldsymbol{s}}\rangle$ is the dipole moment of the distribution of the conventional $\boldsymbol{s}$-current on the spread of a wave packet constructed by superposing Bloch states from a particular band n, and $\boldsymbol{r}_c$ denotes the probability center of the wave packet. $\Omega_n^{li\boldsymbol{s}}=-2\,\mathrm{Im}\sum_{n_1\neq n}v_{nn_1}^lJ_{n_1n}^{i\boldsymbol{s}}/\omega_{nn_1}^2$ is the so called spin Berry curvature when $\boldsymbol{s}$ is spin [25–27], and is thus termed as the $\boldsymbol{s}$-Berry curvature in the following. A similar expression for $J^{i\boldsymbol{s}}$ appeared in Ref. [4], but missed the $\boldsymbol{s}$-Berry curvature contribution to the total dipole density. If $\boldsymbol{s}$ is conserved, $\int f_nJ_n^{i\boldsymbol{s}}$ vanishes, whereas $D^{li\boldsymbol{s}}$ becomes antisymmetric and proportional to the orbital magnetization [32], hence $J^{i\boldsymbol{s}}$ reduces to a circulating magnetization current with vanishing net flow.

The notation $\int$ without integral variable is shorthand for $\int[d\boldsymbol{k}]$, and $[d\boldsymbol{k}]\equiv\sum_n d\boldsymbol{k}/(2\pi)^D$, where $\boldsymbol{k}$ is the crystal momentum ($\hbar=1$), D is the spatial dimensionality. f_n is the Fermi function, and $g_n=k_BT\ln(1-f_n)$ is the state resolved grand potential density, with T as the temperature. $v_{nn_1}^l=\langle u_n|\hat{v}^l|u_{n_1}\rangle$, $\omega_{nn_1}=\varepsilon_n-\varepsilon_{n_1}$, and $|u_n\rangle$ is the periodic part of the Bloch wave with band energy ε_n. The summation over repeated Cartesian indices is implied henceforth.

Meanwhile, the local $\boldsymbol{s}$-generation rate due to $\boldsymbol{s}$ nonconservation takes the form of

$$\boldsymbol{\tau}(\boldsymbol{r})=-\boldsymbol{\nabla}\cdot\boldsymbol{J}^{\boldsymbol{\tau}}(\boldsymbol{r}),\quad J^{i\boldsymbol{\tau}}=D^{i\boldsymbol{\tau}}-\partial_lQ^{il\boldsymbol{\tau}},\qquad(2)$$

where we have used the fact that the value of $\boldsymbol{\tau}$ in the zeroth order of spatial gradients, $\int f_n\boldsymbol{\tau}_n$, vanishes, and

$$\begin{aligned}D^{i\boldsymbol{\tau}}&=\int(f_nd_n^{i\boldsymbol{\tau}}+g_n\Omega_n^{i\boldsymbol{\tau}}),\\Q^{il\boldsymbol{\tau}}&=\int(f_nq_n^{il\boldsymbol{\tau}}+g_n\chi_n^{il\boldsymbol{\tau}}).\end{aligned}\qquad(3)$$

Here $D^{i\boldsymbol{\tau}}$ is the equilibrium $\boldsymbol{\tau}$-dipole density, which involves not only the $\boldsymbol{\tau}$-dipole moment $d_n^{i\boldsymbol{\tau}}$ of each wave packet but also a corresponding Berry curvature $\Omega_n^{i\boldsymbol{\tau}}$. The latter was overlooked previously, but, as will be shown later, plays an important role in ensuring the vanishing net flow of the conserved current. $Q^{il\boldsymbol{\tau}}$ is essentially the $\boldsymbol{\tau}$-quadrupole density $Q^{(il)\boldsymbol{\tau}}=(Q^{il\boldsymbol{\tau}}+Q^{li\boldsymbol{\tau}})/2$ and contributes to the $\boldsymbol{s}$-generation rate as $-\partial_i\partial_lQ^{il\boldsymbol{\tau}}$. Here $q_n^{il\boldsymbol{\tau}}=\mathrm{Re}\langle\frac{1}{2}(\hat{r}-r_c)^i(\hat{r}-r_c)^l\,\hat{\boldsymbol{\tau}}\rangle$ is the $\boldsymbol{\tau}$-quadrupole moment and $\chi_n^{il\boldsymbol{\tau}}=\partial_{k_i}d_n^{l\boldsymbol{s}}-2\,\mathrm{Im}\sum_{n_1\neq n}v_{nn_1}^ld_{n_1n}^{i\boldsymbol{\tau}}/\omega_{nn_1}^2$ is the $\boldsymbol{\tau}$-dipole polarizability of a bulk semiclassical electron, with $d_{n_1n}^{i\boldsymbol{\tau}}$ having the meaning of interband $\boldsymbol{\tau}$-dipole moment [33]. In literatures on the conserved spin current of Bloch electrons, $J^{i\boldsymbol{\tau}}$ is termed as the torque ($\hat{\boldsymbol{\tau}}$ is the torque operator in this context) dipole spin current. Our theory shows that this understanding is inaccurate as the torque quadrupole density is also involved. In fact, it is this quadrupole contribution that cancels out the non-circulating part of the conventional $\boldsymbol{s}$-current and makes the current conserved.

According to the above evaluation, $\boldsymbol{J}^{\boldsymbol{\tau}}$ can be deemed as a $\boldsymbol{s}$-current arising from the nonconservation of $\boldsymbol{s}$ and we can inspect if

$$\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}=\boldsymbol{J}^{\boldsymbol{s}}+\boldsymbol{J}^{\boldsymbol{\tau}}\qquad(4)$$

is a conserved current density. Generally speaking, a conserved current takes the form of $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}=\boldsymbol{\mathcal{J}}_{\mathrm{un}}^{\boldsymbol{s}}+\boldsymbol{\nabla}\times\boldsymbol{M}^{\boldsymbol{s}}$, where $\boldsymbol{\mathcal{J}}_{\mathrm{un}}^{\boldsymbol{s}}$ is the equilibrium current in the uniform case, and $\boldsymbol{M}^{\boldsymbol{s}}$ is referred to as the $\boldsymbol{s}$ orbital magnetization by analogy to the charge orbital magnetization. The developed semiclassical theory enables us to show the conserving nature of $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}$ directly and determine $\boldsymbol{\mathcal{J}}_{\mathrm{un}}^{\boldsymbol{s}}$ and $\boldsymbol{M}^{\boldsymbol{s}}$.

Orbital magnetization and vanishing net flow.— Due to the peculiarity of the $\boldsymbol{s}$-generation rate operator, we get $d_n^{i\boldsymbol{\tau}} = -J_n^{i\boldsymbol{s}} + v_n^i \boldsymbol{s}_n$ and $\Omega_n^{i\boldsymbol{\tau}} = \partial_{k_i}\boldsymbol{s}_n$. One finds that the $\boldsymbol{\tau}$-dipole moment quantifies the deviation of the conventional $\boldsymbol{s}$-current from the classical form of $\boldsymbol{s}$-current due to $\boldsymbol{s}$ nonconservation, and the $\boldsymbol{\tau}$-dipole density $D^{i\boldsymbol{\tau}}$ cancels out the conventional $\boldsymbol{s}$-current $\int f_n J_n^{i\boldsymbol{s}}$. We thus uncover the first important property of the current defined in Eq. (4) that it vanishes in uniform equilibrium

$$\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\mathrm{un}} = 0. \tag{5}$$

This result means that the net flow of $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}$ through any cross section of a sample vanishes, which is a necessary character of a current that can describe transport.

Next, after tedious manipulations of the $\boldsymbol{\tau}$-quadrupole density [31], we get $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}} = \boldsymbol{\nabla}\times\boldsymbol{M}^{\boldsymbol{s}}$ with the $\boldsymbol{s}$ orbital magnetization given by [34]

$$\boldsymbol{M}^{\boldsymbol{s}} = \int (f_n \boldsymbol{m}_n^{\boldsymbol{s}} + g_n \boldsymbol{\Omega}_n \boldsymbol{s}_n). \tag{6}$$

Here $\boldsymbol{m}_n^{\boldsymbol{s}} = \frac{1}{2}\,\mathrm{Re}\sum_{n_1\neq n}\mathcal{A}_{nn_1}\times\boldsymbol{J}^{\boldsymbol{s}}_{n_1 n} + \frac{1}{2}\boldsymbol{d}_n^{\boldsymbol{s}}\times\boldsymbol{v}_n$ accounts for the $\boldsymbol{s}$ orbital magnetic moment carried by each wave packet, with $\mathcal{A}_{nn_1}$ being the $\boldsymbol{k}$-space interband Berry connection, and $\boldsymbol{\Omega}_n$ is the $\boldsymbol{k}$-space Berry curvature. If $\boldsymbol{s}$ is replaced by charge e of Bloch electrons, $\boldsymbol{M}^{\boldsymbol{s}}$ reduces to the charge orbital magnetization [18]. To see the orbital nature of $\boldsymbol{M}^{\boldsymbol{s}}$ for nonconserved $\boldsymbol{s}$, we inspect the content of $\boldsymbol{m}_n^{\boldsymbol{s}}$. The first term of $\boldsymbol{m}_n^{\boldsymbol{s}}$ is equal to $\langle \frac{1}{2}(\hat{\boldsymbol{r}} - \boldsymbol{r}_c)\times\hat{\boldsymbol{J}}^{\boldsymbol{s}}\rangle$, which is the antisymmetric part of the dipole moment of $\boldsymbol{J}^{\boldsymbol{s}}$ and means the circulation of the conventional $\boldsymbol{s}$-current due to the self-rotational motion of a wave packet around its center position. On the other hand, the second term of $\boldsymbol{m}_n^{\boldsymbol{s}}$ signifies remarkably that the center-of-mass motion of a wave packet with a nonvanishing $\boldsymbol{s}$-dipole moment $\boldsymbol{d}_n^{\boldsymbol{s}}$ induces a $\boldsymbol{s}$ orbital magnetic moment. This term is thus reminiscent of the phenomenon in electromagnetism that a moving charge dipole moment results in a charge orbital magnetic moment.

Orbital magnetization in superconductors.—Now we apply the above results to the context of orbital magnetization in superconductors. Intraband spin-singlet pairing without spin-orbit coupling is assumed for illustration. As has been shown recently, the semiclassical theory for Bogoliubov quasiparticle wave packets can be formulated similarly to that for electrons [35, 36]. Such a theory can accommodate slowly-varying perturbations whose length scales are much larger than the superconducting coherence length. The nonlocal pair potential is treated in the continuum limit in our consideration, but may also be dealt with assuming a more general periodic form [35]. To confirm the utility of the semiclassical theory, we show in the Supplemental Material [31] that it readily leads to the quantum thermal Hall and spin quantum Hall (quantized spin conductivity in response to a Zeeman gradient) effects predicted by field theoretical methods in chiral $d+id$ superconductors [37, 38].

The conventional charge current carried by a mean-field quasiparticle is represented by the anticommutator of charge and velocity operators, where the charge operator is e times the third Pauli matrix $\hat{\sigma}^z$ in the electron-hole space [2, 35]: $\hat{\boldsymbol{s}} = e\hat{\sigma}^z$. It is simply given by $e\partial_{\boldsymbol{k}}\xi_{\eta\boldsymbol{k}}$, where ξ_η is the energy of an electron relative to its chemical potential, with η as the Bloch band index [39]. Such a conventional current carried by an ensemble of mean-field quasiparticles, namely $\boldsymbol{J}^{\boldsymbol{s}}$, may not vanish at equilibrium in uniform systems without time-reversal and inversion symmetries.

The source term $\boldsymbol{\tau} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}^{\boldsymbol{\tau}}$ for this conventional current arises from the fact that the mean-field quasiparticles are not eigenstates of charge and embodies the coupling between mean-field condensates and quasiparticles [21–24]. Thus $\boldsymbol{J}^{\boldsymbol{\tau}}$ serves as a condensate backflow, which makes the current conserved and cancels out the net flow of the conventional current carried by mean-field quasiparticles at equilibrium [Eq. (5)].

The orbital magnetization reads

$$\boldsymbol{M} = \int (f_n \boldsymbol{m}_n + e\rho_n g_n \boldsymbol{\Omega}_n). \tag{7}$$

Here and hereafter we omit the superscript $\boldsymbol{s}$ in charge current related quantities. $\rho_n = \sigma_n^z = |\mu_n|^2 - |\nu_n|^2 = \sigma\rho_\eta^0$ is the charge ($e\rho_n$) carried by a mean-field quasiparticle, for which the band index $n = (\eta,\sigma)$, with $\sigma = \pm$ denoting the Bogoliubov bands. $[\mu_{n\boldsymbol{k}}, \nu_{n\boldsymbol{k}}]^T$ is the Bogoliubov wave function, $\rho_\eta^0 = \xi_\eta/|\varepsilon_{\eta\sigma}|$, and $\varepsilon_{\eta\sigma} = \sigma\sqrt{\xi_\eta^2 + |\Delta_\eta|^2}$. The orbital magnetic moment of each quasiparticle is

$$\boldsymbol{m}_n = \frac{1}{2}\,\mathrm{Re}\sum_{n_1\neq n}\mathcal{A}_{nn_1}\times\boldsymbol{J}_{n_1 n} + \frac{1}{2}\boldsymbol{d}_n\times\boldsymbol{v}_n, \tag{8}$$

where

$$\boldsymbol{d}_n = e\mathrm{Re}\langle(\hat{\boldsymbol{r}} - \boldsymbol{r}_c)\,\hat{\sigma}^z\rangle = \frac{e}{2}[(\rho_\eta^0)^2 - 1]\partial_{\boldsymbol{k}}\theta_\eta \tag{9}$$

is the charge dipole moment of a quasiparticle wave packet, with $\theta_\eta = \arg\Delta_\eta$ being the phase of the superconducting gap function in $\boldsymbol{k}$-space. Nonzero dipole moment signifies that the charge distribution on a Bogoliubov wave packet is not centered at the probability center of the wave packet. This charge dipole is a basic property of quasiparticles in unconventional pairing superconductors, and has also been identified recently from a different method [36].

Our account of the orbital magnetization is limited to strongly type-II superconductors where the penetration length is irrelevant (much larger than the length scale considered), such as in two dimensional (2D) systems. In such a case the screening current can be neglected and the consideration of magnetization is simplified. In more complicated cases where the Meissner screening must be taken into account, a theory incorporating the Coulomb interaction is needed. This is left for a future work.

The orbital magnetization [Eq. (7)] is interpreted as a sum of local and global charge circuits of quasiparticle wave packets. The first term of $\boldsymbol{m}_n$ is the circulation of the conventional charge current about the wave-packet center, whereas the second term signifies an orbital magnetic moment induced by a travelling charge dipole (this latter mechanism does not show up for Bloch electrons since their charge center coincides with the wave-packet center). Therefore, $\boldsymbol{m}_n$ can be understood as the local circuit accompanying a moving wave packet. Meanwhile, the Berry curvature term in $\boldsymbol{M}$ results from the global circuit due to the center-of-mass motion of wave packets [32, 40].

A vanishing orbital magnetization $\boldsymbol{M} = \boldsymbol{0}$ is predicted for the case of geometrically trivial electronic bands. In this case one has a particle-hole symmetric two-band model with Bogoliubov band index $\sigma = \pm$, for which the Berry curvature $\boldsymbol{\Omega}_\sigma = -\sigma \partial_{\boldsymbol{k}} \rho^0 \times \partial_{\boldsymbol{k}} \theta/2$ stems from unconventional superconducting pairing. In addition, the first term of $\boldsymbol{m}_n$ vanishes due to the particle-hole symmetry. It is then apparent that the statistical sum of local circuits cancels exactly the global circuit. On the other hand, such a null result is not anticipated for the case of geometrically nontrivial Bloch bands.

Transport in terms of the conserved current.—Next we turn to the nonequilibrium case with a weak uniform electric field $\boldsymbol{E}$, chemical potential gradient $\boldsymbol{\nabla}\mu$ and temperature gradient $\boldsymbol{\nabla}T$. The electric field and the chemical potential gradient are only considered in the context of Bloch electrons (in the perspective of transport, an electric field for Bloch electrons corresponds to a Zeeman gradient applied to a spin-singlet superconductor without spin-orbit coupling [37, 38]). The first observation is that the steady-state linear response of $\int f_n \boldsymbol{\tau}_n$ vanishes if $\hat{\boldsymbol{s}}$ commutes with the position operator. Showing this conclusion entails only a standard linear response analysis in disordered systems [12, 31]. Thus, at nonequilibrium the current defined by Eq. (4) is still conserved.

The intrinsic transport current is found by subtracting the circulating magnetization one: $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}} - \boldsymbol{\nabla} \times \boldsymbol{M}^{\boldsymbol{s}}$, and is quantified by the sum of the $\boldsymbol{s}$-Berry curvature and $\boldsymbol{\tau}$-dipole polarizability: $\Omega_n^{li\boldsymbol{s}} + \chi_n^{il\boldsymbol{\tau}} = \partial_{k_l} d_n^{i\boldsymbol{s}} + \Omega_n^{li} \boldsymbol{s}_n$, which turns out to be expressed by the $\boldsymbol{s}$-dipole moment and $\boldsymbol{k}$-space Berry curvature tensor Ω_n^{il}. The result is

$$\mathcal{J}_{\text{in}}^{i\boldsymbol{s}} = \sigma_{\text{in}}^{il\boldsymbol{s}}(E_l - \partial_l \mu/e) - \alpha_{\text{in}}^{il\boldsymbol{s}} \partial_l T, \tag{10}$$

with

$$\begin{aligned} \sigma_{\text{in}}^{il\boldsymbol{s}} &= e \int f_n (\Omega_n^{li} \boldsymbol{s}_n + \partial_{k_l} d_n^{i\boldsymbol{s}}), \\ \alpha_{\text{in}}^{il\boldsymbol{s}} &= -\int \partial g_n/\partial T (\Omega_n^{li} \boldsymbol{s}_n + \partial_{k_l} d_n^{i\boldsymbol{s}}). \end{aligned} \tag{11}$$

Here $-\partial g_n/\partial T$ is the state resolved entropy density, and the Einstein and Mott relations are ensured. In superconductors, as has been mentioned, the charge current driven by a temperature gradient is considered for strongly type-II case. In the case of conserved spin current of Bloch electrons, the above result for $\sigma_{\text{in}}^{il\boldsymbol{s}}$ unveils the band geometric origin of the intrinsic spin conductivity obtained in the quantum theory [6], and $\alpha_{\text{in}}^{il\boldsymbol{s}}/\sigma_{\text{in}}^{il\boldsymbol{s}}$ is consistent with the intrinsic transport thermal/charge current driven by a Zeeman gradient [41].

Spin transport of Bloch electrons.—In this context $\hat{\boldsymbol{s}}$ is spin, and Eqs. (4) – (6) and (11) solve all the problems mentioned in the Introduction. Moreover, we find that even if the considered spin component is not conserved, the spin conductivity in insulators is of purely Hall type $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}} = \boldsymbol{E} \times \boldsymbol{\sigma}_H^{\boldsymbol{s}}$, a characteristic that is not shared by the conventional spin current. The Hall conductivity is related to the spin orbital magnetization:

$$\boldsymbol{\sigma}_H^{\boldsymbol{s}} = -e \int \boldsymbol{\Omega}_n \boldsymbol{s}_n = e \frac{\partial \boldsymbol{M}^{\boldsymbol{s}}}{\partial \mu}. \tag{12}$$

Our theory also sheds new light on disorder influenced spin transport in terms of $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}$ in metals. Currents stemming from the $\boldsymbol{E}$-field driven off-equilibrium occupation function read $J_{\text{c}}^{i\boldsymbol{s}} = \int \delta f_n J_n^{i\boldsymbol{s}}$ and $J_{\text{c}}^{i\boldsymbol{\tau}} = \int \delta f_n d_n^{i\boldsymbol{\tau}}$ [42]. The resulting conserved current is unexpectedly simple as if the spin were conserved:

$$\mathcal{J}_{\text{c}}^{i\boldsymbol{s}} = \int \delta f_n v_n^i \boldsymbol{s}_n. \tag{13}$$

It may have transverse components in anisotropic systems even if δf_n is determined by the conventional Boltzmann equation with the first Born scattering rate [43]. $\mathcal{J}_{\text{c}}^{i\boldsymbol{s}}$ is time-reversal odd and is relevant to the study of magnetic spin Hall effect [44, 45].

The spin current arises from intra-scattering semiclassics as well. This is a conceptual generalization of the side jump physics rooted in the coordinate shift of an electron wave packet during scattering [28, 29]. When the transported quantity is nonconserved, the key ingredient is replaced by the shift of a $\boldsymbol{s}$-coordinate ($\hat{\boldsymbol{r}}^{\boldsymbol{s}} = \hat{\boldsymbol{s}}\hat{\boldsymbol{r}}$) upon a scattering $n\boldsymbol{k} \to n'\boldsymbol{k}'$. We find [31] $\delta \boldsymbol{r}^{\boldsymbol{s}} = \delta_{\text{dp}} \boldsymbol{r}^{\boldsymbol{s}} + \delta_{\text{sj}} \boldsymbol{r}^{\boldsymbol{s}}$, where $(\delta_{\text{sj}} \boldsymbol{r}^{\boldsymbol{s}})_{\boldsymbol{k}'\boldsymbol{k}}^{n'n} = \boldsymbol{s}_{n'\boldsymbol{k}'} \boldsymbol{\mathcal{A}}_{n'\boldsymbol{k}'} - \boldsymbol{s}_{n\boldsymbol{k}} \boldsymbol{\mathcal{A}}_{n\boldsymbol{k}} - (\boldsymbol{s}_{n'\boldsymbol{k}'} \partial_{\boldsymbol{k}'} + \boldsymbol{s}_{n\boldsymbol{k}} \partial_{\boldsymbol{k}}) \arg V_{\boldsymbol{k}'\boldsymbol{k}}^{n'n}$ is reminiscent of the coordinate shift [28] and reduces to the latter for conserved $\boldsymbol{s}$, with $\arg V$ being the phase of the scattering matrix element, whereas

$$(\delta_{\text{dp}} \boldsymbol{r}^{\boldsymbol{s}})_{\boldsymbol{k}'\boldsymbol{k}}^{n'n} = \boldsymbol{d}_{n'\boldsymbol{k}'}^{\boldsymbol{s}} - \boldsymbol{d}_{n\boldsymbol{k}}^{\boldsymbol{s}} \tag{14}$$

is the intra-scattering change of the $\boldsymbol{s}$-dipole moment. Both parts of $\delta \boldsymbol{r}^{\boldsymbol{s}}$ are independent of the phase choice of Bloch functions, implying two semiclassical contributions to the spin current: $\boldsymbol{\mathcal{J}}_{\text{dp/sj}}^{\boldsymbol{s}} = \int [d\boldsymbol{k}] \delta f_{n\boldsymbol{k}} \boldsymbol{\mathcal{J}}_{\text{dp/sj},n\boldsymbol{k}}^{\boldsymbol{s}}$ [46], where $\boldsymbol{\mathcal{J}}_{\text{dp/sj},n\boldsymbol{k}}^{\boldsymbol{s}} = \int [d\boldsymbol{k}'] P_{\boldsymbol{k}'\boldsymbol{k}}^{n'n} (\delta_{\text{dp/sj}} \boldsymbol{r}^{\boldsymbol{s}})_{\boldsymbol{k}'\boldsymbol{k}}^{n'n}$ is the current carried by each electron and quantifies the accumulation of $\delta \boldsymbol{r}^{\boldsymbol{s}}$ upon possible scattering events per unit time, with $P_{\boldsymbol{k}'\boldsymbol{k}}^{n'n}$ being the first Born scattering rate [29].

Appearing only for nonconserved $\boldsymbol{s}$, $\delta_{\text{dp}} \boldsymbol{r}^{\boldsymbol{s}}$ is noticeable as it originates from scattering but its expression is independent of scattering. It is the difference of a "function of state" $\boldsymbol{d}^{\boldsymbol{s}}_{n\boldsymbol{k}}$ upon a scattering process. This feature has a remarkable influence. By utilizing the relevant Boltzmann equation in the lowest Born order [29], we find that $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm dp} = -eE_l \int f_n \partial_{k_l} \boldsymbol{d}^{\boldsymbol{s}}_n$, cancelling out the $\boldsymbol{s}$-dipole term of the intrinsic $\boldsymbol{s}$-current [Eq. (11)] regardless of the specific form of a weak disorder potential. Hence the band structure dictated current is of Hall type in metals and is connected to the Berry curvature:

$$\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm B} = \boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm in} + \boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm dp} = -e\boldsymbol{E} \times \int f_n \boldsymbol{\Omega}_n \boldsymbol{s}_n. \qquad (15)$$

Gathering the results, the transport conserved current in the semiclassical theory is $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}} = \boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm B} + \boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm sj} + \boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm c}$. As an application, we reveal that the puzzling null results of the conserved spin Hall conductivity in 2D Rashba-type models within the weak disorder regime [12, 13, 47] is due to $s^z_n = 0$ in such models. The $C_{\infty v}$ symmetry forbids spin conductivities with spin polarization along in-plane directions, and $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm B/sj/c} = 0$ for $s = s^z$. In particular, the intrinsic contribution $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm in}$, which reproduces the results of the linear response theory in clean systems [6], is canceled out by $\boldsymbol{\mathcal{J}}^{\boldsymbol{s}}_{\rm dp}$.

Concluding remarks.—The theoretical framework developed may be useful in other subjects of interest, such as the layer pseudospin Hall effect in twisted bilayers [48] and spin Nernst effect in spin-orbit coupled superconductors with Bogoliubov Fermi surfaces [49]. Besides, to generalize the theory to magnon and phonon systems described by bosonic Bogoliubov-de Gennes Hamiltonians is of importance for studying the spin Nernst effect of magnons in a noncollinear antiferromagnetic insulators [50, 51] and that due to magnon-phonon interactions in collinear ferrimagnets [52] as well as the phonon angular momentum Hall effect [53]. In all these subjects there is the issue of the conserved current of nonconserved quantities. In addition, in superconductors the steady-state semiclassical theory need be extended to include time derivatives and gauge fields in order to describe gauge-invariant coupled dynamics of quasiparticles and condensate in the presence of electromagnetic fields [24, 54].

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derivation is presented in Ref. [31]) for interested readers:

$$d_{n_1 n}^{i\boldsymbol{\tau}} = \frac{1}{2}(i\partial_{k_i} + \mathcal{A}_{n_1}^i - \mathcal{A}_n^i)\boldsymbol{\tau}_{n_1 n} - \sum_{n_2 \neq n} \delta_{n_2 n_1} \frac{i\boldsymbol{\tau}_{n_2 n} v_n^i}{\varepsilon_n - \varepsilon_{n_2}} + \frac{1}{2}\sum_{n_2 \neq n_1} \mathcal{A}_{n_1 n_2}^i \boldsymbol{\tau}_{n_2 n} + \frac{1}{2}\sum_{n_2 \neq n} \boldsymbol{\tau}_{n_1 n_2} \mathcal{A}_{n_2 n}^i,$$

where $\mathcal{A}_n^i$ is the $\boldsymbol{k}$-space Berry connection. It can be extended to involve the $\boldsymbol{\tau}$-dipole moment $d_n^{i\boldsymbol{\tau}}$ when $n_1 = n$, hence can be deemed as the interband element of the $\boldsymbol{\tau}$-dipole moment.

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