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Topological Anomalous Skin Effect in Weyl Superconductors

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We show that a Weyl superconductor can absorb light via a novel surface-to-bulk mechanism, which we dub the topological anomalous skin effect. This occurs even in the absence of disorder for a single-band superconductor, and is facilitated by the topological splitting of the Hilbert space into bulk and chiral surface Majorana states. In the clean limit, the effect manifests as a characteristic absorption peak due to surface-bulk transitions. We also consider the effects of bulk disorder, using the Keldysh response theory. For weak disorder, the bulk response is reminiscent of the Mattis-Bardeen result for s -wave superconductors, with strongly suppressed spectral weight below twice the pairing energy, despite the presence of gapless Weyl points. For stronger disorder, the bulk response becomes more Drude-like and the p -wave features disappear. We show that the surface-bulk signal survives when combined with the bulk in the presence of weak disorder. The topological anomalous skin effect can therefore serve as a fingerprint for Weyl superconductivity. We also compute the Meissner response in the slab geometry, incorporating the effect of the surface states.

I. INTRODUCTION

Despite displaying perfect dissipationless conduction at zero frequency, superconductors absorb electromagnetic radiation at finite frequencies. The classical skin depth of a metal is given by $\delta(\omega) = c/\sqrt{2\pi\sigma_{dc}\omega}$, where ω is the radiation frequency and σ_{dc} is the static, zero-frequency conductivity due to impurity scattering [1]. The classical skin depth vanishes in the clean limit. The absorption in a superconductor is associated to a nonzero field penetration depth at finite frequencies due to the pairing, giving rise to an anomalous skin effect and associated optical conductivity [2, 3].

In a topologically trivial one-band superconductor, the dissipative part of the bulk optical conductivity vanishes in the clean limit [4]. This result obtains due to the orthogonality of positive- and negative-energy bulk states and the lack of matrix structure for the current operator, despite the particle-hole hybridization induced by pairing. However, most low-temperature superconductors are measured in the dirty limit, where $\Delta_0 \ll 1/\tau_{el} \ll \varepsilon_F$. Here Δ_0 is the pairing energy, τ_{el} is the lifetime due to elastic impurity scattering, and ε_F is the Fermi energy. For an s -wave superconductor at zero temperature, absorption turns on at $\omega = 2\Delta_0$ according to the famous Mattis-Bardeen result for the dirty limit [2, 4, 5].

What happens when the Hilbert space of a one-band superconductor is split into bulk and surface states by non-trivial topology? In this paper, we will show that a novel surface-bulk absorption can occur in a topological superconductor [6, 7], even for a one-band system in the *clean limit*. We dub this the topological anomalous skin effect. We consider optical absorption by Weyl superconductors (WSCs) [8–27]. WSCs can arise due to bulk $p + ip$ pairing, as exemplified by ^3HeA [28–30]. In static mean-field theory, the quasiparticle spectrum of a WSC exhibits pairs of gapless Weyl nodes in the bulk. Each momentum-separated nodal pair gives rise to a chiral,

two-dimensional (2D) Majorana surface fluid, displaying a Fermi arc connecting the nodes [6, 7, 26, 27, 30]. WSCs could serve as a platform for realizing Majorana zero modes and topological quantum computation [31].

We show that WSCs absorb radiation through optically driven surface-bulk transitions, see Fig. 1. We compute the surface-bulk absorption coefficient for a clean, spinless WSC with $p+ip$ pairing in the slab geometry. For a plane electromagnetic wave with normal incidence upon a crystal face with chiral surface states, we find a relatively narrow (broad) peak around $2\Delta_0$ (below $2\Delta_0$) for electric polarization perpendicular to (along) the Fermi arc, see Fig. 2.

We also consider the effects of disorder on the bulk p -wave state. Using Keldysh response theory [32–34], we derive the semiclassical optical conductivity for a Weyl superconductor. Disorder is treated at the saddle-point level. The surface-bulk transitions giving rise to the topological anomalous skin effect largely involve states away from the Weyl nodes, and therefore we do not expect rare region effects [35–43] to play an important role. For weak disorder, the bulk response is reminiscent of the Mattis-Bardeen result for s -wave superconductors, with strongly suppressed spectral weight below twice the pairing energy, despite the presence of gapless Weyl points. This is consistent with canonical results for dirty anisotropic superconductors [44]. For stronger disorder, the bulk response becomes more Drude-like and the p -wave features disappear. Results are displayed in Fig. 3.

We show that the surface-bulk signal survives when combined with the bulk in the presence of weak disorder, see Fig. 4. The topological anomalous skin effect can therefore serve as a fingerprint for Weyl superconductivity. We also compute the Meissner response in the slab geometry, incorporating the effect of the surface states. In the case of a strong topological superconductor analogous to ^3HeB [28–30], an anomalous power-law-in-temperature dependence was found for the penetration

depth, due to the paramagnetic surface state response [45]. The latter result is surprising because the bulk is fully gapped in that case. For the WSCs studied here, we do not find qualitatively new behavior in the Meissner effect arising from the presence of surface states. This is because power-law temperature-dependence is already expected due to the bulk Weyl nodes.

The Majorana surface fluid in WSCs could also be detected by scanning tunneling microscopy (STM), angle-resolved photoemission spectroscopy (ARPES) [9], and the anomalous thermal Hall effect [8, 46–48]. Effects due to the axial anomaly have also been proposed as signatures for WSCs, including negative thermal magnetoresistance [49] and a T^2 temperature dependence of the axial current [50, 51].

A key difference between Weyl semimetals and Weyl superconductors is that the bulk optical conductivity of the former is nonzero even in the clean limit, due to interband transitions [52–55]. Another key distinction concerns Fermi arc transport [56, 57]. For both fully gapped and gapless topological superconductors lacking spin SU(2) symmetry (classes DIII and D [7, 26, 58]), the coupling of the surface Majorana fluid to electromagnetism is effectively *gravitational*, i.e. the electric density and current operators are formed from components of the

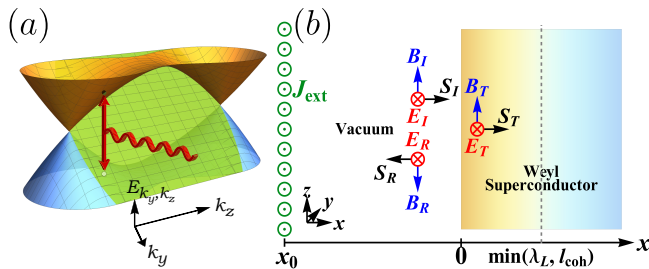


FIG. 1: Geometry for the topological anomalous skin effect. (a) We consider a Weyl superconductor described by Eq. (3.1). Due to bulk $k_x + ik_y$ pairing, a single pair of Weyl points arises at $k_z = \pm k_F$. In the figure, the green plane depicts the dispersion of the chiral surface state $E_{k_y}^s = -\Delta k_y$, for a superconductor occupying the half space $x \geq 0$. Since we consider the surface only at $x = 0$, there is only one branch for the surface states. The topological anomalous skin effect arises via absorption due to optical transitions between surface and bulk states, as indicated by the red vertical arrow in the sketch. (b) We consider a plane electromagnetic wave impacting the Weyl superconductor occupying the $x \geq 0$ half-space at normal incidence. The label I , R and T denote respectively the incident, reflected and transmitted component of the radiation. $(\mathbf{E}, \mathbf{B}, \mathbf{S})$ stands for (E-field, B-field, Poynting vector). A polarization along \hat{y} (i.e. perpendicular to the plane of incidence) is assumed for the electric field \mathbf{E} in this sketch. The topological anomalous skin effect mainly takes place in the yellow region, extending up to the scale of $\min(\lambda_L, l_{\text{coh}})$. Here λ_L denotes the London depth, and l_{coh} is the coherence length (which is the minimal confinement depth for the chiral Majorana surface fluid).

stress tensor for the surface Majorana fluid [45, 59, 60]. For the chiral fluid at the surface of a WSC, this has the consequence that no thermal current flows along the Fermi arc in the presence of perpendicular magnetic flux, different from the surface states of Weyl semimetals [56, 57].

There are only a few bulk candidate materials that might exhibit Weyl superconductivity in nature. Older candidates of WSCs include the Uranium-based ferromagnetic superconductors UGe_2 [64], URhGe [65], and UCoGe [66], in which spin-triplet p -wave pairing is expected. More recent works suggest that the B phase of UPt_3 [18, 19, 26, 47, 67], SrPtAs [46], Praseodymium-based compounds (e.g. $\text{PrOs}_4\text{Sb}_{12}$ and $\text{PrPt}_4\text{Ge}_{12}$ [68]) and YiPtBi [48, 69–71] may host gapless Majorana surface states.

Superconductivity was very recently observed in UTe_2 [72]. There is already extensive experimental evidence pointing towards Weyl superconductivity in this compound [72–82]. UTe_2 shows a relatively high transition temperature of 1.6 K [72] and its superconductivity is suspected to be mediated by ferromagnetic spin-fluctuations in proximity to a quantum critical point [76, 77].

The nearly temperature-independent Knight shift across the transition temperature [72, 82] and an extremely high upper critical field exceeding the Pauli limit [78] strongly support the spin-triplet pairing scenario. Penetration depth, thermal transport, spin-relaxation, and specific heat measurements together demonstrate strong evidence for point nodes lying along the crystallographic a axis [72, 74, 79, 81]. Direct evidence for chiral Majorana surface states was observed in STM experiments with a step-edge setup [80]; indirect evidence was also suggested by measurements of the surface impedance [73] at microwave frequencies. Kerr rotation experiments further suggest time-reversal symmetry breaking in UTe_2 [75].

Despite substantial effort in experimental measurements and first-principle calculations based on density functional theory (DFT) [83–88], consensus on the exact pairing symmetry in UTe_2 is yet still to be reached. On one hand, angular field dependence measurements of specific heat [79] and a Ginzberg-Landau (GL) free energy analysis [89] suggest a d -vector order parameter of the form $\mathbf{d}(\mathbf{k}) = (\mathbf{b} + i\mathbf{c})(k_b + ik_c)$. This would imply pairing of a single spin species (while the other remains unpaired), as in $^3\text{HeA}_1$ [28–30]. On the other hand, there are also analyses based on DFT, GL free energy and point-group symmetries suggesting more complicated scenarios [75, 84–88]. Further investigations are necessary to clarify the underlying pairing symmetry in UTe_2 .

A. Summary of main results

In superconductors, key electromagnetic (EM) responses are the Meissner effect and optical absorption.

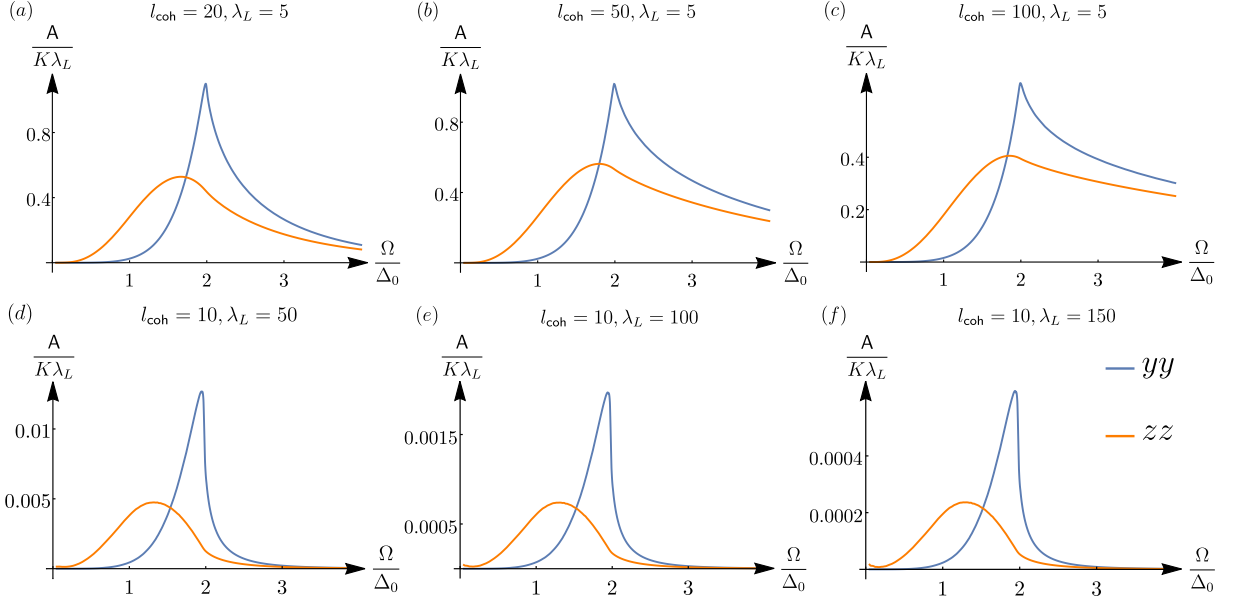


FIG. 2: The topological anomalous skin effect in the clean limit. The panels in this figure plot the frequency (Ω)-dependence of the absorbance $A^{yy}(\Omega)$ (blue) and $A^{zz}(\Omega)$ (orange) due to the surface-bulk transition, Eq. (1.3). Panels show results at zero temperature for different combinations of the coherence length l_{coh} and the diamagnetic (London) penetration depth λ_L . The Weyl nodes lie along the k_z -axis in our model, so that $A^{yy}(\Omega)$ [$A^{zz}(\Omega)$] encodes absorption for electric-field polarization perpendicular (parallel) to the surface Majorana Fermi arc. Parameters used in panels (a)–(c) correspond to the type-I superconductor regime, while those used in panels (d)–(f) correspond to the type-II regime. Here, the frequency is normalized by the pairing gap energy $\Delta_0 = \Delta k_F$; Δ is the $p + ip$ -wave pairing amplitude in the model [Eq. (3.1)]. We set $k_F = 1$ for all the plots.

In the following, we elucidate the interplay of the surface and bulk responses by considering a spinless $p + ip$ model, which is a solid-state analog of ^3HeA [28–30]. This model, with a Hamiltonian described by Eq. (3.1), has a pair of Weyl nodes lying along k_z and no time-reversal symmetry. As a result, it only possesses one branch of chiral surface states, as sketched in Fig. 1(a). It serves as a minimal model for WSCs.

We now summarize the main results of this paper.

1. Meissner effect

In a previous work, we predicted a power-law temperature (T)-dependence in the magnetic penetration depth $\Delta\lambda_L(T) \equiv \lambda_L(T) - \lambda_L(0) \sim T^3$ due to surface states in a strong topological superconductor (TSC), based on a solid-state model analog of ^3HeB [45]. In that case, since the bulk is fully gapped, in the absence of magnetic impurities [90, 91] a power-law T -dependence in $\Delta\lambda_L$ can only arise from the surface states. The combination of power-law temperature dependence in the magnetic penetration depth along with exponential suppression of the specific heat (due to the fully gapped bulk) are hallmarks for strong TSCs [45].

On the other hand, a superconductor with bulk nodes should exhibit power-law temperature dependence in *both* the magnetic penetration depth and the specific heat [2].

Power-law T -dependence in $\Delta\lambda_L$ has also been experimentally observed in WSC candidates [69, 74]. However, for WSCs, because of the gapless excitations in the bulk, this is not necessarily indicative for the presence of surface states. Nevertheless, for WSCs with Weyl nodes lying along a particular k axis (say k_z), we can show by power-counting that the power-law T -dependence from the surface states may still override that from the bulk, depending on the direction of the external magnetic field \mathbf{B} (see Appendix B). In fact, by considering the model WSC in Eq. (3.1) (with one pair of Weyl nodes at $k_z = \pm k_F$), occupying the $x > 0$ half-space and using the same framework outlined in Ref. [45], our calculations reveal that the bulk contribution is

$$\Delta\lambda_L \sim \begin{cases} T^2, & \mathbf{B} \parallel \hat{y}, \\ T^4, & \mathbf{B} \parallel \hat{z}, \end{cases} \quad (1.1)$$

whereas surface state correction is

$$\delta\lambda_L \sim \begin{cases} T^2, & \mathbf{B} \parallel \hat{y}, \\ T^2, & \mathbf{B} \parallel \hat{z}. \end{cases} \quad (1.2)$$

This suggests that at low enough temperature, a T^2 dependence in penetration depth can be an indicator for the presence of surface states, for a magnetic field orientation parallel to the line joining the Weyl nodes. In fact, this could be the case for UTe_2 , since recent exper-

imental evidence suggests that Weyl nodes lie along the a -axis [72, 74, 79, 81].

Nevertheless, more generally, for WSCs with nodes lying along arbitrary directions and/or an arbitrary orientation of the magnetic field, we expect identical T^2 contributions from both the bulk and surface. This calls for an alternative EM response that can possibly distinguish features of the surface and bulk states in WSCs.

2. Optical absorption

We consider a plane EM wave normally incident upon the surface of a WSC at $x = 0$, as shown in Fig. 1(b). The Weyl nodes in our model [Eq. (3.1)] lie along the k_z -axis, giving rise to a chiral surface Majorana band with dispersion $E_{k_y}^s = -\Delta k_y$ [Fig. 1(a)]. The reflectance and absorbance can be obtained by solving the Maxwell's equation (2.1). The material response is encoded in the current-current correlation function in this equation, with diamagnetic and three different paramagnetic contributions, originating (i) purely from the bulk, (ii) purely from the surface, and (iii) from the surface-bulk cross terms. The topological anomalous skin effect arises

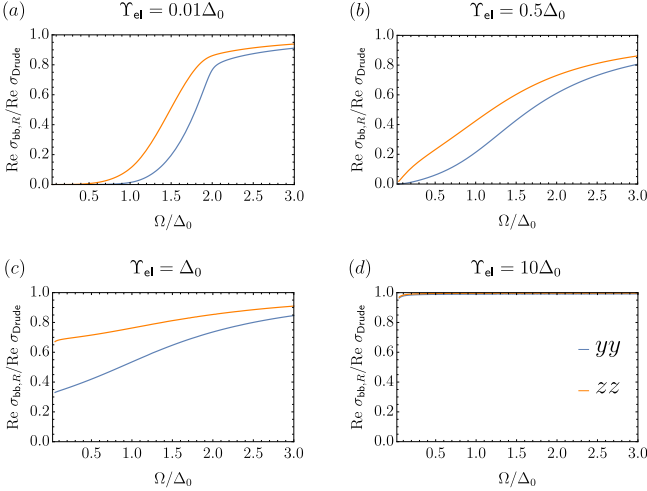


FIG. 3: The bulk optical conductivity of the dirty Weyl superconductor. The panels in this figure plot the normalized optical conductivity $\text{Re } \sigma_{bb,R}^{\mu\nu} / \text{Re } \sigma_{\text{Drude}}^{\mu\nu}$ as a function of the reduced frequency Ω/Δ_0 at zero temperature, with disorder strength (a) $\Upsilon_{\text{el}} = 0.01\Delta_0$ (b) $\Upsilon_{\text{el}} = 0.5\Delta_0$ (c) $\Upsilon_{\text{el}} = \Delta_0$ (d) $\Upsilon_{\text{el}} = 10\Delta_0$. Here $\Upsilon_{\text{el}} = 1/(2\tau_{\text{el}})$, where τ_{el} is the elastic lifetime due to impurity scattering in the normal state, and $\sigma_{\text{Drude}}^{\mu\nu} = \delta^{\mu\nu} \sigma_{\text{dc}} / [1 + (\omega\tau_{\text{el}})^2]$. In the weak disorder regime, the frequency dependence of the optical conductivity is reminiscent of the Mattis-Bardeen result for a dirty s -wave superconductor. The gapless excitations around the Weyl points cause a finite zero-temperature response for $\Omega < 2\Delta_0$. As the disorder strength increases, the optical conductivity gradually approaches the Drude result. This is consistent with the formation of a thermal metallic phase in the dirty limit [40, 48, 94].

from the combination of the diamagnetic response and (iii).

For the clean one-band WSC model in Eq. (3.1), the dissipative part of the optical conductivity in the bulk vanishes, consistent with the standard result for clean s -wave superconductors [4]. Meanwhile, because we only have one branch of surface states [Fig. 1(a)], surface intraband transitions that conserve energy and (nearly) preserve the transverse momentum are impossible. This leaves surface-bulk transitions to dominate in the clean limit. We evaluate the surface-bulk optical conductivity by exploiting the exact eigenstates of the system in the slab geometry.

The zero temperature surface-bulk absorbance $A^{\mu\mu}$ can be written compactly as

$$A^{\mu\mu} = \frac{\Omega}{k_F^2 l_{\text{coh}} c} \hat{\mathcal{F}}^{\mu\mu}(\hat{\Omega}), \quad \hat{\Omega} \equiv \frac{\Omega}{\Delta_0}, \quad (1.3)$$

where Ω is the radiation frequency, k_F is half the separation between the Weyl points, and l_{coh} is the coherence length of the superconductor (which determines the *minimum* for the chiral Majorana surface-state confinement to the WSC-vacuum interface). In Eq. (1.3), $\hat{\mathcal{F}}^{\mu\mu}$ is a dimensionless function of the reduced frequency $\hat{\Omega}$, which is the frequency relative to the pairing energy Δ_0 . The behavior of A is shown in Fig. 2, for different combinations of l_{coh} and the diamagnetic penetration depth λ_L . The components $A^{yy}(\Omega)$ and $A^{zz}(\Omega)$ correspond to the case with y - and z -polarized incident electric fields, respectively; the Weyl nodes lie along the k_z -axis in our model [Fig. 1(a)]. In the type-I regime ($\lambda_L \ll l_{\text{coh}}$) [Fig. 2(a)–(c)], both the yy and zz components of A demonstrate a peak centered around $\Omega = 2\Delta_0$. The broadness of the peaks increases with the coherence length l_{coh} . Larger l_{coh} means more deconfined surface states. In the type-II regime ($\lambda_L \gg l_{\text{coh}}$) [Fig. 2(d)–(f)], the peaks for different polarizations are more distinct. In particular, the yy component still manifests a peak at around $2\Delta_0$, whereas the zz component is peaked slightly above Δ_0 .

We note that the results in Eq. (1.3) and Fig. 2 are appropriate for the strong type-I and type-II limits, where the absorbance $0 < A^{\mu\mu} \ll 1$, but not the intermediate regime with $l_{\text{coh}} \sim \lambda_L$. The absorption is small in the strong type-I limit, because the field penetration is limited to λ_L , which is much smaller than the spatial extent of the surface states (bounded from below by l_{coh}). The absorption is also small in the strong type-II limit, due to the orthogonality of bulk and surface states. The case with $l_{\text{coh}} \sim \lambda_L$ requires the inversion of an integral equation to determine the electric field profile and the absorbance, as discussed in Sec. II. **Combining the type-I and type-II cases, the surface-bulk absorption only extends up to the scale of $\min(\lambda_L, l_{\text{coh}})$, see Fig. 1(b).**

Quenched disorder due to impurities and other defects is inevitable in real materials, and gives rise to a dissipative bulk optical conductivity. Although low-temperature superconductors are typically good metals

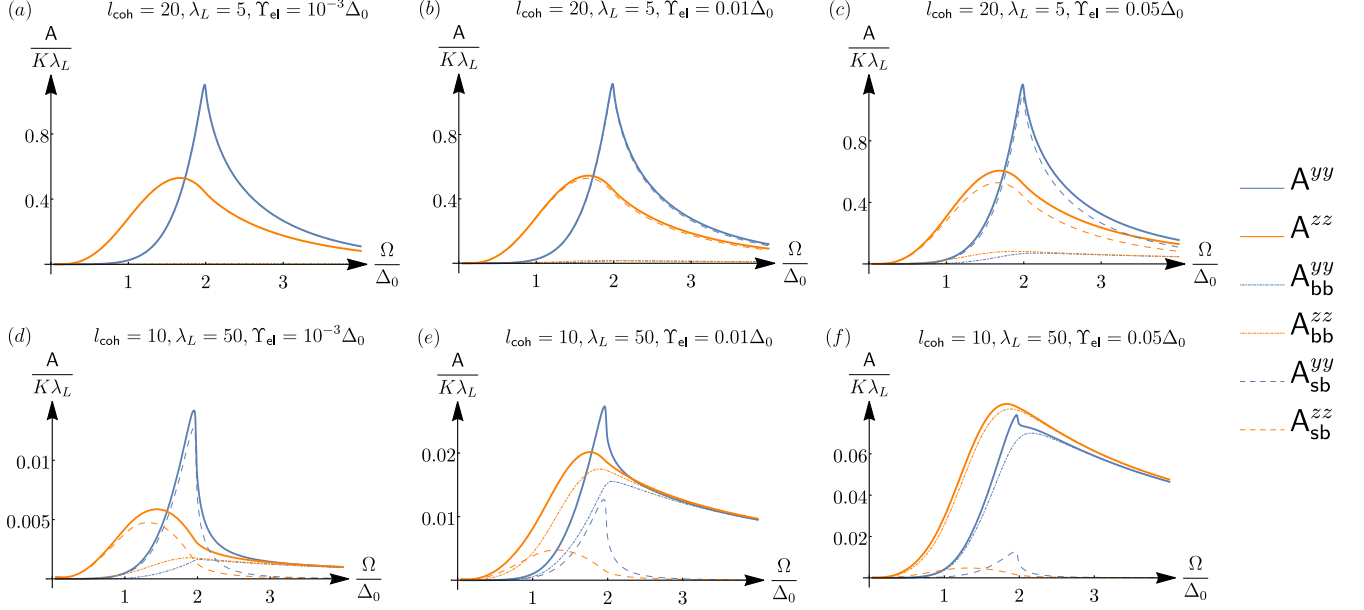


FIG. 4: Topological anomalous skin effect in the presence of weak disorder. The panels in this figure plot the total absorbance (A) as a function of the reduced frequency $\hat{\Omega} = \Omega/\Delta_0$, in the weak-disorder limit with different coherence lengths l_{coh} , London penetration depths λ_L , and disorder strengths Υ_{el} in the (a)–(c) type I and (d)–(f) type II superconductor regimes. In all the panels, orange and blue curves represent the yy and zz components of the absorbance respectively, compare to Fig. 2. Long dashed lines correspond to the surface-bulk contribution, while short dashed lines correspond to the disordered bulk contribution. The total absorbance is depicted by the solid lines. We use $k_F = 1$ for all the plots. In the type I case, the features from the surface-bulk absorption survive up to a relatively large amount of disorder. In the type II case, those features are gradually suppressed by the bulk absorption as the disorder strength increases.

with $\varepsilon_F \tau_{\text{el}} \gg 1$, where ε_F is the Fermi energy and τ_{el} is the lifetime due to elastic impurity scattering, the small T_c means that the superconducting phase typically occurs in the dirty limit with $\Delta_0 \ll 1/\tau_{\text{el}}$. The optical absorption is governed by the classic Mattis-Bardeen result [2, 5].

We use the Keldysh formalism [33, 34] to rederive the s -wave Mattis-Bardeen result via the saddle-point of the matrix field integral [32, 92, 93]. We then derive the matrix field theory appropriate for the finite-frequency response of a disordered $p + ip$ WSC. We compute the bulk Kubo optical conductivity for a WSC with disorder in the saddle-point approximation, equivalent to the self-consistent Born approximation. The zero-temperature bulk conductivity is shown in Fig. 3, for different values of the normal-state scattering rate $\Upsilon_{\text{el}} \equiv 1/(2\tau_{\text{el}})$.

For weak disorder, the optical conductivity shows a Mattis-Bardeen-like frequency dependence [Fig. 3(a)], despite being nonzero even for $\Omega < 2\Delta_0$ due to the low-energy excitations around the Weyl nodes. Here $\Delta_0 = \Delta k_F$ is the pairing energy; Δ is the $p + ip$ -wave pairing amplitude in the model [Eq. (3.1)]. Our results are consistent with previous calculations for anisotropic superconductors [44]. As the disorder strength Υ_{el} increases relative to the pairing energy Δ_0 , the “pseudogap” in

the optical conductivity gradually fills in, approaching the normal-state Drude optical conductivity. Unlike the s -wave case, the dirty limit for the WSC ($\Delta_0 \ll \Upsilon_{\text{el}}$) reduces to that of the normal state, consistent with formation of a thermal diffusive metal for sufficiently strong disorder [40, 48, 94].

Finally, we consider the combined effect of the bulk and the surface-bulk response on the absorbance in the weak disorder limit. Since the topological anomalous skin effect is nonvanishing in the clean limit, we expect weak disorder to modify the surface-bulk response only by slightly broadening its features. To leading order, it is sufficient to compare the contribution from the clean surface-bulk response versus that from the weakly disordered bulk. The result is

$$A^{\mu\mu} = \frac{\Omega\lambda_L}{c} \left\{ \frac{\hat{\mathcal{F}}^{\mu\mu}(\hat{\Omega})}{k_F^2 \lambda_L l_{\text{coh}}} + \left[\frac{2\Omega\tau_{\text{el}}}{1 + (\Omega\tau_{\text{el}})^2} \right] \frac{\text{Re } \sigma_{\text{bb},R}^{\mu\mu}(\Omega)}{\text{Re } \sigma_{\text{Drude}}^{\mu\mu}(\Omega)} \right\}, \quad (1.4)$$

where

$$\text{Re } \sigma_{\text{Drude}}(\Omega) = \sigma_{\text{dc}}/[1 + (\Omega\tau_{\text{el}})^2]$$

is the normal-state optical conductivity. The first term in Eq. (1.4) is the surface-bulk absorption [Eq. (1.3)], while

$\text{Re } \sigma_{\text{bb},R}^{\mu\mu}(\Omega)$ is the bulk optical conductivity of the WSC with disorder.

In Fig. 4, we plot the combined absorbance A for both type I and type II superconductors. The component A^{yy} (A^{zz}) corresponds to the absorbance with an incident electric field polarized along the y (z) direction, respectively. The overall shape of the absorbance versus frequency depends on the relative magnitude of the London penetration depth λ_L , coherence length l_{coh} , and the disorder strength Υ_{el} . In the type-I case, since the external field can only penetrate up to a shallow region at the proximity of the surface, the surface-bulk absorption dominates over that of the bulk for weak enough disorder. The frequency dependence of the absorbance thus mainly follows that of $A(\Omega)$ in the clean case, shown in Fig. 2. In the type-II case, the external field can penetrate deep into the sample and therefore the bulk contribution starts to overtake surface-bulk one as the amount of disorder increases. One would therefore need a relatively clean sample in order to observe the surface-bulk effect in type II WSCs. Due to the sharpness of the yy surface-bulk absorption peak at $2\Delta_0$, its feature can still be seen in the overall absorbance.

B. Outline

The rest of this paper is organized as follows. In Sec. II, we illustrate the geometry of our problem and derive a formal expression for the absorbance via classical electrodynamics. In Sec. III, we evaluate the optical conductivity in the clean limit using linear response theory. In Sec. IV, we derive the optical conductivity for the disordered bulk using the Keldysh theory. The main results of this paper appear already in Sec. IA and Figs. 2–4. We further discuss these and conclude in Sec. V. Technical details are relegated to the Appendices.

II. SETUP OF THE PROBLEM AND OPTICAL ABSORBANCE

For simplicity, we assume that the magnetic permeability μ_m of the material is the same as that for vacuum. Consider an incident EM plane wave that is linearly polarized and propagates along x , as schematically sketched in Fig. 1(b). In the temporal gauge, the electric field is related to the vector potential via $\mathbf{E} = -c^{-1}\partial_t\mathbf{A}$. The electric field of the system is governed by the Maxwell's equation

$$\begin{aligned} (-\partial_x^2 - K^2) E^\mu(x) &= \frac{4\pi}{c} iK J_{\text{ext}}^\mu(x) \\ &\quad - \frac{4\pi}{c^2} \int_0^\infty dx' \Pi_R^{\mu\nu}(\Omega; x, x') E^\nu(x'), \end{aligned} \quad (2.1)$$

where $\mu \in \{y, z\}$, $K = \Omega/c$ is the wavevector of the incident EM wave, $J_{\text{ext}}^\mu(x) = -(cE_0^\mu/2\pi)\delta(x-x_0)$ is a source

current that generates the EM radiation at $x_0 < 0$, and $\Pi_R^{\mu\nu}(\Omega; x, x')$ is the total retarded current-current correlation function of the material, including both the diamagnetic and paramagnetic terms. The second term on the right-hand-side captures the response from the system, consisting of contributions due to bulk-bulk, surface-bulk and surface-surface transitions. For the bulk response, we write

$$\begin{aligned} &\frac{4\pi}{c^2} \Pi_{\text{bb},R}^{\mu\nu}(\Omega; x, x') \\ &= \delta(x-x')\theta(x)\theta(x') \left[\frac{1}{\lambda_L^2} \delta^{\mu\nu} - \frac{4\pi i\Omega}{c^2} \text{Re } \sigma_{\text{bb},R}^{\mu\nu}(\Omega) \right], \end{aligned} \quad (2.2)$$

where $\theta(x)$ is the Heaviside step function, λ_L is the London penetration depth, and $\sigma_{\text{bb},R}^{\mu\nu}$ is the paramagnetic optical conductivity from the bulk. Since our focus is optical absorption, $\text{Im } \sigma_{\text{bb},R}^{\mu\nu}$ will be neglected in the following calculations. For the bulk-bulk paramagnetic response, we neglect the difference between the slab and homogeneous geometries and take $\sigma_{\text{bb},R}^{\mu\nu}(\Omega)$ to be independent of x . This is justified for terahertz (THz) radiation in low-transition-temperature superconductors with modest disorder, since the light wavelength $\lambda \gg \{l_{\text{coh}}, \lambda_L, v_F\tau_{\text{el}}\}$, where l_{coh} is the coherence length and $v_F\tau_{\text{el}}$ is the bulk mean free path due to impurity scattering [4].

As we will see in the next section, for the $p + ip$ WSC that we are going to study, the paramagnetic response function is purely diagonal and the contribution from surface-surface transitions vanishes. In this case, Eq. (2.1) reduces to

$$\begin{aligned} &[-\partial_x^2 - K^2 + K_b^2 \theta(x)] E^\mu(x) \\ &= \frac{4\pi}{c} iK J_{\text{ext}}^\mu - \frac{4\pi}{c^2} \int_0^\infty dx' \Pi_{1,\text{sb},R}^{\mu\nu}(\Omega; x, x') E^\nu(x'), \end{aligned} \quad (2.3)$$

where $K_b^2(\Omega) = \frac{1}{(\lambda_L)^2} - \frac{4\pi i\Omega}{c^2} \sigma_{\text{bb},R}^{\mu\mu}(\Omega)$ captures the response from the bulk, and $\Pi_{1,\text{sb},R}^{\mu\nu}(\Omega; x, x') \equiv \Pi_{1,\text{sb},R}^{\mu\nu}(\Omega, \mathbf{q} \rightarrow 0; x, x')$ is the paramagnetic current-current correlation function due to the transitions between surface and bulk states. Here $\mathbf{q} \rightarrow 0$ is the photon momentum parallel to the interface, which vanishes for normal incidence. To solve for $E^\mu(x)$, we seek a Green's function $G(x, x')$ satisfying

$$[-\partial_x^2 - K^2 + K_b^2 \theta(x)] G(x, x') = \delta(x - x'), \quad (2.4)$$

where $G(x, x')$ is subjected to the boundary conditions

$$G(0^+, x') = G(0^-, x'), \quad (2.5)$$

$$\partial_x G(x, x')|_{x=0^+} = \partial_x G(x, x')|_{x=0^-}. \quad (2.6)$$

The Green's function $G(x, x')$ can be solved by the

method of images. The result is

$$\begin{aligned}
G(x, x') &= \frac{i\theta(-x)\theta(-x')}{2K} \left[e^{iK|x-x'|} + \frac{K-i\mathcal{K}}{K+i\mathcal{K}} e^{-iK(x+x')} \right] \\
&+ \frac{\theta(x)\theta(x')}{2\mathcal{K}} \left[e^{-\mathcal{K}|x-x'|} - \frac{K-i\mathcal{K}}{K+i\mathcal{K}} e^{-\mathcal{K}(x+x')} \right] \\
&+ \frac{\theta(x)\theta(-x')}{-iK+\mathcal{K}} e^{-\mathcal{K}x-iKx'} + \frac{\theta(-x)\theta(x')}{-iK+\mathcal{K}} e^{-\mathcal{K}x'-iKx}, \quad (2.7)
\end{aligned}$$

where $\mathcal{K} \equiv \sqrt{K_b^2 - K^2}$. The electric field can now be expressed as

$$E^\mu(x) = E^{(0)\mu}(x) + \delta E^\mu(x), \quad (2.8)$$

where the bare electric field is

$$E^{(0)\mu}(x) = -2iK E_0^\mu G(x, x_0) \quad (2.9)$$

and the correction due to the surface-bulk response is

$$\delta E^\mu(x) = -\frac{4\pi}{c^2} \int_{x_{1,2}>0} G(x, x_1) \Pi_{1,\text{sb},R}^{\mu\nu}(\Omega; x_1, x_2) E^\nu(x_2). \quad (2.10)$$

In general, the integral equation in (2.8)–(2.10) cannot be solved exactly. However, in the strong type-I or type-II limits for superconductors, the contribution from the surface-bulk response is perturbatively small and explicit results obtainable. For a strong type-I WSC, the coherence length l_{coh} is much greater than λ_L . Since the spatial extent of the surface states is roughly $\sim l_{\text{coh}}$ except when they merge into the bulk, the limited penetration of the EM wave means that the response involving surface states cannot contribute much. On the other hand, for a strong type-II WSC, the coherence length l_{coh} is much smaller than λ_L . As a result, the field decays very slowly within the spatial extent of the surface-bulk correlation function. Owing to the orthogonality between bulk and surface states, the contribution from the surface-bulk response is again small, and would vanish in the limit $\mathcal{K} \rightarrow 0$.

We now focus on the strong type-I and -II limits such that corrections from the surface-bulk term can be treated perturbatively. Formally, Eq. (2.8) can be written as

$$\left[\hat{1} + \frac{4\pi}{c^2} \hat{G} \hat{\Pi}_{1,\text{sb},R}^{\mu\nu} \right] |E^\mu\rangle = |E^{(0)\mu}\rangle \quad (2.11)$$

such that

$$|E^\mu\rangle = \left\{ \left[\hat{1} + \frac{4\pi}{c^2} \hat{G} \hat{\Pi}_{1,\text{sb},R}^{\mu\nu} \right]^{-1} \right\}^{\mu\nu} |E^{(0)\nu}\rangle. \quad (2.12)$$

To leading order, the electric field is just

$$|E^{(1)\mu}\rangle = |E^{(0)\mu}\rangle - \frac{4\pi}{c^2} \hat{G} \hat{\Pi}_{1,\text{sb},R}^{\mu\nu} |E^{(0)\nu}\rangle, \quad (2.13)$$

that is,

$$\begin{aligned}
E^{(1)\mu}(x) &\simeq E_0^\mu \left[\frac{\theta(-x)(e^{iKx} + r e^{-iKx})}{+t\theta(x)e^{-\mathcal{K}x}} \right] e^{-iKx_0} + \delta E^{(1)\mu}(x), \quad (2.14)
\end{aligned}$$

where

$$r = \frac{K-i\mathcal{K}}{K+i\mathcal{K}}, \quad t = \frac{2K}{K+i\mathcal{K}}, \quad (2.15)$$

and

$$\begin{aligned}
\delta E^{(1)\mu}(x) &= -\frac{4\pi}{c^2} \int_{x_{1,2}>0} \hat{G}(x, x_1) \hat{\Pi}_{1,\text{sb},R}^{\mu\nu}(\Omega; x_1, x_2) E^{(0)\nu}(x_2). \quad (2.16)
\end{aligned}$$

In Eq. (2.14), $\delta E(x)$ is the correction term arising from the surface-bulk response. Here, r and t carry the physical meanings of bare reflection and transmission coefficients respectively. $\delta E^{(1)\mu}(x)$ is the surface-bulk correction in first order. As we will demonstrate in the next section, the correlation function $\hat{\Pi}_{1,\text{sb},R}^{\mu\nu}(\Omega; x_1, x_2)$ is non-vanishing only for $\mu = \nu$. Thus, at the proximity of the surface ($x \rightarrow 0$), the correction takes a simple form

$$\delta E^\mu(x \rightarrow 0) = E_0^\mu [\delta r \theta(-x) + \delta t \theta(x)] e^{-iKx_0}, \quad (2.17)$$

where

$$\delta r = \delta t = -\frac{4\pi i}{c^2} \frac{t}{K+i\mathcal{K}} \mathcal{Q}^{\mu\mu}(\Omega), \quad (2.18)$$

$$\mathcal{Q}^{\mu\mu}(\Omega) \equiv \int_{x_{1,2}>0} e^{-\mathcal{K}x_1} \Pi_{1,\text{sb},R}^{\mu\mu}(\Omega; x_1, x_2) e^{-\mathcal{K}x_2}. \quad (2.19)$$

To leading order, the absorbance in the weak disorder limit is simply

$$\begin{aligned}
A^{\mu\mu} &= 1 - |r + \delta r|^2 \quad (2.20) \\
&\simeq K(\lambda_L)^3 \frac{8\pi\Omega}{c^2} \text{Re } \sigma_{\text{bb},R}^{\mu\mu} - \frac{16\pi}{c^2} \frac{K}{\lambda_L^{-2}} \text{Im } \mathcal{Q}^{\mu\mu},
\end{aligned}$$

where we have assumed $\lambda_L^{-1} \gg K$. This is a valid approximation in the THz regime, which is appropriate for probing features of pairing energies Δ_0 in WSC candidate materials. In the above, we also used the fact that $(\Omega/c^2) \text{Re } \sigma_{\text{bb},R}^{\mu\mu} \ll \lambda_L^{-2}$ for weak disorder. We will corroborate this point by explicit calculations in Secs. III and IV. Eq. (2.20) is the key result of this section. Combined with the results of Secs. III and IV, it gives the absorbance expression in Eq. (1.4) of the Introduction.

III. OPTICAL RESPONSE IN THE CLEAN LIMIT

A. Model and some comments on the effective surface theory

We consider a minimal model for a Weyl superconductor (WSC), consisting of spinless electrons with $p + ip$ pairing. This is represented by the following static, mean-field Bogoliubov-de Gennes Hamiltonian

$$H = \frac{1}{2} \int_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} \hat{h}_{\mathbf{k}} \Psi_{\mathbf{k}}, \quad (3.1a)$$

$$\hat{h}_{\mathbf{k}} = \tilde{\varepsilon}_{\mathbf{k}} \hat{\sigma}^3 + \Delta k_x \hat{\sigma}^1 + \Delta k_y \hat{\sigma}^2, \quad (3.1b)$$

$$\tilde{\varepsilon}_{\mathbf{k}} = \frac{\mathbf{k}^2 - k_F^2}{2m},$$

where $\int_{\mathbf{k}} = \int \frac{d^3 k}{(2\pi)^3}$, k_F is the Fermi momentum, Δ is the superconducting order parameter amplitude, $\hat{\sigma}^i$ are the Pauli matrices in particle-hole space, and $\Psi_{\mathbf{k}}^{\dagger} = [c_{\mathbf{k}}^{\dagger}, c_{-\mathbf{k}}^{\dagger}]$. This Hamiltonian possesses the particle-hole symmetry

$$-\hat{M}_P \hat{h}^{\dagger}(-\mathbf{k}) \hat{M}_P = \hat{h}(\mathbf{k}), \quad \hat{M}_P = \hat{\sigma}^1, \quad (3.2)$$

but has no time-reversal symmetry due to the explicit appearance of “ i ” in the pairing. It belongs to class D according to the ten-fold classification scheme [6, 7, 26, 58]. The Weyl nodes of this system are located at $\mathbf{k}_{\text{nodes}} = (0, 0, \pm k_F)$. At the proximity of the Weyl nodes, we can linearize $\hat{h}_{\mathbf{k}}$ such that

$$\hat{h}_{\pm}(\delta \mathbf{k}) = v_{\pm}^x \delta k_x \hat{\sigma}^1 + v_{\pm}^y \delta k_y \hat{\sigma}^2 + v_{\pm}^z \delta k_z \hat{\sigma}^3, \quad (3.3)$$

where \pm correspond to the two Weyl nodes with opposite chirality [27], $\mathbf{v}_{\pm} = (\Delta, \Delta, \pm k_F/m)$ and $\delta \mathbf{k} = \mathbf{k} - \mathbf{k}_{\text{nodes}}$. However, we do not employ the linearized theory in the rest of this Section.

In order to analyze the response related to the surface, we solve for the surface states with dispersion $E_{k_y}^s = -\Delta k_y$ by replacing $k_x \rightarrow -i\partial_x$ in Eq. (3.1b), which is appropriate for the geometry that we are considering in Fig. 1(b). By imposing a hard-wall boundary condition at $x = 0$, we obtain the chiral surface state

$$\psi^s(\mathbf{k}; x) = \frac{\theta(k_F^2 - \mathbf{k}^2)}{\sqrt{N_{\mathbf{k}}^s}} e^{-x/l_{\text{coh}}} \sinh(\kappa_{\mathbf{k}} x) |\sigma^2 = -1\rangle, \quad (3.4)$$

where $\mathbf{k} = (k_y, k_z)$ is the in-plane momentum parallel to the surface, $l_{\text{coh}} \equiv 1/m\Delta$ is the coherence length,

$$\kappa_{\mathbf{k}} \equiv \sqrt{l_{\text{coh}}^{-2} + \mathbf{k}^2 - k_F^2} \quad (3.5)$$

is the wavenumber that controls the confinement of the surface state to the vacuum-WSC interface, $N_{\mathbf{k}}^s \equiv$

$\frac{l_{\text{coh}}^{-2} + \mathbf{k}^2 - k_F^2}{4(k_F^2 - \mathbf{k}^2)l_{\text{coh}}^{-1}}$ is the normalization factor, and $|\sigma^2 = -1\rangle$ denotes the eigenstate of $\hat{\sigma}^2$ with eigenvalue -1 . Notice that there is only one branch of surface states due to time-reversal symmetry breaking. The Heaviside step function $\theta(k_F^2 - \mathbf{k}^2)$ ensures that $\psi_s(\mathbf{k}; x \rightarrow \infty) = 0$ for physical surface state solutions. For $|\mathbf{k}| \rightarrow k_F$ (including the Weyl nodes at $k_z = \pm k_F$), the confinement length for the surface states diverges, signifying the merger with the bulk. The surface states exhibit a “Majorana Fermi arc” for $k_y = 0$, where $E_{k_y}^s = 0$ for $0 \leq |k_z| \leq k_F$.

The bulk states are denoted by $\psi_{\lambda=\pm 1}^b(q, \mathbf{k}; x)$, which are labeled by the transverse momentum \mathbf{k} and standing wave x -momentum $q \geq 0$. Such a state has eigenenergy

$$\lambda E_{q, \mathbf{k}}^b = \lambda \sqrt{\tilde{\varepsilon}_{q, \mathbf{k}}^2 + \Delta^2(q^2 + k_y^2)}, \quad \tilde{\varepsilon}_{q, \mathbf{k}} = \frac{q^2 + \mathbf{k}^2 - k_F^2}{2m}. \quad (3.6)$$

The index $\lambda = \pm$ labels the positive and negative energy bulk states that are related by particle-hole symmetry. Since we consider the weak-pairing BCS limit appropriate to low-temperature solid-state superconductors [95], we have to consider two cases.

- (i) For $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 > 0$, the bulk scattering states are two-fold degenerate for $q \leq q_0 = \sqrt{2k_F^2 - 2k_y^2 - 2k_z^2 - 4m^2\Delta^2}$, as illustrated in Fig. 5(a). For each $q_{\min} < q \leq q_0$, where $q_{\min} = q_0/\sqrt{2}$ minimizes $E_{q, \mathbf{k}}^b$, we have $E_{q, \mathbf{k}}^b = E_{q-, \mathbf{k}}^b$. In this region, we identify two orthonormal states $\Psi_{\lambda}^{b(1)}(q, \mathbf{k}; x)$ and $\Psi_{\lambda}^{b(2)}(q, \mathbf{k}; x)$. For $q > q_0$, degeneracy no longer exists and there is only one bulk state $\psi_{\lambda}^{b>}(q, \mathbf{k}; x)$ for each q .
- (ii) For $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 \leq 0$, there is no degeneracy for all q [Fig. 5(b)]. In this case, the bulk states are just given by $\psi_{\lambda}^{b>}(q, \mathbf{k}; x)$.

The explicit expressions of the bulk scattering states are complicated and thus relegated to Appendix A.

The effect of external fields on the surface can be intuited by incorporating an vector potential \mathbf{A} in Eq. (3.1a), and then projecting it to the low-energy surface states, as in Refs. [45, 59]. The result is

$$H_s = \int_{\mathbf{r}} \left[\frac{1}{2} \eta(\mathbf{r}) \Delta (i\partial_y) \eta(\mathbf{r}) - \frac{1}{c} \mathbf{A} \cdot \mathbf{J} \right], \quad (3.7)$$

where $\mathbf{r} = (y, z)$, $\eta = \eta^{\dagger}$ is the one-component (chiral) Majorana fermion operator on the surface, and

$$\mathbf{J}(\mathbf{r}) = \frac{e}{4m} \int_{\mathbf{r}} \eta(\mathbf{r}) i \overleftrightarrow{\nabla} \eta(\mathbf{r}), \quad (3.8)$$

where $\overleftrightarrow{\nabla} \equiv \overrightarrow{\nabla} - \overleftarrow{\nabla}$ is the left-right derivative. The pairing amplitude Δ sets the “speed of light” for the surface chiral modes.

The Hamiltonian in Eq. (3.7) with $\mathbf{A} = 0$ describes a collection of 1+1-D chiral Majorana fermions, labeled by the continuous index z (since there is no dispersion in this direction). Alternatively, we can view this as a

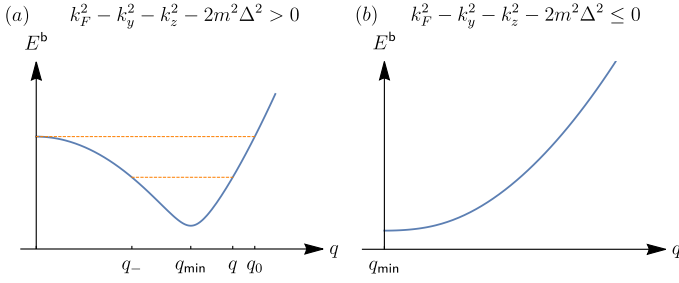


FIG. 5: Schematic illustration of the degeneracy in the bulk band, for bulk eigenenergies $E_{q,\mathbf{k}}^b$ [Eq. (3.6)] plotted versus the standing wave momentum $q \geq 0$. Eigenstates are semi-infinite standing waves in the x -direction [due to the slab geometry, Fig. 1(b)], plane waves transverse to this with momenta $\mathbf{k} = (k_y, k_z)$. (a) For $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 > 0$, there exists a $q_- < q_{\min}$ such that $E_{q,\mathbf{k}}^b = E_{q_-, \mathbf{k}}^b$ for each $q \in (q_{\min}, q_0]$. The minimum of $E_{q,\mathbf{k}}^b$ is located at $q_{\min} = \sqrt{k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2}$. In this region, the two degenerate bulk states are $\Psi_\lambda^{b(1)}(q, \mathbf{k}; x)$ and $\Psi_\lambda^{b(2)}(q, \mathbf{k}; x)$. Beyond q_0 , the bulk states are given by $\psi_\lambda^{b>}(q, \mathbf{k}; x)$. (b) For $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 \leq 0$, $q_{\min} = 0$ and there is no degeneracy for all q . The bulk states are just $\psi_\lambda^{b>}(q, \mathbf{k}; x)$.

many-channel Majorana wire, with channels labeled by the transverse momentum k_z . Formally, the coupling to J^y in the surface theory given by Eq. (3.7) is *gravitational*, i.e. the vector potential couples to

$$J^y \propto (T^{+-} - T^{-+}),$$

where $T^{\pm\pm} \equiv -\pi\eta i(\partial_t \mp \Delta\partial_y)\eta$ are stress tensor components for the 1+1-D chiral Majorana fermions, expressed in lightcone coordinates [96]. The operator J^z on the other hand takes the form of a non-abelian current in the space of k_z channels,

$$J^z(y, z) = \frac{e}{2m} \int_{k_z, k'_z} \eta(y, k_z) (-k'_z) \eta(y, k'_z) e^{i(k_z + k'_z)z}.$$

Although the interpretation in terms of 1+1-D relativistic quantum field theory is interesting, we emphasize that the surface states in Eq. (3.7) cannot be treated in isolation, owing to the gapless nature of the bulk [97]. In particular, we use the full eigenstate spectrum of the 3D Hamiltonian in Eq. (3.1b) in the slab geometry to compute the electromagnetic response.

B. Bulk-bulk transitions

For the clean bulk, the paramagnetic current-current correlation function is given by the bubble

$$\Pi_{1,bb}^{\mu\nu}(i\Omega_m, \mathbf{q}) = -\frac{1}{2} \left(\frac{e}{m}\right)^2 T \sum_{\omega_n} \int_{\mathbf{k}} \left(\mathbf{k} + \frac{\mathbf{q}}{2}\right)^\mu \left(\mathbf{k} + \frac{\mathbf{q}}{2}\right)^\nu \times \text{Tr} \left[\hat{\mathcal{G}}_b(i\omega_n + i\Omega_m, \mathbf{k} + \mathbf{q}) \hat{\mathcal{G}}_b(i\omega_n, \mathbf{k}) \right], \quad (3.9)$$

where T is temperature, $\omega_n = \pi T(2n + 1)$ is the Matsubara frequency, $i\Omega_m$ is the external bosonic frequency, $\mathbf{q} = (q_x, q_y, q_z)$ is the external momentum, and $\mathbf{k} = (k_x, k_y, k_z)$. The bulk Green's function is defined as

$$\hat{\mathcal{G}}_b(i\omega_n, \mathbf{k}) = \frac{1}{-i\omega_n + \hat{h}_{\mathbf{k}}} = \frac{1}{-i\omega_n + \mathbf{b}_{\mathbf{k}} \cdot \hat{\boldsymbol{\sigma}}}, \quad (3.10)$$

where $\mathbf{b}_{\mathbf{k}} = (\Delta k_x, \Delta k_y, \tilde{\epsilon}_{\mathbf{k}})$. The trace can be evaluated readily as

$$\text{Tr} \left[\hat{\mathcal{G}}_b(i\omega_n + i\Omega_m, \mathbf{k} + \mathbf{q}) \hat{\mathcal{G}}_b(i\omega_n, \mathbf{k}) \right] = \frac{2[(i\omega_n + i\Omega_m)(i\omega_n) + \mathbf{b}_{\mathbf{k}+\mathbf{q}} \cdot \mathbf{b}_{\mathbf{k}}]}{[(i\omega_n + i\Omega_m)^2 - b_{\mathbf{k}+\mathbf{q}}^2][(i\omega_n)^2 - b_{\mathbf{k}}^2]}. \quad (3.11)$$

The Matsubara summation can be performed using standard contour integral technique, resulting in

$$\Pi_{1,bb}^{\mu\nu}(i\Omega_m, \mathbf{q}) = \left(\frac{e}{m}\right)^2 \int_{\mathbf{k}} \left(\mathbf{k} + \frac{\mathbf{q}}{2}\right)^\mu \left(\mathbf{k} + \frac{\mathbf{q}}{2}\right)^\nu \times \sum_{\lambda=\pm, \lambda'=\pm} \frac{1}{4} \left(1 + \frac{\mathbf{b}_{\mathbf{k}+\mathbf{q}} \cdot \mathbf{b}_{\mathbf{k}}}{b_{\mathbf{k},\lambda} b_{\mathbf{k}+\mathbf{q},\lambda'}}\right) \frac{f(b_{\mathbf{k}+\mathbf{q},\lambda'}) - f(b_{\mathbf{k},\lambda})}{i\Omega_m - (b_{\mathbf{k}+\mathbf{q},\lambda'} - b_{\mathbf{k},\lambda})}, \quad (3.12)$$

where $f(E) = 1/(1 + e^{E/T})$ is the Fermi distribution function, and $b_{\mathbf{k},\pm} \equiv \pm|\mathbf{b}_{\mathbf{k}}|$. In the $\mathbf{q} \rightarrow \mathbf{0}$ limit, appropriate for the optical conductivity in the THz regime, the above expression vanishes. Since the optical conductivity is related to the current correlation function via

$$\sigma_{bb}^{\mu\nu}(i\Omega_m, \mathbf{q}) = \frac{-1}{i\Omega} \Pi_{1,bb}^{\mu\nu}(i\Omega_m, \mathbf{q}), \quad (3.13)$$

this implies that the paramagnetic contribution to the optical conductivity of the WSC is zero in the clean limit. This conclusion holds regardless of the detailed form of $\mathbf{b}_{\mathbf{k}}$, and is thus valid for all one-band superconductors with Hamiltonian of the form $\hat{h}_{\mathbf{k}} = \mathbf{b}_{\mathbf{k}} \cdot \hat{\boldsymbol{\sigma}}$. This is consistent with the standard result for clean s -wave superconductors [4].

C. Surface-surface transitions

For the model that we are considering, there is only one branch of chiral surface states, with eigenenergy $E_{k_y}^s = -\Delta k_y$. As a result, intraband transitions amongst sur-

face states is not possible and optical absorption purely from the surface is vanishing.

D. Surface-bulk transitions

In order to capture the interplay between the surface and bulk states, we have to implement the open boundary condition along x explicitly and make use of the exact bulk and surface eigenstates given in Appendix A and by Eq. (3.4).

The position-dependent paramagnetic current-current correlation function due to transitions between bulk and surface states can be expressed as

$$\begin{aligned} \Pi_{1,\text{sb}}^{\mu\nu}(i\Omega_m, \mathbf{0}; x, x') &= -\frac{1}{2} \left(\frac{e}{m}\right)^2 T \sum_{\omega_n} \int_{\mathbf{k}} k^\mu k^\nu \\ &\times \text{Tr} \left[\hat{\mathcal{G}}_s(i\omega_n, \mathbf{k}; x, x') \hat{\mathcal{G}}_b(i\omega_n + i\Omega_m, \mathbf{k}; x', x) \right. \\ &\quad \left. + \hat{\mathcal{G}}_b(i\omega_n, \mathbf{k}; x, x') \hat{\mathcal{G}}_s(i\omega_n + i\Omega_m, \mathbf{k}; x', x) \right], \end{aligned} \quad (3.14)$$

where $\mu, \nu \in \{y, z\}$, $\hat{\mathcal{G}}_b(i\omega_n, \mathbf{k}; x, x')$ and $\hat{\mathcal{G}}_s(i\omega_n, \mathbf{k}; x, x')$ respectively denote the bulk and surface Green's functions, and $\mathbf{k} = (k_y, k_z)$. The transverse photon momentum has been sent to zero, appropriate for normal incidence. The Green's functions can be expressed via spectral functions,

$$\hat{\mathcal{G}}_{b,s}(i\omega, \mathbf{k}; x, x') = \int \frac{d\omega'}{2\pi(i\omega - \omega')} \hat{\mathcal{A}}_{b,s}(\omega', \mathbf{k}; x, x'), \quad (3.15)$$

where the bulk spectral function

$$\begin{aligned} \hat{\mathcal{A}}_b(\omega, \mathbf{k}; x, x') &= \int_q \sum_{\lambda=\pm} \frac{-2\kappa_b(\omega)}{(\omega - \lambda E_{q,\mathbf{k}}^b)^2 + \kappa_b^2(\omega)} \psi_\lambda^b(q, \mathbf{k}; x) \psi_\lambda^{b\dagger}(q, \mathbf{k}; x') \\ &\quad (3.16) \end{aligned}$$

and the surface spectral function

$$\hat{\mathcal{A}}_s(\omega, \mathbf{k}; x, x') = \frac{-2\kappa_s(\omega)}{(\omega - E_{k_y}^s)^2 + \kappa_s(\omega)^2} \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x'). \quad (3.17)$$

In Eq. (3.16), the summation $\sum_{\lambda=\pm}$ sums over the eigenstates with eigenenergy $\pm E_{q,\mathbf{k}}^b$ and $\int_q = \int \frac{dq}{2\pi}$ integrates over all scattering wave momenta. κ_b and κ_s are respectively the bulk and surface impurity scattering rates, which can be taken to be 0^+ in the clean limit. Notice that the surface-bulk current-current correlation in Eq. (3.14) depends separately on x and x' due to translational symmetry breaking.

We then substitute Eqs. (3.15)–(3.17) into Eq. (3.14) such that

$$\begin{aligned} \Pi_{1,\text{sb}}^{\mu\nu}(i\Omega_m, \mathbf{0}; x, x') &= \frac{1}{2} \left(\frac{e}{m}\right)^2 \int_{q,\mathbf{k}} d\omega' \int d\omega'' k^\mu k^\nu \\ &\times \left[\frac{f(\omega'') - f(\omega')}{i\Omega_m - (\omega'' - \omega')} \right] \mathcal{I}_{q,\mathbf{k}}(\omega', \omega''; x, x'), \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} \mathcal{I}_{q,\mathbf{k}}(\omega', \omega''; x, x') &= \sum_{\lambda=\pm} \left[\mathcal{D}^s(\omega', E_{k_y}^s) \mathcal{D}^b(\omega'', \lambda E_{q,\mathbf{k}}^b) \Sigma_\lambda^{\text{sb}}(q, \mathbf{k}; x', x) \right. \\ &\quad \left. + \mathcal{D}^s(\omega'', E_{k_y}^s) \mathcal{D}^b(\omega', \lambda E_{q,\mathbf{k}}^b) \Sigma_\lambda^{\text{sb}}(q, \mathbf{k}; x, x') \right], \end{aligned} \quad (3.19)$$

with the broadened Dirac δ -functions

$$\mathcal{D}^s(\omega', E_{k_y}^s) = \frac{\kappa_s(\omega')/\pi}{(\omega' - E_{k_y}^s)^2 + \kappa_s(\omega')^2}, \quad (3.20)$$

$$\mathcal{D}^b(\omega'', \lambda E_{q,\mathbf{k}}^b) = \frac{\kappa_b(\omega'')/\pi}{(\omega'' - \lambda E_{q,\mathbf{k}}^b)^2 + \kappa_b(\omega'')^2}, \quad (3.21)$$

and the double-overlap between surface and bulk states

$$\Sigma_\lambda^{\text{sb}}(q, \mathbf{k}; x, x') = \psi_\lambda^{b\dagger}(q, \mathbf{k}; x) \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x') \psi_\lambda^b(q, \mathbf{k}; x'). \quad (3.22)$$

Specifically, from the bulk states we obtained in Appendix A, we have to consider two cases, due to the double-degeneracy of some bulk states, see Fig. 5.

For $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 > 0$,

$$\Sigma_\lambda^{\text{sb}}(q, \mathbf{k}; x, x') = \begin{cases} 0, & 0 < q < q_{\min}, \\ \left[\begin{aligned} &\Psi_\lambda^{b(1)\dagger}(q, \mathbf{k}; x) \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x') \Psi_\lambda^{b(1)}(q, \mathbf{k}; x') \\ &+ \Psi_\lambda^{b(2)\dagger}(q, \mathbf{k}; x) \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x') \Psi_\lambda^{b(2)}(q, \mathbf{k}; x') \end{aligned} \right], & q_{\min} < q \leq q_0, \\ \psi_\lambda^{b>\dagger}(q, \mathbf{k}; x) \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x') \psi_\lambda^{b>}(q, \mathbf{k}; x'), & q > q_0, \end{cases} \quad (3.23)$$

and for $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 \leq 0$,

$$\begin{aligned} \Sigma_\lambda^{\text{sb}}(q, \mathbf{k}; x, x') &= \psi_\lambda^{b>\dagger}(q, \mathbf{k}; x) \psi^s(\mathbf{k}; x) \psi^{s\dagger}(\mathbf{k}; x') \psi_\lambda^{b>}(q, \mathbf{k}; x'). \end{aligned} \quad (3.24)$$

By performing analytical continuation

$$\Pi_{1,\text{sb},R}^{\mu\nu}(\Omega, \mathbf{0}; x, x') = -\Pi_{1,\text{sb}}^{\mu\nu}(i\Omega_m \rightarrow \Omega + i\eta, \mathbf{0}; x, x') \quad (3.25)$$

where $\eta \rightarrow 0^+$, we obtain the real part of the retarded optical conductivity due to surface-bulk transitions

$$\begin{aligned} \text{Re } \sigma_{\text{sb},R}^{\mu\nu}(\Omega; x, x') &= \frac{-1}{\Omega} \text{Im } \Pi_{1,\text{sb},R}^{\mu\nu}(\Omega, \mathbf{0}; x, x') \\ &= \frac{\pi}{2} \left(\frac{e}{m} \right)^2 \int_{q,\mathbf{k}} \int d\omega k^\mu k^\nu \left[\frac{f(\omega) - f(\omega + \Omega)}{\Omega} \right] \\ &\quad \times \text{Re } \mathcal{I}_{q,\mathbf{k}}(\omega, \omega + \Omega; x, x'). \end{aligned} \quad (3.26)$$

The real part of $\Pi_{1,\text{sb},R}^{\mu\nu}(\Omega, \mathbf{0})$ is irrelevant for optical absorption and therefore will be neglected for simplicity.

In the clean limit, we can convert $\mathcal{D}^s(\omega_1, \omega_2) \rightarrow \delta(\omega_1 - \omega_2)$ and $\mathcal{D}^b(\omega_1, \omega_2) \rightarrow \delta(\omega_1 - \omega_2)$. Using the fact that

$$\Sigma_{\lambda=-1}^{\text{sb}}(q, \mathbf{k}; x, x') = \Sigma_{\lambda=+1}^{\text{sb}}(q, -\mathbf{k}; x', x) \quad (3.27)$$

and performing the ω integral, $\text{Re } \sigma_{\text{sb},R}^{ij}(\Omega; x, x')$ can be simplified as

$$\begin{aligned} \text{Re } \sigma_{\text{sb},R}^{\mu\nu}(\Omega; x, x') &= \frac{\pi}{2} \left(\frac{e}{m} \right)^2 \frac{1}{\Omega} \int_{q,\mathbf{k}} k^\mu k^\nu \\ &\quad \times \left\{ \begin{aligned} &\left[f(E_{k_y}^s) - f(E_{k_y}^s + \Omega) \right] \\ &\times \delta(E_{k_y}^s + \Omega - E_{q,\mathbf{k}}^b) \Sigma_{+1}^{\text{sb}}(q, \mathbf{k}; x', x) \\ &+ \left[f(-E_{q,\mathbf{k}}^b) - f(-E_{q,\mathbf{k}}^b + \Omega) \right] \\ &\times \delta(-E_{q,\mathbf{k}}^b + \Omega - E_{k_y}^s) \Sigma_{+1}^{\text{sb}}(q, -\mathbf{k}; x', x) \end{aligned} \right\}. \end{aligned} \quad (3.28)$$

We can proceed further by considering the $T = 0$ limit, in which the Fermi function becomes $f(E) = \theta(-E)$. In this case, the surface-bulk optical conductivity becomes

$$\begin{aligned} \text{Re } \sigma_{\text{sb},R}^{\mu\nu}(\Omega; x, x') &= 2\pi \left(\frac{e}{m} \right)^2 \frac{1}{\Omega} \int_0^\infty \frac{dq}{2\pi} \int_0^{k_F} \frac{k dk}{2\pi} \\ &\quad \times \delta^{\mu\nu} \int_0^{\pi/2} \frac{d\theta}{2\pi} k^\mu k^\nu \delta(E_{k_y}^s + \Omega - E_{q,\mathbf{k}}^b) \Sigma_{+1}^{\text{sb}}(q, \mathbf{k}; x', x), \end{aligned} \quad (3.29)$$

where we made a change of coordinates $(k_y, k_z) = k(\sin \theta, \cos \theta)$ and repeated indices are not summed over. The constraint $k_y^2 + k_z^2 \leq k_F^2$ for the surface states is now automatically satisfied. By expressing the δ function as

$$\delta(E_{k_y}^s + \Omega - E_{q,\mathbf{k}}^b) = \sum_{\substack{\alpha=\pm 1, \\ q_{\mathbf{k},\Omega}^{(\alpha)} \in \mathbb{R}^+}} \left| \frac{\partial E_{q,\mathbf{k}}^b}{\partial q} \right|^{-1} \delta(q - q_{\mathbf{k},\Omega}^{(\alpha)}), \quad (3.30)$$

where

$$q_{\mathbf{k},\Omega}^{(\pm 1)} = \sqrt{-m^2 \Delta^2 - \kappa_{\mathbf{k}}^2 \pm 2m \sqrt{\Delta^2 \kappa_{\mathbf{k}}^2 - 2\Delta k_y \Omega + \Omega^2}} \quad (3.31)$$

solve $E_{k_y}^s + \Omega - E_{q,\mathbf{k}}^b = 0$, we can eliminate the q integral,

yielding

$$\begin{aligned} \text{Re } \sigma_{\text{sb},R}^{\mu\nu}(\Omega; x, x') &= \frac{\delta^{\mu\nu}}{\Omega} \left(\frac{e^2}{m} \right) \int_0^{k_F} \frac{k dk}{2\pi} \int_0^{\pi/2} \frac{d\theta}{2\pi} \frac{k^\mu k^\nu}{\sqrt{\Delta^2 \kappa_{\mathbf{k}}^2 - 2\Delta k_y \Omega + \Omega^2}} \\ &\quad \times \sum_{\substack{\alpha=\pm 1, \\ q_{\mathbf{k},\Omega}^{(\alpha)} \in \mathbb{R}^+}} \frac{E_{q_{\mathbf{k},\Omega}^{(\alpha)}, \mathbf{k}}^b}{q_{\mathbf{k},\Omega}^{(\alpha)}} \text{Re } \Sigma_{+1}^{\text{sb}}(q_{\mathbf{k},\Omega}^{(\alpha)}, \mathbf{k}; x', x), \end{aligned} \quad (3.32)$$

where the sum over $\alpha = \pm 1$ takes the two roots in Eq. (3.31) into account. The constraint $q_{\mathbf{k},\Omega}^{(\alpha)} \in \mathbb{R}^+$ on allowed standing wave momenta is crucial, and restricts the domains of the remaining k and θ integrations [via Eq. (3.31)]. In Eqs. (3.31) and (3.32), the wavenumber $\kappa_{\mathbf{k}}$ controls the confinement depth of the surface states, see Eqs. (3.4) and (3.5).

From Eq. (2.20), the absorbance due to surface bulk transition is

$$\mathcal{A}_{\text{sb}}^{\mu\mu} = -\frac{16\pi}{c^2} \frac{K}{\lambda_L^{-2}} \text{Im } \mathcal{Q}^{\mu\mu}, \quad (3.33)$$

where

$$\text{Im } \mathcal{Q}^{\mu\mu}(\Omega) = -\Omega \int_{x,x'>0} e^{-\mathcal{K}x} \text{Re } \sigma_{\text{sb},R}^{\mu\mu}(\Omega; x, x') e^{-\mathcal{K}x'} \quad (3.34)$$

and $\mathcal{K} = \sqrt{\lambda_L^{-2} - K^2}$ in the clean limit. Introducing the dimensionless quantities $\hat{k}^\mu = k^\mu/k_F$, $\hat{q} = q/k_F$, $\hat{\kappa}_{\mathbf{k}} = \kappa_{\mathbf{k}}/k_F$ [Eq. (3.5)], and $\hat{\Omega} = \Omega/\Delta k_F = \Omega/\Delta_0$, where Δ_0 is the pairing energy, the absorbance can be expressed as

$$\mathcal{A}^{\mu\mu} = \frac{K}{k_F^2 l_{\text{coh}}} \hat{\mathcal{F}}^{\mu\mu}(\hat{\Omega}), \quad (3.35)$$

where the dimensionless integral $\hat{\mathcal{F}}^{\mu\mu}$ is given by

$$\begin{aligned} \hat{\mathcal{F}}^{\mu\mu}(\hat{\Omega}) &= 24\pi^2 \int_0^1 \frac{\hat{k} d\hat{k}}{2\pi} \int_0^{\pi/2} \frac{d\theta}{2\pi} \frac{\hat{k}^\mu \hat{k}^\mu}{\sqrt{\hat{\kappa}_{\mathbf{k}}^2 - 2\hat{k}_y \hat{\Omega} + \hat{\Omega}^2}} \\ &\quad \times \sum_{\substack{\alpha=\pm 1, \\ \hat{q}_{\mathbf{k},\Omega}^{(\alpha)} \in \mathbb{R}^+}} \left[\frac{-\hat{k}_y + \hat{\Omega}}{\hat{q}_{\mathbf{k},\Omega}^{(\alpha)}} \right] \text{Re } \hat{\mathcal{J}}_{\hat{q}_{\mathbf{k},\Omega}^{(\alpha)}, \hat{\mathbf{k}}}^{\text{sb}}, \end{aligned} \quad (3.36)$$

in which repeated indices are again not summed over. The dimensionless function $\hat{\mathcal{J}}_{\hat{q}_{\mathbf{k},\Omega}^{(\alpha)}, \hat{\mathbf{k}}}^{\text{sb}}$ is due to the integral

$$\hat{\mathcal{J}}_{\hat{q}, \hat{\mathbf{k}}}^{\text{sb}} = k_F^2 l_{\text{coh}} \int_{x,x'>0} e^{-\mathcal{K}x} \Sigma_{+1}^{\text{sb}}(q, \mathbf{k}; x', x) e^{-\mathcal{K}x'}, \quad (3.37)$$

which is just the integrated double-overlap between the surface and bulk states with a London response weight. The detailed form of $\hat{\mathcal{J}}_{\hat{q}, \hat{\mathbf{k}}}^{\text{sb}}$ can be found in Appendix C.

In Fig. 2, we plot the absorbance $\mathcal{A}(\Omega)$ with type I and

type II parameters based on Eq. (3.35). In the type-I case [Figs. 2(a)–(c)], \mathbf{A} displays relatively broad features. Its yy and zz components are both peaked at around $\Omega = 2\Delta_0$. Since the coherence length l_{coh} increases with the spatial extent of the surface state, a larger l_{coh} results in an enhanced overlap between the bulk and surface states. However, at the same time, a larger l_{coh} limits the region of the surface states that can respond to the external field. The magnitude of \mathbf{A} is controlled by the competition between these two effects. Nevertheless, the qualitative features of \mathbf{A} are almost unchanged with increasing l_{coh} except that the tail region at higher frequency is broadened.

On the other hand, in the type-II case [Figs. 2(d)–(f)], the feature of \mathbf{A} is relatively sharp, namely there is a peak at around $2\Delta_0$ for its yy component, and slightly above Δ_0 for its zz component. Interestingly, the positions of these peaks are nearly independent of λ_L in the type II limit. As λ_L increases, the integrated double-overlap in Eq. (3.37) is gradually suppressed due to the orthogonality between the bulk and surface states, resulting in a smaller \mathbf{A} .

In both cases, the peaks in \mathbf{A} are due to the square root van-Hove singularity in its integrand [Eq. (3.36)], weighted by various factors, including the current operator and the surface-bulk double-overlap. Although it is hard to pinpoint the exact position of the peaks analytically, by slicing the integrand at different k and partially

integrating out θ , our numerics reveal that the peaks in $\hat{\mathcal{F}}$ mainly come from the contribution at the vicinity of $k = k_F$, at which the surface states deconfine into the bulk. Roughly speaking, the peak for the yy component is dominated by the contributions at the proximity of $(k_y, k_z) \simeq (k_F, 0)$. In this region, the transitions originate from surface band with energy $\sim -\Delta k_F$ to the bulk band with energy $\sim \Delta k_F$, resulting in a sharp peak at $\Omega = 2\Delta_0$. For the zz component, the situation is more complex.

Our results here reveal that the surface-bulk optical conductivity is in general nonvanishing. Consequently, in the clean limit, the optical absorption of the system is contributed by the surface-bulk transitions.

IV. DISORDERED BULK OPTICAL RESPONSE

A. Keldysh formalism

In this section, we study the bulk optical conductivity of a WSC with quenched disorder, based on the finite-temperature Keldysh response theory [33, 98]. We consider spin-1/2 electrons with $p + ip$ pairing as in ^3HeA [28–30]; the formalism can be directly applied to other symmetries. The Keldysh generating function is [34]

$$Z \equiv \int D\bar{\psi} D\psi \exp \left[i \int_{\omega, \mathbf{x}, \mathbf{x}'} \bar{\psi}(\omega, \mathbf{x}) \hat{G}^{-1}(\omega; \mathbf{x}, \mathbf{x}') \psi(\omega, \mathbf{x}') + i \frac{W}{4} \sum_{a \in \{1, 2\}} \hat{\tau}_{a,a}^3 \int_{t, \mathbf{x}} \left\{ -i \bar{\psi}_a \left[(i \overleftarrow{\nabla}_l) \hat{s}^i \hat{s}^2 \right] \bar{\psi}_a^\top \right\} \left\{ i \psi_a^\top \left[\hat{s}^2 \hat{s}^i (-i \overrightarrow{\nabla}_l) \right] \psi_a \right\} - \frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left[\mathbf{A}_{\text{cl}}(\omega - \omega') \cdot \bar{\psi}(\omega) \hat{\tau}^3 (-i \overleftarrow{\nabla}) \psi(\omega') + \mathbf{A}_{\text{q}}(\omega - \omega') \cdot \bar{\psi}(\omega) (-i \overleftarrow{\nabla}) \psi(\omega') \right] \right], \quad (4.1)$$

where the \hat{s}^i and $\hat{\tau}^j$ respectively denote Pauli matrices acting on the spin-1/2 and Keldysh spaces. In Eq. (4.1), summations over $i \in \{1, 2, 3\}$ and $l \in \{x, y, z\}$ are assumed. The fermionic field $\psi = \psi_{a,s}(t, \mathbf{x})$, where the Keldysh index $a \in \{1, 2\}$ corresponds to the {forward, backward} time contour, and the spin index $s \in \{\uparrow, \downarrow\}$. The noninteracting Green's function in the space-time basis is given by

$$i\hat{G}(t, t'; \mathbf{x}, \mathbf{x}') = \begin{bmatrix} i\hat{G}_T & i\hat{G}_< \\ i\hat{G}_> & i\hat{G}_{\bar{T}} \end{bmatrix} = \begin{bmatrix} \langle T \psi(t, \mathbf{x}) \bar{\psi}(t', \mathbf{x}') \rangle_0 & -\langle \bar{\psi}(t', \mathbf{x}') \psi(t, \mathbf{x}) \rangle_0 \\ \langle \psi(t, \mathbf{x}) \bar{\psi}(t', \mathbf{x}') \rangle_0 & \langle \bar{T} \psi(t, \mathbf{x}) \bar{\psi}(t', \mathbf{x}') \rangle_0 \end{bmatrix}, \quad (4.2)$$

where T and \bar{T} respectively denote the time-ordering and anti-time-ordering operators.

Spin-triplet pairing is mediated by the attractive interaction $W > 0$ in Eq. (4.1). The operator

$$i\psi_a^\top(t, \mathbf{x}) \left[\hat{s}^2 \hat{s}^i (-i \overrightarrow{\nabla}_l) \right] \psi_a(t, \mathbf{x})$$

annihilates a local spin-triplet Cooper pair.

The net vector potential on the forward (backward) part of time contour \mathbf{A}_1 (\mathbf{A}_2) in Eq. (4.1) is expressed in terms of the classical and quantum components of the field, i.e.

$$\mathbf{A}_1 = \mathbf{A}_{\text{cl}} + \mathbf{A}_{\text{q}}, \quad \mathbf{A}_2 = \mathbf{A}_{\text{cl}} - \mathbf{A}_{\text{q}}. \quad (4.3)$$

Decoupling the pairing interaction by Hubbard-Stratonovich transformation, we have

$$Z = \int D\bar{\psi} D\psi D\bar{\Delta} D\Delta \exp \left\{ \begin{aligned} & i \int_{\omega, \mathbf{x}, \mathbf{x}'} \bar{\psi}(\omega, \mathbf{x}) \hat{G}^{-1}(\omega; \mathbf{x}, \mathbf{x}') \psi(\omega, \mathbf{x}') \\ & - \frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left[\mathbf{A}_{\text{cl}}(\omega - \omega') \bar{\psi}(\omega) \hat{\tau}^3 (-i \overleftrightarrow{\nabla}) \psi(\omega') + \mathbf{A}_{\text{q}}(\omega - \omega') \bar{\psi}(\omega) (-i \overleftrightarrow{\nabla}) \psi(\omega') \right] \\ & + i \frac{2}{W} \int_{t, \mathbf{x}} (\Delta_{\text{q}}^{il*} \Delta_{\text{cl}}^{il} + \Delta_{\text{q}}^{il} \Delta_{\text{cl}}^{il*}) \\ & - \frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left\{ \begin{aligned} & \Delta_{\text{cl}}^{il}(\omega + \omega') \left[\bar{\psi}(\omega) (i \overleftrightarrow{\nabla})_l \hat{s}^i \hat{s}^2 \hat{\tau}^3 \bar{\psi}^{\text{T}}(\omega') \right] \\ & + \Delta_{\text{q}}^{il}(\omega + \omega') \left[\bar{\psi}(\omega) (i \overleftrightarrow{\nabla})_l \hat{s}^i \hat{s}^2 \bar{\psi}^{\text{T}}(\omega') \right] \\ & - \Delta_{\text{cl}}^{il*}(\omega + \omega') \left[\psi^{\text{T}}(\omega) \hat{s}^2 \hat{\tau}^3 \hat{s}^i (-i \overleftrightarrow{\nabla})_l \psi(\omega') \right] \\ & - \Delta_{\text{q}}^{il*}(\omega + \omega') \left[\psi^{\text{T}}(\omega) \hat{s}^2 \hat{s}^i (-i \overleftrightarrow{\nabla})_l \psi(\omega') \right] \end{aligned} \right\} \end{aligned} \right\}. \quad (4.4)$$

1. Keldysh and “thermal” rotations

The inverse Green’s function can be written as

$$\hat{G}^{-1}(\omega; \mathbf{x}, \mathbf{x}') = \hat{U}_{\text{LO}} \hat{M}_F(\omega) \hat{G}_{\eta}^{-1}(\omega; \mathbf{x}, \mathbf{x}') \hat{M}_F(\omega) \hat{U}_{\text{LO}}^{\dagger} \hat{\tau}^3, \quad (4.5)$$

where

$$\hat{G}_{\eta}(\omega) = (\omega + i\eta\hat{\tau}^3 - \hat{h}_0)^{-1}, \quad \hat{U}_{\text{LO}} = \frac{1}{\sqrt{2}}(1 + i\hat{\tau}^2), \quad \hat{M}_F(\omega) = \begin{bmatrix} 1 & F(\omega) \\ 0 & -1 \end{bmatrix}_{\tau}, \quad \text{and} \quad F(\omega) = \tanh\left(\frac{\omega}{2T}\right), \quad (4.6)$$

with \hat{h}_0 being the static single particle Hamiltonian and $\eta \rightarrow 0^+$. The diagonal components of \hat{G}_{η} correspond to the retarded and advanced non-interacting Green’s functions. We can remove the distribution function $F(\omega)$ from the non-interacting part of the fermionic action through the following non-unitary transformation [34],

$$\psi(\omega, \mathbf{x}) \rightarrow \hat{\tau}^3 \hat{U}_{\text{LO}} \hat{M}_F(\omega) \psi(\omega, \mathbf{x}), \quad \bar{\psi} \rightarrow \bar{\psi}(\omega, \mathbf{x}) \hat{M}_F(\omega) \hat{U}_{\text{LO}}^{\dagger}. \quad (4.7)$$

The Keldysh function then becomes

$$Z = \int D\bar{\psi} D\psi D\bar{\Delta} D\Delta \exp \left\{ \begin{aligned} & i \int_{\omega, \mathbf{x}, \mathbf{x}'} \bar{\psi}(\omega, \mathbf{x}) \hat{G}_{\eta}^{-1}(\omega; \mathbf{x}, \mathbf{x}') \psi(\omega, \mathbf{x}') \\ & - \frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left[\mathbf{A}_{\text{cl}}(\omega - \omega') \bar{\psi}(\omega) \hat{M}_F(\omega) \hat{M}_F(\omega') (-i \overleftrightarrow{\nabla}) \psi(\omega') \right. \\ & \quad \left. + \mathbf{A}_{\text{q}}(\omega - \omega') \bar{\psi}(\omega) \hat{M}_F(\omega) \hat{\tau}^1 \hat{M}_F(\omega') (-i \overleftrightarrow{\nabla}) \psi(\omega') \right] \\ & + i \frac{2}{W} \int_{t, \mathbf{x}} (\Delta_{\text{q}}^{il*} \Delta_{\text{cl}}^{il} + \Delta_{\text{q}}^{il} \Delta_{\text{cl}}^{il*}) \\ & - \frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left\{ \begin{aligned} & \Delta_{\text{cl}}^{il}(\omega + \omega') \left[\bar{\psi}(\omega) (i \overleftrightarrow{\nabla})_l \hat{s}^i \hat{s}^2 \hat{M}_F(\omega) \hat{\tau}^1 \hat{M}_F^{\text{T}}(\omega') \bar{\psi}^{\text{T}}(\omega') \right] \\ & + \Delta_{\text{q}}^{il}(\omega + \omega') \left[\bar{\psi}(\omega) (i \overleftrightarrow{\nabla})_l \hat{s}^i \hat{s}^2 \hat{M}_F(\omega) \hat{M}_F^{\text{T}}(\omega') \bar{\psi}^{\text{T}}(\omega') \right] \\ & - \Delta_{\text{cl}}^{il*}(\omega + \omega') \left[\psi^{\text{T}}(\omega) \hat{s}^2 \hat{s}^i \hat{M}_F^{\text{T}}(\omega) \hat{\tau}^1 \hat{M}_F(\omega') (-i \overleftrightarrow{\nabla})_l \hat{\tau}^3 \psi(\omega') \right] \\ & - \Delta_{\text{q}}^{il*}(\omega + \omega') \left[\psi^{\text{T}}(\omega) \hat{s}^2 \hat{s}^i \hat{M}_F^{\text{T}}(\omega) \hat{M}_F(\omega') (-i \overleftrightarrow{\nabla})_l \psi(\omega') \right] \end{aligned} \right\} \end{aligned} \right\}, \quad (4.8)$$

where the thermal matrix $\hat{M}_F(\omega)$ now appears solely in the coupling to the superconducting order parameter and the external vector potential.

At the static mean-field level for p -wave, spin-triplet pairing, we take

$$(i\overleftrightarrow{\nabla})_l \Delta_{\text{cl}}^{il}(\omega + \omega') = i\mathbf{d}^i \delta_{\omega+\omega',0}, \quad \Delta_{\text{cl}}^{il*}(-i\overleftrightarrow{\nabla})_l = i(\mathbf{d}^i)^* \delta_{\omega+\omega',0}, \quad \Delta_{\mathbf{q}}(\omega + \omega') = \Delta_{\mathbf{q}}^*(\omega + \omega') = 0, \quad (4.9)$$

where \mathbf{d} is just the (unnormalized) d -vector order parameter [28–30]. Together with the properties

$$\hat{\tau}^1 \hat{M}_F(-\omega) \hat{\tau}^1 = -\hat{M}_F(\omega), \quad \hat{M}_F^{-1}(\omega) = \hat{M}_F(\omega), \quad (4.10)$$

the static mean-field Keldysh function can be recast as

$$Z = \int D\bar{\psi} D\psi \exp \left\{ -\frac{i}{2} \int_{\omega, \omega', \mathbf{x}} \left[\mathbf{A}_{\text{cl}}(\omega - \omega') \bar{\psi}(\omega) \hat{M}_F(\omega) \hat{M}_F(\omega') (-i\overleftrightarrow{\nabla}) \psi(\omega') \right] \right. \\ \left. + \frac{i}{2} \int_{\omega, \mathbf{x}} \left[i\bar{\psi}(\omega) (\mathbf{d} \cdot \hat{\mathbf{s}}) \hat{s}^2 \hat{\tau}^1 \bar{\psi}^\top(-\omega) \right] \right. \\ \left. - i\psi^\top(-\omega) \hat{\tau}^1 \hat{s}^2 (\mathbf{d}^* \cdot \hat{\mathbf{s}}) \psi(\omega) \right\}. \quad (4.11)$$

2. Majorana spinor reformulation

In order to encode both particle-hole and particle-particle channel fluctuations, we recast the action in terms of the (real) Majorana spinor [34]

$$\chi \equiv \begin{bmatrix} \psi \\ \hat{s}^2 \hat{\tau}^1 \hat{\Sigma}^1 \bar{\psi}^\top \end{bmatrix}, \quad (4.12)$$

$$\bar{\chi} = [\bar{\psi} - \psi^\top \hat{s}^2 \hat{\tau}^1 \hat{\Sigma}^1] = -\chi^\top \hat{s}^2 \hat{\sigma}^1 \hat{\tau}^1 \hat{\Sigma}^1, \quad (4.13)$$

which carry discrete particle-hole (σ), Keldysh (τ) and spin (s) indices. In addition, χ possesses a continuous frequency $|\omega|$ index ranging over the positive real axis, and a discrete sign index $\Sigma = \text{sgn}(\omega) \in \{+, -\}$. The Pauli matrix in Eq. (4.13) is defined as $\langle \omega | \hat{\Sigma}^1 | \omega' \rangle = 2\pi \delta(\omega + \omega')$. Eq. (4.11) can be expressed compactly as

$$Z = \int D\chi \exp \left\{ \frac{i}{2} \int_{\mathbf{x}} \bar{\chi} \left[\hat{G}_{\text{BdG}}^{-1} - \frac{1}{2} \hat{\mathbf{A}} \cdot (-i\overleftrightarrow{\nabla}) \right] \chi \right\}, \quad (4.14)$$

where

$$\hat{G}_{\text{BdG}}^{-1} \equiv \hat{\sigma}^3 \hat{\omega} + i\eta \hat{\tau}^3 \hat{\sigma}^3 - \hat{\sigma}^3 \hat{h}_{\text{BdG}}, \quad (4.15)$$

$$\hat{h}_{\text{BdG}} \equiv \begin{bmatrix} \hat{h}_0 & -i\mathbf{d} \cdot \hat{\mathbf{s}} \\ i\mathbf{d}^* \cdot \hat{\mathbf{s}} & -\hat{s}^2 \hat{h}_0^\top \hat{s}^2 \end{bmatrix}_\sigma, \quad (4.16)$$

and

$$\hat{\mathbf{A}}_{\omega, \omega'}(\mathbf{x}) \equiv \mathbf{A}_{\text{cl}}(\omega - \omega', \mathbf{x}) \hat{\gamma}_{\text{cl}}(\omega, \omega') \\ + \mathbf{A}_{\mathbf{q}}(\omega - \omega', \mathbf{x}) \hat{\gamma}_{\mathbf{q}}(\omega, \omega'), \quad (4.17)$$

with

$$\hat{\gamma}_{\text{cl}}(\omega, \omega') = \hat{M}_F(\omega) \hat{\sigma}^3 \hat{M}_F(\omega'), \quad (4.18)$$

$$\hat{\gamma}_{\mathbf{q}}(\omega, \omega') = \hat{M}_F(\omega) \hat{\tau}^1 \hat{\sigma}^3 \hat{M}_F(\omega'). \quad (4.19)$$

B. Disorder averaging and saddle-point equation

For simplicity, we consider only onsite scalar potential disorder and neglect spatial fluctuations of the order parameter Δ . We assume that the static impurity potential $u(\mathbf{x})$ is Gaussian distributed

$$P[u] = \exp \left[-\frac{1}{2g} \int_{\mathbf{x}} u^2(\mathbf{x}) \right], \quad (4.20)$$

where $g = \Upsilon_{\text{el}}/\pi\nu_0$ characterizes the width of the distribution. Here, $\Upsilon_{\text{el}} \equiv 1/(2\tau_{\text{el}})$, τ_{el} is the elastic scattering time, and ν_0 is the density of states per spin species at the Fermi surface. The disordered part of the action is

$$S_{\text{dis}} = \frac{i}{2} \int_{\mathbf{x}} \bar{\chi}(\mathbf{x}) u(\mathbf{x}) \chi(\mathbf{x}) \quad (4.21)$$

We average over the disorder potential u to get

$$\langle e^{-S_{\text{dis}}} \rangle = \int Du P[u] \exp \left[-\frac{i}{2} \int_{\mathbf{x}} \bar{\chi}(\mathbf{x}) u(\mathbf{x}) \chi(\mathbf{x}) \right] \\ = \exp \left\{ -\frac{g}{2} \int_{\mathbf{x}} \text{Tr} \left[\left(\frac{\chi \bar{\chi}}{2} \right) \left(\frac{\chi \bar{\chi}}{2} \right) \right] \right\} \\ = \int D\hat{Q} \exp \left\{ -\frac{1}{4g} \int_{\mathbf{x}} \text{Tr}[\hat{Q}^2] - \frac{1}{2} \int_{\mathbf{x}} \text{Tr}[\hat{Q} \chi \bar{\chi}] \right\}, \quad (4.22)$$

where we have performed a Hubbard-Stratonovich transformation by introducing the matrix field \hat{Q} . By integrating

out the fermionic field χ , the action becomes

$$S = \frac{1}{4g} \int_{\mathbf{x}} \text{Tr}[\hat{Q}^2] - \frac{1}{2} \text{Tr} \log \left[\hat{G}_{\text{BdG}}^{-1} + i\hat{Q} - \hat{\mathbf{A}} \cdot \mathbf{v}_F \right], \quad (4.23)$$

where \hat{G}_{BdG} is the Bogoliubov-de Gennes (BdG) Green's function defined in Eq. (4.15).

Next, we derive the saddle point equation in the absence of the vector potential. By varying the action with respect to \hat{Q} and setting $\delta S / \delta \hat{Q}|_{\hat{Q}=\hat{Q}_{\text{sp}}} = 0$, we have

$$\frac{1}{2g} \hat{Q}_{\text{sp}} = \frac{i}{2} \langle \mathbf{x} | \frac{1}{\hat{G}_{\text{BdG}}^{-1} + i\hat{Q}_{\text{sp}}} | \mathbf{x} \rangle, \quad (4.24)$$

or

$$-\frac{i}{g} \hat{Q}_{\text{sp}} = \int_{\mathbf{k}} \frac{1}{\hat{\sigma}^3 \hat{\omega} + i\eta \hat{\tau}^3 \hat{\sigma}^3 - \hat{\sigma}^3 \hat{h}_{\text{BdG}} + i\hat{Q}_{\text{sp}}}, \quad (4.25)$$

where $\int_{\mathbf{k}} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3}$. Eq. (4.25) is equivalent to the self-consistent Born approximation (SCBA). In the context of disordered Weyl semimetals, it is known that the SCBA cannot capture the nonperturbative effects generated by rare regions [35–39, 41, 42]. For Majorana surface states in WSCs, similar nonperturbative effects induced by rare states are also found numerically [40]. However, these rare states are not expected to qualitatively modify the optical conductivity, and thus we restrict our calculation here to the SCBA.

To solve the saddle point equation [Eq. (4.25)], we employ the *ansatz*

$$\hat{Q}_{\text{sp}} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & -Q_{11} \end{bmatrix}_{\sigma} \hat{\tau}^3. \quad (4.26)$$

We then compute the matrix elements of Eq. (4.25) on both sides. For the WSC that we are studying, the BdG Hamiltonian in Eq. (3.1b) corresponds to a d -vector $\mathbf{d} = i\Delta(k_x + ik_y)\hat{x}$, as in ^3HeA [28–30]. Since the pairing term is odd in \mathbf{k} , the off-diagonal terms of Eq. (4.25) must vanish upon angular integration for self-consistency. Thus, we only have to focus on the diagonal components and solve

$$Q_{11} = ig\nu_0 \int d\tilde{\mathbf{k}} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \frac{-(\omega + iQ_{11})}{-(\omega + iQ_{11})^2 + \Delta_0^2 \sin^2(\theta) + \tilde{\varepsilon}_{\mathbf{k}}^2}, \quad (4.27)$$

where $\Delta_0 = \Delta k_F$ is the pairing energy and we have converted

$$\int_{\mathbf{k}} \simeq \nu_0 \int d\tilde{\mathbf{k}} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi}, \quad (4.28)$$

with $d\Omega_{\hat{\mathbf{k}}} = \sin \theta d\theta d\phi$. The integrals can be done ana-

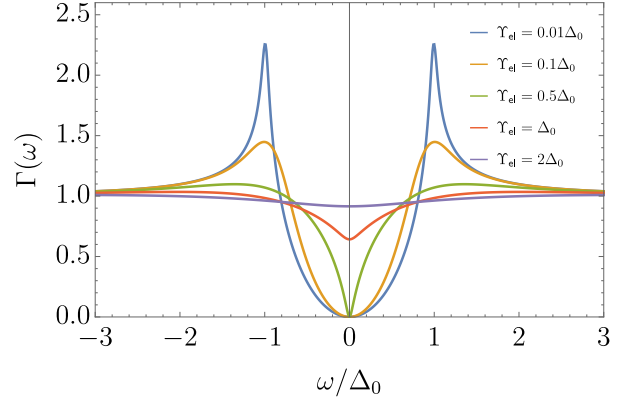


FIG. 6: Plot of the frequency dependence of $\Gamma(\omega)$ based on self-consistently solving Eq. (4.29) at different disorder strengths $\Upsilon_{\text{el}} = 1/2\tau_{\text{el}}$.

lytically, the result is

$$Q_{11} = \frac{1}{2\tau_{\text{el}}} \left(\frac{\omega + iQ_{11}}{\Delta_0} \right) \tanh^{-1} \left(\frac{\Delta_0}{\omega + iQ_{11}} \right). \quad (4.29)$$

The real part of Q_{11} carries the physical meaning of impurity scattering rate, while the imaginary part of it merely renormalizes the quasiparticle dispersion. In the following, we neglect the imaginary part of \hat{Q}_{sp} and just take

$$\hat{Q}_{\text{sp}}(\omega) = \frac{1}{2\tau_{\text{el}}} \hat{\tau}^3 \hat{\sigma}^3 \Gamma(\omega), \quad (4.30)$$

where $\Gamma(\omega) = 2\tau_{\text{el}} \text{Re} Q_{11}$. In particular, in the weak disorder limit, $\Gamma(\omega)$ can be approximated as

$$\Gamma(\omega) \simeq \text{Re} \left[\frac{\omega + i\eta}{\Delta_0} \tanh^{-1} \left(\frac{\Delta_0}{\omega + i\eta} \right) \right], \quad \Upsilon_{\text{el}} \ll \Delta_0, \quad (4.31)$$

where $\eta \rightarrow 0^+$. In Fig. 3, we plot the self-consistent solution of $\Gamma(\omega)$ based on Eq. (4.29).

For weak disorder, $\Gamma(\omega)$ is well-approximated by Eq. (4.31) and has two pronounced coherence peaks located at $\omega = \pm\Delta_0$. The gapless nature of $\Gamma(\omega)$ originates from the nodal points of the system. As the scattering rate Υ_{el} increases, impurities gradually fill the gap and suppress the coherence peaks. In the strong-disorder limit, the p -wave features are completely suppressed and we recover the normal metal saddle point [33, 34]

$$\hat{Q}_{\text{sp}}(\omega) \rightarrow \frac{1}{2\tau_{\text{el}}} \hat{\tau}^3 \hat{\sigma}^3, \quad \Upsilon_{\text{el}} \gg \Delta_0. \quad (4.32)$$

This is consistent with the formation of a thermal metal phase with a nonzero density of quasiparticle states at zero energy [40, 48, 94]. The disorder strength dependence seen here is similar to that in the density of states of quasiparticles in d -wave superconductors [99, 100]. This sharply contrasts with the saddle point solution in s -wave

superconductors, in which the frequency dependence of $\hat{Q}_{\text{sp}}^{\text{s-wave}}$ is independent of Υ_{el} (see Appendix D for a review).

C. Linear response optical conductivity

The retarded linear response conductivity is defined as

$$\sigma_{\text{bb},R}^{\mu\nu}(\Omega) = -\frac{e^2}{i\Omega} \frac{1}{2i} \frac{\delta^2 Z[\mathbf{A}_{\text{cl}}, \mathbf{A}_{\text{q}}]}{\delta A_{\text{cl},\Omega}^\mu \delta A_{\text{q},-\Omega}^\nu} \Big|_{A=0}. \quad (4.33)$$

In the following, we evaluate $\sigma_{\text{bb},R}^{\mu\nu}(\Omega)$ by expanding [via Eq. (4.23)]

$$S_\chi[\hat{Q}] = -\frac{1}{2} \text{Tr} \log \left[\hat{G}_{\text{BdG}}^{-1} + i\hat{Q} - \hat{\mathcal{A}} \cdot \mathbf{v}_F \right] \quad (4.34)$$

up to second order in $\hat{\mathcal{A}}$, and neglecting fluctuations of \hat{Q} by setting $\hat{Q} = \hat{Q}_{\text{sp}}$, as given by Eq.(4.30). Then, up to irrelevant constants, S_χ becomes

$$\begin{aligned} S_\chi[\hat{Q}_{\text{sp}}] &= S_\chi^{(1)}[\hat{Q}_{\text{sp}}] + S_\chi^{(2)}[\hat{Q}_{\text{sp}}] + \mathcal{O}(\hat{\mathcal{A}}^3) \\ &= \frac{1}{2} \text{Tr} \left[\hat{\mathcal{G}}_\Upsilon \hat{\mathcal{A}} \cdot \mathbf{v}_F \right] + \frac{1}{4} \text{Tr} \left[\hat{\mathcal{G}}_\Upsilon \hat{\mathcal{A}} \cdot \mathbf{v}_F \right]^2 + \mathcal{O}(\hat{\mathcal{A}}^3), \end{aligned} \quad (4.35)$$

where

$$\hat{\mathcal{G}}_\Upsilon^{-1} \equiv \hat{G}_{\text{BdG}}^{-1} + i\hat{Q}_{\text{sp}}. \quad (4.36)$$

By taking partial trace over frequency and momentum, the second order term can be expanded as

$$S_\chi^{(2)}[\hat{Q}_{\text{sp}}] = \frac{1}{4} \int_{\mathbf{k}, \mathbf{p}} \int_{\omega, \omega'} \sum_{i,j=\text{cl}, \text{q}} \text{Tr} \left[\hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \omega + \omega') \mathbf{A}_i(\omega, \mathbf{k}) \hat{\gamma}_i(\omega + \omega', \omega') \cdot \mathbf{v}_F(\mathbf{p}) \right. \\ \left. \times \hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \omega') \mathbf{A}_j(-\omega, -\mathbf{k}) \hat{\gamma}_j(\omega', \omega + \omega') \cdot \mathbf{v}_F(\mathbf{p}) \right], \quad (4.37)$$

where we have assumed that \mathbf{k} is small. Upon taking the derivative with respect to the classical and quantum components of the vector potential, we have

$$\frac{\delta^2 S_\chi^{(2)}[\hat{Q}_{\text{sp}}]}{\delta A_{\text{cl},\Omega}^\mu \delta A_{\text{q},-\Omega}^\nu} \Big|_{A=0} = 2 \times 2 \times \frac{v_F^2}{4} \int_{\mathbf{p}, \omega'} \text{Tr} \left[\hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \Omega + \omega') \hat{\gamma}_{\text{cl}}(\Omega + \omega', \omega') \hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \omega') \hat{\gamma}_{\text{q}}(\omega', \Omega + \omega') \right] \hat{v}_F^\mu(\mathbf{p}) \hat{v}_F^\nu(\mathbf{p}), \quad (4.38)$$

where the factor of 2×2 is due to the two-fold derivative and the trace over the spin degree of freedom. To simplify the algebra, we introduce the projection operator $\hat{P}_{\lambda=\pm} \equiv \frac{1}{2}(1 + \lambda \hat{\tau}^3)$ such that

$$\begin{aligned} \frac{\delta^2 S_\chi^{(2)}[\hat{Q}_{\text{sp}}]}{\delta A_{\text{cl},\Omega}^\mu \delta A_{\text{q},-\Omega}^\nu} \Big|_{A=0} &= v_F^2 \int_{\mathbf{p}, \omega'} \sum_{\lambda, \lambda'=\pm} T_{\lambda\lambda'} \hat{v}_F^\mu(\mathbf{p}) \hat{v}_F^\nu(\mathbf{p}), \\ T_{\lambda\lambda'} &\equiv \text{Tr} \left[\hat{P}_\lambda \hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \Omega + \omega') \hat{\gamma}_{\text{cl}}(\Omega + \omega', \omega') \hat{P}_{\lambda'} \hat{\mathcal{G}}_\Upsilon(\mathbf{p}, \omega') \hat{\gamma}_{\text{q}}(\omega', \Omega + \omega') \right]. \end{aligned} \quad (4.39)$$

The trace that appears in the integrand can be evaluated to give

$$T_{++} = F(\omega_2) t_{++}, \quad T_{+-} = [F(\omega_1) - F(\omega_2)] t_{+-}, \quad T_{-+} = 0, \quad T_{--} = -F(\omega_1) t_{--}, \quad (4.40)$$

where

$$t_{\lambda\lambda'} \equiv \frac{2(\tilde{\varepsilon}^2 + \Delta_0^2 \sin^2(\theta) + \varpi_{1,\lambda} \varpi_{2,\lambda'})}{\left[\varpi_{1,\lambda}^2 - \tilde{\varepsilon}^2 - \Delta_0^2 \sin^2(\theta) \right] \left[\varpi_{2,\lambda'}^2 - \tilde{\varepsilon}^2 - \Delta_0^2 \sin^2(\theta) \right]}. \quad (4.41)$$

Here θ is the polar angle in \mathbf{p} space and

$$\varpi_{1,\pm} = \omega_1 \pm \frac{i}{2\tau_{\text{el}}} \Gamma(\omega_1), \quad \varpi_{2,\pm} = \omega_2 \pm \frac{i}{2\tau_{\text{el}}} \Gamma(\omega_2). \quad (4.42)$$

We now convert $\int_{\mathbf{p}} \rightarrow \nu_0 \int d\tilde{\varepsilon} \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}$. The $\tilde{\varepsilon}$ integral can be done by the contour method. The results are

$$I_{++} = \int_{\tilde{\varepsilon}} T_{++} = \frac{2\pi i}{\varpi_{1,+} - \varpi_{2,+}} \left[\frac{\varpi_{1,+} \text{sgn}(\omega_1)}{\sqrt{\varpi_{1,+}^2 - \Delta_0^2 \sin^2(\theta)}} - \frac{\varpi_{2,+} \text{sgn}(\omega_2)}{\sqrt{\varpi_{2,+}^2 - \Delta_0^2 \sin^2(\theta)}} \right] F(\omega_2), \quad (4.43a)$$

$$I_{+-} = \int_{\tilde{\varepsilon}} T_{+-} = \frac{2\pi i}{\varpi_{1,+} - \varpi_{2,-}} \left[\frac{\varpi_{1,+} \text{sgn}(\omega_1)}{\sqrt{\varpi_{1,+}^2 - \Delta_0^2 \sin^2(\theta)}} + \frac{\varpi_{2,-} \text{sgn}(\omega_2)}{\sqrt{\varpi_{2,-}^2 - \Delta_0^2 \sin^2(\theta)}} \right] [F(\omega_1) - F(\omega_2)], \quad (4.43b)$$

$$I_{--} = \int_{\tilde{\varepsilon}} T_{--} = \frac{2\pi i}{\varpi_{1,-} - \varpi_{2,-}} \left[\frac{\varpi_{1,-} \text{sgn}(\omega_1)}{\sqrt{\varpi_{1,-}^2 - \Delta_0^2 \sin^2(\theta)}} - \frac{\varpi_{2,-} \text{sgn}(\omega_2)}{\sqrt{\varpi_{2,-}^2 - \Delta_0^2 \sin^2(\theta)}} \right] F(\omega_1). \quad (4.43c)$$

Notice that $\frac{\delta^2(S_{\text{cl}}^{(1)}[\hat{Q}_{\text{sp}}])^2}{\delta A_{\text{cl},\Omega}^\mu \delta A_{\text{q},-\Omega}^\nu} \Big|_{A=0} = 0$ and therefore does not contribute to $\sigma_{\text{bb},R}^{\mu\nu}(\Omega)$. By combining the results in Eq. (4.33) and Eqs. (4.39)–(4.43), the real part of the optical conductivity can be readily obtained as

$$\text{Re } \sigma_{\text{bb},R}^{\mu\nu}(\Omega) = -\delta^{\mu\nu} \times \frac{e^2}{i\Omega} \frac{1}{2i} \left\{ v_F^2 \nu_0 \int_{\omega'} \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \hat{v}_F^\mu(\mathbf{p}) \hat{v}_F^\nu(\mathbf{p}) \mathcal{S}(\Omega + \omega', \omega', \theta) \times 2\pi [F(\Omega + \omega') - F(\omega')] \right\}, \quad (4.44)$$

where function the \mathcal{S} is defined via

$$\mathcal{S}(\omega_1, \omega_2, \theta) = \text{Im} \left\{ \frac{1}{\varpi_{1,+} - \varpi_{2,+}} \left[\frac{\varpi_{1,+} \text{sgn}(\omega_1)}{\sqrt{\varpi_{1,+}^2 - \Delta_0^2 \sin^2(\theta)}} - \frac{\varpi_{2,+} \text{sgn}(\omega_2)}{\sqrt{\varpi_{2,+}^2 - \Delta_0^2 \sin^2(\theta)}} \right] - \frac{1}{\varpi_{1,+} - \varpi_{2,-}} \left[\frac{\varpi_{1,+} \text{sgn}(\omega_1)}{\sqrt{\varpi_{1,+}^2 - \Delta_0^2 \sin^2(\theta)}} + \frac{\varpi_{2,-} \text{sgn}(\omega_2)}{\sqrt{\varpi_{2,-}^2 - \Delta_0^2 \sin^2(\theta)}} \right] \right\}. \quad (4.45)$$

We can perform the angular integrations exactly to finally obtain

$$\frac{\text{Re } \sigma_{\text{bb},R}^{\mu\nu}(\Omega)}{\sigma_{\text{dc}}} = \frac{3}{2\Omega} \int_{-\infty}^{\infty} d\omega \mathcal{J}^{\mu\nu}(\Omega + \omega, \omega) \times [F(\Omega + \omega) - F(\omega)], \quad (4.46)$$

where $\sigma_{\text{dc}} = e^2(2\nu_0)D$ is the normal state dc conductivity and $D = v_F^2 \tau_{\text{el}}/3$ is the diffusion constant. The kernel in Eq. (4.46) is given by

$$\mathcal{J}^{\mu\nu}(\omega_1, \omega_2) = \Upsilon_{\text{el}} \text{Im} \left\{ \frac{\left[\frac{\Phi^{\mu\nu}\left(\frac{\varpi_{1,+}}{\Delta_0}\right) - \Phi^{\mu\nu}\left(\frac{\varpi_{2,+}}{\Delta_0}\right) }{\varpi_{1,+} - \varpi_{2,+}} \right]}{\left[\frac{\Phi^{\mu\nu}\left(\frac{\varpi_{1,+}}{\Delta_0}\right) - \Phi^{\mu\nu}\left(\frac{\varpi_{2,-}}{\Delta_0}\right) }{\varpi_{1,+} - \varpi_{2,-}} \right]} \right\}, \quad (4.47)$$

where

$$\Phi^{\mu\mu}(v) \equiv \left\{ \frac{v}{4} [-v + (1 + v^2) \coth^{-1} v] \right\}, \quad (4.48a)$$

for $\mu \in \{x, y\}$,

$$\Phi^{zz}(v) \equiv \left\{ \frac{v}{2} [v + (1 - v^2) \coth^{-1} v] \right\}, \quad (4.48b)$$

$$\Phi^{\mu\nu}(\omega) = 0, \quad \mu \neq \nu. \quad (4.48c)$$

Notice that Eq. (4.48) implies that the bulk optical conductivity is purely diagonal. Eqs. (4.46)–(4.48) are the main results for the bulk optical conductivity of a weakly disordered $p+ip$ WSC. Our results are similar to those reported in Ref. [44], despite some differences in the details. In the metallic limit, $\Delta_0 \rightarrow 0$, $\Gamma(\omega) \rightarrow 1$ and $\Phi^{yy}, \Phi^{zz} \rightarrow 1/3$. As a result, $\mathcal{J}^{\mu\nu}(\omega_1, \omega_2) \rightarrow \frac{1}{3} \delta^{\mu\nu} \frac{1}{1 + (\omega_1 - \omega_2)^2 \tau_{\text{el}}^2}$ and we recover the Drude conductivity

$$\text{Re } \sigma_{\text{Drude}}^{\mu\nu}(\Omega) = \delta^{\mu\nu} \sigma_{\text{dc}} / [1 + (\Omega \tau_{\text{el}})^2]. \quad (4.49)$$

In Fig. 3, we plot the zero-temperature optical conductivity based on Eqs. (4.46)–(4.48) with different disorder strengths Υ_{el} . For weak disorder [Fig. 3(a)], the behavior of $\text{Re } \sigma_{\text{bb},R}^{\mu\nu}(\Omega)$ is similar to the Mattis-Bardeen result for s -wave superconductors [5]. This is despite the fact that for the WSC studied here, the response is nonzero even when $\Omega < 2\Delta_0$ due to the point-node nature of the bulk pairing. For large frequencies ($\Omega \gg \Delta_0$), the pairing gap does not play a role anymore and $\text{Re } \sigma_{\text{bb},R}^{\mu\nu}(\Omega)$ approaches $\text{Re } \sigma_{\text{Drude}}^{\mu\nu}(\Omega)$. As the disorder strength becomes stronger, impurities gradually fill the gap and the scattering rate $\sim \Gamma(\omega)$ is no longer gapless, as shown in Fig. 6. Consequently, the zero-frequency response is no longer vanishing and $\text{Re } \sigma_R^{\mu\nu}(\omega)$ becomes more Drude-like [Fig. 3(b)–(d)]. For strong disorder, the optical conductivity approaches the Drude result and the p -wave features are completely suppressed. This makes physical sense because the impurity strength Υ_{el} dominates over the pairing gap Δ_0 in this regime, and the system behaves as a dirty “thermal metal.”

The Mattis-Bardeen optical conductivity for dirty s -

wave superconductors can be efficiently derived within the same Keldysh framework articulated here, as we show in Appendix D.

V. DISCUSSION AND CONCLUSION

To conclude, we studied the optical absorbance in a $p + ip$ Weyl superconductor by examining the contribution from the bulk and surface. Fig. 4 summarizes all of the main results presented in this paper, which were discussed in Sec. I A. In the clean limit, we showed that the absorbance is solely contributed by the surface-bulk response. The frequency dependence of the absorbance depends on both the coherence length and penetration length. On the other hand, in the presence of disorder, we demonstrated that the bulk response is nonzero using Keldysh response theory. The optical conductivity from the bulk is Mattis-Bardeen-like in the weak disorder limit, and becomes more Drude-like as the system becomes dirtier. The overall optical absorbance of the system depends on the interplay between the penetration depth, coherence length and disorder strength. In the weak disorder limit, we found that the surface-bulk absorbance is more pronounced in the type I superconductor regime. For the type II regime, the disordered bulk plays a more important role and one would need a relatively clean system to observe the surface-bulk effect. In both regimes, as long as the disorder strength is weak enough, the characteristic frequency-dependence of the surface-bulk absorbance can provide an additional indicator for the surface states.

In WSC candidate materials, there can in general be multiple pairs of Weyl nodes and thus additional chiral Majorana fluid states. These can wrap around different facets of the sample, depending upon the orientation of the Fermi arcs. Additional arcs provide more channels for the surface-bulk transition and amplify the topological anomalous skin effect. For radiation with a nonzero incident angle, the surface-bulk absorption is still effective as long as the radiation is shone upon a facet with surface states. If polarization of the radiation is parallel to the plane of incidence, we expect to have maximum absorption at the Brewster angle at which the reflectance is minimized [101].

When evaluating the total absorbance, we have restricted ourselves to the weak-disorder regime for simplicity. In general, disorder can influence both the bulk and surface states [42, 102]. On one hand, the chiral Majorana surface states are expected to remain Anderson delocalized along the dispersive direction, perpendicular to surface Fermi arc. On the other hand, bulk states away from the Weyl points are converted by arbitrarily weak disorder into weakly multifractal scattering states, giving rise to thermal quasiparticle diffusion. A simple approach to incorporating disorder into the surface states is to introduce a finite scattering rate κ_b into Eq. (3.21). However, this does not capture the modification of the spatial

profile of the surface wavefunctions (e.g., in along the direction of the Fermi arc), due to the disorder. A more precise theoretical description for analyzing the consequences of disorder on position-dependent surface-bulk responses warrants future investigation.

Unlike s -wave superconductors, p -wave superconductors are not protected by Anderson's theorem [103]. As a result, the pairing gap is vulnerable to disorder and superconductivity can be destroyed. The p -wave features of the system are therefore expected to survive only in the weak-disorder limit.

Our work opens a new door for studying responses that originate from the interplay between bulk and surface states in topological materials. An important extension of this work would be performing material-specific and first-principle calculations in order to make quantitative predictions for the surface-bulk absorption peak in candidate WSC materials, such as UTe_2 [72–88].

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Appendix A: Bulk scattering states and surface states

The bulk scattering states are obtained by solving the Schrödinger's equation with the Hamiltonian given by Eq. (3.1b) in the main text

$$\hat{h}(-i\partial_x, \mathbf{k}) \psi_\lambda^b(q, \mathbf{k}; x) = \lambda E_{q,\mathbf{k}}^b \psi_\lambda^b(q, \mathbf{k}; x), \quad (\text{A1})$$

where $\lambda = \pm 1$, $\mathbf{k} = (k_y, k_z)$, $q \geq 0$ is a standing-wave momentum in the x -direction, and the bulk eigenenergy is

$$E_{q,\mathbf{k}}^b = \sqrt{\tilde{\epsilon}_{q,\mathbf{k}} + \Delta^2(q^2 + k_y^2)}, \quad \tilde{\epsilon}_{q,\mathbf{k}} = \frac{q^2 + \mathbf{k}^2 - k_F^2}{2m}. \quad (\text{A2})$$

By imposing the boundary condition that

$$\psi_\lambda^b(q, \mathbf{k}; x = 0) = 0 \quad (\text{A3})$$

and requiring $\psi_\lambda^b(q, \mathbf{k}; x \rightarrow \infty)$ to be finite, we obtain the bulk states in the following two cases:

- (i) $k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 > 0$

As discussed in Sec. III A, we have to consider degeneracy in this case. For $q_{\min} < q \leq q_0$, we choose the following positive-energy orthonormal states

$$\Psi_{\lambda=+1}^{b(1)}(q, \mathbf{k}; x) = \frac{\psi^{b<}(q, \mathbf{k}; x) + i\phi^{b<}(q, \mathbf{k}; x)}{\sqrt{\mathcal{N}_{q, \mathbf{k}}^{b(1)}}}, \quad (\text{A4})$$

$$\Psi_{\lambda=+1}^{b(2)}(q, \mathbf{k}; x) = \frac{\psi^{b<}(q, \mathbf{k}; x) - i\phi^{b<}(q, \mathbf{k}; x)}{\sqrt{\mathcal{N}_{q, \mathbf{k}}^{b(2)}}}, \quad (\text{A5})$$

where the normalization constants are

$$\frac{1}{\mathcal{N}_{q, \mathbf{k}}^{b(1)}} = \frac{1}{2 - 2\mathfrak{a}(q, \mathbf{k})}, \quad (\text{A6})$$

$$\frac{1}{\mathcal{N}_{q, \mathbf{k}}^{b(2)}} = \frac{1}{2 + 2\mathfrak{a}(q, \mathbf{k})}, \quad (\text{A7})$$

and the function $\mathfrak{a}(q, \mathbf{k})$ will be defined below. $\psi^{b<}(q, \mathbf{k}; x)$ and $\phi^{b<}(q, \mathbf{k}; x)$ are two non-orthogonal but independent positive-energy bulk states given by

$$\psi^{b<}(q; x) = \frac{1}{\sqrt{N_{q, \mathbf{k}}^b}} \left\{ \begin{aligned} & \begin{bmatrix} f_1(q, \mathbf{k}) \\ f_2(q, \mathbf{k}) \end{bmatrix} e^{iqx} + \begin{bmatrix} f_1^*(q, \mathbf{k}) \\ -f_2(q, \mathbf{k}) \end{bmatrix} e^{-iqx} \\ & + \begin{bmatrix} -f_1(q_-, \mathbf{k}) \\ -f_2(q_-, \mathbf{k}) \end{bmatrix} e^{iq_-x} + \begin{bmatrix} -f_1^*(q_-, \mathbf{k}) \\ f_2(q_-, \mathbf{k}) \end{bmatrix} e^{-iq_-x} \end{aligned} \right\} \quad (\text{A8})$$

and

$$\phi^{b<}(q; x) = \frac{1}{\sqrt{M_{q, \mathbf{k}}^b}} \left\{ \begin{aligned} & \begin{bmatrix} -g_2(q, \mathbf{k}) \\ f_1^*(q, \mathbf{k}) \end{bmatrix} e^{iqx} + \begin{bmatrix} g_2(q, \mathbf{k}) \\ f_1(q, \mathbf{k}) \end{bmatrix} e^{-iqx} \\ & + \begin{bmatrix} g_2(q_-, \mathbf{k}) \\ -f_1^*(q_-, \mathbf{k}) \end{bmatrix} e^{iq_-x} + \begin{bmatrix} -g_2(q_-, \mathbf{k}) \\ -f_1(q_-, \mathbf{k}) \end{bmatrix} e^{-iq_-x} \end{aligned} \right\}. \quad (\text{A9})$$

Here, we have defined the following functions

$$f_1(q, \mathbf{k}) = \frac{(ik_y - q)m\Delta}{q}, \quad (\text{A10})$$

$$f_2(q, \mathbf{k}) = \frac{m(\tilde{\varepsilon}_{q, \mathbf{k}} - E_{q, \mathbf{k}}^b)}{q}, \quad (\text{A11})$$

$$g_2(q, \mathbf{k}) = \frac{m(\tilde{\varepsilon}_{q, \mathbf{k}} + E_{q, \mathbf{k}}^b)}{q}, \quad (\text{A12})$$

$$q_-(q, \mathbf{k}) = \sqrt{2k_F^2 - 2k_y^2 - 2k_z^2 - q^2 - 4m^2\Delta^2}, \quad (\text{A13})$$

and the normalization constants are

$$\frac{1}{N_{q, \mathbf{k}}^b} = \left[\frac{|f_1(q, \mathbf{k})|^2 + f_2(q, \mathbf{k})^2}{+ \frac{q_-}{q} (|f_1(q_-, \mathbf{k})|^2 + f_2(q_-, \mathbf{k})^2)} \right]^{-1}, \quad (\text{A14})$$

$$\frac{1}{M_{q, \mathbf{k}}^b} = \left[\frac{|f_1(q, \mathbf{k})|^2 + g_2(q, \mathbf{k})^2}{+ \frac{q_-}{q} (|f_1(q_-, \mathbf{k})|^2 + g_2(q_-, \mathbf{k})^2)} \right]^{-1}. \quad (\text{A15})$$

$\psi^{b<}(q, \mathbf{k}; x)$ and $\phi^{b<}(q, \mathbf{k}; x)$ satisfy the following properties

$$\int_0^\infty dx \psi^{b<\dagger}(q', \mathbf{k}; x) \psi^{b<}(q, \mathbf{k}; x) = 2\pi\delta(q - q'), \quad (\text{A16})$$

$$\int_0^\infty dx \phi^{b<\dagger}(q', \mathbf{k}; x) \phi^{b<}(q, \mathbf{k}; x) = 2\pi\delta(q - q'), \quad (\text{A17})$$

and have nonzero overlap

$$\int_0^\infty dx \psi^{b<\dagger}(q', \mathbf{k}; x) \phi^{b<}(q, \mathbf{k}; x) = i\mathfrak{a}(q) 2\pi\delta(q - q'), \quad (\text{A18})$$

where

$$\mathfrak{a}(q, \mathbf{k}) = \frac{\text{Im } f_1(q, \mathbf{k})}{\sqrt{N_{q, \mathbf{k}}^b M_{q, \mathbf{k}}^b}} [g_2(q, \mathbf{k}) - f_2(q, \mathbf{k})] \left[1 + \frac{q}{q_-} \right]. \quad (\text{A19})$$

The states $\Psi_\lambda^{b(1)\dagger}$ and $\Psi_\lambda^{b(2)\dagger}$ are constructed in such a way that they are orthonormal according to

$$\int_0^\infty dx \Psi_\lambda^{b(i)\dagger}(q', \mathbf{k}; x) \Psi_\lambda^{b(j)}(q, \mathbf{k}; x) = 2\pi\delta(q - q')\delta_{ij}. \quad (\text{A20})$$

For $q > q_0$, there is no degeneracy. The bulk state in this case is

$$\psi_{+1}^{b>}(q, \mathbf{k}; x) = \frac{1}{\sqrt{N_{q, \mathbf{k}}^{b>}}} \left\{ \begin{aligned} & \mathfrak{c}_{+, q, \mathbf{k}} \begin{bmatrix} u_{q, \mathbf{k}} \\ v_{q, \mathbf{k}} e^{i\phi_{q, \mathbf{k}}} \end{bmatrix} e^{iqx} \\ & + \mathfrak{c}_{-, q, \mathbf{k}} \begin{bmatrix} u_{q, \mathbf{k}} \\ -v_{q, \mathbf{k}} e^{-i\phi_{q, \mathbf{k}}} \end{bmatrix} e^{-iqx} \\ & - \begin{bmatrix} \Theta_{q, \mathbf{k}} \\ \Phi_{q, \mathbf{k}} \end{bmatrix} e^{-\lambda_{q, \mathbf{k}} x} \end{aligned} \right\}, \quad (\text{A21})$$

where

$$u_{q, \mathbf{k}} = \sqrt{\frac{1}{2} \left(1 + \frac{\tilde{\varepsilon}_{q, \mathbf{k}}}{E_{q, \mathbf{k}}^b} \right)}, \quad (\text{A22})$$

$$v_{q, \mathbf{k}} = \sqrt{\frac{1}{2} \left(1 - \frac{\tilde{\varepsilon}_{q, \mathbf{k}}}{E_{q, \mathbf{k}}^b} \right)}, \quad (\text{A23})$$

$$\lambda_{q, \mathbf{k}} = \sqrt{q^2 + 2k_y^2 + 2k_z^2 - 2k_F^2 + 4m^2\Delta^2}, \quad (\text{A24})$$

$$\phi_{q, \mathbf{k}} = \arg(q + ik_y), \quad (\text{A25})$$

$$\mathfrak{c}_{\pm, q, \mathbf{k}} = \frac{m}{2} \left[\frac{\frac{\Delta(\lambda_{q, \mathbf{k}} - k_y) e^{\mp i\phi_{q, \mathbf{k}}}}{u_{q, \mathbf{k}}}}{\mp i \frac{\tilde{\varepsilon}_{q, \mathbf{k}} + E_{q, \mathbf{k}}^b + 2m\Delta^2}{v_{q, \mathbf{k}}}} \right], \quad (\text{A26})$$

$$\Theta_{q, \mathbf{k}} = (\mathfrak{c}_{+, q, \mathbf{k}} + \mathfrak{c}_{-, q, \mathbf{k}}) u_{q, \mathbf{k}}, \quad (\text{A27})$$

$$\Phi_{q, \mathbf{k}} = (-\mathfrak{c}_{-, q, \mathbf{k}} e^{-i\phi_{q, \mathbf{k}}} + \mathfrak{c}_{+, q, \mathbf{k}} e^{i\phi_{q, \mathbf{k}}}) v_{q, \mathbf{k}}, \quad (\text{A28})$$

and

$$\frac{1}{N_{q,\mathbf{k}}^{\text{b>}}} = \frac{2}{|\mathbf{c}_{+,q,\mathbf{k}}|^2 + |\mathbf{c}_{-,q,\mathbf{k}}|^2} \quad (\text{A29})$$

is the normalization constant.

In all the cases, the negative energy solution is related to the positive one via particle-hole symmetry

$$\psi_{-1}^{\text{b}}(q, \mathbf{k}; x) = \hat{\sigma}^1 [\psi_{+1}^{\text{b}}(q, -\mathbf{k}; x)]^*. \quad (\text{A30})$$

$$(ii) \quad k_F^2 - k_y^2 - k_z^2 - 2m^2\Delta^2 \leq 0$$

In this regime, there is no degeneracy for all $q > 0$. The bulk states are simply given by Eq. (A21).

Appendix B: Meissner effect and temperature dependence of the penetration depth

In this Appendix, we corroborate the discussion of the Meissner effect in WSCs in Sec. IA 1 by considering the spinless WSC model defined by Eq. (3.1a). We compute the temperature T -dependence of the correction to the London penetration depth due to the presence of surface states. Since the T -dependence of the penetration depth originates from the paramagnetic current-current correlation function, it is sufficient to study the T -dependence of the latter [45].

1. Temperature dependence due to the bulk response

The yy component of the bulk paramagnetic current-current correlation function is given by

$$\begin{aligned} \Pi_{1,\text{bb}}^{yy}(i\Omega_n = 0, \mathbf{q} = 0) \\ = -\frac{1}{2} \left(\frac{e}{m}\right)^2 T \sum_{\omega_n} \int_{\mathbf{k}} \text{Tr} \left[k_y \hat{\mathcal{G}}_{\text{b}}(i\omega_n, \mathbf{k}) k_y \hat{\mathcal{G}}_{\text{b}}(i\omega_n, \mathbf{k}) \right] \\ = \frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{2} \int_{\mathbf{k}} k_y^2 \text{sech}^2 \left(\frac{\beta E_{\mathbf{k}}^{\text{b}}}{2} \right), \end{aligned} \quad (\text{B1})$$

where $\int_{\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3}$, $\beta = 1/T$ and $E_{\mathbf{k}}^{\text{b}} = \sqrt{\tilde{\varepsilon}_{\mathbf{k}}^2 + \Delta^2(k_x^2 + k_y^2)}$. At low temperature, the dominate contribution comes from the Weyl nodes. We can thus linearize $E_{\mathbf{k}}^{\text{b}}$ [cf. Eq. (3.3)] and rescale the coordinates to remove the velocity anisotropy such that

$$\begin{aligned} \Pi_{1,\text{bb}}^{yy}(i\Omega_n = 0, \mathbf{q} = 0) \\ = \frac{e^2}{2m^2} \frac{\beta}{2} \frac{1}{\Delta^4(k_F/m)} \int_{\delta\tilde{\mathbf{k}}} \delta\tilde{k}_y^2 \text{sech}^2 \left(\frac{\beta\delta\tilde{k}}{2} \right) \times 2, \end{aligned} \quad (\text{B2})$$

where the factor of 2 accounts for the two Weyl nodes and $\delta\tilde{\mathbf{k}}$ is rescaled by the velocity. Performing the integral and analytical continuation, we obtain the yy component

retarded current-current correlation function

$$\Pi_{\text{bb},1,R}^{yy}(0, \mathbf{0}) = -\frac{e^2}{2m} \frac{7\pi^2}{45} \left(\frac{1}{\Delta^4 k_F \beta^4} \right) \sim T^4. \quad (\text{B3})$$

Similarly, we can show that the zz component is

$$\Pi_{1,\text{bb},R}^{zz}(0, \mathbf{0}) = -\frac{e^2}{2m} \left(\frac{4k_F}{\pi^2 \beta^2 \Delta^2} \right) \sim T^2. \quad (\text{B4})$$

2. Temperature dependence due to the surface response

We now turn to the paramagnetic current-current correlation function due to the surface states. The yy component is given by

$$\begin{aligned} \Pi_{1,\text{ss}}^{yy}(i\Omega_n = 0, \mathbf{q} = 0; x, x') \\ = -\frac{T}{2} \left(\frac{e}{m}\right)^2 \\ \times \sum_{\omega_n} \int_{\mathbf{k}} \text{Tr} \left[k_y \hat{\mathcal{G}}_{\text{s}}(i\omega_n, \mathbf{k}; x, x') k_y \hat{\mathcal{G}}_{\text{s}}(i\omega_n, \mathbf{k}; x', x) \right] \\ \simeq \frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{4} \int_{\mathbf{k}} k_y^2 \text{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) \Sigma^{\text{ss}}(\mathbf{k}; x, x'), \end{aligned} \quad (\text{B5})$$

where $\mathbf{k} = (k_y, k_z)$ and

$$\Sigma^{\text{ss}}(\mathbf{k}; x, x') = |\psi^{\text{s}}(\mathbf{k}; x)|^2 |\psi^{\text{s}}(\mathbf{k}; x')|^2. \quad (\text{B6})$$

The surface state $\psi^{\text{s}}(\mathbf{k}; x)$ is given by Eq. (3.4). In the $T \rightarrow 0$ limit, we can approximate the retarded correlation function as

$$\begin{aligned} \Pi_{1,\text{ss},R}^{yy}(0, \mathbf{0}; x, x') \\ \simeq -\frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{4} \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} \int_{-\sqrt{k_F^2 - k_z^2}}^{\sqrt{k_F^2 - k_z^2}} \frac{dk_y}{2\pi} k_y^2 \\ \times \text{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) \Sigma^{\text{ss}}(k_y = 0, k_z; x, x') \\ \simeq -\left(\frac{e}{m}\right)^2 \frac{\beta}{2(2\pi)^2} \int_0^\infty dk_y k_y^2 \text{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) \\ \times \int_0^{k_F} dk_z \Sigma^{\text{ss}}(k_y = 0, k_z; x, x') \\ = -\left(\frac{e}{m}\right)^2 \frac{1}{12\beta^2 \Delta^3} \int_0^{k_F} dk_z \Sigma^{\text{ss}}(k_y = 0, k_z; x, x') \\ \sim T^2. \end{aligned} \quad (\text{B7})$$

The zz component can be evaluated in a similar manner, but one has to be more careful. We write

$$\Pi_{1,\text{ss},R}^{zz}(0, \mathbf{0}; x, x') \simeq S_1 + S_2, \quad (\text{B8})$$

where

$$S_1 \equiv -\frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{4} \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} \int_{-\sqrt{k_F^2 - k_z^2}}^{\sqrt{k_F^2 - k_z^2}} \frac{dk_y}{2\pi} \times k_z^2 \operatorname{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) \Sigma^{\text{ss}}(k_y = 0, k_z; x, x') \quad (\text{B9})$$

and

$$S_2 \equiv -\frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{4} \int_{-k_F}^{k_F} \frac{dk_z}{2\pi} \int_{-\sqrt{k_F^2 - k_z^2}}^{\sqrt{k_F^2 - k_z^2}} \frac{dk_y}{2\pi} \times k_z^2 \operatorname{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) \left(\frac{1}{2} \frac{\partial^2 \Sigma^{\text{ss}}}{\partial k_y^2} \Big|_{k_y=0} k_y^2 \right). \quad (\text{B10})$$

First, consider

$$\begin{aligned} S_1 &= -\left(\frac{e}{m}\right)^2 \frac{\beta}{(2)(4)(2\pi)^2} \frac{2(4)}{\beta \Delta} \int_0^{k_F} dk_z k_z^2 \\ &\quad \times \tanh \left(\frac{\beta \Delta}{2} \sqrt{k_F^2 - k_z^2} \right) \Sigma^{\text{ss}}(k_y = 0, k_z; x, x') \\ &\simeq -\left(\frac{e}{m}\right)^2 \frac{1}{4\pi^2 \Delta} \int_0^{k_F} dk_z k_z^2 \left(1 - 2e^{-\beta \Delta \sqrt{k_F^2 - k_z^2}} \right) \\ &\quad \times \Sigma^{\text{ss}}(k_y = 0, k_z; x, x'). \end{aligned} \quad (\text{B11})$$

The first term just gives us a T -independent constant. One can show that the second term gives a strongly sub-leading T^6 dependence. Meanwhile, for $T \rightarrow 0$,

$$\begin{aligned} S_2 &\simeq -\frac{1}{2} \left(\frac{e}{m}\right)^2 \frac{\beta}{2(2\pi)^2} \int_0^{k_F} dk_z k_z^2 \frac{\partial^2 \Sigma^{\text{ss}}}{\partial k_y^2} \Big|_{k_y=0} \\ &\quad \times \int_0^\infty dk_y \operatorname{sech}^2 \left(\frac{\beta \Delta k_y}{2} \right) k_y^2 \\ &= -\frac{1}{24} \left(\frac{e}{m}\right)^2 \frac{1}{\beta^2 \Delta^3} \int_0^{k_F} dk_z k_z^2 \frac{\partial^2 \Sigma^{\text{ss}}}{\partial k_y^2} \Big|_{k_y=0} \\ &\sim T^2. \end{aligned} \quad (\text{B12})$$

Hence, for $T \rightarrow 0$,

$$\Pi_{1,\text{ss},R}^{zz}(0, \mathbf{0}; x, x') = S_1 + S_2 \sim T^2, \quad (\text{B13})$$

meaning that the zz component of the surface current-current correlation function only renormalizes the coefficient of T^2 power law from the bulk [cf. Eq. (B4)].

3. Temperature dependence due to the surface-bulk cross terms

We can also study the effects of the surface-bulk term by computing $\Pi_{1,\text{sb},R}^{\mu\nu}(0, \mathbf{0}; x, x')$ using the exact bulk and surface states (as we did for the topological anomalous skin effect computed in Sec. III D), but the exact expression is unwieldy so we omit it here. By focusing on the $T \rightarrow 0$ limit, we find that

$$\Pi_{1,\text{sb},R}^{yy}(0, \mathbf{0}; x, x') \sim T^4, \quad (\text{B14})$$

$$\Pi_{1,\text{sb},R}^{zz}(0, \mathbf{0}; x, x') \sim T^2, \quad (\text{B15})$$

i.e. the surface-bulk cross-terms take the same T -dependence as the bulk terms [Eq. (1.1)] and thus do not give new power-laws in T .

Appendix C: The kernel $\hat{J}_{q,\mathbf{k}}$ in the surface-bulk optical conductivity

The integral appeared in Eq. (3.37) of the main text

$$\hat{J}_{\hat{q},\hat{\mathbf{k}}}^{\text{sb}} = k_F^2 l_{\text{coh}} \int_{x,x'>0} e^{-\mathcal{K}x} \Sigma_{+1}^{\text{sb}}(q, \mathbf{k}; x', x) e^{-\mathcal{K}x'} \quad (\text{C1})$$

can be evaluated exactly. Due to the possible degeneracy in the bulk band discussed in Sec. III A, we consider the following two cases:

$$(i) \quad k_F^2 - k_y^2 - k_z^2 - 2m^2 \Delta^2 > 0$$

$$\hat{J}_{\hat{q},\hat{\mathbf{k}}}^{\text{sb}} = 2 \left(1 - \hat{\mathbf{k}}^2 \right) \theta \left(1 - \hat{\mathbf{k}}^2 \right) \times \begin{cases} 0, & 0 < \hat{q} < \hat{q}_{\min}, \\ \sum_{i=1}^2 \left| \sum_{j=1}^4 \mathbf{j}_{<}^{(i,j)}(\hat{q}, \hat{\mathbf{k}}) \right|^2, & \hat{q}_{\min} < \hat{q} \leq \hat{q}_0, \\ \left| \sum_{j=1}^3 \mathbf{j}_{>}^{(j)}(\hat{q}, \hat{\mathbf{k}}) \right|^2, & \hat{q}_0 < \hat{q}, \end{cases} \quad (\text{C2})$$

where

$$\hat{k}^\mu = k^\mu/k_F, \quad \hat{q} = q/k_F, \quad (\text{C3})$$

$$j_{<}^{(1,1)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(\hat{q}, \hat{\mathbf{k}})}{\sqrt{\mathcal{N}_{q,\mathbf{k}}^{\text{b}(1)}}} \left\{ \frac{1}{\sqrt{N_{q,\mathbf{k}}^{\text{b}}}} [if_1^*(q, \mathbf{k}) + f_2(q, \mathbf{k})] + \frac{i}{\sqrt{M_{q,\mathbf{k}}^{\text{b}}}} [-ig_2(q, \mathbf{k}) + f_1^*(q, \mathbf{k})] \right\}, \quad (\text{C4})$$

$$j_{<}^{(1,3)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(\hat{q}, \hat{\mathbf{k}})}{\sqrt{\mathcal{N}_{q,\mathbf{k}}^{\text{b}(1)}}} \left\{ \frac{1}{\sqrt{N_{q,\mathbf{k}}^{\text{b}}}} [-if_1^*(q_-, \mathbf{k}) - f_2(q_-, \mathbf{k})] + \frac{i}{\sqrt{M_{q,\mathbf{k}}^{\text{b}}}} [ig_2(q_-, \mathbf{k}) - f_1(q_-, \mathbf{k})] \right\}, \quad (\text{C5})$$

$$j_{<}^{(2,1)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(\hat{q}, \hat{\mathbf{k}})}{\sqrt{\mathcal{N}_{q,\mathbf{k}}^{\text{b}(2)}}} \left\{ \frac{1}{\sqrt{N_{q,\mathbf{k}}^{\text{b}}}} [if_1^*(q, \mathbf{k}) + f_2(q, \mathbf{k})] - \frac{i}{\sqrt{M_{q,\mathbf{k}}^{\text{b}}}} [-ig_2(q, \mathbf{k}) + f_1^*(q, \mathbf{k})] \right\}, \quad (\text{C6})$$

$$j_{<}^{(2,3)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(\hat{q}, \hat{\mathbf{k}})}{\sqrt{\mathcal{N}_{q,\mathbf{k}}^{\text{b}(2)}}} \left\{ \frac{1}{\sqrt{N_{q,\mathbf{k}}^{\text{b}}}} [-if_1^*(q_-, \mathbf{k}) - f_2(q_-, \mathbf{k})] - \frac{i}{\sqrt{M_{q,\mathbf{k}}^{\text{b}}}} [ig_2(q_-, \mathbf{k}) - f_1(q_-, \mathbf{k})] \right\}, \quad (\text{C7})$$

and where

$$j_{<}^{(1,2)}(\hat{q}, \hat{\mathbf{k}}) = -j_{<}^{(1,1)*}(\hat{q}, \hat{\mathbf{k}}), \quad (\text{C8})$$

$$j_{<}^{(1,4)}(\hat{q}, \hat{\mathbf{k}}) = -j_{<}^{(1,3)*}(\hat{q}, \hat{\mathbf{k}}), \quad (\text{C9})$$

$$j_{<}^{(2,2)}(\hat{q}, \hat{\mathbf{k}}) = -j_{<}^{(2,1)*}(\hat{q}, \hat{\mathbf{k}}), \quad (\text{C10})$$

$$j_{<}^{(2,4)}(\hat{q}, \hat{\mathbf{k}}) = -j_{<}^{(2,3)*}(\hat{q}, \hat{\mathbf{k}}). \quad (\text{C11})$$

We also define

$$R(\hat{q}, \hat{\mathbf{k}}) = \frac{1}{\left[i\hat{q} + (k_F l_{\text{coh}})^{-1} + \hat{\mathcal{K}} \right]^2 - \hat{\kappa}_{\hat{\mathbf{k}}}^2} \quad (\text{C12})$$

where

$$\hat{\kappa}_{\hat{\mathbf{k}}} = \kappa_{\mathbf{k}}/k_F, \quad \hat{\mathcal{K}} = \mathcal{K}/k_F, \quad (\text{C13})$$

$$(ii) \quad k_F^2 - k_y^2 - k_z^2 - 2m^2 \Delta^2 \leq 0$$

$$\hat{J}_{\hat{q}, \hat{\mathbf{k}}}^{\text{sb}} = 2 \left(1 - \hat{\mathbf{k}}^2 \right) \theta \left(1 - \hat{\mathbf{k}}^2 \right) \left| \sum_{j=1}^3 j_{>}^{(j)}(\hat{q}, \hat{\mathbf{k}}) \right|^2, \quad (\text{C14})$$

where

$$j_{>}^{(1)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(-\hat{q}, \hat{\mathbf{k}})}{\sqrt{N_{q,\mathbf{k}}^{\text{b}>}}} \mathbf{c}_{+,q,\mathbf{k}} (u_{q,\mathbf{k}} + iv_{q,\mathbf{k}} e^{i\phi_{q,\mathbf{k}}}), \quad (\text{C15})$$

$$j_{>}^{(2)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(\hat{q}, \hat{\mathbf{k}})}{\sqrt{N_{q,\mathbf{k}}^{\text{b}>}}} \mathbf{c}_{-,q,\mathbf{k}} (u_{q,\mathbf{k}} - iv_{q,\mathbf{k}} e^{-i\phi_{q,\mathbf{k}}}), \quad (\text{C16})$$

$$j_{>}^{(3)}(\hat{q}, \hat{\mathbf{k}}) = \frac{R(-i\hat{\lambda}_{\hat{q}, \hat{\mathbf{k}}}, \hat{\mathbf{k}})}{\sqrt{N_{q,\mathbf{k}}^{\text{b}>}}} (\Theta_{q,\mathbf{k}} + i\Phi_{q,\mathbf{k}}), \quad (\text{C17})$$

where

$$\hat{\lambda}_{\hat{q}, \hat{\mathbf{k}}} = \lambda_{q,\mathbf{k}}/k_F. \quad (\text{C18})$$

Appendix D: Mattis-Bardeen formula in dirty s -wave superconductors

In this Appendix, we compute the s -wave superconductor saddle point $\hat{Q}_{\text{sp}}^{\text{s-wave}}$ and rederive the Mattis-Bardeen optical conductivity in the dirty limit [5], using the Keldysh formalism.

One can repeat the derivation presented in Sec. IV A by introducing BCS singlet pairing interactions [34]. At the static mean-field level, the saddle point equation is still given by Eq. (4.25), but now with $\hat{h}_{\text{BdG}} = \tilde{\epsilon}_{\mathbf{k}} \hat{\sigma}^3 + \Delta_0 \hat{\sigma}^1$, where Δ_0 is the s -wave BCS gap. An important observation is that in this case, the denominator of Eq. (4.25) can

be diagonalized via [32, 92]

$$\hat{U}_\omega^{-1} [\hat{\sigma}^3 \hat{\omega} + i\eta \hat{\tau}^3 \hat{\sigma}^3 + \hat{\sigma}^3 (\Delta_0 \hat{\sigma}^1)] \hat{U}_\omega = \hat{\Xi}_\omega, \quad (\text{D1})$$

with $\hat{\Xi}_\omega$ a momentum-independent diagonal matrix. As a result, the Green's function can be diagonalized by \hat{U}_ω as well,

$$\hat{\mathcal{G}}_\Gamma^{\text{s-wave}} = \frac{1}{\hat{\sigma}^3 \hat{\omega} + i\eta \hat{\tau}^3 \hat{\sigma}^3 - \hat{\sigma}^3 \hat{h}_{\text{BdG}} + i\hat{Q}_{\text{sp}}^{\text{s-wave}}} = \hat{U}_\omega \left[\frac{1}{\hat{\Xi}_\omega - \tilde{\varepsilon}_{\mathbf{k}} + i\hat{U}_\omega^{-1} \hat{Q}_{\text{sp}}^{\text{s-wave}} \hat{U}_\omega} \right] \hat{U}_\omega^{-1}. \quad (\text{D2})$$

Using the knowledge of the normal metal saddle point $\hat{Q}_{\text{sp}}^{\text{metal}} = \frac{1}{2\tau_{\text{el}}} \hat{\tau}^3 \hat{\sigma}^3$, we have

$$\hat{Q}_{\text{sp}}^{\text{s-wave}}(\omega) = \frac{1}{2\tau_{\text{el}}} \hat{U}_\omega \hat{\tau}^3 \hat{\sigma}^3 \hat{U}_\omega^{-1} = \frac{1}{2\tau_{\text{el}}} \frac{\text{sgn}(\omega) \hat{\tau}^3}{\sqrt{(\omega + i\eta \hat{\tau}^3)^2 - \Delta_0^2}} \begin{bmatrix} \omega & -\Delta_0 \\ \Delta_0 & -\omega \end{bmatrix}_\sigma. \quad (\text{D3})$$

In the $\Delta_0 = 0$ limit, we recover the normal metal saddle point $\hat{Q}_{\text{sp}}^{\text{metal}}$. Note that $\hat{Q}_{\text{sp}}^{\text{s-wave}}$ can alternatively be obtained by solving the Usadel equation [33].

The linear response optical conductivity is again given by Eq. (4.33). At the saddle point level, we again just expand the action up to second of $\hat{\mathcal{A}}$ and evaluate the expression in Eq. (4.38), but with $\hat{\mathcal{G}}_\Gamma \rightarrow \hat{\mathcal{G}}_\Gamma^{\text{s-wave}}$ given by Eq. (D2). By converting $\int_{\mathbf{p}} \rightarrow \nu_0 \int d\tilde{\varepsilon} \int \frac{d\Omega_{\mathbf{p}}}{4\pi}$ and using the integral

$$\int_{\mathbf{p}} \frac{v_F^\mu(\mathbf{p}) v_F^\nu(\mathbf{p})}{[\Xi_{\omega+\omega'} \sigma^3 + i\Upsilon_{\text{el}} \tau^3 \sigma^3 - \tilde{\varepsilon}_{\mathbf{p}}] [\Xi_{\omega'} \sigma^{3'} + i\Upsilon_{\text{el}} \tau^{3'} \sigma^{3'} - \tilde{\varepsilon}_{\mathbf{p}}]} \simeq 2\pi\nu_0 D \left(1 - \delta_{\tau^3 \sigma^3, \tau^{3'} \sigma^{3'}} \right), \quad (\text{D4})$$

valid in the diffusive regime, we simplify Eq. (4.38) in the s -wave case, and obtain the optical conductivity in the dirty limit

$$\sigma_{\text{bb},R}^{s,\mu\nu}(\Omega) = -\frac{e^2}{i\Omega} \frac{1}{2i} \left\{ \frac{\pi\nu_0 D}{4} \int_{\omega'} \text{Tr} \left[\hat{U}_{\Omega+\omega'} \hat{\tau}^3 \hat{\sigma}^3 \hat{U}_{\Omega+\omega'}^{-1} \hat{\gamma}_{\text{cl}}(\Omega + \omega', \omega') \hat{U}_{\omega'} \hat{\tau}^3 \hat{\sigma}^3 \hat{U}_{\omega'}^{-1} \hat{\gamma}_{\text{q}}(\omega', \Omega + \omega') \right] \right\} \delta^{\mu\nu}, \quad (\text{D5})$$

where $\int_{\omega'} = \int \frac{d\omega'}{2\pi}$. By recognizing that $\frac{1}{2\tau_{\text{el}}} \hat{U}_\omega \hat{\tau}^3 \hat{\sigma}^3 \hat{U}_\omega^{-1}$ is nothing but the s -wave saddle point $\hat{Q}_{\text{sp}}^{\text{s-wave}}$, we arrive the Mattis-Bardeen result

$$\sigma_{\text{bb},R}^{s,\mu\nu}(\Omega) = \frac{\pi\sigma_{\text{dc}}}{\Omega} \left\{ \int_{\omega'} \left\{ \frac{[\Delta_0^2 + (\Omega + \omega')\omega'] \text{sgn}(\omega') \text{sgn}(\Omega + \omega')}{\sqrt{(\Omega + \omega')^2 - \Delta_0^2} \sqrt{\omega'^2 - \Delta_0^2}} \right\} [F(\Omega + \omega') - F(\omega')] \right\} \delta^{\mu\nu}, \quad (\text{D6})$$

where it is understood that the integral over ω' excludes regions with $\omega'^2 - \Delta_0^2 < 0$ and $(\Omega + \omega')^2 - \Delta_0^2 < 0$. Here, $\sigma_{\text{dc}} = e^2(2\nu_0)D$ is the Drude conductivity in the normal state. In the $T = 0$ limit, the above integral can be performed analytically, resulting in

$$\sigma_{\text{bb},R}^{s,\mu\nu}(\Omega) = \sigma_{\text{dc}} \left[\left(1 + \frac{2\Delta_0}{\Omega} \right) E \left(\frac{\Omega - 2\Delta_0}{\Omega + 2\Delta_0} \right) - \frac{4\Delta_0}{\Omega} K \left(\frac{\Omega - 2\Delta_0}{\Omega + 2\Delta_0} \right) \right] \theta(\Omega - 2\Delta_0) \delta^{\mu\nu}, \quad (\text{D7})$$

where E and K are complete elliptical integrals. We note that a similar derivation based on the \hat{Q} -matrix non-linear σ model in the Matsubara formalism also appeared in a recent study [93].

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