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Tunable Fragile Topology in Floquet Systems

Rui-Xing Zhang^{1,2,*} and Zhi-Cheng Yang^{2,3,†}

¹Condensed Matter Theory Center, Department of Physics,
University of Maryland, College Park, Maryland 20742-4111, USA

²Joint Quantum Institute, University of Maryland, College Park, MD 20742, USA

³Joint Center for Quantum Information and Computer Science,
University of Maryland, College Park, MD 20742, USA

We extend the notion of fragile topology to periodically-driven systems. We demonstrate driving-induced fragile topology in two different models, namely, the Floquet honeycomb model and the Floquet π -flux square-lattice model. In both cases, we discover a rich phase diagram that includes Floquet fragile topological phases protected by crystalline rotation or mirror symmetries, Floquet Chern insulators, and trivial atomic phases, with distinct boundary features. Remarkably, the transitions between different phases can be feasibly achieved by simply tuning the driving amplitudes, which is a unique feature of driving-enabled topological phenomena. Moreover, corner-localized fractional charges are identified as a “smoking-gun” signal of fragile topology in our systems. Our work paves the way for studying and realizing fragile topology in Floquet systems.

Introduction.— Topological band insulators usually differ from trivial insulators by the existence of anomalous gapless boundary modes, a manifestation of their non-trivial bulk topology [1, 2]. Various momentum-space topological invariants have been proposed to diagnose possible topology for a given set of energy bands, e.g. the Chern number [3] and the $\mathbb{Z}_2$ index [4] etc. It was recently realized that band topology follows from a unified real-space definition in terms of an obstruction towards describing topological bands using exponentially localized and symmetric Wannier functions [5, 6]. In contrast, trivial/atomic insulators are always capable of being “Wannierized”. This definition has led to the concept of fragile topology, where a set of bands are Wannier obstructed by themselves, yet the obstruction can be removed upon coupling to certain additional trivial bands [7–17]. Practically, fragile topological bands are proposed to exist in magic-angle twisted bilayer graphene [18–20], and are engineered in photonic[21] and phononic systems [22].

Apart from static systems, topological phenomena also exist in systems far from equilibrium [23, 24]. A prototypical example is periodically driven systems that are described by Floquet theory. Floquet engineering can enable nontrivial band topology in statically trivial systems [25–28] and even achieve exotic topological phases without any static counterparts [29–32]. Existing studies, however, have only considered Floquet systems with stable Wannier obstructions [33]. It is then natural to ask whether fragile topology exists in an out-of-equilibrium setting, and whether Floquet engineering can realize such phases in statically trivial systems.

In this work, we provide an affirmative answer to the above questions by constructing two explicit lattice models, the Floquet honeycomb model and the Floquet π -flux model, to demonstrate driving-enabled Floquet fragile topology. By tuning the driving amplitudes, both models realize a variety of phases including Floquet fragile topological phases protected by crystalline rotation or mirror

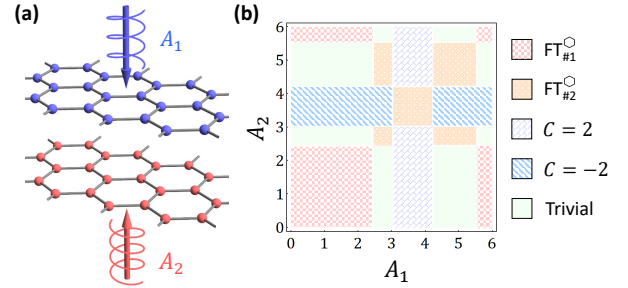


FIG. 1. (a) A schematic of the Floquet honeycomb model. “Atoms” with different colors are driven by gauge fields with different chiralities and amplitudes. (b) Phase diagram of the Floquet honeycomb model upon tuning the driving amplitudes A_1 and A_2 [34].

symmetries, Floquet Chern insulators, and trivial atomic phases. We characterize the fragile topological phases via atomic decompositions as well as the Wilson loop technique. As opposed to other phases in our models, all fragile phases carry additional higher-order topology with fractional corner charges, which serves as a unique and unambiguous boundary feature for fragile topology.

Floquet honeycomb model.— Consider a two-dimensional honeycomb lattice with two species of spinless s -orbital electrons per site:

$$H_{\square}(\tau) = - \sum_{\langle i,j \rangle, \alpha} t_{\alpha}(\tau) c_{i,\alpha}^{\dagger} c_{j,\alpha} + \Delta \sum_i c_{i,1}^{\dagger} c_{i,2} + \text{H.c.}, \quad (1)$$

where $\alpha = 1, 2$ labels the species, and the hoppings $t_{\alpha}(\tau)$ are time-dependent. The time-dependence in t_{α} originates from coupling to a time-periodic gauge field $\mathcal{A}_{\alpha}(\tau) = A_{\alpha}(\cos \omega \tau, (-1)^{\alpha} \sin \omega \tau)$ via the Peierls substitution: $t_{\alpha}(\tau) = t_0 \exp[-i \int_{\mathbf{r}_i}^{\mathbf{r}_j} \mathcal{A}_{\alpha}(\tau) \cdot d\mathbf{r}]$. Here A_{α} and ω are the amplitudes and frequency of the drive. This dynamic gauge field is physically equivalent to that of circularly polarized light [e.g. left-handed for $\alpha = 1$ and right-

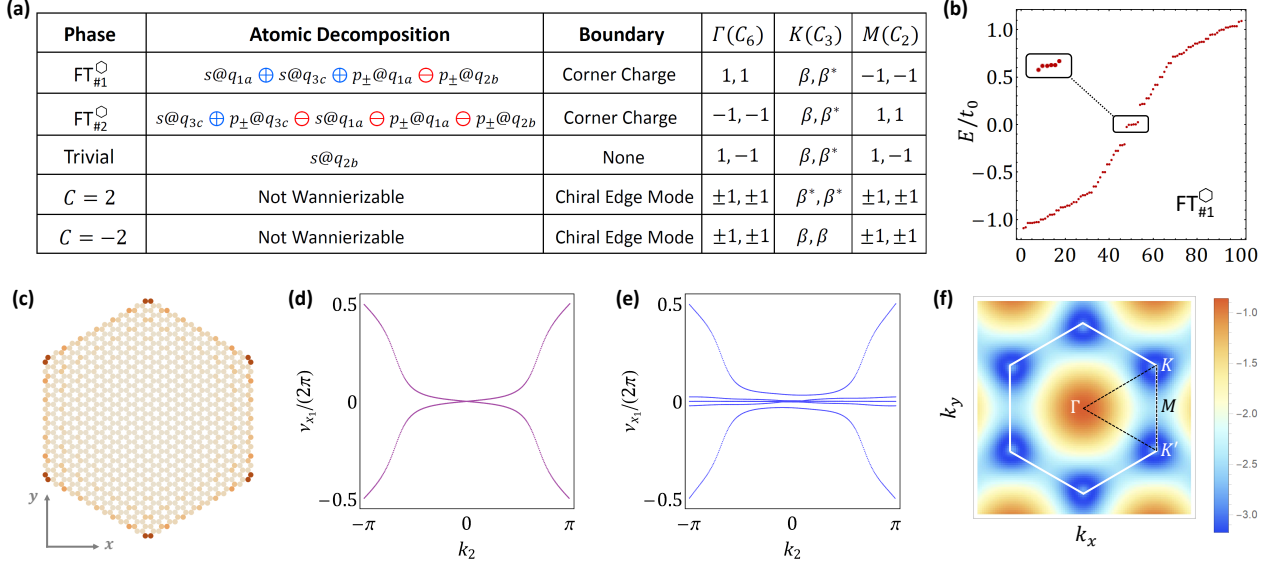


FIG. 2. (a) Atomic decompositions, boundary signatures, and symmetry data for topologically distinct phases in the Floquet honeycomb model. Here the notation $s@q_{1a}$ denotes an s orbital placed at maximal Wyckoff position $1a$ (the hexagon center), and similarly for all other notations; $\beta = e^{i\frac{2\pi}{3}}$. (b) and (c) Fractional corner charge modes for $\text{FT}_{\#1}^{\circ}$ [35]. (d) Windings of the Wilson loop spectrum of $\text{FT}_{\#1}^{\circ}$. k_2 is the crystal momentum along the reciprocal lattice vector. (e) Coupling to a pair of $p_{\pm}$ orbitals placed at the hexagon corners, $p_{\pm}@q_{2b}$, unwinds the Wilson loop for $\text{FT}_{\#1}^{\circ}$. (f) A non-vanishing $\log(\det[S(\mathbf{k})])$ throughout the BZ implies no Wannier obstruction for the combined system: $\text{FT}_{\#1}^{\circ} \oplus (p_{\pm}@q_{2b})$.

handed for $\alpha = 2$, as illustrated in Fig. 1(a)], and can be engineered in various experimental setups [25, 27, 36].

While $H_{\circ}(\tau)$ generally breaks all its crystalline symmetries at a fixed time τ , its corresponding effective Floquet Hamiltonian $H_{\circ}^F$ generating stroboscopic time-evolutions actually restores the six-fold rotation symmetry C_6 in the high-frequency limit $\omega \gg |t_0|$ [37], where

$$H_{\circ}^F = h_{\tilde{t}}(A_{\alpha}) + h_{\Delta} + h_{\lambda}(A_{\alpha}, \omega) + \mathcal{O}(\omega^{-2}). \quad (2)$$

Here $h_{\tilde{t}}$ and h_{Δ} take the same form as in Eq. 1, now with the renormalized nearest-neighbor (NN) hopping $\tilde{t}_{\alpha} = t_0 \mathcal{J}_0(A_{\alpha})$, where $\mathcal{J}_n(x)$ is the n -th Bessel function of the first kind. h_{λ} is a driving-induced Haldane-like next-nearest-neighbor (NNN) hopping term that explicitly breaks time-reversal symmetry and opens up a bulk energy gap [38]: $h_{\lambda}(A_{\alpha}, \omega) = i \sum_{\alpha} \sum_{\langle\langle i, j \rangle\rangle} \mu_{ij} \lambda_{\alpha} c_{i, \alpha}^{\dagger} c_{j, \alpha}$. If an electron hops to its NNN (counter-)clockwise around the hexagon center, it picks up a phase $\mu_{ij} = +1(-1)$. The NNN hopping can be analytically obtained as $\lambda_{\alpha} = -2t_0^2 \sum_{n=1}^{\infty} \sin \frac{2\pi n}{3} \mathcal{J}_n^2(A_{\alpha}) / (n\omega)$ [24]. Crucially, since $\tilde{t}_{\alpha}$ and λ_{α} depend on the driving amplitudes A_{α} through the oscillating Bessel functions, their signs can be *individually* controlled by tuning A_{α} , which is the key for realizing various topological phases shown in Fig. 1(b). In particular, when $\lambda_1 \lambda_2 < 0$, $H_{\circ}^F$ achieves two inequivalent fragile topological phases (dubbed $\text{FT}_{\#1}^{\circ}$ and $\text{FT}_{\#2}^{\circ}$) with Wannier obstructions that can nevertheless be removed upon coupling to certain atomic bands.

To understand the nature of the Wannier obstruction, we take $\text{FT}_{\#1}^{\circ}$ with $\tilde{t}_{1,2} > 0$ as an example. A similar analysis can be performed for the $\text{FT}_{\#2}^{\circ}$ with $\tilde{t}_{1,2} < 0$. The first evidence of the “fragile” Wannier obstruction manifests in the symmetry eigenvalues of the two occupied bands at high-symmetry momenta, as shown in Fig. 2(a). In particular, we find that no two-band atomic insulator on a honeycomb lattice shares the same set of symmetry data as $\text{FT}_{\#1}^{\circ}$ [37]. This clearly demonstrates an obstruction towards adiabatically connecting $\text{FT}_{\#1}^{\circ}$ to an atomic system with the same amount of degrees of freedom.

We now show that such Wannier obstruction in $\text{FT}_{\#1}^{\circ}$ can be removed upon adding additional atomic orbitals. Consider coupling $\text{FT}_{\#1}^{\circ}$ with a pair of $p_{\pm}$ orbitals placed at the hexagon corners (i.e. maximal Wyckoff position q_{2b}), which we denote as “ $p_{\pm}@q_{2b}$ ” for short. Here $p_{\pm} = p_x \pm ip_y$. This composite system $\text{FT}_{\#1}^{\circ} \oplus (p_{\pm}@q_{2b})$ shares the same symmetry data as the following combination of atomic insulators: $(s@q_{1a}) \oplus (s@q_{3c}) \oplus (p_{\pm}@q_{1a})$, where q_{1a} and q_{3c} denote the maximal Wyckoff positions at the hexagon center and the center of each hexagon edge, respectively. This implies that the composite system $\text{FT}_{\#1}^{\circ} \oplus (p_{\pm}@q_{2b})$ can be Wannierized with no obstruction. We further study the Wilson loop spectrum of $\text{FT}_{\#1}^{\circ}$, as shown in Fig. 2(d). The Wilson loop spectrum is gapless and features nontrivial windings, a direct implication of Wannier obstruction [10, 39]. On the

other hand, when coupled to additional orbitals $p_{\pm}@q_{2b}$, the Wilson loop for the composite system indeed unwinds and becomes gapped, as shown in Fig. 2(e). This again signals the Wannierizability of the composite system $\text{FT}_{\#1}^{\circ} \oplus (p_{\pm}@q_{2b})$.

Finally, we construct a set of localized trial Wannier basis $|w_l(\mathbf{r})\rangle$ ($l = 1, \dots, 6$) for $(s@q_{1a}) \oplus (s@q_{3c}) \oplus (p_{\pm}@q_{1a})$, by superposing the tight-binding basis of $H_{\square}^F$. We provide the explicit expressions of $|w_l(\mathbf{k})\rangle$ and the Fourier transform of $|w_l(\mathbf{r})\rangle$ in [37]. We now follow the procedure in Ref. [5] and define an overlap matrix $\mathcal{S}_{ll'}(\mathbf{k}) = \langle \Phi_l(\mathbf{k}) | \Phi_{l'}(\mathbf{k}) \rangle$, where $|\Phi_l(\mathbf{k})\rangle$ is constructed by projecting our trial basis $|w_l(\mathbf{k})\rangle$ onto the occupied states of $\text{FTI}_{\#1} \oplus (p_{\pm}@q_{2b})$. The matrix $\mathcal{S}(\mathbf{k})$ is necessary for obtaining an orthonormal set of Bloch states from the projected trial basis $|\Phi_l(\mathbf{k})\rangle$, using the Löwdin orthonormalization procedure. For such a procedure to succeed, the determinant of $\mathcal{S}(\mathbf{k})$ must be nonzero everywhere in the Brillouin zone (BZ), i.e. $\mathcal{S}(\mathbf{k})$ is invertible. In Fig. 2(f), we map out (the logarithm of) $\det[\mathcal{S}(\mathbf{k})]$ for $\text{FTI}_{\#1} \oplus (p_{\pm}@q_{2b})$ and find $\det[\mathcal{S}(\mathbf{k})] \neq 0$ throughout the BZ, indicating no obstruction towards a Wannier representation with $|w_l(\mathbf{k})\rangle$ for our target system. This unambiguously proves the Wannierizability of $\text{FT}_{\#1}^{\circ} \oplus (p_{\pm}@q_{2b})$ and its adiabatic equivalence to $(s@q_{1a}) \oplus (s@q_{3c}) \oplus (p_{\pm}@q_{1a})$. Formally, $\text{FT}_{\#1}^{\circ}$ can be decomposed into a superposition of atomic insulators as

$$\text{FT}_{\#1}^{\circ} \equiv (s@q_{1a}) \oplus (s@q_{3c}) \oplus (p_{\pm}@q_{1a}) \ominus (p_{\pm}@q_{2b}), \quad (3)$$

where the atomic subtraction “ $\ominus$ ” indicates the fragile topological nature of $\text{FT}_{\#1}^{\circ}$. The physical reason for the Wannierizability of $\text{FT}_{\#1}^{\circ}$ upon adding $p_{\pm}$ orbitals at $2b$ locations is that it is now possible to superpose these orbitals to form new basis states, such that these new basis states transform under symmetry in the same way as an atomic insulator formed by the first three orbitals in the atomic decomposition (3). We emphasize that the fragile topology here is protected by the C_6 symmetry, resembling a Floquet version of the “shift insulator” [12]. In [37], we show that breaking C_6 down to C_3 will necessarily spoil the fragile topology in our system.

When $\lambda_1\lambda_2 < 0$ and $t_1t_2 < 0$, $H_{\square}^F$ is trivially atomic, equivalent to $s@q_{2b}$, with a gapped boundary. When $\lambda_1\lambda_2 > 0$, both electrons species carry identical Chern numbers $|\mathcal{C}_{1,2}| = 1$, leading to a Chern insulator phase for $H_{\square}^F$ with $\mathcal{C} = \pm 2$. Such a Chern insulator hosts a pair of chiral edge modes circulating along the system boundary, with the chirality (i.e. the sign of $\mathcal{C}_{\alpha}$) determined by the sign of λ_{α} .

Remarkably, both fragile phases are in fact *higher-order topological*, which feature in-gap fractional corner charges $\frac{e}{6} \pmod{e}$ in a finite system with C_6 symmetric boundary, as is numerically confirmed in Fig. 2(b)&(c). The fractional quantization of the corner charge can be understood from our atomic decomposition, along with

the fact that atomic insulators with orbitals on q_{2b} and q_{3c} host corner charges of $\frac{2e}{3}$ and $\frac{e}{2}$, respectively [40]. The presence of robust corner charge modes clearly distinguishes $\text{FT}_{\#1,2}^{\circ}$ from both the Chern insulator phases and the trivial phase, and thus serves as a clear experimental indicator of fragile topology in our model.

Floquet π -flux model.— We now present a second example with tunable fragile topology under periodic driving. Consider a square lattice with π -flux penetrating each elementary plaquette, with the Hamiltonian $h_{\square} = \sum_{\langle ij \rangle} t_{ij} c_i^{\dagger} c_j + \text{H.c.}$. We choose the gauge such that the NN hoppings $t_{ij} = t$ on horizontal bonds, and $t_{ij} = \pm it$ on vertical bonds, as shown in Fig. 3(a). In the absence of driving, the energy spectrum of the π -flux model takes the form: $E_{\mathbf{k}} = \pm 2t \sqrt{\cos^2 k_x + \sin^2 k_y}$ where $\mathbf{k} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi)$. The spectrum has two Dirac nodes at high-symmetry points X and M in the BZ. Similarly to the honeycomb model, the π -flux model is coupled to a time-dependent gauge field $Ae^{i\omega\tau}$ via the Peierls substitution. In the high-frequency limit, we show that the effective Floquet Hamiltonian to order $1/\omega$ is given by [37]:

$$h_{\square}^F = \sum_{\langle ij \rangle} t_{1,ij} c_i^{\dagger} c_j + \sum_{\langle\langle ij \rangle\rangle} t_{2,ij} c_i^{\dagger} c_j + \text{H.c.} + \mathcal{O}(\omega^{-2}), \quad (4)$$

where the renormalized NN hopping and the driving-induced NNN hopping are given by $t_{1,ij} = t_{ij} \mathcal{J}_0(A)$ and $t_{2,ij} = \frac{4\eta_{ij}t^2}{\omega} \sum_{m>0} \frac{\mathcal{J}_m^2(A)}{m} \sin(\frac{\pi}{2}m)$. The NNN hopping has alternating signs $\eta_{ij} = \pm 1$ as illustrated in Fig. 3(a). In the presence of t_2 , each triangle in Fig. 3 has a flux $\pm \frac{\pi}{2}$ which breaks time-reversal symmetry and opens up a gap at the Dirac points, yielding two bands carrying Chern numbers $\mathcal{C} = \pm 1$ [41].

Consider coupling two copies of the π -flux model with s -orbital electrons of *opposite* spins under oppositely polarized drives:

$$H_{\square}^F = h_{\square,\uparrow}^F(t, A_1 e^{-i\omega\tau}) + h_{\square,\downarrow}^F(-t, A_2 e^{i\omega\tau}) + H_g, \quad (5)$$

where $\pm t$ denotes the NN hoppings of the undriven model, and H_g is the coupling between two copies, whose explicit form will be specified below. Hamiltonian (5) has a C_2 rotation symmetry with the rotation center being the bond center. Once again, the signs and values of NN and NNN hoppings in each copy $\{t_{1\uparrow}, t_{2\uparrow}, t_{1\downarrow}, t_{2\downarrow}\}$ can be varied by tuning the two driving amplitudes A_1 and A_2 , such that Hamiltonian (5) realizes a variety of phases with distinct boundary signatures. In Fig. 3(b)&(c), we show the phase diagram of Hamiltonian (3) upon tuning A_1 and A_2 , as well as their boundary signatures.

For $t_{1\uparrow}t_{1\downarrow} < 0$, $t_{2\uparrow}t_{2\downarrow} < 0$, Hamiltonian (5) realizes C_2 -protected fragile topology. This further splits into two distinct phases which we label as $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$, as shown in Fig. 3(b). Since t_2 has opposite signs in the two copies, the two occupied bands carry Chern numbers $\mathcal{C} = \pm 1$, and host two counter-propagating chiral

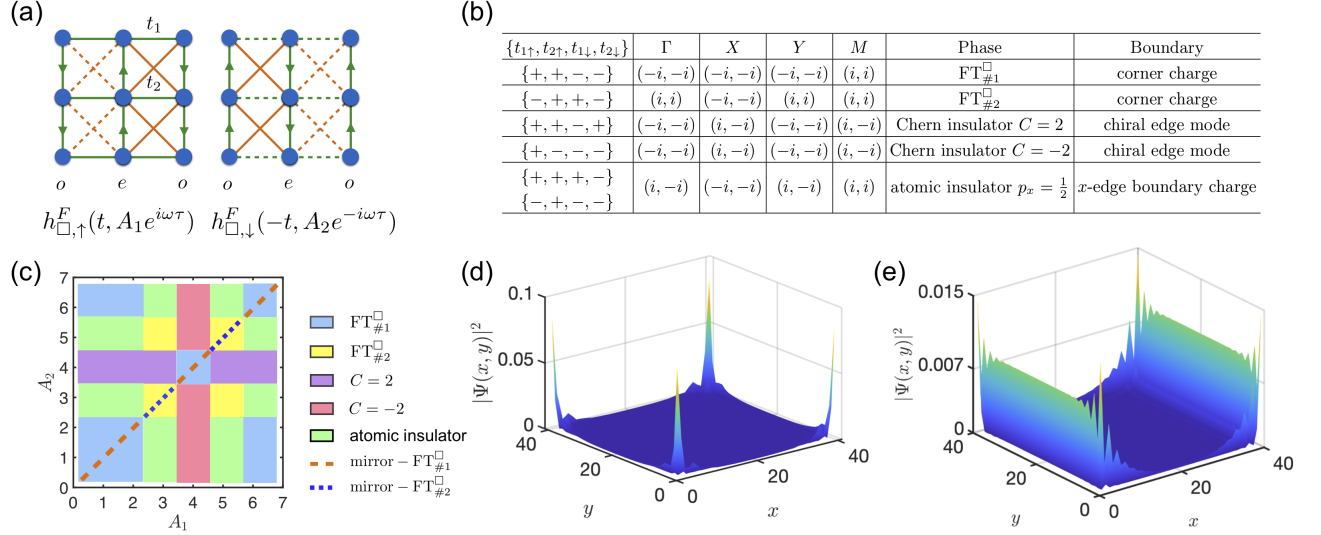


FIG. 3. (a) FT insulator constructed by coupling two copies of the π -flux model with opposite spins and oppositely polarized drives. Fermions hopping across the solid lines, dotted lines, and along the arrows will pick up a phase of 0, π , and $\frac{\pi}{2}$, respectively. (b) C_2 eigenvalues at high symmetry momenta, the corresponding phases, as well as their boundary signatures. $\{\pm\}$ denotes the sign of each hopping term. (c) Phase diagram of Hamiltonian (5) upon tuning A_1 and A_2 . (d) Corner charge in the mirror- $\text{FT}_{\#1}^{\square}$ phase. (e) Boundary charge in the atomic insulator phase with $\mathbf{P} = (\frac{1}{2}, 0)$.

edge modes. However, since the total Chern number vanishes, these chiral edge modes can be gapped out by adding C_2 -preserving couplings between the two copies $H_g = \sum_{i \in \Lambda_o} g_1 (c_{i\uparrow}^\dagger c_{i\downarrow} - c_{i+\hat{x}\uparrow}^\dagger c_{i+\hat{x}\downarrow}) + g_2 (c_{i\uparrow}^\dagger c_{i+\hat{x}\downarrow} - c_{i\downarrow}^\dagger c_{i+\hat{x}\uparrow}) + \text{H.c.}$, where Λ_o denotes the sublattice belonging to odd columns, and site $i + \hat{x}$ belongs to even columns. By constructing the low-energy effective edge theory [37], one can show that the edge spectrum indeed becomes gapped upon adding H_g . Hamiltonian (5) thus realizes an insulator without anomalous edge modes. Nonetheless, it suffers from an obstruction towards a Wannier representation. In Fig. 3(b), we list the C_2 eigenvalues at high symmetry momenta for $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$. We find that these symmetry representations cannot be realized with any two-band atomic insulator and hence the system is Wannier obstructed. The Wannier obstruction can also be detected from the Wilson loop spectrum. In fact, the symmetry data of $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$ in Fig. 3(b) enforce nontrivial windings of the Wilson loop spectrum [42], which we verify numerically [37].

Next, we show that the Wannier obstruction is fragile and can be removed upon adding atomic orbitals to Hamiltonian (5). Consider coupling $\text{FT}_{\#1}^{\square}$ to two spin-up orbitals per unit cell at C_2 -symmetric positions: $(0, \frac{1}{4})$ away from the odd sublattice and $(0, -\frac{1}{4})$ away from the even sublattice. We find that the new composite system shares the same symmetry data as the atomic insulator: $(s_\uparrow @ q_{1a}) \oplus (s_\uparrow @ q_{1b}) \oplus (s_\uparrow @ q_{1c}) \oplus (s_\downarrow @ q_{1d})$, indicating that the composite system has no Wannier obstruction. Here the maximal Wyckoff positions for a C_2 -symmetric unit

cell are: $q_{1a} = (0, 0)$, $q_{1b} = (\frac{1}{2}, 0)$, $q_{1c} = (0, \frac{1}{2})$, $q_{1d} = (\frac{1}{2}, \frac{1}{2})$. Furthermore, we show in [37] that the Wilson loop unwinds upon coupling to the added atomic orbitals. Taken together, the Wannier obstructions in $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$ are indeed fragile.

Along the line $A_1 = A_2$ in Fig. 3(c), the system has two additional mirror symmetries M_x and M_y [43]. Under the basis $(c_{o\uparrow}^\dagger, c_{e\uparrow}^\dagger, c_{o\downarrow}^\dagger, c_{e\downarrow}^\dagger)$, these two symmetries can be written as: $M_x = -i\sigma^y \otimes \gamma^y$ and $M_y = i\sigma^z \otimes \gamma^x$ with $[M_x, M_y] = 0$, where the Pauli matrices σ^a and γ^b act on sublattice and spin degrees of freedom respectively. The mirror symmetries protect *two distinct new fragile topological phases*: mirror- $\text{FT}_{\#1}^{\square}$ and mirror- $\text{FT}_{\#2}^{\square}$, whose M_x and M_y representations cannot be realized in any atomic insulator. We provide the atomic decompositions in the presence of mirror symmetries in [37]. Remarkably, mirror- $\text{FT}_{\#1}^{\square}$ and mirror- $\text{FT}_{\#2}^{\square}$ host four corner modes with charge $\frac{e}{4} \pmod{e}$ [Fig. 3(d)], which can be seen from the atomic decomposition and the filling anomaly [40]. These corner charges persist for $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$ when A_1 and A_2 are slightly off the mirror-symmetric line. Nonetheless, when $M_{x,y}$ are strongly broken, C_2 symmetry alone will generically protect only two corner charges $\frac{e}{2} \pmod{e}$ for $\text{FT}_{\#1}^{\square}$ and $\text{FT}_{\#2}^{\square}$.

When $t_{2\uparrow}t_{2\downarrow} > 0$, the occupied bands of $H_{\square}$ have a net Chern number $C = \pm 2$, hence realize a Chern insulator with two chiral edge modes at the boundary. Finally, when $t_{1\uparrow}t_{1\downarrow} > 0$, $t_{2\uparrow}t_{2\downarrow} < 0$, $H_{\square}$ realizes an atomic insulator $(s_\uparrow @ q_{1c}) \oplus (s_\downarrow @ q_{1d})$, which has a nonzero bulk polarization $\mathbf{P} = (\frac{1}{2}, 0)$. Consequently, this atomic insu-

lator has charge accumulation along edges parallel to the y -direction, but not on the other edges [Fig. 3(e)]. Therefore, similar to the honeycomb model, different phases of the π -flux model can also be diagnosed solely from their boundary signatures.

Discussions - We present two examples of Floquet systems exhibiting driving-induced tunable Floquet fragile topology with characteristic boundary features. Floquet systems are advantageous in their tunability, which offers a unique opportunity to explore the topological quantum criticality for fragile topology in experiments. Although both of our models are statically semimetallic, one may as well start from static gapped systems. Meanwhile, general fragile topological phases are recently proposed to host in-gap spectral flows under twisted boundary conditions in real space [16, 22]. For Floquet fragile topological systems, we expect that the Floquet drive acts as an on-off controller of the twisted-boundary-induced spectral flows. A detailed discussion will be left for future studies.

Floquet Haldane phase, the key ingredient of our honeycomb model, has already been proposed and realized in a variety of experimental settings, including photonic waveguides [27], cold atoms [36], and acoustic crystals [44]. To realize our Floquet honeycomb model, for example, one can stack two layers of the acoustic Floquet Haldane system proposed in Ref. [44] with layer-dependent drives, and further introduce an additional acoustic waveguide at each site to bridge between the layers, which mimics the interlayer coupling Δ . With the state-of-the-art fabrication techniques in the metamaterial platforms, it is conceivable that fragile topology in out-of-equilibrium systems that we propose in this work can be realized experimentally in the near future.

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* ruixing@umd.edu

† zcyang@umd.edu

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