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Anna Keselman, Laimei Nie, and Erez Berg

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Scrambling and Lyapunov Exponent in Spatially Extended Systems

Anna Keselman,[1,2] Laimei Nie,[3,4] and Erez Berg[5]

[1]Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030
[2]Microsoft Station Q, Santa Barbara, California 93106-6105, USA
[3]Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, IL 60637, USA
[4]Department of Physics and Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
[5]Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

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Scrambling of information in a quantum many-body system, quantified by the out-of-time-ordered correlator (OTOC), is a key manifestation of quantum chaos. A regime of exponential growth in the OTOC, characterized by a Lyapunov exponent, has so far mostly been observed in systems with a high-dimensional local Hilbert space and in weakly-coupled systems. Here, we propose a general criterion for the existence of a well-defined regime of exponential growth of the OTOC in spatially extended systems with local interactions. In such systems, we show that a parametrically long period of exponential growth requires the butterfly velocity to be much larger than the Lyapunov exponent times a microscopic length scale, such as the lattice spacing. As an explicit example, we study a random unitary circuit with tunable interactions. In this model, we show that in the weakly interacting limit the above criterion is satisfied, and there is a prolonged window of exponential growth. Our results are based on numerical simulations of both Clifford and universal random circuits supported by an analytical treatment.

Introduction.- Many-body quantum chaos has recently attracted an increasing amount of attention thanks to its connections with quantum thermalization [1, 2], many-body localization [3, 4], and black hole physics [5–8]. Among the many operational diagnostics of quantum chaos [8–18], the scrambling of local quantum information, typically quantified by out-of-time-ordered correlators (OTOC) [5–7, 19–29], aims to capture the growth of complexity of local operators under Heisenberg time evolution. In large-N systems such as the Sachdev-Ye-Kitaev model and conformal field theories with large central charge and holographic duals, the OTOC has been shown to exhibit a regime of exponential growth characterized by the quantum Lyapunov exponent $\lambda_{\rm L}$ [5, 7, 30–32]. The scrambling time, which determines the time window for the exponential growth, is parametrically long in the large-N limit.

OTOCs in quantum many-body lattice systems with a finite-dimensional Hilbert space and local interactions have also been studied extensively [19–21, 27, 33–38]. In generic situations, no regime of exponential growth was found [19, 21, 37]. A special case which does exhibit exponential growth is the weak coupling regime [24, 39–42]. However, it remained unclear what controls the scrambling time in systems with a local structure.

In this work, we propose a general criterion for the existence of a time period of exponential growth of the OTOC, which is applicable for any system with spatial structure and local interactions. We argue that a parametrically long scrambling time can arise as a result of a competition between the exponential growth of the OTOC locally, and the rapid growth of the number of accessible degrees of freedom. The latter growth rate is set by the butterfly velocity $v_{\rm B}$ [43], which is defined as the velocity of the propagation of the operator front. The existence of a parametrically long regime of exponential growth is thus possible in the limit of large $v_{\rm B}/(\lambda_{\rm L} a)$ ratio [see Eq. (3)], where a is a microscopic length scale, to be discussed below.

We demonstrate this principle using a random unitary circuit model. Such models have been employed to draw insights into the dynamical properties of deterministic quantum systems [9, 11, 14, 19–22, 44, 45]. Previous works on (1+1)D random unitary circuits showed that the front of the OTOC travels ballistically with a diffusive broadening, and no extended exponential regime was found [19, 21]. Here, we introduce a random circuit model with a tunable parameter, that plays the role of the interaction strength. We provide both analytical results and numerical verifications of the existence of an extended exponential growth regime in the limit of weak interaction. Our analysis further reveals the full structure of the OTOC during the entire evolution, including a crossover to a saturated regime at late times. In the limit of strong interactions, we recover the behavior observed in previous studies [21].

Below, we start by defining the integrated OTOC, which is suitable to characterize operator growth in systems with spatial structure. We then introduce our random unitary circuit model with tunable interactions. Focusing on a special type of a Clifford circuit, we demonstrate the existence of a regime of exponential growth, and characterize the crossover time to the saturated regime. Numerical results for this model are complemented by analytic rate equations. We then consider more generic circuits, and show that our main results remain unchanged.

Scrambling time in systems with local interactions.- We consider a finite-dimensional system defined on a lattice,

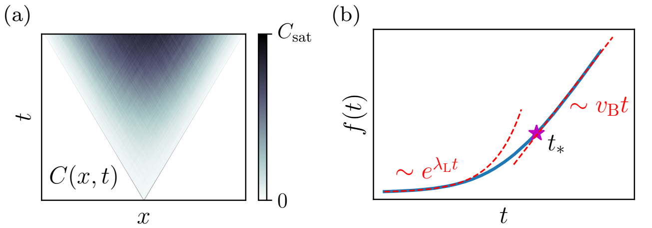

FIG. 1. Time evolution of the (a) local and (b) integrated OTOC in the Clifford circuit (see text), demonstrating the behavior expected in a generic system. The early time growth is characterized by a Lyapunov exponent $\lambda_{\rm L}$, while at late times the growth of the iOTOC is linear, with a rate proportional to the butterfly velocity $v_{\rm B}$ and the saturation value, $C_{\rm sat}$. The crossover time between the two regimes (the scrambling time) is indicated by t_*.

where each lattice site contains a degree of freedom with a finite-dimensional Hilbert space. The OTOC of two local operators, W_i, V_j, acting on sites i, j respectively, is given by

$$C_{i,j}(t) = -\left\langle [W_i(t), V_j]^2 \right\rangle . \qquad (1)$$

Under random unitary circuit evolution, temperature is ill-defined, and therefore we take the expectation value above with respect to an infinite temperature distribution. For simplicity, we will focus on the one-dimensional case and address higher dimensions later. In a generic scenario, upon time evolution, the support of the operator $W_i(t)$ grows ballistically, forming a light cone with the front propagating at the butterfly velocity $v_{\rm B}$. The OTOC above becomes non-zero once the site j enters this light cone. Following an early exponential growth regime, the value of $C_{i,j}(t)$ must saturate at late times, since it is bounded due to the finite dimension of the local Hilbert space. This behavior is shown in Fig. 1(a), for a random circuit model to be described below.

While the structure and dynamics of the local OTOC (1) are interesting on their own right, here we focus on the global properties of the scrambling dynamics. To this end, we introduce the integrated OTOC (iOTOC),

$$f(t) = \frac{1}{L}\sum_{i,j} C_{i,j}(t) = -\frac{1}{L}\sum_{i,j}\left\langle [W_i(t), V_j]^2 \right\rangle , \qquad (2)$$

where a summation is performed over all the lattice sites [46]. The iOTOC measures the expectation value of the "size" of the operator [47, 48], washing out any transients and details of spatial structure, and thus simplifying the identification and characterization of the scrambling time. Similarly to the local OTOC, at early times, iOTOC may exhibit an exponential growth with a Lyapunov exponent $f(t) \sim e^{\lambda_{\rm L} t}$. At late times, when the OTOC in the bulk of the system reaches its saturation value $C_{\rm sat}$, the iOTOC crosses over to a linear growth regime (due to the linear growth of the light cone), $f(t) \sim C_{\rm sat} v_{\rm B} t$.

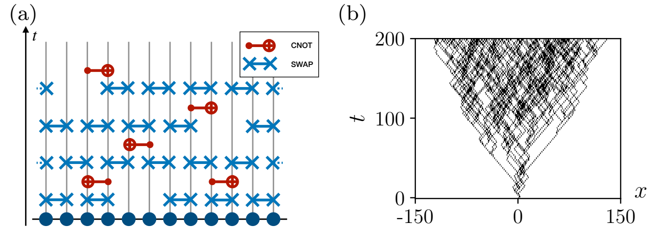

FIG. 2. (a) Schematic representation of the Clifford unitary circuit (see text). (b) Density of σ^z operators in the operator string $\sigma_0^z(t)$, for a single realization of the circuit with $p = 0.9, r = 0.05$.

Assuming a single crossover time t_* (the scrambling time) between these two regimes, t_* can be obtained from

$$e^{\lambda_{\rm L} t_*} \sim c_{\rm sat} v_{\rm B} t_* \Rightarrow t_* \sim \frac{1}{\lambda_{\rm L}} \log \frac{c_{\rm sat} v_{\rm B}}{\lambda_{\rm L}} , \qquad (3)$$

where we introduced the OTOC density at saturation $c_{\rm sat} = C_{\rm sat}/a$ (a is the lattice spacing) and dropped corrections to t_* that are of higher order in $c_{\rm sat} v_{\rm B}/\lambda_{\rm L}$. We thus find that a parametrically long scrambling time is expected when $v_{\rm B}/\lambda_{\rm L}$ diverges. Note that in the limit of $\lambda_{\rm L} \to 0$, both $1/\lambda_{\rm L}$ and t_* diverge, however there still is a parametric separation between the two time scales due to the logarithmic enhancement of the latter with respect to the former, allowing for a clear exponential regime to be present. The iOTOC, exhibiting this behavior, is plotted in Fig. 1(b).

Random Clifford circuit model.- To demonstrate the arguments above, we study the scrambling dynamics in a random unitary circuit. We start with the simplest model in which the general behavior discussed above can be observed, and which is amenable to both analytical and large scale numerical analysis. A study of a more generic version of the circuit, which concurs with the results obtained here, is presented later. The structure of the circuit is shown in Fig. 2(a). Every site hosts a single qubit. At every odd (even) time step a set of SWAP gates is applied on the odd (even) bonds, with probability $0 \le p \le 1$ on each bond. A SWAP gate interchanges the state of the two qubits it acts on, and can be written explicitly as $\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+ + (1+\sigma_1^z\sigma_2^z)/2$, where σ^z and $\sigma^\pm$ are Pauli operators. Then, a set of CNOT gates is applied on a fraction $0 \le r \le 1/2$ of all the bonds. The bonds are chosen such that only configurations where no two bonds share a site are allowed and each such configuration is equally probable. The role of each qubit (control or target) is chosen randomly and independently for each CNOT gate.

Note that the circuit consists of Clifford gates only, and therefore it can be simulated classically [49], allowing us to explore large systems and long times. Upon Clifford evolution, an operator $\sigma_i^\alpha(t=0) = \sigma_i^\alpha$, with $\alpha = x, y, z$ remains a single operator string of Pauli operators. In particular, for the circuit structure described above, when the operator $\sigma_i^z(t)$ is considered, the corresponding operator string at times $t > 0$ consists only of σ^z and identity operators. Fig. 2(b) shows the density of σ^z operators as a function of time, for a single realization of the circuit.

To understand the evolution of operators consider first the limit of $r = 0$ (no CNOT gates), and $p = 1$ (SWAP gates are applied on all odd / even bonds at each time step). In this case, a single-site operator located on an odd (even) site propagates ballistically to the right (left) with velocity $v_{B,0} = 1$. Decreasing p can be thought of as introducing disorder, as a missing SWAP gate results in a back-scattering of an operator, flipping its velocity. This gives rise to a diffusive propagation with diffusion constant $D \sim v_{B,0}^2 \tau_p$, where $\tau_p \sim (1-p)^{-1}$ is the characteristic time between consecutive back-scatterings.

Next, consider $r > 0$, and for concreteness focus on the evolution of a σ^z operator. The action of a CNOT gate on operators (when the first qubit is the control qubit and the second one is the target) is given by

$$\sigma^z \otimes \mathbb{1} \to \sigma^z \otimes \mathbb{1},\ \mathbb{1} \otimes \sigma^z \to \sigma^z \otimes \sigma^z,\ \sigma^z \otimes \sigma^z \to \mathbb{1} \otimes \sigma^z. \quad (4)$$

This process can be thought of as a scattering event due to interactions, which increases the support of a local operator. Denoting the scattering time due to the CNOT gates as $\tau_r \sim r^{-1}$, we note that in the diffusive case ($p < 1$), a finite r gives rise to a finite butterfly velocity $v_B \sim \sqrt{D/\tau_r} \sim \sqrt{r/(1-p)}$ [40].

We are interested in the evolution of the OTOC. Writing the operator string corresponding to $\sigma_i^\alpha(t)$ explicitly as $\otimes_k \sigma_k^{\alpha_k(t)}$ (where k runs over all the sites, and σ^0 is identity), the commutator $\left[\sigma_i^\alpha(t), \sigma_j^\beta\right]$ is given by $\left(\otimes_{k\neq j} \sigma_k^{\alpha_k(t)}\right) \otimes \left[\sigma_j^{\alpha_j(t)}, \sigma_j^\beta\right]$. Since $\left(\sigma_k^{\alpha_k(t)}\right)^2 = \mathbb{1}$, the commutator squared is simply $\left[\sigma_j^{\alpha_j(t)}, \sigma_j^\beta\right]^2 = 4(\delta_{\beta,\alpha_j(t)} - 1)\mathbb{1}$ for $\alpha_j(t) \neq 0$ (i.e. $\sigma_j^{\alpha_j(t)} \neq \mathbb{1}$). In particular, the expectation value is state independent. Performing the summation over j in Eq. (2) with $W_i = \sigma_i^\alpha/\sqrt{2}$ and $V_j = \sigma_j^\beta/\sqrt{2}$, we find $f(t) = \sum_j (1-\delta_{\beta,\alpha_j(t)})(1-\delta_{\alpha_j(t),0})$, i.e. the number of (non-identity) Pauli operators in $\sigma_i^\alpha(t)$ which are different from σ^β. Below, we consider the OTOC between $\sigma_i^z(t)$ and σ_j^x , so that $f(t)$ amounts to the number of σ^z operators in the string $\sigma_i^z(t)$ at time t. Note that the commutator $[\sigma_i^z(t), \sigma_j^z]$ vanishes identically for any realization of the circuit. In fact, any product state in the z basis remains un-entangled upon evolution with the circuit above. However, as shown later, our main results hold also in more generic circuit models.

Numerical results for integrated OTOCs and crossover time.- We now study the behavior of the iOTOC in this model as function of the circuit parameters. To this end, we perform numerical simulations of the operator dynamics, calculating the local and integrated OTOC. As was already mentioned, the density of the CNOT gates, r, is the parameter that sets the growth rate of the support of a local operator in the circuit and leads to scrambling. Therefore, we expect the Lyapunov exponent λ_L to be directly determined by r. In the limit $r \ll 1$ (and hence large v_B/λ_L), we expect an extended time regime in which the growth of the iOTOC is exponential with a well-defined Lyapunov exponent. We find that this is indeed the case both for the ballistic and the diffusive parameter regime, as can be seen in the insets of Figs. 3(a,c). To further analyze the crossover time, and its scaling with λ_L and v_B, we look at the average OTOC density, namely the iOTOC, $f(t)$, divided by the size of the light cone, $2v_B t$ (see Figs. 3(a,c)). At late times we expect this quantity to approach the saturation value of the OTOC in the bulk, which we find to be $1/2$ and independent of r in the regime $r \ll 1$. (This value is in agreement with the expectation from the analytic rate equation analysis presented later on.) We define the crossover time t_* as the time at which the averaged OTOC density reaches half of its saturation value. In Figs. 3(b,d) we show that the crossover time extracted as above, indeed obeys the scaling expected from Eq. (3) both for the ballistic circuit with $p = 1$ and the diffusive one with $p = 0.9$. The results were obtained by averaging over $4 \cdot 10^3$ (10^4) realizations for the ballistic (diffusive) case.

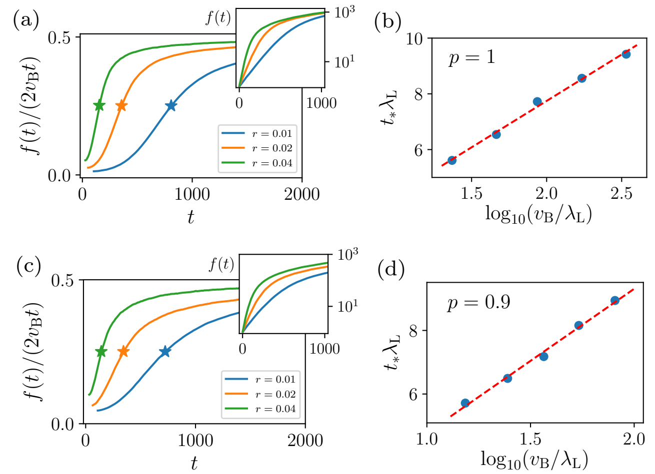

FIG. 3. Numerical results for the OTOC in the Clifford circuit, for the ballistic case ($p = 1$) in (a,b), and for the diffusive case ($p = 0.9$) in (c,d). (a,c) The OTOC density, $f(t)/(2v_B t)$, for different values of r as function of time. The crossover time, t_*, when the OTOC density reaches half of the saturation value, is indicated by a star. Insets: $f(t)$ on a log scale. (b,d) Scaling of t_* with λ_L (extracted as the slope of $\log[f(t)]$ vs. t at early times) and v_B (extracted from the spatial profile of the operator density at late times). Data points shown correspond to $r = 0.003, 0.006, 0.012, 0.024, 0.048$ in (b) and $r = 0.002, 0.004, 0.008, 0.016, 0.032$ in (d).

Master equation for integrated OTOC.- To gain further insights on the scrambling process in the model described above, we now derive analytic rate equations for the iOTOC. We consider the limit of small but finite

r, and $1-p \ll 1$, such that the scattering events due to the CNOT gates are dilute and can be assumed to be uncorrelated [50]. This assumption is analogous to the molecular chaos hypothesis.

At time step t, the number of CNOT gates applied within the light cone of an operator $\sigma_i^z(t)$ is $N_{\mathrm{CNOT}} = 2rv_{\mathrm{B}}t$. Consider what happens to the total number of σ^z operators in the operator string upon application of a CNOT gate. From (4), we see that this number increases by one if the target (but not the control) site hosts a σ^z operator, while if both sites host a σ^z operator, the number decreases by one. Denote the fraction of non-identity operators in the operator string of $\sigma_i^z(t)$, within its light cone, by $q \equiv f(t)/(2v_{\mathrm{B}}t)$. Assuming the probabilities of different sites to host a σ^z operator are independent, the probabilities for the processes which increase or decrease the number of non-identity operators in the string are given by $q(1-q)$ and q^2, respectively. Thus, the change in the number of σ^z operators in the operator string in a single time step is given by $N_{\mathrm{CNOT}}\left(q(1-q)-q^2\right)$. Recalling that the iOTOC is given simply by the number of non-identity operators in the operator string, as discussed above, we find that the rate equation for the iOTOC is (treating the time as continuous)

$$\frac{df}{dt} = rf(t)\left(1-\frac{f(t)}{v_{\mathrm{B}}t}\right). \quad (5)$$

This equation admits a solution of the form

$$f(t) = \frac{g_0 e^{rt}}{1+g_0\frac{r}{v_{\mathrm{B}}}\left[\mathrm{Ei}(rt)-\mathrm{Ei}(1)\right]}, \quad (6)$$

where $\mathrm{Ei}(rt)$ is the exponential integral, and g_0 is a constant set by the initial conditions. At early times, we see that indeed $f(t) \sim e^{\lambda_{\mathrm{L}}t}$, with a Lyapunov exponent set by the CNOT gates density, $\lambda_{\mathrm{L}} = r$. At late times, using the asymptotic expansion for the exponential integral, we find $f(t) \simeq v_{\mathrm{B}}t/(1+(rt)^{-1})$, i.e. the slope asymptotically approaches the butterfly velocity. The average OTOC density, $f(t)/(2v_{\mathrm{B}}t)$ thus tends to $1/2$ as observed numerically (see Figs. 3(a,c)). The crossover time, t_*, at which the OTOC density reaches a finite fraction of the saturation value, is given (to leading order) by $e^{rt_*} \sim v_{\mathrm{B}}t_*$ in agreement with Eq. (3) and as observed numerically. Although the focus of our discussion here was on the iOTOC, in which the spatial structure is washed out, in the Supplementary Material (SM) [50] we discuss hydrodynamic equations capturing the spatial structure and discuss their validity. We observe a crossover from a propagation in which the front maintains its shape to a regime where the front broadens diffusively.

Generalizations of the random circuit model.- As noted previously, the circuit model considered above is a special type of a Clifford circuit, in which both operator entanglement and state entanglement do not grow upon time evolution. We now demonstrate that our results do not rely on either of this properties. For simplicity, we restrict the analysis below to the ballistic case, i.e. $p = 1$.

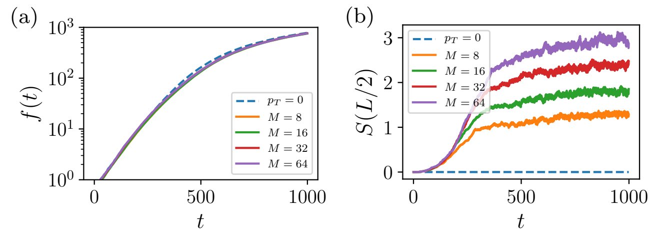

FIG. 4. (a) Integrated OTOC for the generalized circuit model, shown on a log-scale, and (b) operator entanglement entropy across the middle bond in the system, as function of time, for different bond dimensions M. Averaging over 300 realizations of the circuit is performed in each case. For $p_T = 0$, the model becomes a Clifford circuit, for which the operator entanglement remains zero at all times.

First, consider a generalization of the circuit, in which the standard CNOT gate is replaced by a CNOT operation where the basis for both the control and the target qubits is chosen to be the x,y or z basis randomly and independently for each of the two qubits. Namely

$$U_{CNOT}^{\alpha,\beta} = \frac{\mathbb{1}_1+\sigma_1^{\alpha}}{2}\otimes\mathbb{1}_2 + \frac{\mathbb{1}_1-\sigma_1^{\alpha}}{2}\otimes e^{i\frac{\pi}{4}\sigma_2^{\beta}}, \quad (7)$$

where $\alpha,\beta \in x,y,z$. While this remains a Clifford circuit, the entanglement of a state generically grows with time in this case. We consider the averaged OTOC $\propto \sum_{\mu,\nu=x,y,z}\left\langle[\sigma_i^{\mu}(t),\sigma_j^{\nu}]^2\right\rangle$. The averaged iOTOC obtained for this model, for CNOT gate density $r = 0.01$, is plotted in Fig. 4(a) (dashed blue line). It can be seen that a prolonged regime of exponential growth is present, similarly to the simplified model. Additional results for this model, and in particular a verification of the scaling in Eq. (3), are given in the SM [50].

We next consider a further generalization to a non-Clifford circuit. In this circuit operators evolve into superpositions of operators as in a generic quantum system and the operator entanglement grows with time. In our non-Clifford circuit, at each time step of the evolution, following the application of CNOT gates, a T gate (i.e. a $\pi/4$ phase gate, around a randomly chosen axis) is applied with probability p_T at each site. To calculate the OTOCs in presence of T gates we perform the time evolution of operators using a matrix product state (MPS) [51] representation of the operator string, employing the ITensor library [52]. Due to the exponential growth of operator entanglement exact simulations are limited to short times. To go to longer times we perform truncation of the MPS bond dimension. In Fig. 4(a) we plot the iOTOC for $r = 0.01$ and T gate density $p_T = 0.01$, for different maximal bond dimensions. The respective operator entanglement that builds up in the system is shown in Fig. 4(b). We see that although the operator entanglement in the system is now non-zero, the iOTOCs are essentially unmodified. Note that evolution up to times $t \sim 300$ is carried out without any truncation, and is thus exact.

Discussion.- In this work we proposed a criterion for the existence of a Lyapunov exponent in many-body systems with local interactions and a finite dimensional on-site Hilbert space. Having a parametrically long scrambling time (where the OTOC is exponentially growing, and hence λ_{L} is well-defined) requires the ratio v_B/λ_{L} to be large. This condition is naturally fulfilled in weakly coupled systems. Whether the condition is satisfied in other situations, e.g., in generic strongly-coupled systems in the low-temperature limit, remains to be seen.

Our condition is demonstrated in an explicit one-dimensional random unitary circuit model, where we have verified the relation between the scrambling time and v_B/λ_{L}. However, we expect the results to carry over to higher dimensions. Since the number of sites in the light cone grows as $(v_{\mathrm{B}}t)^d$ in the d-dimensional case, the late-time iOTOC scales as $f(t) \sim t^d$. Therefore, the scrambling time is enhanced by a factor of d relative to the one-dimensional case.

Finally, we note that other probes for scrambling have been proposed, in particular the growth of state and operator entanglement [9–14, 19, 21, 53, 54]. In our Clifford circuit, we find an exponential growth of the OTOC despite the fact that the operator entanglement (as well as the state entanglement in the special circuit described above) do not grow, indicating that the existence of a Lyapunov exponent captures a different aspect of scrambling.

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