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Unwinding Fermionic SPT Phases: Supersymmetry Extension

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Abstract

We show how 1+1-dimensional fermionic symmetry-protected topological states (SPTs, i.e. nontrivial short-range entangled gapped phases of quantum matter whose boundary exhibits 't Hooft anomaly and whose bulk cannot be deformed into a trivial tensor product state under finite-depth local unitary transformations only in the presence of global symmetries), indeed can be *unwound* to a trivial state by enlarging the Hilbert space via adding extra degrees of freedom and suitably extending the global symmetries. The extended projective global symmetry on the boundary can become supersymmetric in a specific sense, i.e., it contains group elements that do *not* commute with the fermion number parity $(-1)^F$, while the anti-unitary time-reversal symmetry becomes fractionalized. This also means we can uplift and remove certain exotic fermionic anomalies (e.g., “parity” anomaly in time-reversal or reflection symmetry) via appropriate *supersymmetry extensions* in terms of *group extensions*. We work out explicit examples for multi-layers of 1+1d Majorana fermion chains, then comment on models with Sachdev-Ye-Kitaev (SYK) interactions, intrinsic fermionic *gapless* SPTs protected by supersymmetry, and generalizations to higher spacetime dimensions via a cobordism theory.

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1 Overview and summary of results

1.1 Classification of gapped phases of matter

A central goal of condensed matter physics is to enumerate and understand the properties of various phases of matter. The systems of interest are generally many-body quantum systems in the infinite system size limit described by local interactions and typically invariant under some group of global symmetries. Long-range ordering is one common way a system can transit to a non-trivial phase. The paramagnet-ferromagnet transition of magnets and the superfluid transition via Bose-Einstein condensation are prominent examples of such an ordering and are well described by the Ginzburg-Landau framework based on *spontaneous symmetry breaking* (SSB) and characterized by local order parameters. The discovery of quantum Hall phases, which were beyond this framework heralded a new era in condensed matter and was followed by intense theoretical and experimental research attempting to get a sense of the various possible phases of matter (classification) and their distinguishing properties (characterization) which continues to this day. This program has been particularly successful in the case of *gapped phases* of matter i.e. when the systems of interest have a non-zero spectral gap separating the lowest energy states from the excited states.

Gapped phases can be divided into two categories- *non-invertible* and *invertible* phases. Loosely speaking, a physical system \mathcal{P} belongs to an invertible phase if there exists an inverse system, \mathcal{P}^{inv} such that the composite system produced by *stacking* \mathcal{P} and \mathcal{P}^{inv} , belongs to the *trivial* phase i.e. can be connected, without encountering a phase transition, to a trivial system \mathcal{P}_0 which has always has a unique trivial ground state i.e. product state for bosons or Slater determinant state for fermions. The collection of invertible phases forms an *Abelian group*, with group multiplication arising from stacking and the identity element being the trivial phase. Examples of invertible phases include paramagnets, integer quantum Hall states, insulators and superconductors. On the other hand, any phase that is not invertible is called a non-invertible phase. Examples of non-invertible phases include all gapped phases with long range order arising through SSB like ferromagnets as well as fractional quantum Hall phases and spin liquids. In this work, we will be concerned with a *specific* class of invertible phases in the presence of global symmetries.

Given a physical system with some global symmetries G belonging to a non-trivial invertible phase, if we explicitly break global symmetries, one of two things can happen- the phase can become trivial (eg: topological insulators), or the phase can continue to remain non-trivial (eg: integer quantum Hall phases). The latter phases form a *subgroup* of the group of invertible phases and are called Symmetry-Protected-Topological states or Symmetry-Protected-Trivial states or simply SPT states (SPTs) [1, 2] and will be the focus of this work.

Let us state the above in the language of groups. Let \mathcal{I}_G be the Abelian group of invertible phases. Explicitly breaking symmetries maps each element of \mathcal{I}_G to \mathcal{I}_0 , the group of invertible phases with *no symmetries*. It is easy to see that breaking symmetry induces a *surjective group*

homomorphism, π

$$\mathcal{I}_G \xrightarrow{\pi} \mathcal{I}_0. \quad (1.1)$$

SPT phases are those elements of \mathcal{I}_G that land on the trivial element of \mathcal{I}_0 under the symmetry breaking map i.e. π . In other words, SPT phases (which we will call \mathcal{S}_G) correspond to the *kernel* of π .

$$\mathcal{S}_G = \ker \pi \quad (1.2)$$

These simple arguments reveals a rich structure in the space of gapped physical systems- for any dimension, for either bosons or fermions with global symmetry G , \mathcal{S}_G , \mathcal{I}_G , and \mathcal{I}_0 can be arranged in the following *short exact sequence*¹

$$1 \rightarrow \mathcal{S}_G \xrightarrow{i} \mathcal{I}_G \xrightarrow{\pi} \mathcal{I}_0 \rightarrow 1 \quad (1.3)$$

Let us briefly review some important properties of SPT phases:

1. If symmetries are disregarded, all SPT phases become trivial. This means that we can adiabatically connect the ground state to a tensor product state using a finite time-evolution operator generated by a local Hamiltonian which can be represented as a finite depth unitary circuit (FDUC). Following the terminology of our previous work [3] based on the method of symmetry extension [4], we call this *unwinding*.² An FDUC is a locality preserving unitary operator that be written as a product of a finite number of layers of *ultra-local unitaries* schematically shown in Fig 1.
2. Boundaries of SPT phases are always non-trivial i.e. are gapless or have some form of long-range order if gapped. This *persistent ordering* [7] is due to the presence of an 't Hooft *anomaly* for the global symmetry G that forbids a trivial phase.

Although not immediately obvious, the above two properties are intimately connected — the different gapped boundary terminations of SPT phases can tell us how to construct FDUC to unwind an SPT phase. Let us focus on symmetry breaking boundary terminations of SPT phases. If a G -SPT phase can be terminated by spontaneously breaking G down to one of its subgroups, G_{sub} on the boundary, this means that the same SPT phase can be unwound by a FDUC that is invariant under G_{sub} . Note that breaking all symmetries is one specific case of the above paradigm corresponding to G_{sub} being the trivial group.

¹An exact sequence is a sequence of homomorphisms $\{\phi_n\}$ between groups $\{A_n\}$ of the form

$$\dots A_n \xrightarrow{\phi_n} A_{n+1} \xrightarrow{\phi_{n+1}} \dots$$

such that $\text{im} \phi_n = \ker \phi_{n+1}$. A short exact sequence is an exact sequence of the form

$$1 \rightarrow C \xrightarrow{i} B \xrightarrow{\pi} A \rightarrow 1$$

where i is injective and π is surjective as a consequence of exactness.

²A more mathematical way to interpret our approach on unwinding of SPTs [3, 4] can be that pulling back trivialization of fiber bundle via fibrations [5]. These have physical consequences of (1) removing 't Hooft anomalies or (2) constructing symmetry-extended topological boundaries, via symmetry extensions [3, 4]. The 't Hooft anomalies are the obstruction of gauging global symmetries; here we concern the global symmetries including time-reversal or reflection symmetry, in addition to spacetime or internal symmetries. These anomalies are generalized “parity” anomalies — more accurately involving time-reversal \mathbb{Z}_2^T or reflection symmetry \mathbb{Z}_2 instead of the spatial parity. In fact, Ref. [5, 6] provide further mathematical insights of the problems dealt in [3, 4].

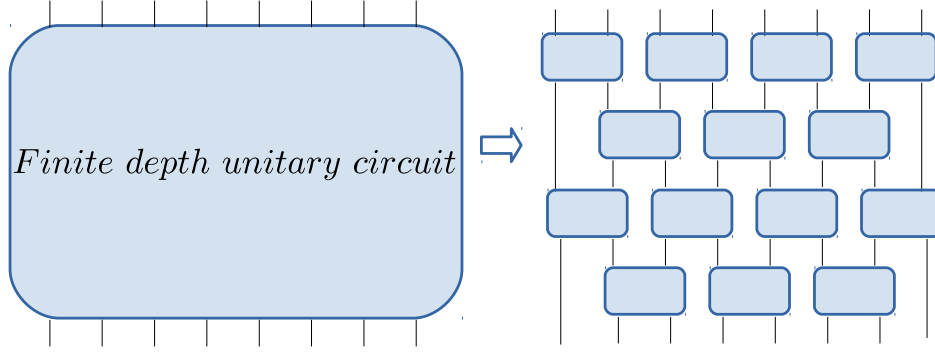


Figure 1: Finite depth unitary circuit (FDUC). The operator can be expressed as a product of finite number of layers (vertically separated in the figure, indicating the time direction). Each layer is a product of unitary operators with strictly local support (square blobs in the figure) i.e. acts on the Hilbert space of a finite number of spins.

A second boundary termination, particularly insightful for 3+1d bulk systems,³ is when no global symmetry is broken but the boundary is topologically ordered. The anomalous nature of the boundary symmetry is seen in the way it acts on the anyons of the theory which is impossible in a purely 2+1d system [8, 68] (see the review [2] and references within for an overview). For a specific class of bosonic SPT phases (those classified by group cohomology), the authors of [4] were able to place the understanding of symmetric gapped boundary terminations in a systematic framework. They prove that given a system belonging to a non-trivial $d + 1$ -dimensional bosonic G -SPT phase with criteria:

1. bosonic SPTs classified by group cohomology.
2. G is a finite group, including both unitary and/or anti-unitary symmetries.
3. $d \geq 1$ for d is the spatial dimension. (Namely the spacetime dimension $d + 1 \geq 2$.)

there always exists a larger group, \tilde{G} such that, starting with a $(d - 1) + 1$ dimensional \tilde{G} -symmetric invariant theory and gauging a particular subgroup, K satisfying $\tilde{G}/K \cong G$, we are left with a K gauge theory with G symmetry realized anomalously and can form a boundary termination for the G -SPT. This in turn also teaches us something else very interesting- *the G -SPT phase can be unwound, not by breaking symmetry but by extending to \tilde{G} !* In other words, the 't Hooft anomaly of G -symmetry of a boundary theory (in a $d - 1$ dimensional space and 1-dimensional time) can be trivialized by extending the system to \tilde{G} -symmetry. In Ref. [3], this was demonstrated explicitly in the case of 1+1d for various groups G by constructing FDUCs invariant under an appropriately identified \tilde{G} .

This naturally brings about the following questions — *can there be a systematic framework to understand symmetric gapped boundary terminations for SPT phases not classified by group cohomology?* and *can these be obtained by symmetry extension?* In particular, this includes all *intrinsically fermionic* SPT phases i.e. which cannot be constructed by stacking a non-trivial bosonic SPT phase with a trivial fermionic phase. In this work, for 1+1d fermionic SPT phases, we provide evidence that the answer to both the above italicized questions is in the affirmative.

³Our convention on the dimensions is that the one, two, three spatial dimensions are written as 1d, 2d, 3d, etc. We denote their spacetime dimensions as 1+1d, 2+1d, 3+1d, etc.

1.2 Summary of results and outline

We focus only on on-site, unitary and anti-unitary symmetries (including various time-reversal symmetries) in this article. We find that just like bosonic SPT phases, 1+1d fermionic SPT phases can be unwound by an FDUC invariant under an extended symmetry. For intrinsically fermionic SPT phases, we find that the extended symmetry has an interesting property — the fermion parity is not at the center of the symmetry group, i.e. there exists elements of the symmetry group that *do not commute with fermion parity* locally. We term this as supersymmetry extension. This is our main result which we demonstrate using several examples. Our results are demonstrated using exactly solvable lattice models and constructing explicit FDUCs.

This article is organized as follows: we warm up by considering a bosonic example in Sec. 2 which sets the language and the general ideas that are detailed in Ref. [3]. In Sec. 3, we consider non-trivial free-fermion models of SPT phases in 1+1d corresponding to classes BDI, AIII and DIII and demonstrate how they can be unwound to a trivial phase by symmetry extension. In Sec. 4, we study an interacting example of a fermionic SPT phase with time-reversal symmetry and demonstrate its unwinding. In Sec. 5, we justify our usage of the term *supersymmetry extension* and show that the extended symmetry has an intimate correspondence to the supersymmetric quantum mechanics. In Sec. 6, we make comments about generalization in the continuum, suitable for quantum field theory (QFT) formulation. In Sec. 7, we explore generalizations to higher dimensions and relations to a cobordism theory (from Sec. 7.1 to Sec. 7.7) and the implications to intrinsically fermionic *gapless* SPTs (Sec. 7.8), and finally conclude with possible future directions in (Sec. 7.9). Additional details are relegated to the Appendices.

2 Warm up: a bosonic example — the Haldane phase

In this section, we warm-up by considering a bosonic example to set the language as well as list the results that we generalize to fermionic systems in this work. Additional examples as well as general results for bosonic SPTs can be found in [3].

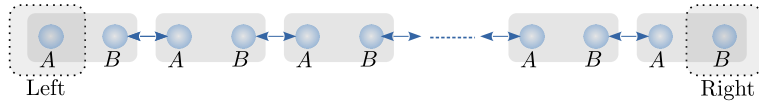


Figure 2: Model Hamiltonian belonging to the Haldane phase with open boundary conditions. The $SU(2)$ doublet written as the $SU(2)$ representation $J = \frac{1}{2}$ qubits at both ends are indicated.

2.1 The Hamiltonian, symmetry fractionalization, and classification

We consider a 1+1d bosonic spin chain belonging to the Haldane phase protected by an on-site $SO(3)$ symmetry. The Hilbert space on each site is a pair of qubits each of which transforms as the $J = \frac{1}{2}$ representation of the $SU(2)$ group. Together, each site forms a $\frac{1}{2} \otimes \frac{1}{2} \cong 1 \oplus 0$ reducible representation that is faithful to $SO(3)$. The Hamiltonian with ℓ sites and $SO(3)$ symmetry

operators are

$$H = - \sum_{j=1}^{\ell} \vec{\sigma}_{B,j} \cdot \vec{\sigma}_{A,j+1}, \quad (2.1)$$

$$U(\hat{n}, \theta) = \prod_{j=1}^{\ell} \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{A,j} \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{B,j}. \quad (2.2)$$

For eq (2.2), two SU(2) symmetry operators (two of $J = \frac{1}{2}$) act on the same site j makes the SO(3) symmetry operator (for $J = 0 \oplus 1$). The model by Affleck, Kennedy, Lieb and Tasaki (AKLT) is obtained by projecting out the $J = 0$ sector on each site and only retaining $J = 1$ [10]. We will choose not to perform this projection and work with the exactly solvable model of eq. (2.2). With periodic boundary conditions, eq. (2.2) has the following unique ground state which has SO(3) singlets on each bond connecting the sites.

$$|\text{GS}\rangle = \prod_{j=1}^{\ell} \frac{|\uparrow_{B,j} \downarrow_{A,(j|\ell)+1}\rangle - |\downarrow_{B,j} \uparrow_{A,(j|\ell)+1}\rangle}{\sqrt{2}}, \quad (2.3)$$

where, $(j|\ell) \equiv j \bmod \ell$. With any symmetry preserving open boundary conditions, the ground state degeneracy is four (GSD = 4), i.e. any ground state can be written as a vector in a four-dimensional Hilbert subspace of ground states:

$$|\text{GS}\rangle = \left(\prod_{j=1}^{\ell} \frac{|\uparrow_{B,j} \downarrow_{A,j+1}\rangle - |\downarrow_{B,j} \uparrow_{A,j+1}\rangle}{\sqrt{2}} \right) \otimes |\psi\rangle, \quad (2.4)$$

$$|\psi\rangle = \sum_{\alpha=\uparrow,\downarrow} \sum_{\beta=\uparrow,\downarrow} \psi_{\alpha,\beta} |\alpha_{A,1}\rangle |\beta_{B,\ell}\rangle, \quad \sum_{\alpha=\uparrow,\downarrow} \sum_{\beta=\uparrow,\downarrow} |\psi_{\alpha,\beta}|^2 = 1. \quad (2.5)$$

We can define a Hermitian projection operator $\mathbf{P}_{\text{GS}} = \mathbf{P}_{\text{GS}}^{\dagger}$ onto the ground space that has support on the ends of the chain

$$\mathbf{P}_{\text{GS}} = \mathbf{P}_{\text{left}} \otimes |s\rangle\langle s| \otimes \mathbf{P}_{\text{right}} \quad (2.6)$$

$$\mathbf{P}_{\text{left}} = \sum_{\alpha=\uparrow,\downarrow} |\alpha_{A,1}\rangle\langle\alpha_{A,1}|, \quad \mathbf{P}_{\text{right}} = \sum_{\alpha=\uparrow,\downarrow} |\alpha_{B,\ell}\rangle\langle\alpha_{B,\ell}|, \quad (2.7)$$

$$|s\rangle = \left(\prod_{j=1}^{\ell} \frac{|\uparrow_{B,j} \downarrow_{A,j+1}\rangle - |\downarrow_{B,j} \uparrow_{A,j+1}\rangle}{\sqrt{2}} \right). \quad (2.8)$$

\mathbf{P}_{GS} has support on the boundaries of the chain. This allows us to study the *fractionalization* of the SO(3) symmetry on the boundaries at low energies.

$$\begin{aligned} \mathbf{P}_{\text{GS}} U(\hat{n}, \theta) \mathbf{P}_{\text{GS}} &= \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{A,1} \otimes \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{B,\ell} \\ &= V(\hat{n}, \theta)_{\text{left}} \otimes V(\hat{n}, \theta)_{\text{right}}. \end{aligned} \quad (2.9)$$

The matrix symmetry representation of each of the boundary modes corresponds to $J = \frac{1}{2}$ which is a fraction of the quantum numbers allowed by the bulk symmetry group SO(3). Furthermore, the representation is faithful to the group SU(2) which is a non-trivial extension of SO(3) by \mathbb{Z}_2 . This is an example demonstration of the emergence of extended symmetries on the boundaries of SPT phases [4].

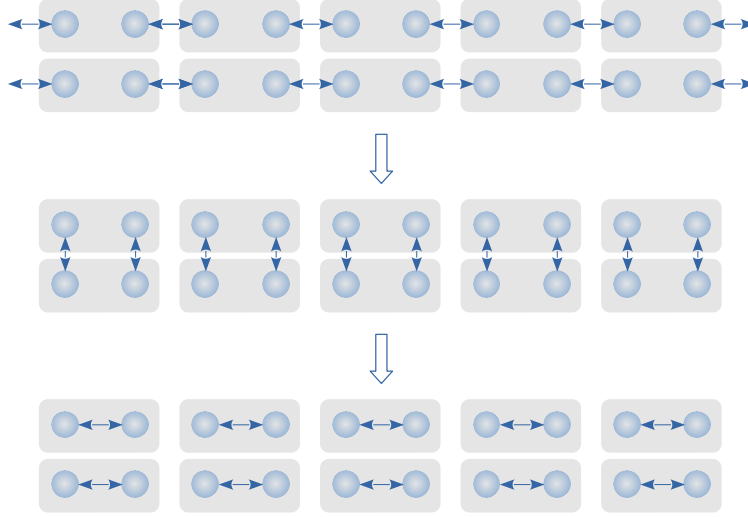


Figure 3: Unwinding two-layers of eq. (2.10) via the two-layer FDUC $W = W_2W_1$ in eq (2.14).

The non-trivial nature of the Haldane phase comes from the fact that there does not exist an $\text{SO}(3)$ invariant FDUC that can map the ground state to a product state. If we consider two layers of the Hamiltonian eq. (2.2) however,

$$H^\Pi = - \sum_{j=1}^{\ell} \vec{\sigma}_{B,j} \cdot \vec{\sigma}_{A,j+1} - \sum_{j=1}^{\ell} \vec{\tau}_{B,j} \cdot \vec{\tau}_{A,j+1}, \quad (2.10)$$

$$U(\hat{n}, \theta) = \prod_{j=1}^{\ell} \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{j,A} \exp \left(i \hat{n} \cdot \vec{\sigma} \frac{\theta}{2} \right)_{j,B} \exp \left(i \hat{n} \cdot \vec{\tau} \frac{\theta}{2} \right)_{j,A} \exp \left(i \hat{n} \cdot \vec{\tau} \frac{\theta}{2} \right)_{j,B}. \quad (2.11)$$

it is easy to see that it can be *unwound* to one whose ground state is a product state as shown in Fig. 3 using a two-layer FDUC $W = W_2W_1$:

$$WH^\Pi W^\dagger = - \sum_{j=1}^{\ell} \vec{\sigma}_{A,j} \cdot \vec{\sigma}_{B,j} - \sum_{j=1}^{\ell} \vec{\tau}_{A,j} \cdot \vec{\tau}_{B,j}, \quad (2.12)$$

$$W = W_2W_1, \quad (2.13)$$

$$W_1 = \prod_{j=1}^{\ell} \frac{1}{2} (\mathbb{1} + \vec{\sigma}_{B,j} \cdot \vec{\tau}_{A,j+1}), \quad W_2 = \prod_{j=1}^{\ell} \frac{1}{2} (\mathbb{1} + \vec{\sigma}_{A,j} \cdot \vec{\tau}_{B,j}). \quad (2.14)$$

The bosonic SWAP operator $\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{\tau})$ swaps the basis states of the σ and τ qubits. What we have reproduced is the well-known \mathbb{Z}_2 classification of the Haldane phase [1].

2.2 Unwinding by symmetry extension

We now demonstrate that by elevating the extended boundary symmetry ($\text{SU}(2)$) to the bulk, we can unwind the single layer Hamiltonian eq. (2.2). To do this, we first add an extra $J = \frac{1}{2}$ qubit on each site (which we label “C”). We also add a trivial Hamiltonian that dimerizes the C qubits

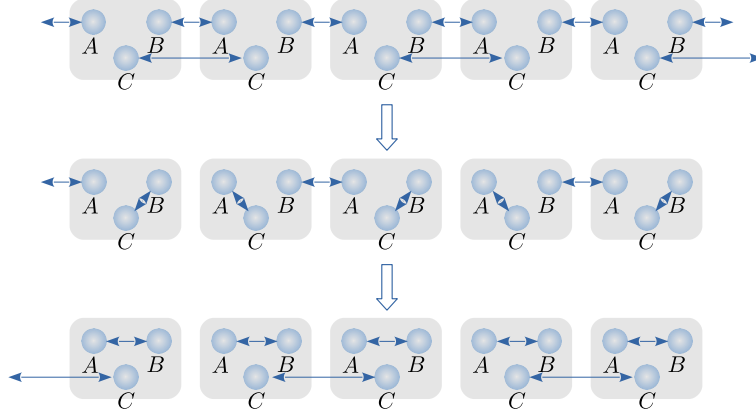


Figure 4: Unwinding the extended Hamiltonian eq. (2.15) state by extending the bulk on-site symmetry to $SU(2)$, via $W = W_2 W_1$ in two steps: W_1 and W_2 in eq (2.18).

in the ground state ⁴.

$$H = - \sum_{j=1}^{\ell} \vec{\sigma}_{B,j} \cdot \vec{\sigma}_{A,j+1} - \sum_{\text{odd } j} \vec{\sigma}_{C,j} \cdot \vec{\sigma}_{C,j+1}. \quad (2.15)$$

This Hamiltonian can be unwound as shown in fig. 4 using a two-layer FDUC

$$WHW^\dagger = - \sum_{j=1}^{\ell} \vec{\sigma}_{A,j} \cdot \vec{\sigma}_{B,j} - \sum_{\text{even } j} \vec{\sigma}_{C,j} \cdot \vec{\sigma}_{C,j+1}, \quad (2.16)$$

$$W = W_2 W_1 \quad (2.17)$$

$$W_1 = \prod_{\text{odd } j} \frac{1}{2} (\mathbb{1} + \vec{\sigma}_{C,j} \cdot \vec{\sigma}_{A,j+1}), \quad W_2 = \prod_{\text{odd } j} \frac{1}{2} (\mathbb{1} + \vec{\sigma}_{A,j} \cdot \vec{\sigma}_{C,j}). \quad (2.18)$$

2.3 Boundary Hamiltonians and protected degeneracy for the Haldane chain: Generalized Sachdev-Ye interactions

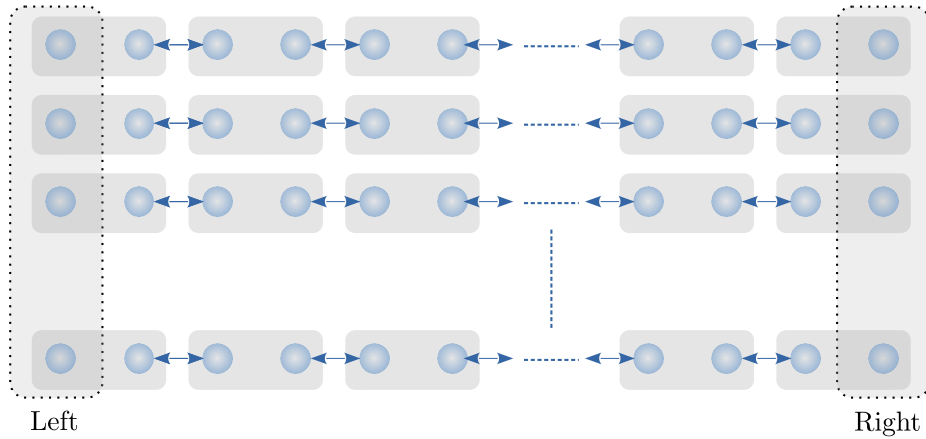


Figure 5: Multiple copies of the Hamiltonian eq. (2.2)

⁴ An aside: When the on-site representation is half-odd-integer as in eq. (2.15), the Hamiltonian has to explicitly break the lattice translation symmetry if we want to have a unique ground state. This is a consequence of the Lieb-Schultz-Mattis theorem [11, 12]

The classification of the Haldane phase tells us that even copies of Hamiltonian eq. (2.2) belong to the trivial phase and odd number of copies belong to the Haldane phase. This result can also be obtained by attempting to lift the boundary degeneracy by introducing interactions. Let us consider N copies of the Hamiltonian eq. (2.2). The boundary Hilbert space consists of N of bosonic $J = \frac{1}{2}$ qubits which we shall label $m = 1, \dots, N$. The most general symmetry preserving Hamiltonian we can write down is as follows

$$H_{\text{bdry}} = \sum_{a,b=1}^N J_{ab} \vec{\sigma}_a \cdot \vec{\sigma}_b + \sum_{a,b,c,d=1}^N K_{abcd} (\vec{\sigma}_a \cdot \vec{\sigma}_b) (\vec{\sigma}_c \cdot \vec{\sigma}_d) + \dots \quad (2.19)$$

The leading term in the above Hamiltonian with J_{ab} drawn from a random distribution is the model by Sachdev and Ye [13]. We are concerned with the question of when there is a possibility that eq. (2.19) can have a unique ground state. This can be answered from some basic representation theoretic facts of $\text{SU}(2)$. The total Hilbert space of N spins forms a reducible representation of $\text{SU}(2)$ corresponding to the Clebsch-Gordan (CG) decomposition

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \dots \otimes \frac{1}{2} \quad (\text{N terms}). \quad (2.20)$$

The only way to obtain a symmetric unique ground state is if the Hilbert space contains a $J = 0$ irreducible representation (irrep) in the above CG decomposition. If N is even, this is possible. For example, consider $N = 2$:

$$\frac{1}{2} \otimes \frac{1}{2} \cong 1 \oplus 0. \quad (2.21)$$

If N is odd however, we only obtain half-odd-integer values of J in the CG decomposition. For example, for $N = 3$,

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \cong \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}. \quad (2.22)$$

This recovers the \mathbb{Z}_2 classification.

2.4 Formal explanation

We now provide a formal way of understanding the above results following for the mathematically inclined reader [4]. In Sec. 2.1, the boundary spin $J = \frac{1}{2}$ is a projective representation of $\text{SO}(3)$, which is in fact the faithful representation of $\text{SU}(2)$. This can also be interpreted as the group extension as a short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{SU}(2)|_{\text{boundary}} \xrightarrow{r} \text{SO}(3)|_{\text{bulk}} \longrightarrow 1, \quad (2.23)$$

as the boundary-bulk symmetry extension, with the following implications:

1. *Topological invariant:* The 1+1d Haldane chain is characterized by a 1+1d topological invariant given by a nontrivial group element of the cohomology group $H^2(\text{BSO}(3), \text{U}(1)) = \mathbb{Z}_2$ [1] also the cobordism group $\Omega_2^{\text{SO}}(\text{BSO}(3)) = \mathbb{Z}_2$ (see Sec. 5.5.2 of [14].) Precisely the 1+1d topological invariant contains the second Stiefel-Whitney (SW) class w_2 of the associated vector bundle V of $\text{SO}(3)$, as

$$\exp(i\pi \int_{M^2} w_2(V_{\text{SO}(3)})) \quad (2.24)$$

(which is known as the group cocycle in a group cohomology, or the cobordism invariant in a cobordism theory). The eq (2.24) is a partition function of the low energy physics of Haldane chain on a 1+1d spacetime manifold M^2 .

For example, we can construct a nontrivial manifold generator: take the base manifold $M^2 = S^2$ as a 2-sphere, the map $S^2 \rightarrow \text{BSO}(3)$ determines an $\text{SO}(3)$ bundle over S^2 . Recall the homotopy group $\pi_2(\text{BSO}(3)) = \pi_1(\text{SO}(3)) = \mathbb{Z}_2$. The homotopy class of the map $S^2 \rightarrow \text{BSO}(3) \rightarrow \text{B}^2\mathbb{Z}_2$ is given by a nontrivial generator exactly $w_2(V_{\text{SO}(3)})$ that can be pull back to the base manifold S^2 .

2. *Trivialization of topological invariant:* This group extension eq (2.25) actually solves a problem about the trivialization of topological invariant via the symmetry extension — once we pull back the $\text{SO}(3)$ bundle to the $\text{SU}(2)$ bundle via the pull back r^* , then the 1+1d topological invariant of Haldane chain $r^* \exp(i\pi \int w_2(V_{\text{SO}(3)})) \sim 1$ becomes trivialized because $r^* w_2(V_{\text{SO}(3)}) = w_2(V_{\text{SU}(2)}) = 0$ (the trivialization means that it is a group coboundary as a trivial element in the cohomology group (in $H^2(\text{BSU}(2), \mathbb{U}(1)) = 0$), also a trivial identity invariant in the cobordism theory (in $\Omega_2^{\text{SO}}(\text{BSU}(2)) = 0$)).

This implies that we can in fact add an extra $\text{SU}(2)$ doublet $J = \frac{1}{2}$ on each of the boundary of Haldane chain, then we can add $\text{SU}(2)$ -invariant interaction terms to the boundary Hamiltonian to gap the degenerate $\text{GSD} = 4$ states, left with only a single unique ground state ($\text{GSD} = 1$) preserving the extended $\text{SU}(2)$ symmetry (but not preserving the $\text{SO}(3)$ symmetry), see [3].

3. *Trivialization of 't Hooft anomaly:* The 1+1d bulk topological invariant from $H^2(\text{BSO}(3), \mathbb{U}(1)) = \Omega_2^{\text{SO}}(\text{BSO}(3)) = \mathbb{Z}_2$ classifies its 0+1d boundary 't Hooft anomaly of the $\text{SO}(3)$ global symmetry. The $w_2(V_{\text{SO}(3)})$ regarded as the $\text{SO}(3)$ background field, once turned on, can detect the 't Hooft anomaly.

It is shown that the eq (2.25), with the suitable condition mentioned above, says that the 0+1d 't Hooft anomaly of the $\text{SO}(3)$ global symmetry becomes anomaly-free once we pull back the theory to the $\text{SU}(2)$ symmetry in 0+1d [4].

4. *Bulk trivialization:* If we consider the trivialization of topological invariant not only at the boundary but also at the full bulk via the unwinding method in Sec. 2.2. Then we also write the group extension as the bulk-bulk symmetry extension:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{SU}(2)|_{\text{bulk}} \xrightarrow{r} \text{SO}(3)|_{\text{bulk}} \longrightarrow 1. \quad (2.25)$$

The above bosonic SPTs serve as a guiding principle, for the readers to follow. In the following sections, we will explore the generalization to fermionic SPTs counterpart. We provide many examples in Sec. 3 and Sec. 4, then we provide a formal explanation of the fermionic examples in Sec. 6.

3 Unwinding SPT phases of fermions

In this section, we present the key results of the paper. Fermionic invertible phases as well as SPT phases were originally discovered in *non-interacting* systems i.e. Hamiltonians are quadratic in fermion creation and annihilation operators. Prominent examples are the integer quantum Hall states, topological insulators and superconductors with various symmetries. In a monumental series of works [15, 16], all free-fermion invertible phases have been classified in all dimensions. The

Symmetry Class	\mathcal{I}_G	\mathcal{S}_G	Internal Symmetry	Spacetime-Internal Symmetry G	$\text{TP}_2(G)$	$\frac{\text{TP}_2(G)}{\text{TP}_2(\text{Spin})}$
D	\mathbb{Z}_2	0	\mathbb{Z}_2^F	Spin	\mathbb{Z}_2	0
BDI	\mathbb{Z}	$2\mathbb{Z}$	$\mathbb{Z}_2^T \times \mathbb{Z}_2^F$	Pin^-	\mathbb{Z}_8	\mathbb{Z}_4
DIII	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_4^{TF}	Pin^+	\mathbb{Z}_2	\mathbb{Z}_2
AIII	\mathbb{Z}	\mathbb{Z}	$\text{U}(1)^F \times \mathbb{Z}_2^T$	Pin^c	\mathbb{Z}_4	\mathbb{Z}_4
CII	\mathbb{Z}	\mathbb{Z}	$\text{SU}(2)^F \times \mathbb{Z}_2^T$	$\text{Pin}^- \times_{\mathbb{Z}_2^F} \text{SU}(2)$	\mathbb{Z}_2	\mathbb{Z}_2

Table 1: For 1+1d fermionic systems, on the left-hand side (l.h.s) table, we show the free fermion classification of invertible phases, \mathcal{I}_G and the subgroup of SPT phases, \mathcal{S}_G of a given symmetry class. On the right-hand side (r.h.s) table, we show the corresponding symmetry group G suitable for the interacting systems (interacting analogous of symmetry class), their interacting fermion classification of invertible topological phases (TP) denoted as a version of cobordism group $\text{TP}_2(G)$ in Freed-Hopkins's [24], and interacting fermion classification of SPT phases denoted as $\frac{\text{TP}_2(G)}{\text{TP}_2(\text{Spin})}$.

The $\frac{\text{TP}_2(G)}{\text{TP}_2(\text{Spin})}$ means that the invertible topological phases (TP) of symmetry group G mod out the invertible fermionic topological order $\text{TP}_2(\text{Spin})$ that is long-range entangled (LRE). Thus the $\frac{\text{TP}_2(G)}{\text{TP}_2(\text{Spin})}$ is the short-range entangled (SRE) fermionic SPT phases. Note that $\text{U}(1)^F \times \mathbb{Z}_2^T$ is equivalent to $\frac{\text{U}(1)^F \times \mathbb{Z}_4^T}{\mathbb{Z}_2^F}$ via a redefinition of the time reversal symmetry (see Sec. 3.4). See also Table 3 for a systematic web of these 1+1d fermionic SPTs. The cobordism group data is explained in more details in Sec. 3.6.

program of extending the classification to interacting fermions in any dimension is under progress, although so far the best known mathematical framework is known to be the generalization of group supercohomology [17–22], and the cobordism group classification [23–26] and [14]

In the coming sections, we will demonstrate unwinding 1+1 d fermionic SPT phases by extended symmetries similar to section 2.2 and ref. [3] with examples of models from fermion SPT phases with a *free-limit* i.e. have representatives which are quadratic Hamiltonians and whose symmetries can be placed in one of the symmetry classes (see ref. [27] for an accessible review on symmetry classes). This is only for convenience. We focus on the following five symmetry classes — D, BDI, DIII, AII, and CII and pick a particular representative symmetry for each class.⁵ In Table 1, we list the group of invertible phases and the subgroup of SPT phases. Following Sec. 1.1, we use the convention:

- (1). For the free fermion symmetry classes and their group classification of phases, we denote the group of invertible phases as \mathcal{I}_G and the group of SPT phases, which forms a subgroup of \mathcal{I}_G , as \mathcal{S}_G .
- (2). For the interacting fermion systems and their group classification of phases, we can obtain the group classification from a cobordism group $\text{TP}_d(G)$ defined in [24] for invertible topological phases with a G -global symmetry. Here G is the full spacetime-internal global symmetry in the continuum limit, see examples of G listed in Table 1. The low energy physics of invertible topological phases are described by the invertible topological field theory (iTQFT) [24] which is also known as SPT invariants [28]. Based on Wen's definition [29], part of the invertible topological phases are long-ranged entangled invertible topological orders, while another part are short-ranged

⁵The A and C symmetry classes have no SPT phases in 1+1d. The AI and CI have also SPT phases but they are only bosonic phases (in fact, Haldane spin-1 chain) up to stacking gapped fermionic product states. For bosonic SPTs, we can unwind by the approach of [3, 4]. See the later Table 3 for the systematic web of SPTs related by a symmetry group embedding.

entangled invertible SPTs. For example, any layer of Kitaev chains is an invertible topological phase: a single Kitaev chain is a *long-ranged entangled* invertible topological order, while two or more layers of Kitaev chains are *short-ranged entangled* invertible SPTs.

We can generate a representative of each 1+1d free-fermion invertible phase for every class by considering an appropriate number of layers of Kitaev's Majorana chains [30] and identifying an appropriate symmetry belonging to the symmetry class in consideration. SPT phases on the other hand turn out to be those invertible phases which can be represented by an even number of Majorana chains.⁶ We relegate a mathematical and formal explanation on the classifications of invertible phases vs SPT phases via the cobordism group data in the end of section in Sec. 3.6.

3.1 Toy model Hamiltonian of Majorana fermionic chains

Let us start by writing down the Hamiltonian for the Kitaev's Majorana chain [30]. The Hilbert space is of one complex fermion per unit site on a 1d lattice which can be represented using two Majorana operators, $\gamma, \bar{\gamma}$ satisfying $\gamma_i^\dagger = \gamma_i, \bar{\gamma}_i^\dagger = \bar{\gamma}_i, \{\gamma_i, \gamma_j\} = \{\bar{\gamma}_i, \bar{\gamma}_j\} = 2\delta_{ij}$, and $\{\gamma_i, \bar{\gamma}_j\} = 0$. Majorana operators, which are Hermitian, can be defined in terms of creation and annihilation operators of the complex fermions, ψ_i and ψ_i^\dagger as⁷

$$\gamma_i = (\psi_i^\dagger + \psi_i), \quad \bar{\gamma}_i = i(\psi_i^\dagger - \psi_i). \quad (3.1)$$

Kitaev's Majorana chain [30] has the Hamiltonian:

$$H_{\text{Kitaev}} = -i \sum_j \gamma_j \bar{\gamma}_{j+1}. \quad (3.2)$$



Figure 6: Graphical representation of Kitaev's Majorana chain of eq (3.2).

Let us consider two layers of Majorana chains with their flavor indices $\varsigma = \uparrow, \downarrow$:

$$H_{\text{Kitaev}}^{\text{II}} = -i \sum_{\varsigma=\uparrow,\downarrow} \sum_j \gamma_{\varsigma,j} \bar{\gamma}_{\varsigma,j+1}. \quad (3.3)$$

⁶By Wen's definition [4], a single layer of Kitaev chain is protected by no symmetry except by the fermion parity, this means that in fact

an odd number of Kitaev chains is a 1+1d *long-range entangled state* as a 1+1d *fermionic invertible topological order*.

Based on the Wen's definition using the local unitary quantum circuit, Kitaev chain is long-range entangled (LRE) but not short-range entangled (SRE) because it cannot be deformed to a trivial product state by breaking all symmetries (but maintaining the fermion parity \mathbb{Z}_2^F as a fermionic system). In contrast,

an even number of Kitaev chains is a 1+1d *short-range entangled state* as a 1+1d *fermionic invertible SPT state*.

because it can be deformed to a trivial product state by breaking all symmetries (but maintaining the fermion parity \mathbb{Z}_2^F as a fermionic system).

⁷Throughout the article, we denote the imaginary number as i with $i^2 = -1$. The complex fermion operators obey $\{\psi_i, \psi_j^\dagger\} = \delta_{ij}$, $\{\psi_i, \psi_j\} = 0$, and $\{\psi_i^\dagger, \psi_j^\dagger\} = 0$. We have:

$$\psi_i = \frac{1}{2}(\gamma_i + i\bar{\gamma}_i), \quad \psi_i^\dagger = \frac{1}{2}(\gamma_i - i\bar{\gamma}_i).$$

For convenience, we now perform a change-of-basis on the Hamiltonian of eq (3.3) using the unitary

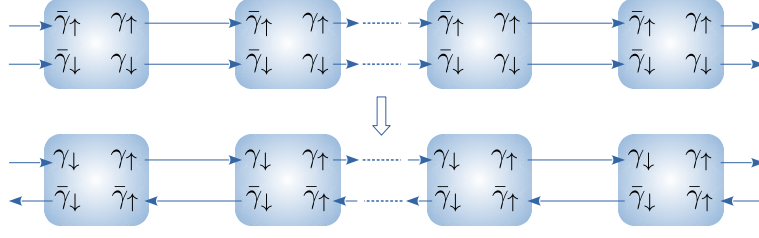


Figure 7: Fermionic SWAP operator and two layers of Kitaev's Majorana chains: Two layers of Majorana chains before and after change of basis by the fermionic SWAP operator $\text{SWAP}_f = M = \prod_j \frac{1}{\sqrt{2}} (1 + \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j})$ in eq (3.4). The M is the fermionic analog of the SWAP operator of bosonic spin systems (i.e., the bosonic SWAP operator in Fig. 3 and in Ref. [3]'s Fig. 7). The Hamiltonian before the basis changing is eq (3.3), after the basis changing is $H_{\text{gen}} = M H_{\text{Kitaev}}^{\text{II}} M^\dagger = -i \sum_j (\gamma_{\uparrow,j} \gamma_{\downarrow,j+1} - \bar{\gamma}_{\uparrow,j} \bar{\gamma}_{\downarrow,j+1})$ in eq (3.5).

operator M ,

$$M = \prod_j \exp\left(\frac{\pi}{4} \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}\right) = \prod_j \frac{1}{\sqrt{2}} (1 + \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}). \quad (3.4)$$

The new Hamiltonian H_{gen} under the change of basis (see Fig. 7) becomes:⁸

$$H_{\text{gen}} = M H_{\text{Kitaev}}^{\text{II}} M^\dagger = -i \sum_j (\gamma_{\uparrow,j} \gamma_{\downarrow,j+1} - \bar{\gamma}_{\uparrow,j} \bar{\gamma}_{\downarrow,j+1}). \quad (3.5)$$

The M in eq (3.4) and in Fig. 7 is the fermionic analog of the SWAP operator of bosonic spin systems (such as the bosonic SWAP operator in Ref. [3]'s Fig. 7).⁹ The Hamiltonian H_{gen} of eq (3.5) is the proper form of generator of two layers of Kitaev chains, which will be a primary object of focus of this section and throughout our work.

3.2 Class BDI: $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry or Pin^-

We start with class BDI with representative global symmetry $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ where \mathbb{Z}_2^T is generated by the anti-unitary operator \mathcal{T} satisfying $\mathcal{T}^2 = 1$ and \mathbb{Z}_2^F is the fermion parity group generated by $P_F = (-1)^F$ where F is the total fermion number. In the absence of interactions, it is known that there are an infinite number of invertible phases [16] indexed by an integer $k \in \mathbb{Z}$ representing the number of ‘dangling’ Majorana modes on each end i.e. $\mathcal{J}_G \cong \mathbb{Z}$. This reduces to \mathbb{Z}_8 in the presence of interactions [31, 32]. A representative of the generator of these phases is the Majorana chain of

⁸In terms of complex fermion bases eq (3.1), we have $H_{\text{Kitaev}}^{\text{II}} = + \sum_{\varsigma=\uparrow,\downarrow} \sum_j (\psi_{\varsigma,j}^\dagger + \psi_{\varsigma,j}) (\psi_{\varsigma,j+1}^\dagger - \psi_{\varsigma,j+1})$, while $H_{\text{gen}} = M H_{\text{Kitaev}}^{\text{II}} M^\dagger = -i \sum_j ((\psi_{\uparrow,j}^\dagger + \psi_{\uparrow,j}) (\psi_{\downarrow,j+1}^\dagger + \psi_{\downarrow,j+1}) + (\psi_{\uparrow,j}^\dagger - \psi_{\uparrow,j}) (\psi_{\downarrow,j+1}^\dagger - \psi_{\downarrow,j+1}))$.

⁹The bosonic SWAP operator $\text{SWAP}_b \equiv \frac{1}{2} (1 + \vec{\sigma}_A \cdot \vec{\sigma}_B)$ exchanges the basis states $|\uparrow\rangle, |\downarrow\rangle$ of two qubit Hilbert spaces, A and B , and satisfies $(\text{SWAP}_b)^2 = 1$. On the other hand, the fermionic SWAP operator

$$\text{SWAP}_f \equiv \frac{1}{\sqrt{2}} (1 + \gamma_A \gamma_B) = \exp\left(\frac{\pi}{4} \gamma_A \gamma_B\right)$$

acts on two Majorana operators (γ_A, γ_B) to give

$$\text{SWAP}_f \cdot (\gamma_A, \gamma_B) \cdot \text{SWAP}_f^{-1} = (-\gamma_B, \gamma_A).$$

Note that $(\text{SWAP}_f)^2 = \gamma_A \gamma_B$, $(\text{SWAP}_f)^4 = -1$, and $(\text{SWAP}_f)^8 = 1$.

eq (3.2) with symmetry operator

$$\mathcal{T} = \mathcal{K} \quad (3.6)$$

which is a complex conjugation which acts as

$$\mathcal{K}\gamma\mathcal{K} = \gamma, \quad \mathcal{K}\bar{\gamma}\mathcal{K} = -\bar{\gamma}, \quad \mathcal{K}i\mathcal{K} = -i, \quad (3.7)$$

and

$$P_F = (-1)^F = \prod_j i\bar{\gamma}_j\gamma_j. \quad (3.8)$$

The subgroup of SPT phases are those corresponding to even members $k \in 2\mathbb{Z}$ and are non-trivial only in the presence of symmetries $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$. The SPT phases also correspond to also the even members $k = 0, 2, 4, 6 \in \mathbb{Z}_8$ in the interacting SPT classification (See Table 1). A generator of the SPT phases (corresponding to $k = 2$) is two layers of the Majorana chains, eq (3.3) or its equivalent representation in eq (3.5) with the new time-reversal \mathbb{Z}_2^T symmetry operators $\tilde{\mathcal{T}}$ under the eq (3.4) basis-change via a unitary M :

$$\tilde{\mathcal{T}} \equiv M\mathcal{T}M^\dagger = M\mathcal{K}M^\dagger = \left(\prod_j (\gamma_{\downarrow,j}\bar{\gamma}_{\uparrow,j}) \right) \mathcal{K}. \quad (3.9)$$

The form of fermion parity P_F stays unchanged under basis change $MP_F = P_F M$.

3.2.1 Gapping the boundary: Extension to a non-Abelian $\mathbb{D}_8^{F,T}$

Let us see how we can gap the boundary when we have open boundary conditions for Hamiltonian (3.5). Fig. 8 shows that the boundary has four fold ground state degeneracy (GSD = 4) coming from the two dimensional Hilbert space from two Majorana modes (i.e., one complex fermion) on each boundary site, L and R

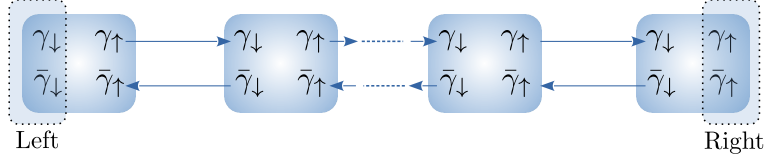


Figure 8: Basis-changed 2-layer Majorana chain with left (L) and right (R) open boundaries, with a symmetry as in eq (3.9), we have four fold ground state degeneracy (GSD = 4) with two degenerate Majorana modes on each end, emphasized in the box frames.

Fig. 9 shows another boundary termination of the Hamiltonian of eq. 3.5 by adding two extra Majorana modes on each end (L and R) leading to a unique ground state. From Fig. 9, we can read-off the effective *extended* boundary symmetry operators on the left (L) and right (R) boundaries (on the site $j = 0$ and $j = \ell + 1$) omitting site indices below:

$$\mathcal{T}_L = \bar{\gamma}_{\uparrow}\mathcal{K}, \quad P_{F,L} = i\bar{\gamma}_{\uparrow}\gamma_{\uparrow}, \quad \mathcal{T}_L^2 = -1, P_{F,L}^2 = 1, \quad \mathcal{T}_L P_{F,L} = -P_{F,L} \mathcal{T}_L. \quad (3.10)$$

$$\mathcal{T}_R = \gamma_{\downarrow}\mathcal{K}, \quad P_{F,R} = i\bar{\gamma}_{\downarrow}\gamma_{\downarrow}, \quad \mathcal{T}_R^2 = P_{F,R}^2 = 1, \quad \mathcal{T}_L P_{F,R} = -P_{F,R} \mathcal{T}_R. \quad (3.11)$$

We can also show:

$$\mathcal{T}_L \gamma_{\uparrow} \mathcal{T}_L^{-1} = -\gamma_{\uparrow}, \quad \mathcal{T}_L \bar{\gamma}_{\uparrow} \mathcal{T}_L^{-1} = -\bar{\gamma}_{\uparrow}. \quad (3.12)$$

$$\mathcal{T}_R \gamma_{\downarrow} \mathcal{T}_R^{-1} = \gamma_{\downarrow}, \quad \mathcal{T}_R \bar{\gamma}_{\downarrow} \mathcal{T}_R^{-1} = \bar{\gamma}_{\downarrow}. \quad (3.13)$$

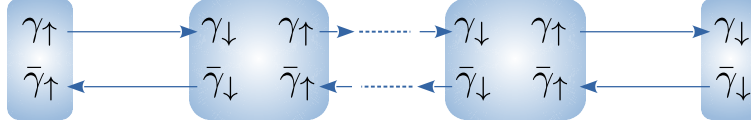


Figure 9: Basis-changed 2-layer Majorana chain from Fig. 8 with two open boundaries, but now each of the boundary couples to additional two Majorana modes on both the left L and right R (the $j = 1$ site couples to a new site $j = 0$, $j = \ell$ site couples to a new site $j = \ell + 1$ respectively). We can modify H_{gen} to $H_{\text{gen}} + H_{\text{gen,bdry}}$ by adding local Hamiltonian terms to two boundaries as $H_{\text{gen,bdry}} = -i(\gamma_{\uparrow,0}\gamma_{\downarrow,1} - \bar{\gamma}_{\uparrow,0}\bar{\gamma}_{\downarrow,1}) - i(\gamma_{\uparrow,\ell}\gamma_{\downarrow,\ell+1} - \bar{\gamma}_{\uparrow,\ell}\bar{\gamma}_{\downarrow,\ell+1})$. The whole system becomes gapped but with only a unique ground state ($\text{GSD} = 1$).

It can be easily checked that the boundary symmetry operators generate a faithful representation of the dihedral group of order 8 known as \mathbb{D}_8 which is a group with two generators satisfying the following relation:

$$\mathbb{D}_8 \equiv \mathbb{Z}_4 \rtimes \mathbb{Z}_2 \equiv \langle a, x | a^4 = x^2 = 1, xax = a^{-1} \rangle. \quad (3.14)$$

On the left (L) end, $a = \mathcal{T}_L$, $x = P_{F,L}$, giving us

$$\mathbb{D}_8^{F,T}|_{\text{boundary}} = \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F \equiv \langle \mathcal{T}_L, P_{F,L} | \mathcal{T}_L^4 = P_{F,L}^2 = 1, P_{F,L}\mathcal{T}_L P_{F,L} = \mathcal{T}_L^{-1} \rangle.$$

On the right (R) end, $a = \mathcal{T}_R P_{F,R}$, $x = P_{F,R}$, giving us

$$\mathbb{D}_8^{F,T}|_{\text{boundary}} = \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F \equiv \langle (\mathcal{T}_R P_{F,R}), P_{F,R} | (\mathcal{T}_R P_{F,R})^4 = (P_{F,R})^2 = 1, P_{F,R}(\mathcal{T}_R P_{F,R}) P_{F,R} = (\mathcal{T}_R P_{F,R})^{-1} \rangle.$$

Importantly, we observe that the fermion parity ($P_{F,L}$ or $P_{F,R}$) is *not* in the center of $\mathbb{D}_8^{F,T} = \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F$ with \mathbb{Z}_4^T generated by a and \mathbb{Z}_2^F generated by x . This is the extended symmetry that will help us to unwind the bulk.¹⁰

In particular, given the bulk onsite symmetry $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ (where time reversal \mathbb{Z}_2^T and fermion parity \mathbb{Z}_2^F commutes), we can extend the symmetry transformation on the boundary to $\mathbb{D}_8^{F,T}$ written as a short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{D}_8^{F,T}|_{\text{boundary}} \longrightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^F|_{\text{bulk}} \longrightarrow 1. \quad (3.15)$$

See footnote 10 for our conventions. The normal subgroup $N = \mathbb{Z}_2 = \{1, -1\}$ is at the center of \mathbb{D}_8 on both ends (left and right), while the quotient group $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ contains four commutative elements: $N\{1, a, x, ax\}$ for $\mathbb{D}_8^{F,T}$.

Some more comments:

¹⁰Some comments about our conventions and notations:

1. Here the $\mathbb{D}_8^{F,T}$ in our conventions is the left (L) or right (R) boundary versions of \mathbb{D}_8 in eq (3.14). The upper indices (F, T) emphasize that fermion parity and time reversal are included, but they do *not* commute in the extended groups.
2. In general, when we write a group $G^{F,T}$, the fermion parity (P_F) and time-reversal (T) do not necessarily commute in G .
3. G^{TF} means that G contains time-reversal (T) which commutes with and generates fermion parity $\mathcal{T}^2 = (-1)^F$.
4. G^{TB} means G contains the time-reversal (T) which satisfies $\mathcal{T}^2 = (-1)$.

1. We can explicitly check that the fully gapped system in Fig. 9 preserves the global symmetry. Not only the bulk Hamiltonian term H_{gen} respects the bulk $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry, but the boundary Hamiltonian $H_{\text{gen,bdry}} = H_{\text{gen,bdry},L} + H_{\text{gen,bdry},R} = -i(\gamma_{\uparrow,0}\gamma_{\downarrow,1} - \bar{\gamma}_{\uparrow,0}\bar{\gamma}_{\downarrow,1}) - i(\gamma_{\uparrow,\ell}\gamma_{\downarrow,\ell+1} - \bar{\gamma}_{\uparrow,\ell}\bar{\gamma}_{\downarrow,\ell+1})$ also respects the boundary $\mathbb{D}_8^{F,T}$ symmetry. More precisely, for the time reversal symmetry on the boundary Hamiltonian:

On the left end L , we have the symmetry operator from \mathcal{T}_L on the site 0 and $\tilde{\mathcal{T}}$ on the site 1, which together act on the left end $H_{\text{gen,bdry},L}$.

On the right end R , we have the symmetry operator from \mathcal{T}_R on the site $\ell + 1$ and $\tilde{\mathcal{T}}$ on the site ℓ , which together act on the right end $H_{\text{gen,bdry},R}$.

The system of Fig. 9 is fully gapped with a unique ground state ($\text{GSD} = 1$), preserving the bulk $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry and the boundary $\mathbb{D}_8^{F,T}$ symmetry.

2. We should emphasize that there are at least three different ways to look at \mathcal{T} symmetries on the boundary depending on (1) the boundary truncation types of quantum systems and (2) the Hilbert space projection \mathbf{P}_{GS} to ground state subspace (Below we omit the boundary site j labels, $j = 1$ for the left L and $j = \ell$ for the right R , throughout):

- (a) On boundaries of Fig. 8, if we look at \mathcal{T}_L and \mathcal{T}_R acting on whole boundary sites of $j = 1$ and $j = \ell$ respectively without \mathbf{P}_{GS} projection, then we have:

$$\mathcal{T}_L = \gamma_{\downarrow}\bar{\gamma}_{\uparrow}\mathcal{K}, \quad \mathcal{T}_R = \gamma_{\downarrow}\bar{\gamma}_{\uparrow}\mathcal{K}.$$

- (b) On boundaries of Fig. 9, if we look at \mathcal{T}_L and \mathcal{T}_R acting on extra boundary sites of $j = 0$ and $j = \ell + 1$ respectively (with or without \mathbf{P}_{GS} projection), then we have:

$$\mathcal{T}_L = \bar{\gamma}_{\uparrow}\mathcal{K}, \quad \mathcal{T}_R = \gamma_{\downarrow}\mathcal{K}.$$

- (c) On boundaries of Fig. 8, if we look at \mathcal{T}_L and \mathcal{T}_R acting on whole boundary sites of $j = 1$ and $j = \ell$ respectively with \mathbf{P}_{GS} projection to dangling zero energy modes, then we have:

$$\mathcal{T}_L = \gamma_{\downarrow}\mathcal{K}, \quad \mathcal{T}_R = \bar{\gamma}_{\uparrow}\mathcal{K}.$$

3. There is a special *supersymmetric* feature for a 2 layer of Kitaev chains which holds when the layer number $N = 2 \pmod 4$ only.

- When the layer number $N = 2 \pmod 4$, we have two Majorana zero modes (with two operators $\gamma, \bar{\gamma}$) with $\text{GSD} = 2$ on each end of system (with the left and right ends combined to give $\text{GSD} = 4$). In fact, for this $\text{GSD} = 2$, on one end, the Hilbert space \mathcal{H} must have one bosonic ground state $|B\rangle$ and one fermionic ground state $|F\rangle$. We denote the $\text{GSD} = 2$ ground state subspace as $\mathcal{H} = \{|B\rangle, |F\rangle\} = \mathcal{H}_B \oplus \mathcal{H}_F$, say $|B\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|F\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Consider the L boundary in Fig. 8, we can write down the operators acting on the two-dimensional Hilbert space \mathcal{H} explicitly:

$$\begin{aligned} \gamma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x, \quad \bar{\gamma} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y, \quad P_F = -i\gamma\bar{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z. \\ \psi &= \frac{1}{2}(\gamma + i\bar{\gamma}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^\dagger = \frac{1}{2}(\gamma - i\bar{\gamma}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{T} = \bar{\gamma}\mathcal{K} = \sigma_y\mathcal{K}. \end{aligned} \quad (3.16)$$

The bosonic $|B\rangle$ and fermionic $|F\rangle$ have the distinct fermion parity P_F (even vs odd).

The $\text{GSD} = 2$ on each end is protected by time-reversal \mathcal{T} symmetry: because to gap any ground state, the only possible Hamiltonian is

$$H' \propto -i\gamma\bar{\gamma} = P_F \quad (3.17)$$

which is disallowed by a global symmetry \mathcal{T} due to the fact H' does not preserve \mathcal{T} . (i.e., \mathcal{T} does not commute with P_F , namely $\mathcal{T}P_F\mathcal{T}^{-1} = -P_F$ or $P_F\mathcal{T}P_F = -\mathcal{T}$.)

We will revisit the underlying supersymmetric quantum mechanics in Sec. 5 in depth.

- When the layer number $N = 4$ (or a nonzero $N = 0 \pmod{4}$), we have four Majorana zero modes (with four operators $\gamma_1, \bar{\gamma}_1, \gamma_2, \bar{\gamma}_2$) with GSD = 4 on one end (with the left and right ends combined to give GSD = 16). However, we can introduce a four Majorana interaction with a \mathcal{T} -symmetric Hamiltonian

$$H' = \alpha \gamma_1 \bar{\gamma}_1 \gamma_2 \bar{\gamma}_2 = -\alpha (P_{F,1})(P_{F,2}) = -\alpha P_F \quad (3.18)$$

where $P_{F,i} = -i\gamma_i \bar{\gamma}_i$ is the fermion parity for the i -th complex fermion ψ_i sector. This H' is \mathcal{T} -symmetric because $\mathcal{T}(P_{F,1}P_{F,2})\mathcal{T}^{-1} = (-P_{F,1})(-P_{F,2}) = (P_{F,1}P_{F,2})$. Overall, we can gap two ground states out of GSD = 4 to leave with only GSD = 2 on one end. By turning on $H' = \alpha \gamma_1 \bar{\gamma}_1 \gamma_2 \bar{\gamma}_2$, depend on the sign of α , we are left with only either *two bosonic states* ($\alpha > 0$, so ground states with $P_F = +1$) or *two fermionic states* ($\alpha < 0$, so ground states with $P_F = -1$).

In summary, the $N = 0 \pmod{4}$ may be left with two ground states but always with the same fermion parity. In contrast, only $N = 2 \pmod{4}$, we have two ground states with the different fermion parity, hinting the supersymmetric quantum mechanics to be revealed in Sec. 5.

3.2.2 Unwinding the bulk: Extension to a non-Abelian $\mathbb{D}_8^{F,T}$

In order to unwind the bulk, we follow the similar steps as in Sec. 2.2, see Fig 10:

1. The first step, we extend the Hilbert space by adding one complex fermion per site (which corresponds to Majorana operators γ and $\bar{\gamma}$) with dynamics given by a dimerizing Hamiltonian:

$$H_o = i \sum_{\text{odd } j} (\gamma_j \gamma_{j+1} - \bar{\gamma}_j \bar{\gamma}_{j+1}). \quad (3.19)$$

The extended Hamiltonian and symmetry operators are (with H_{gen} in eq (3.5))

$$H_{\text{ext}} = H_{\text{gen}} + H_o. \quad (3.20)$$

$$\mathcal{T}_{\text{ext}} = \left(\prod_{\text{odd } j} (\gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \gamma_j \right) \left(\prod_{\text{even } j} (\gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \bar{\gamma}_j \right) \mathcal{K}. \quad (3.21)$$

$$P_F = \prod_j (i \bar{\gamma}_{\downarrow,j} \gamma_{\downarrow,j}) (i \bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j}) (i \bar{\gamma}_j \gamma_j). \quad (3.22)$$

In the extended 1+1d bulk, we have the extended $\mathbb{D}_8^{F,T}$ symmetry on-site in the bulk:

$$\mathbb{D}_8^{F,T}|_{\text{bulk}} \equiv \langle \mathcal{T}_{\text{ext}}, P_F | \mathcal{T}_{\text{ext}}^4 = P_F^2 = 1, P_F \mathcal{T}_{\text{ext}} P_F = \mathcal{T}_{\text{ext}}^{-1} = -\mathcal{T}_{\text{ext}} \rangle. \quad (3.23)$$

2. The second and third steps, we show the Hamiltonian can be unwound as shown in Fig 10 using a 2-layer \mathbb{D}_8 -symmetry invariant FDUC $W = W_2 W_1$ as

$$W H_{\text{ext}} W^\dagger = (W_2 W_1) H_{\text{ext}} (W_2 W_1)^\dagger = -i \sum_j (\gamma_{\downarrow,j} \gamma_{\uparrow,j} - \bar{\gamma}_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) + H_e, \quad (3.24)$$

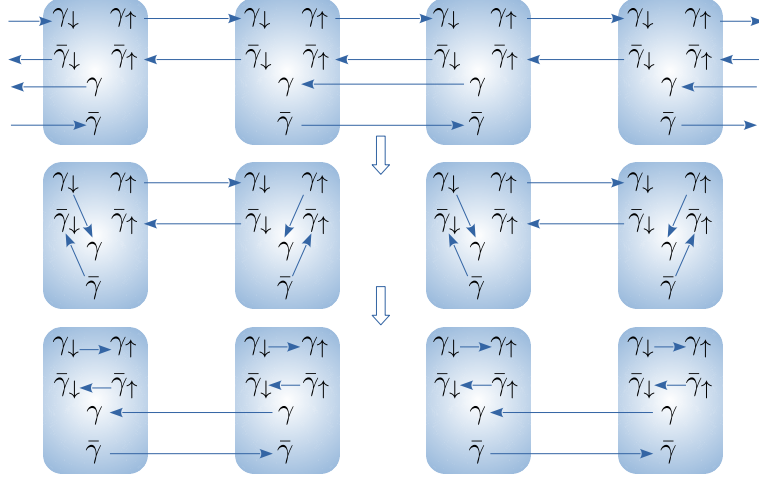


Figure 10: Unwinding the Hamiltonian of the basis-changed 2-layer Majorana chain (eq (3.20) for BDI class) by three steps. We show four sites (as even, odd, even, and odd sites) from the left to the right. Step 1: we add one extra complex fermion per site to extend the Hilbert space. Step 2: we perform a local unitary transformation W_1 on each of two neighbor sites (a neighbor pair of odd and even sites). Step 3: we perform a local unitary transformation W_2 on each of sites, while the odd and even sites have different transformations. The final form of Hamiltonian becomes eq (3.24), while the ground state becomes a tensor product state, which is a trivial tensor product state respect to effective sites (with each effective site combining two neighbor [an even and an odd] sites). See Footnote 4: We have the final ground state breaking the translation symmetry with a double unit cell due to the LSM constraint.

such that $\mathcal{T}_{\text{ext}} W_l \mathcal{T}_{\text{ext}}^{-1} = W_l$ and $P_F W_l P_F^{-1} = W_l$ with $l = 1, 2$, where,

$$W_1 = \prod_{\text{odd } j} \left(\frac{1 + \gamma_j \gamma_{\downarrow, j+1}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_j \bar{\gamma}_{\downarrow, j+1}}{\sqrt{2}} \right). \quad (3.25)$$

$$W_2 = \prod_{\text{odd } j} \left(\frac{1 + \gamma_j \gamma_{\downarrow, j}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_j \bar{\gamma}_{\downarrow, j}}{\sqrt{2}} \right) \prod_{\text{even } j} \left(\frac{1 + \gamma_{\uparrow, j} \gamma_j}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_{\uparrow, j} \bar{\gamma}_j}{\sqrt{2}} \right). \quad (3.26)$$

$$H_e = i \sum_{\text{even } j} (\gamma_j \gamma_{j+1} - \bar{\gamma}_j \bar{\gamma}_{j+1}). \quad (3.27)$$

The deformed Hamiltonian eq (3.24) under FDUC is exactly what we aim for, which shows that under the *enlarged* Hilbert space and *extended* global symmetry, its ground state can be deformed to a trivial tensor product state (See the outcome state in Fig. 10)! In short, the bulk unwinding follows also the symmetry extension:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{D}_8^{F,T} |_{\text{bulk}} \longrightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^F |_{\text{bulk}} \longrightarrow 1. \quad (3.28)$$

3.3 Class DIII: \mathbb{Z}_4^{TF} symmetry or Pin^+

We now consider 1+1d superconductors with a *conventional* \mathbb{Z}_4^{TF} time-reversal invariance, which is known as the Class DIII for the free fermion systems. We work with the same Hilbert space as the previous subsection i.e. two species of fermions per unit site. The 2-layer Majorana chain with a Hamiltonian $H_{\text{Kitaev}}^{\text{II}}$ eq (3.3) is in fact the generator of $\nu = 1 \in \mathbb{Z}_2$ class of the interacting SPTs

with \mathbb{Z}_4^{TF} -symmetry (see Table 1). The \mathcal{T} transformation generating \mathbb{Z}_4^{TF} obey

$$\mathcal{T} = \prod_j \left(\frac{1 + \gamma_{\uparrow,j} \gamma_{\downarrow,j}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_{\uparrow,j} \bar{\gamma}_{\downarrow,j}}{\sqrt{2}} \right) \mathcal{K} = \prod_j e^{\frac{\pi}{4} \gamma_{\uparrow,j} \gamma_{\downarrow,j}} e^{\frac{\pi}{4} \bar{\gamma}_{\uparrow,j} \bar{\gamma}_{\downarrow,j}} \mathcal{K}. \quad (3.29)$$

$$\mathcal{T}^2 = P_F = (-1)^F = \prod_j \prod_{\varsigma=\uparrow,\downarrow} i \bar{\gamma}_{\varsigma,j} \gamma_{\varsigma,j}. \quad \mathcal{T}^4 = 1. \quad (3.30)$$

This means $\mathbb{Z}_2^F \subset \mathbb{Z}_4^{TF}$, the fermion parity generated by P_F is a normal subgroup of time reversal \mathbb{Z}_4^{TF} , thus these *bulk* global symmetries form a short exact sequence:

$$1 \rightarrow \mathbb{Z}_2^F \rightarrow \mathbb{Z}_4^{TF} \rightarrow \mathbb{Z}_2^T \rightarrow 1. \quad (3.31)$$

It can easily be checked that the Hamiltonian of eq (3.3) is invariant under the time-reversal symmetry defined above. Once again, we use the basis change via eq (3.4)'s M to get the Hamiltonian H_{gen} of eq (3.5) and the following basis-changed time-reversal operator:

$$\tilde{\mathcal{T}} \equiv M \mathcal{T} M^\dagger = \prod_j \gamma_{\downarrow,j} \left(\frac{1 - \gamma_{\downarrow,j} \bar{\gamma}_{\downarrow,j}}{\sqrt{2}} \right) \bar{\gamma}_{\uparrow,j} \left(\frac{1 + \gamma_{\uparrow,j} \bar{\gamma}_{\uparrow,j}}{\sqrt{2}} \right) \mathcal{K}. \quad (3.32)$$

The fermion parity $\tilde{P}_F = M P_F M^\dagger = P_F$ stays unchanged.

3.3.1 Gapping the boundary: Extension to a non-Abelian $\mathbb{M}_{16}^{F,T}$

To find the symmetry extension, let us now look at the symmetry transformation on the edges. We have

$$\mathcal{T}_L = \bar{\gamma}_\uparrow \left(\frac{1 + \gamma_\uparrow \bar{\gamma}_\uparrow}{\sqrt{2}} \right) \mathcal{K} = \bar{\gamma}_\uparrow e^{\frac{\pi}{4} \gamma_\uparrow \bar{\gamma}_\uparrow} \mathcal{K}, \quad P_{F,L} = i \bar{\gamma}_\uparrow \gamma_\uparrow. \quad (3.33)$$

$$\mathcal{T}_R = \gamma_\downarrow \left(\frac{1 - \gamma_\downarrow \bar{\gamma}_\downarrow}{\sqrt{2}} \right) \mathcal{K} = \gamma_\downarrow e^{-\frac{\pi}{4} \gamma_\downarrow \bar{\gamma}_\downarrow} \mathcal{K}, \quad P_{F,R} = i \bar{\gamma}_\downarrow \gamma_\downarrow. \quad (3.34)$$

We can also show:

$$\mathcal{T}_L \gamma_\uparrow \mathcal{T}_L^{-1} = -\bar{\gamma}_\uparrow, \quad \mathcal{T}_L \bar{\gamma}_\uparrow \mathcal{T}_L^{-1} = \gamma_\uparrow. \quad (3.35)$$

$$\mathcal{T}_R \gamma_\downarrow \mathcal{T}_R^{-1} = -\bar{\gamma}_\downarrow, \quad \mathcal{T}_R \bar{\gamma}_\downarrow \mathcal{T}_R^{-1} = \gamma_\downarrow. \quad (3.36)$$

The 0+1d edge symmetry operators form a finite non-Abelian group \mathbb{M}_{16} [33] of order 16:¹¹

$$\mathbb{M}_{16}^{F,T}|_{\text{boundary}} \equiv \langle \mathcal{T}_{L/R}, P_{F,L/R} \mid \mathcal{T}_{L/R}^4 = -1, \mathcal{T}_{L/R}^8 = P_{F,L/R}^2 = 1, P_{F,L/R} \mathcal{T}_{L/R} P_{F,L/R} = \mathcal{T}_{L/R}^5 \rangle. \quad (3.37)$$

which again has the fermion parity $P_{F,L/R}$ not at its center $Z(\mathbb{M}_{16}^{F,T}) \cong \mathbb{Z}_4 = \{1, \mathcal{T}_{L/R}^2, \mathcal{T}_{L/R}^4, \mathcal{T}_{L/R}^6\}$. There is a short exact sequence for \mathbb{M}_{16} as a central extension

$$1 \longrightarrow (Z(\mathbb{M}_{16}) \cong \mathbb{Z}_4) \longrightarrow (\mathbb{M}_{16}) \longrightarrow (\mathbb{Z}_2)^2 \longrightarrow 1, \quad (3.38)$$

where the quotient group can be regarded as $(\mathbb{Z}_2)^2 = Z(\mathbb{M}_{16})\{1, \mathcal{T}_{L/R}, P_{F,L/R}, \mathcal{T}_{L/R} P_{F,L/R}\}$. But the short exact sequence that relates the extended symmetry on the boundary $\mathbb{M}_{16}^{F,T}|_{\text{boundary}}$ and the original symmetry group $\mathbb{Z}_4^{TF}|_{\text{bulk}}$, is the following

$$1 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{M}_{16}^{F,T}|_{\text{boundary}} \longrightarrow \mathbb{Z}_4^{TF}|_{\text{bulk}} \longrightarrow 1, \quad (3.39)$$

¹¹Note that $\mathcal{K}^2 = 1$, we can show $\mathcal{T}_L^2 = \gamma_\uparrow \bar{\gamma}_\uparrow = -i P_{F,L}$, $\mathcal{T}_R^2 = \gamma_\downarrow \bar{\gamma}_\downarrow = -i P_{F,R}$, $\mathcal{T}_L^4 = \mathcal{T}_R^4 = -1$, and $\mathcal{T}_L^8 = \mathcal{T}_R^8 = 1$, while $P_{F,L/R} \mathcal{T}_{L/R} P_{F,L/R} = -\mathcal{T}_{L/R} = \mathcal{T}_{L/R}^5$

where the normal subgroup $N = \mathbb{Z}_4 = \{1, \mathcal{T}_{L/R}^2 P_{F,L/R}, \mathcal{T}_{L/R}^4, \mathcal{T}_{L/R}^6 P_{F,L/R}\}$ is not the center $Z(\mathbb{M}_{16}^{F,T})$. Thus eq (3.39) is the non-central extension. The quotient group can be regarded as $\mathbb{Z}_4^{TF} = N \setminus \{1, \mathcal{T}_{L/R}, \mathcal{T}_{L/R}^2, \mathcal{T}_{L/R}^3\}$.

In fact the boundary fractionalized symmetry with $\mathcal{T}^4 = -1$ and $\mathcal{T}^8 = +1$ has been noticed by a remarkable work by Gu in Ref. [34]. We further point out that the full boundary fractionalized symmetry group forms a non-Abelian $\mathbb{M}_{16}^{F,T}$ which contains a subgroup \mathbb{Z}_8^T with the fractionalized time reversal symmetry.

3.3.2 Unwinding the bulk: Extension to a non-Abelian $\mathbb{M}_{16}^{F,T}$

The DIII Hamiltonian can be unwound using the *same* Hilbert space extension and the *same* 2-layer FDUC,

$$W = W_2 W_1$$

as for class BDI (eq (3.24)-eq (3.26)). It is easy to check that W is also invariant under the extended $\mathbb{M}_{16}^{F,T}$ symmetry generator \mathcal{T}_{ext} and the new P_F defined below

$$\begin{aligned} \mathcal{T}_{\text{ext}} &= \prod_{\text{odd } j} \gamma_{\downarrow,j} \left(\frac{1 - \gamma_{\downarrow,j} \bar{\gamma}_{\downarrow,j}}{\sqrt{2}} \right) \bar{\gamma}_{\uparrow,j} \left(\frac{1 + \gamma_{\uparrow,j} \bar{\gamma}_{\uparrow,j}}{\sqrt{2}} \right) \gamma_j \left(\frac{1 - \gamma_j \bar{\gamma}_j}{\sqrt{2}} \right) \\ &\cdot \prod_{\text{even } j} \gamma_{\downarrow,j} \left(\frac{1 - \gamma_{\downarrow,j} \bar{\gamma}_{\downarrow,j}}{\sqrt{2}} \right) \bar{\gamma}_{\uparrow,j} \left(\frac{1 + \gamma_{\uparrow,j} \bar{\gamma}_{\uparrow,j}}{\sqrt{2}} \right) \bar{\gamma}_j \left(\frac{1 + \gamma_j \bar{\gamma}_j}{\sqrt{2}} \right) \mathcal{K}. \end{aligned} \quad (3.40)$$

$$P_F = \prod_j (i \bar{\gamma}_{\downarrow,j} \gamma_{\downarrow,j}) (i \bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j}) (i \bar{\gamma}_j \gamma_j). \quad (3.41)$$

The extended bulk symmetry operators form again a non-Abelian \mathbb{M}_{16} [33] of order 16:¹²

$$\mathbb{M}_{16}^{F,T}|_{\text{bulk}} \equiv \langle \mathcal{T}_{\text{ext}}, P_F \mid \mathcal{T}_{\text{ext}}^4 = -1, \mathcal{T}_{\text{ext}}^8 = P_F^2 = 1, P_F \mathcal{T}_{\text{ext}} P_F = \mathcal{T}_{\text{ext}}^5 \rangle. \quad (3.42)$$

The short exact sequence that relates the extended and original symmetry groups in the bulk again is the following

$$1 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{M}_{16}^{F,T}|_{\text{bulk}} \longrightarrow \mathbb{Z}_4^{TF}|_{\text{bulk}} \longrightarrow 1. \quad (3.43)$$

where the normal subgroup $\mathbb{Z}_4 = \{1, \mathcal{T}_{\text{ext}}^2 P_F, \mathcal{T}_{\text{ext}}^4, \mathcal{T}_{\text{ext}}^6 P_F\}$ is not the center $Z(\mathbb{M}_{16}^{F,T})$. The quotient group \mathbb{Z}_4^{TF} contains $\{1, \mathcal{T}_{\text{ext}}, \mathcal{T}_{\text{ext}}^2, \mathcal{T}_{\text{ext}}^3\}$.

3.4 Class AIII: $U(1)^F \times \mathbb{Z}_2^T$ or $\frac{U(1)^F \times \mathbb{Z}_4^T}{\mathbb{Z}_2^F}$ symmetry or Pin^c

Let us look at another example for free fermion Hamiltonian AIII class, which can be chosen to be $U(1)^F \times \mathbb{Z}_2^T$ or $\frac{U(1)^F \times \mathbb{Z}_4^T}{\mathbb{Z}_2^F}$ symmetry, see Table 1.¹³ For $U(1)^F \times \mathbb{Z}_2^T$ symmetry, a representative of

¹²Note that $\mathcal{K}^2 = 1$, we can show $\mathcal{T}_{\text{ext}}^2 = \prod_j (\gamma_{\downarrow,j} \bar{\gamma}_{\downarrow,j}) (\gamma_{\uparrow,j} \bar{\gamma}_{\uparrow,j}) (\gamma_j \bar{\gamma}_j)$, $\mathcal{T}_{\text{ext}}^4 = -1$, and $\mathcal{T}_{\text{ext}}^8 = 1$, also $P_F \mathcal{T}_{\text{ext}} P_F = -\mathcal{T}_{\text{ext}} = \mathcal{T}_{\text{ext}}^5$.

¹³The time reversal \mathcal{T} with $\mathcal{T}^2 = +1$ in $U(1)^F \times \mathbb{Z}_2^T$ can be redefined as a new time reversal $\mathcal{T}' = \exp(i\pi/2)\mathcal{T} = i\mathcal{T}$ with $\mathcal{T}'^2 = -1$ and $\mathcal{T}'^4 = +1$ in $\frac{U(1)^F \times \mathbb{Z}_4^T}{\mathbb{Z}_2^F}$. Here the $\exp(i\pi/2) = i$ is a $\pi/2$ rotation of the $U(1)$ symmetry transformation which is also related to the π rotation of fermion since the 2π rotation of fermion gives a spin-statistics (-1) from \mathbb{Z}_2^F .

class AIII systems is a superconductor with time reversal $\mathcal{T} = \mathcal{K}$ and a spin-rotation $U(1)$ symmetry

$$V(\theta) = \prod_j \exp \left(\frac{\theta}{2} (\gamma_{\downarrow,j} \gamma_{\uparrow,j} + \bar{\gamma}_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \right). \quad (3.44)$$

It can be easily checked that the Hamiltonian $H_{\text{Kitaev}}^{\text{II}}$ of eq (3.3) is invariant under the symmetries defined above. Once again, we use the basis change to get the Hamiltonian H_{gen} of eq (3.5) and the following basis-changed symmetry operators:

$$\tilde{\mathcal{T}} \equiv M \mathcal{T} M^\dagger = M \mathcal{K} M^\dagger = \left(\prod_j (\gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \right) \mathcal{K}. \quad (3.45)$$

$$\tilde{V}(\theta) \equiv M V(\theta) M^\dagger = \prod_j \exp \left(\frac{\theta}{2} (\bar{\gamma}_{\downarrow,j} \gamma_{\downarrow,j} - \bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j}) \right) = \prod_j \exp \left(i\theta ((N_{F,\downarrow})_j - (N_{F,\uparrow})_j) \right). \quad (3.46)$$

Here we use the fact for the flavor \uparrow or \downarrow (abbreviated as \uparrow / \downarrow),¹⁴

$$(N_{F,\uparrow/\downarrow})_j \equiv \psi_{\uparrow/\downarrow,j}^\dagger \psi_{\uparrow/\downarrow,j} = \frac{1 - i\bar{\gamma}_{\uparrow/\downarrow,j} \gamma_{\uparrow/\downarrow,j}}{2} = \frac{1 - (P_{F,\uparrow/\downarrow})_j}{2}.$$

Note that eq (3.46) is in the form of S_z spin-rotational $U(1)$ symmetry generator $\prod_j \exp(-i\theta(S_z)_j)$ of the spin-1/2 system (a fundamental representation **2** of $SU(2)$), with a periodicity $\theta \sim \theta + 2\pi$, so $\theta \in [0, 2\pi)$.

3.4.1 Gapping the boundary: Extension to a non-Abelian $\frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2}$

Let us now look at the symmetry transformation on the edges, on the left (L) and right (R) boundaries:

$$V_L(\theta) = \exp \left(\frac{\theta}{2} \bar{\gamma}_{\downarrow} \gamma_{\downarrow} \right), \quad V_R(\theta) = \exp \left(-\frac{\theta}{2} \bar{\gamma}_{\uparrow} \gamma_{\uparrow} \right), \quad (3.47)$$

$$\mathcal{T}_L = \gamma_{\downarrow} \mathcal{K}, \quad \mathcal{T}_R = \bar{\gamma}_{\uparrow} \mathcal{K}, \quad (3.48)$$

$$P_{F,L} = i\bar{\gamma}_{\downarrow} \gamma_{\downarrow}, \quad P_{F,R} = i\bar{\gamma}_{\uparrow} \gamma_{\uparrow}. \quad (3.49)$$

Some comments are in order:

- The bulk $U(1)^F$ symmetry eq (3.46) acting on the two complex fermions per site has a 2π periodicity: $\theta \in [0, 2\pi)$. Note that eq (3.46) at $\theta = \pi$, we have effectively $\tilde{V}(\theta = \pi) = P_F$; this generates a $\mathbb{Z}_2^F = \{\tilde{V}(\theta = 0), \tilde{V}(\theta = \pi)\} = \{1, P_F\}$, which is a normal subgroup of $U(1)^F$.
- The boundary $U(1)$ symmetry eq (3.47) acting on the one complex fermion per site has a 4π periodicity: $\theta \in [0, 4\pi)$. Note that this boundary $U(1)$ does *not* contain $\mathbb{Z}_2^F = \{1, P_F\}$, since eq (3.47)'s generator $\exp(\frac{\theta}{2} \bar{\gamma} \gamma) = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2}) \bar{\gamma} \gamma$ can *not* contain eq (3.49)'s $P_F = i\bar{\gamma} \gamma$. Instead eq (3.47) of the L/R boundary contains a $\mathbb{Z}_2 = \{1, V_{L/R}(\theta = 2\pi)\} = \{1, -1\}$.

¹⁴We remark that the left-handed side expression fermion number $(N_{F,\uparrow/\downarrow})_j$ has eigenvalues in $\{0, 1\}$ for states with a complex fermion empty or filled. In contrast, $i\bar{\gamma}_{\uparrow/\downarrow,j} \gamma_{\uparrow/\downarrow,j} = (P_{F,\uparrow/\downarrow})_j$ is the fermion parity on the site j for the flavor \uparrow or \downarrow , which $(P_{F,\uparrow/\downarrow})_j$ has eigenvalues in $\{1, -1\}$. So the right-handed side $\frac{1 - (P_{F,\uparrow/\downarrow})_j}{2}$ also indeed has eigenvalues in $\{0, 1\}$ exactly matching $(N_{F,\uparrow/\downarrow})_j$.

- On the boundary, still there is a $\mathbb{D}_8^{F,T}$ finite group symmetry generated by $\mathcal{T}_{L/R}$ and $P_{F,L/R}$, similar to Sec. 3.2.1. There is a normal subgroup $\mathbb{Z}_2 = \{1, -1\}$ shared by both the boundary symmetry $U(1)$ and $\mathbb{D}_8^{F,T} \equiv \mathbb{Z}_2^F \times \mathbb{Z}_4^T$ with the new fractionalized \mathbb{Z}_4^T . We can explicitly check the boundary $U(1)$ commutes with other generators of $\mathbb{D}_8^{F,T}$.

In summary of the above, so the edge symmetry operators form the boundary extended symmetry group $\tilde{G}|_{\text{boundary}}$, which can be written into the short exact sequence:¹⁵

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2} \Big|_{\text{boundary}} \rightarrow U(1)^F \times \mathbb{Z}_2^T \Big|_{\text{bulk}} \rightarrow 1. \quad (3.50)$$

$\supseteq \mathbb{D}_8^{F,T} = \mathbb{Z}_2^F \times \mathbb{Z}_4^T$ $\supseteq \mathbb{Z}_2^F \times \mathbb{Z}_2^T$

In the underset of groups $\tilde{G}|_{\text{boundary}}$ and $G|_{\text{bulk}}$, we indicate that they contain the subgroups ($\supseteq \dots$) established in Sec. 3.2.1.

3.4.2 Unwinding the bulk: Extension to a non-Abelian $\frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2}$

The AIII Hamiltonian can be unwound using the same Hilbert space extension and 2-layer FDUC, $W = W_2 W_1$ as for class BDI (eq (3.24)-eq (3.27)). It is easy to check that W is also invariant under the extended $\frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2}$ symmetry generators defined below

$$V_{\text{ext}}(\theta) = \prod_{\text{odd } j} \exp\left(\frac{\theta}{2}(\bar{\gamma}_{\downarrow,j}\gamma_{\downarrow,j} - \bar{\gamma}_{\uparrow,j}\gamma_{\uparrow,j} + \bar{\gamma}_j\gamma_j)\right) \prod_{\text{even } j} \exp\left(\frac{\theta}{2}(\bar{\gamma}_{\downarrow,j}\gamma_{\downarrow,j} - \bar{\gamma}_{\uparrow,j}\gamma_{\uparrow,j} - \bar{\gamma}_j\gamma_j)\right), \quad (3.51)$$

$$\mathcal{T}_{\text{ext}} = \prod_{\text{odd } j} \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j} \gamma_j \prod_{\text{even } j} \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j} \bar{\gamma}_j \mathcal{K}, \quad (3.52)$$

$$P_F = \prod_j (i \bar{\gamma}_{\downarrow,j} \gamma_{\downarrow,j}) (i \bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j}) (i \bar{\gamma}_j \gamma_j). \quad (3.53)$$

In short, the bulk unwinding follows also the symmetry extension:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2} \Big|_{\text{bulk}} \longrightarrow U(1)^F \times \mathbb{Z}_2^T \Big|_{\text{bulk}} \longrightarrow 1. \quad (3.54)$$

3.5 Class CII: $SU(2)^F \times \mathbb{Z}_2^T$ symmetry or $\text{Pin}^- \times_{\mathbb{Z}_2^F} SU(2)^F$

The 1+1d CII fermion phase has a nontrivial class $\nu_{\text{CII}} \in \mathbb{Z}_2$ which actually corresponds to a bosonic SPT phase with a \mathbb{Z}_2^T time reversal symmetry tensor product with a trivial fermionic gapped phase, and can be unwound by an extended symmetry that has a fermion parity at the center, see the

¹⁵To check some of the above mentioned equalities, we use the fact $\exp(i\theta X) = \cos(\theta X) + i \sin(\theta X)$ for a matrix operator X . For example, we have $\exp(\frac{\theta}{2} \bar{\gamma} \gamma) = \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2}) \bar{\gamma} \gamma$.

• Note that for eq (3.46) of the bulk, $\tilde{V}(\theta = \pi) = \prod_j \exp(i\pi((N_{F,\uparrow})_j + (N_{F,\downarrow})_j)) = (-1)^{((N_{F,\uparrow})_j + (N_{F,\downarrow})_j)} = P_F = \prod_j (i \bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j}) (i \bar{\gamma}_{\downarrow,j} \gamma_{\downarrow,j})$ generates the fermion parity \mathbb{Z}_2^F of the bulk system.

• Note that for eq (3.47) of the L or R boundary, $V_{L/R}(\theta = 2\pi) = \exp\left(\pi \bar{\gamma}_{\downarrow/\uparrow} \gamma_{\downarrow/\uparrow}\right) = (-1)^{i(\bar{\gamma}_{\downarrow/\uparrow} \gamma_{\downarrow/\uparrow})} = (-1)^{P_{F,L/R}} = -1$, generates a \mathbb{Z}_2 but *not* the fermion parity \mathbb{Z}_2^F of the effective boundary.

detailed discussions in [3]. In terms of the CII class symmetry realization, other than $SU(2)^F \times \mathbb{Z}_2^T$, it can also be realized as $\frac{(U(1) \times \mathbb{Z}_4^C)}{\mathbb{Z}_2^F}$ symmetry [3, 25]. Let us interpret how to gap the boundary and unwind the bulk via the following symmetry extensions.

If we extend the \mathbb{Z}_2^T via a new species of fermions with a fermion parity \mathbb{Z}_2^{F+} to \mathbb{Z}_4^{TF+} with $\mathcal{T}^2 = (-1)^{F+}$, we have the symmetry extensions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_2^{F+} & \longrightarrow & SU(2)^F \times \mathbb{Z}_4^{TF+} & \longrightarrow & SU(2)^F \times \mathbb{Z}_2^T \longrightarrow 1. \\ 1 & \longrightarrow & \mathbb{Z}_2^{F+} & \longrightarrow & \frac{(U(1) \times \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_4^{TF+} & \longrightarrow & \frac{(U(1) \times \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_2^T \longrightarrow 1. \\ 1 & \longrightarrow & \mathbb{Z}_2^{F+} & \longrightarrow & \mathbb{Z}_4^{TF+} & \longrightarrow & \mathbb{Z}_2^T \longrightarrow 1. \end{array} \quad (3.55)$$

If we extend the \mathbb{Z}_2^T via a bosonic \mathbb{Z}_2^B to \mathbb{Z}_4^{TB} symmetry with $\mathcal{T}^2 = -1$ as a bosonic (-1) , we have the symmetry extensions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_2^B & \longrightarrow & SU(2)^F \times \mathbb{Z}_4^{TB} & \longrightarrow & SU(2)^F \times \mathbb{Z}_2^T \longrightarrow 1. \\ 1 & \longrightarrow & \mathbb{Z}_2^B & \longrightarrow & \frac{(U(1) \times \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_4^{TB} & \longrightarrow & \frac{(U(1) \times \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_2^T \longrightarrow 1. \\ 1 & \longrightarrow & \mathbb{Z}_2^B & \longrightarrow & \mathbb{Z}_4^{TB} & \longrightarrow & \mathbb{Z}_2^T \longrightarrow 1. \end{array} \quad (3.56)$$

The quantum mechanical lattice constructions of the above are done in [3]. Formally, we can also understand it from trivializing the 2d topological term in the bordism group $\Omega_2^{\text{Pin}^- \times_{\mathbb{Z}_2^F} SU(2)^F}$, which will be explained in Sec. 6.

3.6 Formal explanation on the classifications of invertible phases vs SPT phases

- *Long-ranged entangled (LRE) invertible fermionic topological orders* (protected by no other symmetry, except the fermion parity \mathbb{Z}_2^F essential for fermionic systems): To obtain its classification, we can read from $TP_d(\text{Spin})$ with the group $G = \text{Spin}(d)$ symmetry. Note that $\text{Spin}(d)/\text{SO}(d) = \mathbb{Z}_2^F$ with the $\text{SO}(d)$ is the rotational symmetry for the continuum limit of the tangent manifold of spacetime manifold.
- *Short-ranged entangled (SRE) invertible SPTs*: To obtain its classification, we can consider the quotient group $\frac{TP_d(G)}{TP_d(\text{Spin})}$. Namely, the invertible fermionic phases classified by $TP_d(G)$ mod out the LRE invertible fermionic topological orders in $TP_d(\text{Spin})$, indeed are left with only SRE invertible SPTs [26]. In fact, for interacting systems, for the finite group subclasses of the SRE SPT classification (known as the torsion [tors] classes in mathematics), we can rewrite the SRE fermionic SPT finite-subgroup-part classification following [26] as the torsion part of the quotient group $\frac{TP_d(G)}{TP_d(\text{Spin})}$:

$$\left[\frac{TP_d(G)}{TP_d(\text{Spin})} \right]_{\text{tors}} = \frac{[TP_d(G)]_{\text{tors}}}{[TP_d(\text{Spin})]_{\text{tors}}} = \frac{[\Omega_d^G]_{\text{tors}}}{[\Omega_d^{\text{Spin}}(pt)]_{\text{tors}}}. \quad (3.57)$$

Here we use the fact that the torsion (finite subgroup) part of cobordism and bordism groups are the same $[TP_d(G)]_{\text{tors}} = [\Omega_d^G]_{\text{tors}}$; this eq (3.57) relates to the d -th bordism group Ω_d . This expression in eq (3.57) left out other infinite group \mathbb{Z} classes:

- Some of \mathbb{Z} classes are known as the free classes in mathematics correspond to LRE invertible topological orders by Wen's definition [29], captured by

$$[TP_d(\text{Spin})]_{\text{free}}. \quad (3.58)$$

- Other \mathbb{Z} classes are captured by the free part of quotient group $\frac{\text{TP}_d(G)}{\text{TP}_d(\text{Spin})}$ as

$$\left[\frac{\text{TP}_d(G)}{\text{TP}_d(\text{Spin})} \right]_{\text{free}}, \quad (3.59)$$

which happen especially in an odd dimensional spacetime and correspond to the \mathbb{Z} classes of SRE invertible SPTs whose boundary exhibits the *perturbative local anomalies*. For example, in 2+1d, there is a \mathbb{Z} class of SRE invertible SPTs protected by U(1) symmetry, for both bosonic or fermionic systems [14].

- In contrast, the other \mathbb{Z}_n classes (the torsion) from $[\text{TP}_d(G)]_{\text{tors}}$ correspond to invertible topological phases whose boundary exhibits the *non-perturbative global anomalies*. This $[\text{TP}_d(G)]_{\text{tors}}$ includes both the $[\text{TP}_d(\text{Spin})]_{\text{tors}}$ as classes of LRE invertible topological orders and the $[\frac{\text{TP}_d(G)}{\text{TP}_d(\text{Spin})}]_{\text{tors}}$ as SRE invertible SPTs. The *pt* is a point as a certain classifying space.

4 Interacting models for even $\nu \in \mathbb{Z}_8$ classes with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry or Pin⁻

We now consider an interacting model of time-reversal invariant fermions with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry belonging to the same $k = 2$ phase as the model in section 3.2 of the \mathbb{Z}_8 classification. This is a basis-changed version of the model introduced by Tantivasadakarn and Vishwanath [35] (see Appendix A for details of the basis-change operations). The Hilbert space for the model consists of two species of fermions ($\varsigma = \uparrow, \downarrow$: each complex fermions beget two Majorana fermions, γ and $\bar{\gamma}$) on the sites and qubits (σ) on the links of a one-dimensional lattice. Using the Majorana operators as defined in the previous sections, the Hamiltonian is written as

$$H = -i \sum_j [\sigma_{j,j+1}^z (\bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j} + \bar{\gamma}_{\downarrow,j+1} \gamma_{\downarrow,j+1}) + \sigma_{j,j+1}^x \bar{\gamma}_{\uparrow,j} \gamma_{\downarrow,j+1}]. \quad (4.1)$$

The above Hamiltonian is also represented graphically in Fig 11.

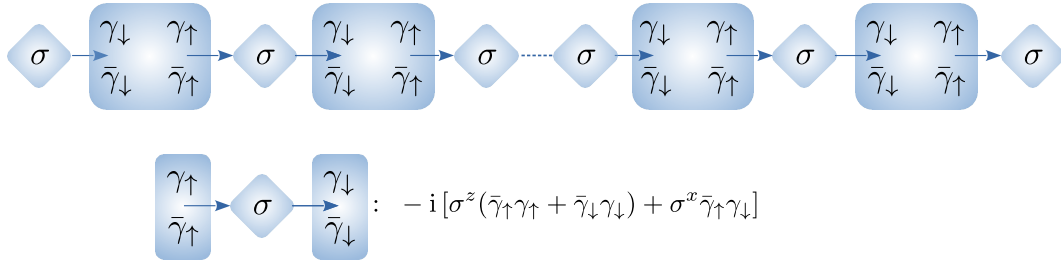


Figure 11: The interacting $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ invariant SPT Hamiltonian of eq. (4.1). The squares consist of two complex fermions, labeled \uparrow and \downarrow and reside on the sites of the lattice. The diamonds represent the qubits σ and reside on the links of the lattice.

The time-reversal and fermion parity operators are

$$\mathcal{T} = \left(\prod_j (i \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \sigma_{j,j+1}^x \right) \mathcal{K}, \quad P_F = \prod_j \prod_{\varsigma=\uparrow,\downarrow} i \bar{\gamma}_{\varsigma,j} \gamma_{\varsigma,j}, \quad (4.2)$$

where \mathcal{K} is the complex conjugation operation as before. The boundary symmetry operators are as follows:

$$\mathcal{T}_L = \gamma_{\downarrow}\mathcal{K}, \quad \mathcal{T}_R = i\bar{\gamma}_{\uparrow}\mathcal{K} \quad (4.3)$$

$$P_{F,L} = i\bar{\gamma}_{\downarrow}\gamma_{\downarrow}, \quad P_{F,R} = i\bar{\gamma}_{\uparrow}\gamma_{\uparrow} \quad (4.4)$$

On both ends, the symmetry operators generate the group $\mathbb{D}_8^{F,T}$ which does not have fermion parity at the center

$$\mathbb{D}_8^{F,T} \equiv \langle (\mathcal{T}_{L/R}P_{F,L/R}), P_{F,L/R} \mid \mathcal{T}_{L/R}^2 = P_{F,L/R}^2 = 1, P_{F,L/R}\mathcal{T}_{L/R}P_{F,L/R} = -\mathcal{T}_{L/R} \rangle. \quad (4.5)$$

We now proceed to extend the Hilbert space and unwind the Hamiltonian using a $\mathbb{D}_8^{F,T}$ invariant quantum circuit. The extended Hilbert space we choose consists of an additional qubit per link $(\tau_{j,j+1})$, an additional qubit per site (σ_j) and an additional fermion per site $(\gamma_j, \bar{\gamma}_j)$. The extended Hamiltonian we consider is

$$H_{\text{ext}} = H - \sum_j \sigma_j^x - i \sum_{\text{odd } j} [\tau_{j,j+1}^z (\bar{\gamma}_j \gamma_j + \bar{\gamma}_{j+1} \gamma_{j+1}) + \tau_{j,j+1}^x \gamma_j \bar{\gamma}_{j+1}] - \sum_{\text{even } j} \tau_{j,j+1}^x. \quad (4.6)$$

This extended Hamiltonian is schematically shown in Fig 12. The Hamiltonian H_{ext} is unwound

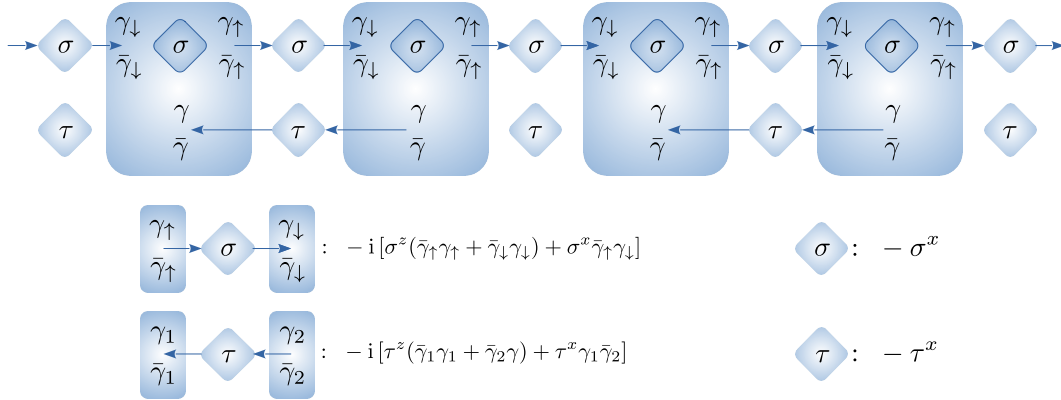


Figure 12: Extended $\mathbb{D}_8 \equiv \mathbb{D}_8^{F,T}$ invariant SPT Hamiltonian of eq. (4.6). The squares consist of three complex fermions and one qubit and reside on the sites of the lattice. Two other qubits (diamonds not enclosed in squares) reside on the links of the lattice.

to H_{triv} as shown in Fig 13 using the FDUC, W as follows

$$WH_{\text{ext}}W^\dagger = H_{\text{triv}}. \quad (4.7)$$

$$H_{\text{triv}} = - \sum_j \left[\sigma_j^z \left(\sum_{\varsigma=\uparrow,\downarrow} i\bar{\gamma}_{\varsigma,j} \gamma_{\varsigma,j} \right) + i\sigma_j^x \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j} + \sigma_{j,j+1}^x \right] - i \sum_{\text{odd } j} [\tau_{j,j+1}^z (\bar{\gamma}_j \gamma_j + \bar{\gamma}_{j+1} \gamma_{j+1}) + \tau_{j,j+1}^x \gamma_j \bar{\gamma}_{j+1}] - \sum_{\text{even } j} \tau_{j,j+1}^x. \quad (4.8)$$

$$W = W_2 W_1. \quad (4.9)$$

$$W_1 = \prod_{\text{odd } j} \text{SWAP}(\sigma_j, \sigma_{j,j+1}) \text{SWAP}(\tau_{j,j+1}, \sigma_{j+1}) \left(\frac{1 + \gamma_j \gamma_{\downarrow,j+1}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_j \bar{\gamma}_{\downarrow,j+1}}{\sqrt{2}} \right).$$

$$W_2 = \prod_{\text{even } j} \left[\text{SWAP}(\sigma_{j,j+1}, \tau_{j,j+1}) \left(\frac{1 + \gamma_j \gamma_{\uparrow,j}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_j \bar{\gamma}_{\uparrow,j}}{\sqrt{2}} \right) \right] \prod_{\text{odd } j} \left[\left(\frac{1 + \gamma_j \gamma_{\downarrow,j}}{\sqrt{2}} \right) \left(\frac{1 + \bar{\gamma}_j \bar{\gamma}_{\downarrow,j}}{\sqrt{2}} \right) \right].$$

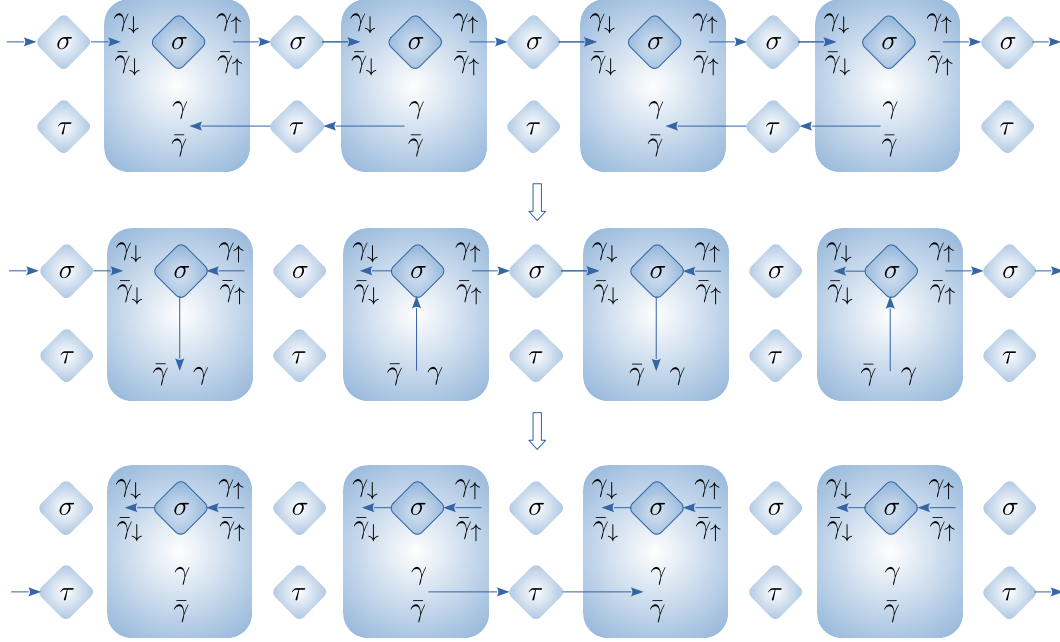


Figure 13: Unwinding the interacting $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ invariant SPT Hamiltonian eq. (4.6). The unwinding is done in two steps using a series of bosonic and fermionic swap operators which commute with the extended symmetry.

It can be verified that the extended Hamiltonian as well as the unwinding FDUC are invariant under the extended symmetry operator transformation

$$\begin{aligned}
\mathcal{T}_{\text{ext}} &= \prod_{\text{odd } j} (i\gamma_{\downarrow,j}\bar{\gamma}_{\uparrow,j}\gamma_j \sigma_j^x \sigma_{j,j+1}^x \tau_{j,j+1}^x) \prod_{\text{even } j} (i\gamma_{\downarrow,j}\bar{\gamma}_{\uparrow,j}i\bar{\gamma}_j \sigma_j^x \sigma_{j,j+1}^x \tau_{j,j+1}^x) \mathcal{K} \\
&= \prod_{\text{all } j} (i\gamma_{\downarrow,j}\bar{\gamma}_{\uparrow,j} \sigma_j^x \sigma_{j,j+1}^x \tau_{j,j+1}^x) \left(\prod_{\text{odd } j} \gamma_j \right) \left(\prod_{\text{even } j} i\bar{\gamma}_j \right) \mathcal{K}.
\end{aligned} \tag{4.10}$$

5 Supersymmetric quantum mechanics

In this section, we detail the relationship between the extended symmetries and supersymmetric quantum mechanics. More details can be found in ref [36]. Supersymmetric systems are characterized by the presence of *supercharges*- symmetry operators which are fermionic i.e. anti-commute with fermion parity and square to the generators of space-time translations [37]. The simplest of supersymmetric systems are supersymmetric quantum-mechanical (SUSY QM) systems where supercharges only square to the generator of time-translations i.e. the Hamiltonian. More specifically, a Hamiltonian H is a SUSY QM system if there exists $\alpha = 1 \dots \mathcal{N}$ Hermitian supercharges Q_α satisfying the following algebra [38–41]

$$\{Q_\alpha, Q_\beta\} = 2H\delta_{\alpha,\beta}, \quad [Q_\alpha, H] = \{Q_\alpha, P_F\} = 0. \quad (5.1)$$

If we consider the *fractionalized* time-reversal symmetry on the boundaries,

- from \mathbb{Z}_2^T ($\mathcal{T}^2 = +1$ for BDI) to a fractionalized \mathbb{Z}_4^T ($\mathcal{T}^4 = +1$ in Sec. 3.2), and
- from \mathbb{Z}_4^{TF} ($\mathcal{T}^4 = +1$ for DIII) to a fractionalized \mathbb{Z}_8^T ($\mathcal{T}^8 = +1$ in Sec. 3.3),

their anti-commutation with the fermion parity P_F operator,

$$\{\mathcal{T}, P_F\} \equiv \mathcal{T}P_F + P_F\mathcal{T} = 0,$$

will result in any compatible Hamiltonian being a SUSY QM system. As a result, we refer to the presence of any symmetry operators which do not commute with fermion parity as supersymmetric.

5.1 General setting for SUSY quantum mechanics

Let us see the emergence of SUSY QM from global symmetries. Consider a fermionic system described by some Hamiltonian operator H . This means that the Hamiltonian commutes with a fermion parity \mathbb{Z}_2^F symmetry, generated by $P_F \equiv (-1)^F$, so $P_F H P_F^{-1} = H$. Let $\{V_g\}$ be a collection of symmetry operators, including P_F that commute with H . Recall that P_F also grades states and operators as bosonic and fermionic as follows

$$P_f|b\rangle = +|b\rangle, \quad P_F O_b P_F = +O_b \quad (5.2)$$

$$P_f|f\rangle = +|f\rangle, \quad P_F O_f P_F = -O_f \quad (5.3)$$

We show that if there exist symmetry operators which are fermionic i.e. anti-commutes with P_F , H is a SUSY QM system with at least $\mathcal{N} = 2$ supercharges. Let \mathcal{T} be a such a symmetry of H i.e. $\mathcal{T} H \mathcal{T}^{-1} = H$, and $P_F \mathcal{T} P_F = -\mathcal{T}$. This implies that any eigenstate with non-zero eigenvalue (ensured by a constant shift of H) is degenerate and contains an even number of bosonic and fermionic states. The diagonalized Hamiltonian can be schematically written, with μ the eigenvalue labels, as

$$H = \sum_{\mu} \sum_{a_{\mu}} E_{\mu} (|\mu, a_{\mu}, +\rangle \langle \mu, a_{\mu}, +| + |\mu, a_{\mu}, -\rangle \langle \mu, a_{\mu}, -|). \quad (5.4)$$

The $+$ or $-$ specifies the $(-1)^F$ eigenvalue which correspond to bosonic or fermionic sector and a_{μ} keeps track of additional degeneracies. Let us consider the minimal setting when there are no other degeneracies in the system. For concreteness, let us also take $\mathcal{T}^2 = 1$ so we have a \mathbb{Z}_2^T symmetry

(in fact, \mathbb{Z}_2^T can be more general, *either unitary or anti-unitary* in this section). We assume that the Hamiltonian is shifted so as to ensure that all energies are positive-definite. The eigenstates are organized as

$$H|\mu, \pm\rangle = E_\mu|\mu, \pm\rangle, \quad P_F|\mu, \pm\rangle = \pm|\mu, \pm\rangle. \quad (5.5)$$

It is clear that \mathcal{T} switches the parity

$$\mathcal{T}|\mu, \pm\rangle = |\mu, \mp\rangle. \quad (5.6)$$

Following ref [42], we can now define the following operators

$$Q_\mu \equiv \sqrt{E_\mu}|\mu, +\rangle\langle\mu, -|, \quad Q_\mu^\dagger \equiv \sqrt{E_\mu}|\mu, -\rangle\langle\mu, +|, \quad (5.7)$$

using which we can define the following two Hermitian *supercharges*

$$Q_+ \equiv \sum_\mu Q_\mu + Q_\mu^\dagger. \quad (5.8)$$

$$Q_- \equiv \sum_\mu -i \left(Q_\mu - Q_\mu^\dagger \right). \quad (5.9)$$

We also have the fermion parity P_F and identity $\mathbb{1}$ operators as:

$$P_F \equiv \sum_\mu |\mu, +\rangle\langle\mu, +| - |\mu, -\rangle\langle\mu, -|, \quad \mathbb{1} \equiv \sum_\mu |\mu, +\rangle\langle\mu, +| + |\mu, -\rangle\langle\mu, -|. \quad (5.10)$$

It can be checked that the supercharges satisfy the algebra of eq. (5.1) (below $\alpha, \beta \in \{\pm\}$) [38, 41] i.e.

1. Supersymmetry anti-commutes with the \mathbb{Z}_2^F fermion parity symmetry, generated by $P_F \equiv (-1)^F$:

$$\{Q_\alpha, P_F\} \equiv Q_\alpha P_F + P_F Q_\alpha = 0. \quad (5.11)$$

2. Supercharges commute with Hamiltonian thus supersymmetry is a global symmetry:

$$[Q_\alpha, H] \equiv Q_\alpha H - H Q_\alpha = 0. \quad (5.12)$$

3. The anti-commutator for supersymmetric quantum mechanics obeys:

$$\{Q_\alpha, Q_\beta\} \equiv Q_\alpha Q_\beta + Q_\alpha Q_\beta = 2H\delta_{\alpha\beta}. \quad (5.13)$$

This means that we have an $\mathcal{N} = 2$ SUSY QM system at hand. The value of \mathcal{N} could be larger depending on the additional symmetries but some SUSY is guaranteed. The symmetry \mathcal{T} acts on the supercharges as follows.¹⁶ If \mathcal{T} is a unitary \mathbb{Z}_2 symmetry, we can choose

$$\mathcal{T} : Q_\pm \mapsto \mathcal{T} Q_\pm \mathcal{T}^{-1} = \pm Q_\pm. \quad (5.16)$$

If the \mathcal{T} is an anti-unitary \mathbb{Z}_2 symmetry, we can choose \mathcal{T} acts as an identity operator on the supercharges:

$$\mathcal{T} : Q_\pm \mapsto \mathcal{T} Q_\pm \mathcal{T}^{-1} = +Q_\pm. \quad (5.17)$$

Generally, the original symmetries acts by permuting the supercharges.

¹⁶In general, for each energy E_μ sector, we have a 2-dimensional Hilbert space $\mathcal{H}_\mu = \mathcal{H}_{B,\mu} \oplus \mathcal{H}_{F,\mu}$ where the bosonic sector $\mathcal{H}_{B,\mu}$ is spanned by the vector $\{|\mu, +\rangle\}$, the fermionic sector $\mathcal{H}_{F,\mu}$ is spanned by the vector $\{|\mu, -\rangle\}$. Overall, the total Hilbert space is even dimensional:

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F = \bigoplus_\mu \mathcal{H}_\mu = \bigoplus_\mu (\mathcal{H}_{B,\mu} \oplus \mathcal{H}_{F,\mu}). \quad (5.14)$$

We can write all the aforementioned operators H, P_F, \mathcal{T} , and Q_\pm respect to the 2-dimensional Hilbert space \mathcal{H}_μ associated to each energy E_μ . We can denote the projection of the H, P_F, \mathcal{T} , and Q_\pm to a 2-dimensional Hilbert

5.2 $\mathcal{N} = 2$ SUSY on the boundary of 1+1d fermionic SPT phase with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry: Generalized Sachdev-Ye-Kitaev interactions

Recall that one-dimensional SPT phases have symmetries that are realized *projectively*. For *intrinsically* fermionic SPT phases i.e phases that cannot be thought of as a stack of a bosonic SPT phase, tensor product with a trivial gapped fermionic system, some of the projective symmetry generators do not commute with fermion parity [43]. Due to the arguments of the previous subsection, this leads to the boundary Hamiltonian being supersymmetric. Let us consider a specific phase for concreteness. We again consider the $\nu = 2 \bmod 8$ phase of time-reversal invariant superconductors where \mathcal{T} which acts anti-unitarily and has the property $\mathcal{T}^2 = 1$. This belongs to class BDI and (in the presence of interactions) has a \mathbb{Z}_8 classification. We again consider the Hamiltonian in eq (3.5):

$$H = -i \sum_{j=1}^{\ell} (\gamma_{\uparrow,j} \gamma_{\downarrow,j+1} - \bar{\gamma}_{\uparrow,j} \bar{\gamma}_{\downarrow,j+1}). \quad (5.18)$$

Recall the symmetries are

$$\mathcal{T} = \left(\prod_j (\gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \right) \mathcal{K}, \quad P_F = \left(\prod_j \prod_{\varsigma=\uparrow,\downarrow} i \bar{\gamma}_{\varsigma,j} \gamma_{\varsigma,j} \right). \quad (5.19)$$

\mathcal{K} is the complex conjugation which acts on the Majorana operators as

$$\mathcal{K} \gamma \mathcal{K} = \gamma, \quad \mathcal{K} \bar{\gamma} \mathcal{K} = -\bar{\gamma}, \quad \mathcal{K} i \mathcal{K} = -i. \quad (5.20)$$

Let us now consider a boundary termination as shown in Fig. 8. There are 2 dangling Majorana zero modes on each (left or right) end of Fig. 8, each end has a 2-dimensional Hilbert space. For Fig. 8, the symmetries acting on the boundary sites are the same as the bulk sites. However, if we only consider how the *effective symmetries* acting on the ground state subspace which associated with the boundary zero modes, we can do the ground state (GS) projection $\mathbf{P}_{\text{GS}} = \mathbf{P}_{\text{GS}}^\dagger$ on \mathcal{T} which factorize eq (5.19) to left and right boundary symmetry operators:

$$\mathbf{P}_{\text{GS}} \mathcal{T} \mathbf{P}_{\text{GS}} \simeq \mathcal{T}_L \otimes \mathcal{T}_R.$$

Here *effective* \mathcal{T}_L and \mathcal{T}_R act only on the left and right Majorana zero modes of Fig. 8 as:

$$\mathcal{T}_L = \gamma_{\downarrow} \mathcal{K}, \quad P_{F,L} = i \bar{\gamma}_{\downarrow} \gamma_{\downarrow}, \quad \mathcal{T}_L^2 = P_{F,L}^2 = 1, \quad \mathcal{T}_L P_{F,L} = -P_{F,L} \mathcal{T}_L. \quad (5.21)$$

$$\mathcal{T}_R = \bar{\gamma}_{\uparrow} \mathcal{K}, \quad P_{F,R} = i \bar{\gamma}_{\uparrow} \gamma_{\uparrow}, \quad \mathcal{T}_R^2 = -1, \quad P_{F,R}^2 = 1, \quad \mathcal{T}_R P_{F,R} = -P_{F,R} \mathcal{T}_R. \quad (5.22)$$

It can be easily checked that the boundary symmetry operators generate a faithful representation of dihedral group of order 8, $\mathbb{D}_8 \equiv \mathbb{D}_8^{F,T}$, which is a group with two generators satisfying the following

space $\mathcal{H}_\mu \equiv \{|\mu, +\rangle, |\mu, -\rangle\}$ as $(H)_\mu, (P_F)_\mu, (Q_\pm)_\mu$, and $(\mathcal{T})_\mu$. Then we have:

$$\begin{aligned} (H)_\mu &= E_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_\mu = E_\mu (\sigma_0)_\mu, \quad (P_F)_\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_\mu \equiv (\sigma_z)_\mu, \\ (Q_+)_\mu &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_\mu \equiv (\sigma_x)_\mu, \quad (Q_-)_\mu = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_\mu \equiv (\sigma_y)_\mu, \quad Q_\mu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_\mu, \quad Q_\mu^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\mu. \\ (\mathcal{T})_\mu &= \begin{cases} (n_x(Q_+)_\mu + n_y(Q_-)_\mu) = (n_x(\sigma_x)_\mu + n_y(\sigma_y)_\mu), & \text{if } \mathcal{T} \text{ is unitary.} \\ (n_x(Q_+)_\mu + n_y(Q_-)_\mu)K = (n_x(\sigma_x)_\mu + n_y(\sigma_y)_\mu)K, & \text{if } \mathcal{T} \text{ is anti-unitary.} \end{cases} \end{aligned} \quad (5.15)$$

Here $(n_x, n_y) \in S^1 \subset \mathbb{R}^2$ is a unit vector so $n_x^2 + n_y^2 = 1$, and K is a complex conjugation operator. It is amusing to notice such a $(\mathcal{T})_\mu$ is in fact a linear combination $(n_x(Q_+)_\mu + n_y(Q_-)_\mu)$. In the case if \mathcal{T} is an anti-unitary time-reversal symmetry, interestingly we have $(\mathcal{T})_\mu$ proportional to SUSY charge linear combination $(Q_+)_\mu$ and $(Q_-)_\mu$ up to a complex conjugation K . In the main text, we choose $(n_x, n_y) = (1, 0)$ to derive eq (5.16) and eq (5.17).

relation: $\langle a, x | a^4 = x^2 = 1, xax = a^{-1} \rangle$. On the left end, $a = \mathcal{T}_L P_{F,L}$, $x = P_{F,L}$ and on the right, $a = \mathcal{T}_R$, $x = P_{F,R}$.¹⁷ Observe that fermion parity is not in the center of $\mathbb{D}_8^{F,T}$. This is going to be crucial. From now on, we focus only on the left boundary and drop the \downarrow index. The effective symmetry can be simply written as

$$\mathcal{T} = \gamma \mathcal{K}, \quad P_F = i \bar{\gamma} \gamma. \quad (5.23)$$

The symmetry acts on the Majorana operators trivially:

$$\mathcal{T} : \begin{pmatrix} \gamma \\ \bar{\gamma} \end{pmatrix} \mapsto \begin{pmatrix} \gamma \\ \bar{\gamma} \end{pmatrix}. \quad (5.24)$$

However, it is anti-linear $\mathcal{T} : i \mapsto -i$. As a result, the only possible boundary Hamiltonian term in this Hilbert space,

$$H \propto -i \gamma \bar{\gamma} = P_F, \quad (5.25)$$

is disallowed by \mathcal{T} symmetry. Therefore, the only allowed boundary Hamiltonian is proportional to the identity operator,

$$H = c \mathbb{1}. \quad (5.26)$$

As shown in [36], with c being positive (with no loss of generality), this has an $\mathcal{N} = 2$ SUSY with supercharges

$$Q_+ = \sqrt{c} \gamma, \quad Q_- = \sqrt{c} \bar{\gamma}. \quad (5.27)$$

We can increase the possible boundary dynamics by *layering* trivial phases on the bulk and there-

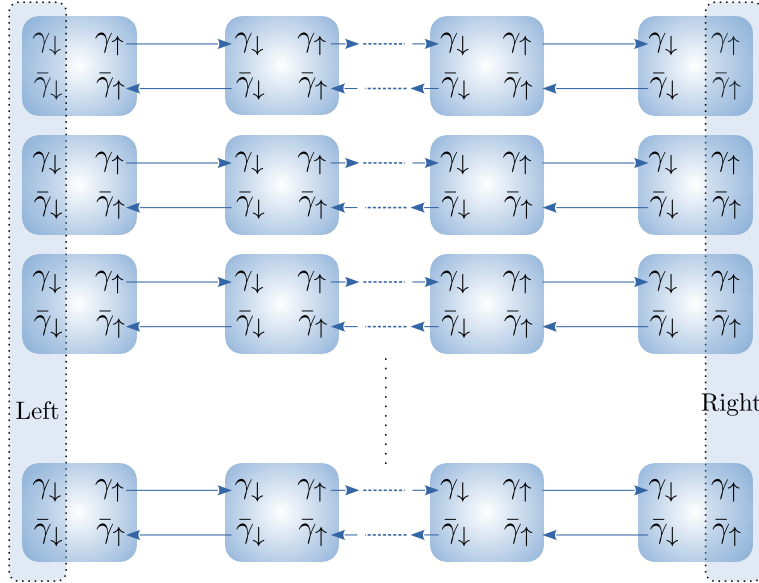


Figure 14: Representative of the $\nu = 2$ phase with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry consisting of multiple layers — here $(4N + 1)$ layers for the $\nu = 2$ phase. We can introduce SYK model Hamiltonian on the boundary, similar to the setup of [44].

fore increasing the boundary by mod 8 layers of Kitaev chains. Since the Hamiltonian eq. (5.18)

¹⁷Readers may be puzzled by why that the boundary \mathcal{T}_L and \mathcal{T}_R symmetries act oppositely in comparison between those of eq (3.10) and eq (3.11) versus those of eq (5.21) and eq (5.22). Recall Sec. 3.2.1's remark 2 that there are three different ways to look at \mathcal{T} symmetries on the boundary depending on (1) the boundary truncation types of quantum systems and (2) the Hilbert space projection \mathbf{P}_{GS} to ground state subspace. Here we do the remark 2's (c) case, with the \mathbf{P}_{GS} projection to dangling zero energy modes.

is the $\nu = 2 \bmod 8$ member, this means that we can add multiples of four copies of the Hamiltonian eq. (5.18) to the bulk and preserve the same phase. The bulk Hamiltonian and symmetry operators correspond to this layered system shown in Fig. 14 is

$$H = -i \sum_{j=1}^{\ell} \sum_{a=1}^{4N+1} (\gamma_{\uparrow,a,j} \gamma_{\downarrow,a,j+1} - \bar{\gamma}_{\uparrow,a,j} \bar{\gamma}_{\downarrow,a,j+1}). \quad (5.28)$$

$$\mathcal{T} = \left(\prod_{a=1}^{4N+1} \prod_{j=1}^{\ell} (\gamma_{\downarrow,a,j} \bar{\gamma}_{\uparrow,a,j}) \right) \mathcal{K}. \quad (5.29)$$

$$P_F = \left(\prod_{a=1}^{4N+1} \prod_{j=1}^{\ell} \prod_{\varsigma=\uparrow,\downarrow} i \bar{\gamma}_{\varsigma,a,j} \gamma_{\varsigma,j} \right). \quad (5.30)$$

We now focus on the boundary. The Hilbert space can be acted by a total $8N + 2$ Majorana operators: $\{\gamma_a, \bar{\gamma}_a\}$ for $a = 1, \dots, 4N + 1$. The symmetry operators are

$$\mathcal{T} = \prod_{a=1}^{4N+1} \gamma_a \mathcal{K}, \quad P_F = \prod_{a=1}^{4N+1} i \bar{\gamma}_a \gamma_a. \quad (5.31)$$

It can be checked that the operators again form a $\mathbb{D}_8^{F,T}$ group where the fermion parity is not at the center subgroup because \mathcal{T} anti-commutes with P_F .

$$\mathcal{T}^2 = P_F^2 = 1, \quad \mathcal{T} P_F = -P_F \mathcal{T}. \quad (5.32)$$

However, we can have a boundary Hamiltonian that is not quadratic, but can consist of Majorana operators coupled in multiples of 4:

$$H = \sum_{a,b,c,d} J_{a,b,c,d}^1 \gamma_a \gamma_b \gamma_c \gamma_d + J_{a,b,c,d}^2 \bar{\gamma}_a \gamma_b \gamma_c \gamma_d + J_{a,b,c,d}^3 \bar{\gamma}_a \bar{\gamma}_b \gamma_c \gamma_d + J_{a,b,c,d}^4 \bar{\gamma}_a \bar{\gamma}_b \bar{\gamma}_c \gamma_d + J_{a,b,c,d}^5 \bar{\gamma}_a \bar{\gamma}_b \bar{\gamma}_c \bar{\gamma}_d \\ + K_{a,b,c,d,e,f,g,h}^1 \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f \gamma_g \gamma_h + K_{a,b,c,d,e,f,g,h}^2 \bar{\gamma}_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f \gamma_g \gamma_h + \dots \quad (5.33)$$

This Hamiltonian H is time reversal invariant: $\mathcal{T} H \mathcal{T}^{-1} = H$. As shown in [44], if the couplings are restricted to the quartic operators and J_{abcd}^i are drawn from a random distribution, this is the famous Sachdev-Ye-Kitaev (SYK) model ([13, 45] also the later expositions in [46, 47]). The SUSY nature of the SYK model was shown by the authors of [42]. This SUSY nature still holds if we add any number of $4n$ -Majorana operators (such as quartic, or octatonic as shown in eq. (5.33)) to the Hamiltonian all of which preserve the \mathcal{T} and P_F symmetry. The logic is precisely what was sketched out in the previous subsection and detailed in [42]: once we have the anti-commutative $\mathcal{T} P_F = -P_F \mathcal{T}$, this implies that we must have two-fold degenerate spectra from the equal number of bosonic sectors $\{|\mu, +\rangle\}$ and fermionic sectors $\{|\mu, -\rangle\}$ at every energy level E_μ , each with a distinct fermion parity $P_F = +1$ and $P_F = -1$. We denote the number of states

$$n_{B,\mu} = n_{F,\mu} = 1 \quad (5.34)$$

for each E_μ . By a constant shift, we can ensure that the Hamiltonian has positive definite energies and can be written as

$$H = \sum_{\mu} E_\mu (|\mu, +\rangle \langle \mu, +| + |\mu, -\rangle \langle \mu, -|), \quad (5.35)$$

where

$$H|\mu, \pm\rangle = E_\mu |\mu, \pm\rangle, \quad P_F |\mu, \pm\rangle = \pm |\mu, \pm\rangle. \quad (5.36)$$

and \mathcal{T} switches the parity

$$\mathcal{T}|\mu, \pm\rangle = |\mu, \mp\rangle. \quad (5.37)$$

We can now define the two *supercharges*

$$Q_+ = \sum_{\mu} Q_{\mu} + Q_{\mu}^{\dagger}, \quad Q_- = \sum_{\mu} -i (Q_{\mu} - Q_{\mu}^{\dagger}), \quad (5.38)$$

where $Q_{\mu} \equiv \sqrt{E_{\mu}}|\mu, +\rangle\langle\mu, -|$ and $Q_{\mu}^{\dagger} \equiv \sqrt{E_{\mu}}|\mu, -\rangle\langle\mu, +|$, and check that the supercharges satisfy the algebra of eq. (5.1)

$$\{Q_{\alpha}, Q_{\beta}\} = 2H\delta_{\alpha\beta}, \quad [Q_{\alpha}, H] = \{Q_{\alpha}, P_F\} = 0 \quad (5.39)$$

This means that we have an $\mathcal{N} = 2$ SUSY QM system on the boundary of these fermionic SPT phases. The action of \mathcal{T} on the supercharges is as shown in eq. (5.17) i.e.

$$\mathcal{T} : Q_{\pm} \mapsto \mathcal{T}Q_{\pm}\mathcal{T}^{-1} = +Q_{\pm}. \quad (5.40)$$

5.3 $\mathcal{N} = 2$ SUSY on the boundary of 1+1d fermionic SPT phase with \mathbb{Z}_4^{TF} symmetry

In Sec. 5.2, we had shown that the Sec. 3.2's model with a bulk $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ -symmetric $8N + 2$ layers of 1+1d Kitaev chains, the boundary exhibits the $\mathcal{N} = 2$ SUSY quantum mechanics. In this section, we can also take the Sec. 3.3's model with a bulk \mathbb{Z}_4^{TF} -symmetric $8N + 2$ layers of Kitaev chains, to analyze its boundary energy spectrum property. Follow the Remark 3 in Sec. 3.2.1, when the number of Kitaev chains is $2 \bmod 4$ and when $\mathcal{T}P_F = -P_F\mathcal{T}$, we have at least two fold generate ground states (with GSD = 2), one bosonic $|B\rangle$ and one fermionic $|F\rangle$. In general, we have the equal number of bosonic eigenstate $\{|\mu, +\rangle\}$ and fermionic eigenstate $\{|\mu, -\rangle\}$ for each eigenvalue E_{μ} . We again have $n_{B,\mu} = n_{F,\mu} = 1$ as in eq (5.34) for each E_{μ} . Thus we can repeat the analysis Sec. 5.2 again reaching the same conclusion that we have $\mathcal{N} = 2$ SUSY quantum mechanics, but with a different bulk symmetry \mathbb{Z}_4^{TF} and the boundary has $\mathcal{T}^4 = -1$ and $\mathcal{T}^8 = +1$. This would also result in a different action on the supercharges.

5.4 Systematic framework in 1+1d

The original fractionalized global symmetry such as \mathcal{T} acts on the supercharges Q_{α} , thus the \mathcal{T} is within a subgroup of the R symmetry that permutes the supercharges Q_{α} . Note that the usual R symmetry is a unitary symmetry, here we need to generalize R symmetry to include both unitary and anti-unitary symmetry (to include time-reversal).

In general, we only have the $\mathcal{N} = 2$ SUSY quantum mechanics when the layer number $k = 2 \bmod 4$. We can summarize our result in Table 2.

We remark again that the fact that the SYK model can be thought of as a termination of fermionic SPT phases has been observed by You, Ludwig and Xu [44]. Also the SUSY QM property of the SYK model was studied by Behrends and Béri [42]. However, our interpretation of SUSY is tightly connected to the generalization of the symmetry extension.

Cartan class	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
BDI ($\nu \in \mathbb{Z}_8$)	-	$\mathcal{N} = 2$ ($\nu = 2$)	-	$\mathcal{N} = 0$ ($\nu = 4$)	-	$\mathcal{N} = 2$ ($\nu = 6$)	-
DIII ($\nu \in \mathbb{Z}_2$)	-	$\mathcal{N} = 2$ ($\nu = 1$)	-		-		-
AI, CI, CII ($\nu \in \mathbb{Z}_2$)	-		-	$\mathcal{N} = 0$ ($\nu = 1$)	-		-
AIII ($\nu \in \mathbb{Z}_4$)	-	$\mathcal{N} = 2$ ($\nu = 1$)	-	$\mathcal{N} = 0$ ($\nu = 2$)	-	$\mathcal{N} = 2$ ($\nu = 3$)	-

Table 2: We show the relations between the SUSY number \mathcal{N} , the global symmetry (Cartan symmetry class or Wigner-Dyson-Altland-Zirnbauer symmetry class) and their SPT classes ν , and the number k of Majorana zero modes. For $k \in \mathbb{Z}_{\text{odd}}^+$, the - implies that those models are not studied in our framework. The blanks mean those classes are either not the generators of the SPT classification, or they are already determined by the smaller k of SPT class (other filled data).

6 Generalizations: from the lattice to continuum

In Sec. 6, We will make some refinement about the group extension (for a quick overview see Appendix B.1) and properties of bosonic/fermionic/time-reversal/spacetime symmetries and group extensions (for a quick overview see Appendix B.2) that we used.¹⁸ In Table 3, we show a symmetry embedding web of 1+1d fermionic invertible topological phases with their symmetry groups (relevant for 10 Cartan symmetry classes), and the number of layers of Kitaev chains to construct the minimal phase. In Table 4, we summarize the result of fermionic symmetry group extension presented in Sec. 3 and 4, also those in [3], for fermionic SPT phases in 1+1d.

We also summarize the bundle constraints for various symmetry groups and corresponding spacetime structures in Table 6 following Ref. [48].

In the following subsections, we show by examples, based on the knowledge of Appendix B.1 and B.2, how to convert the discrete group version of symmetry extensions (Table 4) to the continuous group version of symmetry extensions (Table 5), suitable for the continuum quantum field theory [QFT]).

6.1 1+1d: Lift $\text{Pin}^-(d)$ to $\text{EPin}(d)$ for $\nu_{\text{BDI}} = 4 \in \mathbb{Z}_8$ class

The symmetry extension $\mathbb{Z}_2 \rightarrow \tilde{G} = \mathbb{Z}_4^T \times \mathbb{Z}_2^F \rightarrow G = \mathbb{Z}_2^T \times \mathbb{Z}_2^F$ in Table 4, reduces BDI class from the interacting \mathbb{Z}_8 classification to \mathbb{Z}_4 classification. The $\tilde{G} = \mathbb{Z}_4^T \times \mathbb{Z}_2^F$ in fact can have at least three physical interpretations of the symmetry realization, which in turn give also three physical interpretations of $\mathbb{Z}_2 \rightarrow \text{EPin}(d) \rightarrow \text{Pin}^-(d)$. The $\text{EPin}(d)$ is firstly introduced by Ref. [48].¹⁹ The three interpretations are:

¹⁸We relegate the two subsections B.1 and B.2 to Appendices due to their heavy mathematical formality. However, the readers are strongly encouraged to read Appendix B.1 and B.2 first before proceed to Sec. 6.

¹⁹Below we encounter and define three versions of EPin called $\text{EPin}^{BF-}(d)$, $\text{EPin}^{BF+}(d)$, and $\text{EPin}^{F+F-}(d)$. In fact the $\text{EPin}^{F+F-}(d)$ is denoted as $\text{EPin}(d)$ in Ref. [48]. It is shown in Ref. [48] and in this section that the three versions of EPin are the same group.

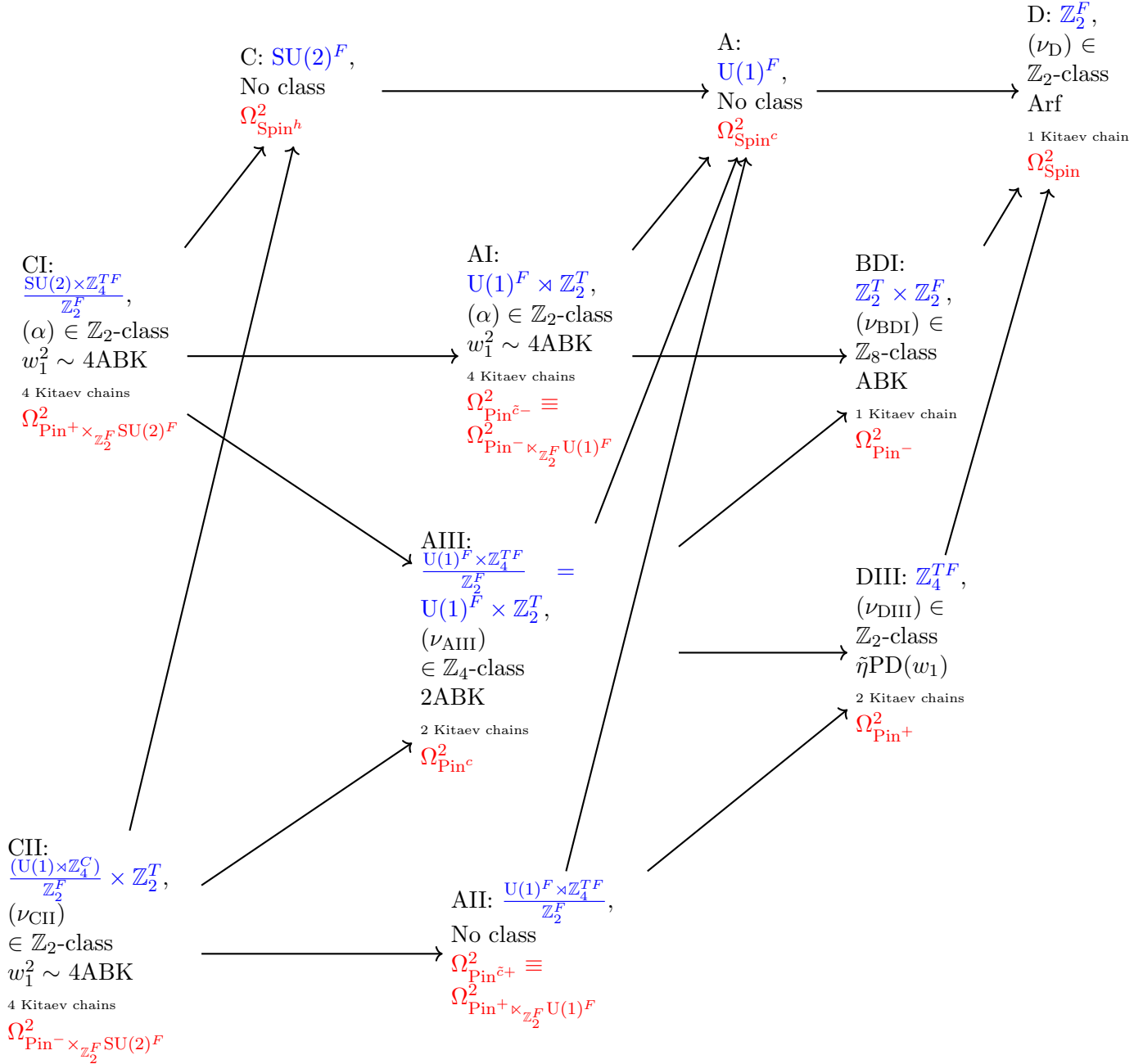


Table 3: The symmetry embedding web of 1+1d fermionic invertible topological phases relevant for 10 Cartan symmetry classes (we follow the notations of Table 2 in [3] and Table 4 in [25] for 3+1D cases). The web suggests the maps between the nontrivial classes of their classifications of SPTs (namely, cobordism invariants). The web can also suggest a possible symmetry group extension to unwind the SPT states. We include topological invariants from the 2d Pin^- bordism invariant Arf-Brown-Kervaire (ABK), and 1d spin bordism invariant $\tilde{\eta}$. The PD means the Poincaré dual. The “ n Kitaev chains” mean the generator of invertible phases of given symmetries.

(1). One interpretation of Table 4’s involves a $\mathcal{T}^2 = -1$ boson B and a $\mathcal{T}^2 = 1$ fermion F_- :

$$\mathbb{Z}_2^B \rightarrow \mathbb{Z}_4^{TB} \times \mathbb{Z}_2^{F-} \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^{F-},$$

Cartan class	Symmetry G	$N \rightarrow \tilde{G} \rightarrow G$	Reduced classification of 1+1d G -iTQFT or 0+1d G -anomaly in \tilde{G}
BDI	$\mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \times \mathbb{Z}_2^F \rightarrow G$	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_4$
		$\mathbb{Z}_2 \rightarrow \mathbb{D}_8^{F,T} \rightarrow G$	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_2$
DIII	\mathbb{Z}_4^{TF}	$\mathbb{Z}_4 \rightarrow \mathbb{M}_{16}^{F,T} \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$
AIII	$U(1)^F \times \mathbb{Z}_2^T$	$\mathbb{Z}_2 \rightarrow U(1)^F \times \mathbb{Z}_4^T \rightarrow G$	$\mathbb{Z}_4 \Rightarrow \mathbb{Z}_2$
		$\mathbb{Z}_2 \rightarrow \frac{U(1) \times \mathbb{D}_8^{F,T}}{\mathbb{Z}_2} \rightarrow G$	$\mathbb{Z}_4 \Rightarrow 0$
CII	$\frac{(U(1)^F \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_2^T$	$\mathbb{Z}_2 \rightarrow \frac{(U(1)^F \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \times \mathbb{Z}_4^T \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$
	$SU(2)^F \times \mathbb{Z}_2^T$	$\mathbb{Z}_2 \rightarrow SU(2)^F \times \mathbb{Z}_4^T \rightarrow G$	$\mathbb{Z}_2 \Rightarrow 0$

Table 4: Summary of unwinding 1+1d fermionic G -SPT of G -invertible topological field theory (iTQFT) phases via the symmetry extension $1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ and the change of classification by symmetry extension. It turns out that the group extension of G by \mathbb{Z}_2 has the $\mathbb{Z}_2 = \mathbb{Z}_2^B$ as a bosonic symmetry acting on any state by multiplying a sign $\{1, -1\}$. So the \mathbb{Z}_4^T in the \tilde{G} is in fact $\mathbb{Z}_4^T = \mathbb{Z}_4^{TB}$ obtained from $1 \rightarrow \mathbb{Z}_2^B \rightarrow \mathbb{Z}_4^{TB} \rightarrow \mathbb{Z}_2^T \rightarrow 1$, which by including (Lorentz to Euclidean) spacetime symmetry becomes $1 \rightarrow \mathbb{Z}_2^B \rightarrow E(d) \rightarrow O(d) \rightarrow 1$, with the $E(d) = \mathbb{Z}_2^B \rtimes O(d) = SO(d) \rtimes \mathbb{Z}_4^{TB}$, see details in Sec. B.2.

Cartan class	Spacetime-Internal Symmetry G_{Tot}	Symmetry Extension to \tilde{G}_{Tot}	Reduced classification of 1+1d G -iTQFT or 0+1d G -anomaly in \tilde{G}
BDI	Pin^-	EPin	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_4$
		$\mathbb{D}_8^{F,T} \text{Pin}$	$\mathbb{Z}_8 \Rightarrow \mathbb{Z}_2$
DIII	Pin^+	$\mathbb{M}_{16}^{F,T} \text{Pin}$	$\mathbb{Z}_2 \Rightarrow 0$
AIII	$\text{Pin}^c = \text{Pin}^- \times_{\mathbb{Z}_2^F} U(1)^F$ $= \text{Pin}^+ \times_{\mathbb{Z}_2^F} U(1)^F$	$\text{EPin} \times_{\mathbb{Z}_2^F} U(1)^F$	$\mathbb{Z}_4 \Rightarrow \mathbb{Z}_2$
		$\mathbb{D}_8^{F,T} \text{Pin} \times_{\mathbb{Z}_2^F} U(1)^F$	$\mathbb{Z}_4 \Rightarrow 0$
CII	$\text{Pin}^- \times_{\mathbb{Z}_2^F} \frac{(U(1)^F \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F}$	$\text{EPin} \times_{\mathbb{Z}_2^F} \frac{(U(1)^F \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F}$	$\mathbb{Z}_2 \Rightarrow 0$
	$\text{Pin}^- \times_{\mathbb{Z}_2^F} SU(2)^F$	$\text{EPin} \times_{\mathbb{Z}_2^F} SU(2)^F$	$\mathbb{Z}_2 \Rightarrow 0$

Table 5: The *continuum* version description of Table 4. The extended \tilde{G}_{Tot} includes the EPin group (firstly introduced in Ref. [48] and reviewed in Sec. 6.1), the $\mathbb{D}_8^{F,T} \text{Pin}$ and $\mathbb{M}_{16}^{F,T} \text{Pin}$ (that we will introduce in Sec. 6.3 and Sec. 6.4).

by including $SO(d)$ which becomes Table 5's

$$\mathbb{Z}_2^B \rightarrow \text{EPin}^{BF-}(d) \equiv \left((\mathbb{Z}_2^{F-} \rtimes SO(d)) \times \mathbb{Z}_2^B \right) \rtimes \mathbb{Z}_2^{T_E} \rightarrow \text{Pin}^-(d). \quad (6.1)$$

Note that $\text{EPin}^{BF-}(d) \equiv \left((\mathbb{Z}_2^{F-} \rtimes SO(d)) \times \mathbb{Z}_2^B \right) \rtimes \mathbb{Z}_2^{T_E} = \left(\text{Spin}(d) \times \mathbb{Z}_2^B \right) \rtimes \mathbb{Z}_2^{T_E}$ has the properties *specifically* here:

- $(\mathbb{Z}_2^{F-} \rtimes SO(d)) \rtimes \mathbb{Z}_2^{T_E} = \text{Spin}(d) \rtimes \mathbb{Z}_2^{T_E} = \text{Pin}^-(d)$.
- In Euclidean signature: $\mathbb{Z}_2^B \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_4^{E_B}$ and $\mathbb{Z}_2^{F-} \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_4^{E_B F-}$.
- In Lorentzian signature: $\mathbb{Z}_2^B \rtimes \mathbb{Z}_2^T = \mathbb{Z}_4^{TB}$ and $\mathbb{Z}_2^{F-} \rtimes \mathbb{Z}_2^T = \mathbb{Z}_2^T \times \mathbb{Z}_2^{F-}$.
- $(SO(d) \times \mathbb{Z}_2^B) \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_2^B \rtimes O(d) = SO(d) \rtimes \mathbb{Z}_4^{E_B} = E(d)$.

Group	w_1	w_1^2	w_2
$\text{SO}(d)$	0	0	free
$\text{Spin}(d)$	0	0	0
$\text{O}(d)$	free	free	free
$\text{E}(d)$	free	0	free
$\text{Pin}^+(d)$	free	free	0
$\text{Pin}^-(d)$	free	$w_1^2 + w_2 = 0$	
$\text{EPin}(d)$	free	0	0

Table 6: Follow Ref. [48], we list down the (spacetime) bundle constraints for various space-time structures. Here we denote $w_j \equiv w_j(TM)$ are the j -th Stiefel-Whitney (SW) classes of the tangent bundle TM of the spacetime manifold M . The product of cohomology classes as $w_1^2 \equiv w_1(TM) \cup w_1(TM)$ is the cup product. The free here means no restriction which can be a nontrivial cohomology class.

- (2). One interpretation of Table 4's involves two types of fermions, F_+ with $\mathcal{T}^2 = -1$ and F_- with $\mathcal{T}^2 = 1$:

$$\mathbb{Z}_2^{F_+} \rightarrow \mathbb{Z}_4^{TF_+} \times \mathbb{Z}_2^{F_-} \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^{F_-},$$

by including $\text{SO}(d)$ which becomes Table 5's

$$\mathbb{Z}_2^{F_+} \rightarrow \text{EPin}^F(d) \equiv \left(\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-} \right) \rtimes \left(\text{SO}(d) \rtimes \mathbb{Z}_2^T \right) \rightarrow \text{Pin}^-(d). \quad (6.2)$$

Note that $\text{EPin}^F(d) \equiv \left(\mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{F_-} \right) \rtimes \left(\text{SO}(d) \rtimes \mathbb{Z}_2^T \right)$ has the properties *specifically* here:

- $\mathbb{Z}_2^{F_+} \rtimes \text{SO}(d) \cong \mathbb{Z}_2^{F_-} \rtimes \text{SO}(d) \cong \text{Spin}(d)$.
- In Euclidean signature: $\mathbb{Z}_2^{F_+} \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_2^{F_+} \times \mathbb{Z}_2^{T_E}$ and $\mathbb{Z}_2^{F_-} \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_4^{T_E F_-}$.
- In Lorentzian signature: $\mathbb{Z}_2^{F_+} \rtimes \mathbb{Z}_2^T = \mathbb{Z}_4^{TF_+}$ and $\mathbb{Z}_2^{F_-} \rtimes \mathbb{Z}_2^T = \mathbb{Z}_2^T \times \mathbb{Z}_2^{F_-}$.
- $\mathbb{Z}_2^{F_\pm} \rtimes \left(\text{SO}(d) \rtimes \mathbb{Z}_2^{T_E} \right) = \text{Pin}^\pm(d)$.

- (3). Another interpretation of Table 4's involves a $\mathcal{T}^2 = -1$ boson B and a $\mathcal{T}^2 = -1$ fermion F_+ :

$$\mathbb{Z}_2^{F_+} \rightarrow \frac{\mathbb{Z}_4^{TB} \times \mathbb{Z}_4^{TF_+}}{\mathbb{Z}_2^T} \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^{F_-},$$

by including $\text{SO}(d)$ which becomes Table 5's

$$\mathbb{Z}_2^B \rightarrow \text{EPin}^{BF_+}(d) \equiv \left((\mathbb{Z}_2^{F_+} \rtimes \text{SO}(d)) \times \mathbb{Z}_2^B \right) \rtimes \mathbb{Z}_2^{T_E} \rightarrow \text{Pin}^-(d). \quad (6.3)$$

Note that $\text{EPin}^{BF_+}(d) \equiv \left((\mathbb{Z}_2^{F_+} \rtimes \text{SO}(d)) \times \mathbb{Z}_2^B \right) \rtimes \mathbb{Z}_2^{T_E}$ has the properties *specifically* here:

- $\text{Pin}^+(d) = (\mathbb{Z}_2^{F_+} \rtimes \text{SO}(d)) \rtimes \mathbb{Z}_2^{T_E} = \text{Spin}(d) \rtimes \mathbb{Z}_2^{T_E} = \frac{\text{Spin}(d) \rtimes \mathbb{Z}_4^{T_E F_+}}{\mathbb{Z}_2^F}$.
- In Euclidean signature: $\mathbb{Z}_2^B \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_4^{T_E B}$ and $\mathbb{Z}_2^{F_+} \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_2^{T_E} \times \mathbb{Z}_2^{F_+}$.
- In Lorentzian signature: $\mathbb{Z}_2^B \rtimes \mathbb{Z}_2^T = \mathbb{Z}_4^{TB}$ and $\mathbb{Z}_2^{F_+} \rtimes \mathbb{Z}_2^T = \mathbb{Z}_4^{TF_+}$.
- $(\text{SO}(d) \times \mathbb{Z}_2^B) \rtimes \mathbb{Z}_2^{T_E} = \mathbb{Z}_2^B \rtimes \text{O}(d) = \text{SO}(d) \rtimes \mathbb{Z}_4^{T_E B} = \text{E}(d)$.

Ref. [48] finds that the three interpretations and expressions eq (6.1), (6.2), and (6.3) are equivalent by relabeling the group elements, so we can generally write all of them as the same as the EPin extension:

$$\mathbb{Z}_2 \rightarrow \text{EPin}(d) \rightarrow \text{Pin}^-(d). \quad (6.4)$$

Our claim is that the $1+1\text{d } \mathbb{Z}_2^T \times \mathbb{Z}_2^F$ fermionic SPTs classified by $\Omega_2^{\text{Pin}^-} = \mathbb{Z}_8$ with $\nu_{\text{BDI}} \in \mathbb{Z}_8$ class 2d topological invariants over a 2-manifold M^2 with Pin^- structure

$$\exp(i \frac{2\pi\nu_{\text{BDI}}}{8} \text{ABK})|_{M^2} \quad (6.5)$$

can be trivialized at $\nu_{\text{BDI}} = 4$ by an extension from Pin^- to EPin. The reason is that 4ABK (4 layers of Kitaev fermionic chain) equals to the topological invariant of the first Stiefel-Whitney classes $w_1^2 \equiv w_1(TM)^2$ (1 layer of Haldane spin chain) [3].²⁰ Namely, the two topological invariants are related

$$\exp(i \frac{2\pi 4}{8} \text{ABK})|_{M^2} \simeq \exp(i\pi(w_1)^2)|_{M^2} \quad (6.6)$$

up to a trivially gapped fermionic sector specifying spin structures.

Let us show the eq (6.4)'s EPin can trivialize the eq (6.6). First, the EPin structure requires both Pin^+ and Pin^- to hold, while Pin^+ requires $w_2 = 0$ and Pin^- requires $w_2 + w_1^2 = 0$, so EPin requires $w_2 = w_1^2 = 0$, but all can have nontrivial w_1 to be on unorientable manifolds. (See Ref. [48] and Table 6 for a summary of the bundle constraints for various symmetry groups and corresponding spacetime structures.) Second, eq (6.6) also implies $\exp(i\pi(w_1)^2) = \exp(i\pi w_2)$ in 2d Pin^- , while both expressions become trivial as 1 in EPin. Thus EPin as a symmetry extension of Pin^- trivializes eq (6.4) for $\nu_{\text{BDI}} = 4$.

We can also write down how the symmetry operator S of $\text{EPin}(d)$ group acts on the state vectors of Hilbert space

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_{F_+} \oplus \mathcal{H}_{F_-},$$

spanned by three sectors, the bosonic sector $|B\rangle$ with $T^2 = +1$, the fermionic sector $|F_+\rangle$ with $T^2 = -1$, and the fermionic sector $|F_-\rangle$ with $T^2 = +1$. Naively one looks for the larger symmetry including $O(d)$, $\text{Pin}^+(d)$, and $\text{Pin}^-(d)$ for three sectors in the continuum, in a more general manifold on M^{d+1} . But for quantum mechanics, it is more practical to define the Hilbert space on a flat space on a \mathbb{R}^{d-1} with a continuous time t on \mathbb{R}^1 . Then we find the operator S acts on state vectors living on the \mathbb{R}^{d-1} generating the following groups for each sector:

$$\begin{pmatrix} O(d-1) \times \mathbb{Z}_2^T & 0 & 0 \\ 0 & \text{Pin}^+(d-1) \times \mathbb{Z}_2^F \mathbb{Z}_4^{TF} & 0 \\ 0 & 0 & \text{Pin}^-(d-1) \times \mathbb{Z}_2^T \end{pmatrix} \text{ acts on } \begin{pmatrix} |B\rangle \\ |F_+\rangle \\ |F_-\rangle \end{pmatrix}. \quad (6.7)$$

Note that \mathbb{Z}_2^{F+} and \mathbb{Z}_2^{F-} have the two generators

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ act on } \begin{pmatrix} |B\rangle \\ |F_+\rangle \\ |F_-\rangle \end{pmatrix},$$

²⁰Here we denote $w_j \equiv w_j(TM)$ are the j -th Stiefel-Whitney (SW) classes of the tangent bundle TM of the spacetime manifold M . For the SW class for the vector bundle V_G associated with certain group G (where G can be gauge group or global symmetry group), we denote it as $w_j(V_G)$ or simply $w_j(G)$. All the product notations between cohomology classes are cup products, such as $w_i w_j := w_i(TM) w_j(TM) = w_i(TM) \cup w_j(TM)$.

each sits at the \mathbb{Z}_2 center of Spin group within Pin^+ and Pin^- groups for each. If we choose $|B\rangle$ with $T^2 = -1$, we must change eq (6.7)'s $\text{O}(d-1) \times \mathbb{Z}_2^T$ to $\text{E}(d-1) \times_{\mathbb{Z}_2^B} \mathbb{Z}_4^{TB}$. This eq (6.7) is our interpretation of $\text{EPin}(d)$ acting on the assumed Hilbert space.

In summary, eq (6.4) on $\mathbb{Z}_2 \rightarrow \text{EPin}(d) \rightarrow \text{Pin}^-(d)$ gives the mathematical way to understand the unwinding of 4 layers of Kitaev chains via the symmetry extension $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \times \mathbb{Z}_2^F \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^F$ demonstrated in Ref. [3]'s Sec. 4.

6.2 1+1d: Lift $\text{O}(d)$ to $\text{E}(d)$ for $\nu_{\text{CII}} = 1 \in \mathbb{Z}_2$ or even $\nu_{\text{AIII}} \in \mathbb{Z}_4$ class

In fact, the previous extension method in Sec. 6.1 also works for 1+1d SPTs with the same topological invariants $\exp(i\pi(w_1)^2)$ but with other symmetry groups of Table 4, such as CII and AIII classes. In the discrete version, we have $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4^T \rightarrow \mathbb{Z}_2^T$, converting to a continuum language becomes $\mathbb{Z}_2 \rightarrow \text{E}(d) \rightarrow \text{O}(d)$.

For example, Sec. 3.5's CII class symmetry extension eq (3.55) and eq (3.56), and AIII even $\nu_{\text{AIII}} \in \mathbb{Z}_4$ class studied in Ref. [3]'s Sec. 4, can be understood by including the spacetime symmetry in the continuum as the following extensions $N \rightarrow \tilde{G}_{\text{Tot}} \rightarrow G_{\text{Tot}}$:

$$\begin{aligned} \text{CII: } \mathbb{Z}_2 &\rightarrow \text{EPin} \times_{\mathbb{Z}_2^F} \text{SU}(2)^F \rightarrow \text{Pin}^- \times_{\mathbb{Z}_2^F} \text{SU}(2)^F. \\ \mathbb{Z}_2 &\rightarrow \text{EPin} \times_{\mathbb{Z}_2^F} \frac{(\text{U}(1) \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F} \rightarrow \text{Pin}^- \times_{\mathbb{Z}_2^F} \frac{(\text{U}(1) \rtimes \mathbb{Z}_4^C)}{\mathbb{Z}_2^F}. \\ \text{AIII: } \mathbb{Z}_2 &\rightarrow \text{EPin} \times_{\mathbb{Z}_2^F} \text{U}(1)^F \rightarrow \text{Pin}^- \times_{\mathbb{Z}_2^F} \text{U}(1)^F. \\ \text{In principle: } \mathbb{Z}_2 &\rightarrow \text{E}(d) \rightarrow \text{O}(d). \end{aligned} \quad (6.8)$$

Here $d = 2$. The above extension works because the $\exp(i\pi(w_1)^2)$ is a nontrivial topological invariant in Ω_2^{O} but becomes trivial by pulling it back to Ω_2^{E} due to the constraint $(w_1)^2 = 0$ for $\text{E}(d)$ structure, see Table 6. More generally, the cocycle $(w_1)^2$ becomes a coboundary, thus $(w_1)^2 = \delta\beta_1$, which splits into one lower dimensional 1-cochain β_1 under the pullback trivialization.²¹

6.3 1+1d supersymmetry extension: Lift Pin^- to $\mathbb{D}_8^{F,T} \text{Pin}$ for even $\nu_{\text{BDI}} \in \mathbb{Z}_8$ class

We have given a discrete group interpretation of $\mathbb{D}_8^{F,T}$ of Table 4 and the extension:

$$\mathbb{Z}_2^B \rightarrow (\mathbb{D}_8^{F,T} \equiv \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F) \rightarrow \mathbb{Z}_2^T \times \mathbb{Z}_2^F.$$

Now we will like to include the continuous $\text{SO}(d)$ sector in order to generalize our formulation to the continuum QFT. Note that $\mathbb{D}_8^{F,T} \text{Pin}(d)$ contains the ingredients

$$\begin{cases} \mathbb{Z}_2^F \rtimes \text{SO}(d) = \text{Spin}(d), \\ \text{SO}(d) \rtimes \mathbb{Z}_4^{TB} = \text{E}(d), \\ \mathbb{D}_8^{F,T} \equiv \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F. \end{cases} \quad (6.9)$$

However, there is a *caveat*: The short exact sequence

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \tilde{G}_{\text{Tot}} \rightarrow \text{Pin}^-(d) \rightarrow 1$$

is classified by $\tilde{G}_{\text{Tot}} = \mathbb{Z}_2 \rtimes_{\rho, \varphi} \text{Pin}^-(d)$ controlled by no $\rho \in G_{\text{Tot}} \rightarrow \text{Aut}(\mathbb{Z}_2) = 0$, but only by a 2-cocycle $\varphi \in H^2(\text{BPin}^-(d), \mathbb{Z}_2) = \mathbb{Z}_2$:

²¹More generally in a multiplicative notation, in terms of 2-cocycle with $\text{U}(1)$ coefficient: $\omega_2 = (-1)^{f(w_1)^2}$, we mean the $\omega_2 = \delta\alpha_1$ can split into one lower dimensional 1-cochain α_1 under the pullback trivialization.

- The trivial φ gives $\tilde{G}_{\text{Tot}} = \mathbb{Z}_2 \times \text{Pin}^-(d)$.
- The nontrivial $\varphi = w_1^2$ gives the $\tilde{G}_{\text{Tot}} = \text{EPin}(d)$.

So naively \tilde{G}_{Tot} has only two solutions, $\mathbb{Z}_2 \times \text{Pin}^-(d)$ and EPin , there is *no* $\mathbb{D}_8^{F,T} \text{Pin}$. How can we make sense the extension from $\text{Pin}^-(d)$ to $\mathbb{D}_8^{F,T} \text{Pin}(d)$?²²

The answer comes to the subtle fact that the time-reversal \mathcal{T} in $\mathbb{D}_8^{F,T}$ has to switch between bosonic and fermionic sectors, thus a *supersymmetry extension* comes in. We can write down how the symmetry operator S of $\mathbb{D}_8^{F,T} \text{Pin}$ group acts on the state vectors of Hilbert space

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F,$$

spanned by two sectors, the bosonic sector $|B\rangle$ with $T^2 = -1$, the fermionic sector $|F\rangle$ with $T^2 = -1$. Again it is practical to define the Hilbert space on a flat space on a \mathbb{R}^{d-1} with a continuous time t on \mathbb{R}^1 . Then we find the S acts on state vectors living on the \mathbb{R}^{d-1} generating the following groups for each sector:²³

1. The unitary \mathbb{Z}_2^F generator is

$$P_F = (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z, \text{ which acts on } \begin{pmatrix} |B\rangle \\ |F\rangle \end{pmatrix}. \quad (6.11)$$

We omit a square identity matrix $\mathbb{1}$ that acts on the whole Hilbert space of $|B\rangle$ and $|F\rangle$ (fermionic and bosonic sectors have equal Hilbert space dimensions), so precisely $P_F = \sigma_z \otimes \mathbb{1}$.

2. The anti-unitary \mathbb{Z}_4^T generator, with complex conjugation K , is

$$\mathcal{T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} K = \sigma_y K, \text{ which acts on } \begin{pmatrix} |B\rangle \\ |F\rangle \end{pmatrix}, \quad (6.12)$$

or precisely $\mathcal{T} = (\sigma_y \otimes \mathbb{1})K$. The \mathcal{T} switches between $|B\rangle$ and $|F\rangle$.

3. The $\mathbb{D}_8^{F,T} \equiv \mathbb{Z}_4^{TB} \rtimes \mathbb{Z}_2^{F-}$ is generated by $\{\mathcal{T}, P_F | \mathcal{T}^4 = P_F^2 = 1, P_F \mathcal{T} P_F = \mathcal{T}^3 = -\mathcal{T}\}$. Note that $\mathcal{T}^2 = -1$ for all bosonic and fermionic states.
4. The Spin group has elements S_{rot} acting on the Hilbert space associated with a \mathbb{R}^{d-1} space:

$$S_{\text{rot}} = \begin{pmatrix} \text{SO}(d-1) & 0 \\ 0 & \text{Spin}(d-1) \end{pmatrix} \text{ acts on } \begin{pmatrix} |B\rangle \\ |F\rangle \end{pmatrix}. \quad (6.13)$$

The P_F is the \mathbb{Z}_2 center of Spin group.

²²We should clarify that our $\mathbb{D}_8^{F,T} \text{Pin}(d)$ is distinct from the $\text{DPin}(d)$ studied in [49, 50]. The $\mathbb{D}_8^{F,T} \text{Pin}(d)$ contains a $\mathbb{D}_8^{F,T} \equiv \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^F$, where fermion parity is not at the $\mathbb{D}_8^{F,T}$'s center. The $\text{DPin}(d)$ contains a different \mathbb{D}_8 which is $\mathbb{D}_8 = (\mathbb{Z}_2^{F+} \times \mathbb{Z}_2^{F-}) \rtimes_{\rho,0} \mathbb{Z}_2^T$ where ρ is a nontrivial \mathbb{Z}_2^T action on $\text{Aut}(\mathbb{Z}_2^{F+} \times \mathbb{Z}_2^{F-})$ with two kinds of fermion parity $\mathbb{Z}_2^{F+} \times \mathbb{Z}_2^{F-}$ at the \mathbb{D}_8 's center. In fact, in this article, we will not pursue a Euclidean spacetime realization of $\mathbb{D}_8^{F,T} \text{Pin}(d)$. We do not yet know whether the full version $\mathbb{D}_8^{F,T} \text{Pin}(d)$ (with the full spacetime rotation $\text{SO}(d)$) exists mathematically. Since we do not specify the full dd spacetime rotational part, we will denote it as $\mathbb{D}_8^{F,T} \text{Pin}$ from now on. We shall only discuss the $\mathbb{D}_8^{F,T} \text{Pin}$ symmetry generator acts on the states in the Hilbert space.

²³ We represent the complex and Majorana fermionic operators acting on 2-dimensional Hilbert space $\begin{pmatrix} |B\rangle \\ |F\rangle \end{pmatrix}$ as:

$$\begin{aligned} \psi &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv \sigma_- \equiv \frac{(\sigma_x + i\sigma_y)}{2}, & \psi^\dagger &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv \sigma_+ \equiv \frac{(\sigma_x - i\sigma_y)}{2}, & F &\equiv N_F \equiv \psi^\dagger \psi \equiv \sigma_+ \sigma_- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \\ \gamma &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x, & \bar{\gamma} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y, & P_F &= (-1)^F = i\bar{\gamma}\gamma = -i\gamma\bar{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z. \end{aligned} \quad (6.10)$$

The following matrix representations of higher-dimensional Hilbert spaces are the direct parallel story of 2-dimensional Hilbert space.

Combining the group elements P_F , \mathcal{T} , and S_{rot} together, we generate the full structure of $\mathbb{D}_8^{F,T}\text{Pin}$ that acts on the assume Hilbert space $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ associated with the \mathbb{R}^{d-1} space. We name such an extension as a *supersymmetry extension* when we require some of the group elements (here \mathcal{T}) permutes bosonic and fermionic sectors. Formally, we propose that by pulling back Pin^- to $\mathbb{D}_8^{F,T}\text{Pin}$ via the *supersymmetry extension*, we can trivialize the even $\nu \in \Omega_2^{\text{Pin}^-} = \mathbb{Z}_8$ whose cobordism invariant eq (6.5) can be written as:

$$\exp(i \frac{2\pi 2}{8} \text{ABK})|_{M^2} \simeq \exp(i \frac{2\pi}{4} \eta' \text{PD}(w_1(TM)))|_{M^2}. \quad (6.14)$$

The η' is the 1d cobordism invariant of $\Omega_1^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \mathbb{Z}_4$, which is a \mathbb{Z}_4 extended version of the $\tilde{\eta}$ (which is the 1d invariant of $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$). See more discussions about cobordism invariants in Sec. 7.1.

6.4 1+1d supersymmetry extension: Lift Pin^+ to $\mathbb{M}_{16}^{F,T}\text{Pin}$ for odd $\nu_{\text{DIII}} \in \mathbb{Z}_2$ class

We have given a discrete group interpretation of $\mathbb{M}_{16}^{F,T}$ of Table 4 via an extension:

$$\mathbb{Z}_4 \rightarrow (\mathbb{M}_{16}^{F,T} \equiv \mathbb{Z}_4^{TF} \rtimes \mathbb{Z}_4) \rightarrow \mathbb{Z}_4^{TF}.$$

Similar to Sec. 6.3, we write down how the symmetry operator S of $\mathbb{M}_{16}^{F,T}\text{Pin}$ group acts on the state vectors of Hilbert space²⁴

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F,$$

spanned by two sectors, the bosonic sector $|B\rangle$ and the fermionic sector $|F\rangle$ both with $T^4 = -1$ and $T^8 = +1$. Again it is practical to define the Hilbert space on a flat space on a \mathbb{R}^{d-1} with a continuous time t on \mathbb{R}^1 . Then we find the S acts on state vectors living on the \mathbb{R}^{d-1} generating the following groups for each sector:

1. The unitary \mathbb{Z}_2^F generator is the same as eq (6.11).
2. The anti-unitary \mathbb{Z}_8^T generator, with complex conjugation K , is

$$\mathcal{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} K = (\frac{\sigma_x - \sigma_y}{\sqrt{2}}) K, \text{ which acts on } \begin{pmatrix} |B\rangle \\ |F\rangle \end{pmatrix}, \quad (6.15)$$

or precisely $\mathcal{T} = ((\frac{\sigma_x - \sigma_y}{\sqrt{2}}) \otimes \mathbb{1}) K$. The \mathcal{T} switches between $|B\rangle$ and $|F\rangle$.

3. The $\mathbb{M}_{16}^{F,T}$ is generated by $\{\mathcal{T}, P_F | \mathcal{T}^8 = P_F^2 = 1, P_F \mathcal{T} P_F = \mathcal{T}^5 = -\mathcal{T}\}$. Note that $\mathcal{T}^2 = -i P_F = -i(\sigma_z \otimes \mathbb{1})$, while $\mathcal{T}^4 = -1$ for all bosonic and fermionic states.
4. The Spin group has elements S_{rot} the same as eq (6.13), with P_F its \mathbb{Z}_2 center.

Combining the group elements P_F , \mathcal{T} , and S_{rot} together, we generate the full structure of $\mathbb{M}_{16}^{F,T}\text{Pin}$ that acts on the assume Hilbert space $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ associated with the \mathbb{R}^{d-1} space. Again such an extension is a *supersymmetry extension* when we require some group element (here \mathcal{T}) permutes bosonic and fermionic sectors. Formally, we propose that we can trivialize the odd $\nu \in \Omega_2^{\text{Pin}^+} = \mathbb{Z}_2$ class whose cobordism invariant is the same as in eq (6.14), by pulling back Pin^+ to $\mathbb{M}_{16}^{F,T}\text{Pin}$ via a *supersymmetry extension*.

²⁴Similar to Footnote 22, in this article, we will not pursue a Euclidean spacetime realization of $\mathbb{M}_{16}^{F,T}\text{Pin}(d)$. We do not yet know whether the full version $\mathbb{M}_{16}^{F,T}\text{Pin}(d)$ (with the full spacetime rotation $\text{SO}(d)$) exists mathematically. Since we do not specify the full dd spacetime rotational part, we will denote it as $\mathbb{M}_{16}^{F,T}\text{Pin}$ from now on. We shall only discuss the $\mathbb{M}_{16}^{F,T}\text{Pin}$ symmetry generator acts on the states in the Hilbert space.

7 Conclusion and Higher Dimensional Generalization

Let us discuss the higher dimensional generalization of the *fermionic symmetry extension* and *supersymmetry extension*.

Earlier we have studied $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ (Pin^-) and \mathbb{Z}_4^{TF} (Pin^+) and their *fermionic symmetry extension* and *supersymmetry (SUSY) extension* via a finite group extension. In fact, there are maps known as Smith homomorphisms in any dimension relating the following SPT classes and symmetry groups (see [23, 51, 52] and also [26, 48]):

$$(n+3)\text{d unitary } \mathbb{Z}_2 \times \mathbb{Z}_2^F \Rightarrow (n+2)\text{d anti-unitary } \mathbb{Z}_2^T \times \mathbb{Z}_2^F \Rightarrow (n+1)\text{d unitary } \mathbb{Z}_4^F \Rightarrow n\text{d anti-unitary } \mathbb{Z}_4^{TF} \Rightarrow \dots \quad (7.1)$$

By an adding SO rotational symmetry in terms of continuous groups, eq (7.1) becomes

$$(n+3)\text{d Spin} \times \mathbb{Z}_2 \Rightarrow (n+2)\text{d Pin}^- \Rightarrow (n+1)\text{d Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4 \Rightarrow n\text{d Pin}^+ \Rightarrow \dots \quad (7.2)$$

Smith homomorphism maps include the maps between SPT classifications of these symmetries [23, 48, 51, 52]:

$$\begin{aligned} \Omega_8^{\text{Pin}^+} = \mathbb{Z}_{32} \times \mathbb{Z}_2 &\rightarrow \Omega_7^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_{16}, \\ \Omega_7^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_{16} &\rightarrow \Omega_6^{\text{Pin}^-} = \mathbb{Z}_{16}, \\ \Omega_6^{\text{Pin}^-} = \mathbb{Z}_{16} &\rightarrow \Omega_5^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_{16}, \\ \Omega_5^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_{16} &\rightarrow \Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}, \\ \Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16} &\rightarrow \Omega_3^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_8, \\ \Omega_3^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_8 &\rightarrow \Omega_2^{\text{Pin}^-} = \mathbb{Z}_8, \\ \Omega_2^{\text{Pin}^-} = \mathbb{Z}_8 &\rightarrow \Omega_1^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_4, \\ \Omega_1^{\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4} = \mathbb{Z}_4 &\rightarrow \Omega_0^{\text{Pin}^+} = \mathbb{Z}_2. \end{aligned} \quad (7.3)$$

In fact, as found in Ref. [48] and [53], the \mathbb{Z}_2 extensions can trivialize many SPT classes — thus this symmetry extension can help to construct the symmetry-extended gapped boundaries (lifting all boundary zero modes):

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{EPin}(d) \rightarrow \text{Pin}^\pm \rightarrow 1. \quad (7.4)$$

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}(d) \times \mathbb{Z}_4 \rightarrow \begin{array}{c} \text{Spin}(d) \times \mathbb{Z}_2 \\ \text{or} \\ \frac{\text{Spin}(d) \times \mathbb{Z}_4}{\mathbb{Z}_2} \end{array} \rightarrow 1. \quad (7.5)$$

In the following subsections, we make some comments about these extensions and the generalization to supersymmetry extensions, for various dimensions.

7.1 1+1d

For $d = 2$ or $1 + 1\text{d}$: In the bulk part of our work, we had focused on $d = 2$ or $1 + 1\text{d}$ cases. We have $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ SPTs classified by $\Omega_2^{\text{Pin}^-} = \mathbb{Z}_8$, which involves 3d SPT cobordism invariant with $\nu \in \mathbb{Z}_8$ [23]:

$$\exp\left(\frac{2\pi i \nu}{8} \int_{M^2} \text{ABK}\right). \quad (7.6)$$

- When $\nu = 4$, we can show that the four layers of Kitaev fermionic chain ($\nu = 4$) has SPT invariants 4ABK relates to the bosonic Haldane spin chain $w_1(TM)^2$ up to a trivial gapped fermionic sector,

$$\exp\left(\frac{2\pi i 4}{8} \int_{M^2} \text{ABK}\right) = \exp\left(\pi i \int_{M^2} w_1(TM)^2\right).$$

such as in Pin^- which becomes trivialized $w_1(TM)^2 = \delta\mu$ as a coboundary in EPin , via the group extension $1 \rightarrow \mathbb{Z}_2 \rightarrow \text{EPin}(d) \rightarrow \text{Pin}^- \rightarrow 1$, see the result of [48], Sec. 6.1, and Sec. 6.2.

- When $\nu = 2$, we can show that 2ABK is related to $w_1(TM)\eta' \equiv \eta' \text{PD}(w_1(TM))$ via

$$\exp\left(\frac{2\pi i 2}{8} \int_{M^2} \text{ABK}\right) = \exp\left(\frac{\pi i}{2} \int_{M^2} \eta' \text{PD}(w_1(TM))\right).$$

The η' is the 1d cobordism invariant of $\Omega_1^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \mathbb{Z}_4$, a \mathbb{Z}_4 enhancement of the $\tilde{\eta}$. The $\tilde{\eta}$ is the 1d cobordism invariant of $\Omega_1^{\text{Spin}} = \mathbb{Z}_2$ which assigns different spin structures with different values:

$$\begin{cases} \text{For the periodic (Ramond) boundary condition, } \tilde{\eta} = 1. \\ \text{For the anti-periodic (Neveu-Schwarz) boundary condition, } \tilde{\eta} = 0. \end{cases}$$

The $\eta' \text{PD}(w_1(TM))$ is also a \mathbb{Z}_4 class like η' . In Sec. 3.2, we have shown this 2 ABK invariant can be trivialized by a *SUSY extension* involving a discrete $\mathbb{D}_8^{F,T}$ that we introduce (or we denote it as a $\mathbb{D}_8^{F,T} \text{Pin}$ in the continuum).

- When $\nu = 1$, our method does not apply. But Dijkgraaf-Witten [54] suggests that 2d topological gravity can lift up a single Majorana zero mode and thus trivialize the Kitaev chain. Other related 10 Cartan symmetry classes and 2d topological gravity are explored subsequently by Stanford-Witten [55].

7.2 2+1d

For $d = 3$ or $2 + 1d$: we have $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ SPTs classified by $\Omega_3^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_8$, which involves 3d SPT cobordism invariant with $\nu \in \mathbb{Z}_8$ [23, 56]:

$$\exp\left(\frac{2\pi i \nu}{8} \int_{M^3} \text{ABK}(\text{PD}(A))\right), \quad (7.7)$$

where ABK is the Arf-Brown-Kervaire (ABK) invariant and A is a \mathbb{Z}_2 gauge field so it is a mod 2 cohomology class $A \in H^1(M, \mathbb{Z}_2)$.

- When $\nu = 4$, Ref. [4, 48] shows that the 4ABK(PD(A)) is related to A^3 . This is the same generator as a 2+1d bosonic SPTs with \mathbb{Z}_2 symmetry given by $H^3(B\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$, namely up to a trivial gapped fermionic sector

$$\exp\left(\frac{2\pi i 4}{8} \int_{M^3} \text{ABK}(\text{PD}(A))\right) = \exp\left(\pi i \int_{M^3} A^3\right).$$

- When $\nu = 2$, the 2ABK(PD(A)) is related to $A^2\eta' \equiv \eta'(\text{PD}(A^2))$. Namely,

$$\exp\left(\frac{2\pi i 2}{8} \int_{M^3} \text{ABK}(\text{PD}(A))\right) = \exp\left(\frac{\pi i}{2} \int_{M^3} \eta'(\text{PD}(A^2))\right).$$

Again, the η' is the 1d cobordism invariant of $\Omega_1^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \mathbb{Z}_4$. By pulling back $\text{Spin} \times \mathbb{Z}_2$ to $\text{Spin} \times \mathbb{Z}_4$, we can trivialize A^2 by a \mathbb{Z}_2 extension, thus we also trivialize all the even classes $\nu = 4, 2$ at once [48, 53].

- When $\nu = 1$ or odd, it will be interesting to check whether our *SUSY extension* method applies.

7.3 3+1d

For $d = 4$ or $3+1d$: We have \mathbb{Z}_4^{TF} SPTs classified by $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$, which involves 4d SPT cobordism invariant with $\nu \in \mathbb{Z}_{16}$ [18, 23]:

$$\exp\left(\frac{2\pi i \nu}{16} \int_{M^4} \eta\right), \quad (7.8)$$

where η is the famous eta invariant.

- When $\nu = 8$, the 8η is related to $w_1(TM)^4$ up to trivial gapped fermions, namely

$$\exp\left(\frac{2\pi i 8}{16} \int_{M^4} \eta\right) = \exp\left(\pi i \int_{M^4} w_1(TM)^4\right).$$

- When $\nu = 4$, the 4η is related to $w_1(TM)^3 \eta' \equiv \eta'(\text{PD}(w_1(TM)^3))$, namely

$$\exp\left(\frac{2\pi i 4}{16} \int_{M^4} \eta\right) = \exp\left(\frac{\pi i}{2} \int_{M^4} \eta'(\text{PD}(w_1(TM)^3))\right).$$

- When $\nu = 2$, the 2η is related to $w_1(TM)^2 \text{ABK} \equiv \text{ABK}(\text{PD}(w_1(TM)^2))$, namely

$$\exp\left(\frac{2\pi i 2}{16} \int_{M^4} \eta\right) = \exp\left(\frac{\pi i}{4} \int_{M^4} \text{ABK}(\text{PD}(w_1(TM)^2))\right).$$

By pulling back Pin^+ to EPin , we can trivialize $w_1(TM)^2$ by a \mathbb{Z}_2 extension, thus we also trivialize all the even classes $\nu = 8, 4, 2$ at once [48, 53].

- When $\nu = 1$ or odd, it will be interesting to check whether our *SUSY extension* method applies.

7.4 4+1d

For $d = 5$ or $4+1d$: We have \mathbb{Z}_4^F SPTs classified by $\Omega_5^{\text{Spin} \times \mathbb{Z}_2 \mathbb{Z}_4} = \mathbb{Z}_{16}$, which involves 5d SPT cobordism invariant with $\nu \in \mathbb{Z}_{16}$ [26, 48, 51]:

$$\exp\left(\frac{2\pi i \nu}{16} \int_{M^5} \eta(\text{PD}(A))\right). \quad (7.9)$$

- When $\nu = 8$, the $8\eta(\text{PD}(A))$ is related to A^5 up to trivial gapped fermions, namely

$$\exp\left(\frac{2\pi i 8}{16} \int_{M^5} \eta(\text{PD}(A))\right) = \exp\left(\pi i \int_{M^5} A^5\right).$$

- When $\nu = 4$, the $4\eta(\text{PD}(A))$ is related to $A^4 \eta' \equiv \eta'(\text{PD}(A^4))$, namely

$$\exp\left(\frac{2\pi i 4}{16} \int_{M^5} \eta(\text{PD}(A))\right) = \exp\left(\frac{\pi i}{2} \int_{M^5} \eta'(\text{PD}(A^4))\right).$$

- When $\nu = 2$, the $2\eta(\text{PD}(A))$ is related to $A^3 \text{Arf} \equiv \text{ABK}(\text{PD}(A^3))$, namely

$$\exp\left(\frac{2\pi i 2}{16} \int_{M^5} \eta(\text{PD}(A))\right) = \exp\left(\frac{\pi i}{4} \int_{M^5} \text{ABK}(\text{PD}(A^3))\right).$$

By pulling back $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ to $\text{Spin} \times \mathbb{Z}_4$, we can trivialize A^3 by a \mathbb{Z}_2 extension, thus we also trivialize all the even classes $\nu = 8, 4, 2$ at once [48, 53].

- When $\nu = 1$ or odd, it will be interesting to check whether our *SUSY extension* method applies.

These results of 4d \mathbb{Z}_{16} 't Hooft anomalies have applications to the 3+1d Beyond the Standard Model physics and non-perturbative interactions [57–61]. Our work suggests that the \mathbb{Z}_2 symmetry extension [48, 53] and possible *SUSY extensions* can help to compensate the $\nu \in \mathbb{Z}_{16}$ 't Hooft anomaly via:

- ⎧ a 3+1d symmetry-extended TQFT by trivializing the $\nu \in \mathbb{Z}_{16}$ anomaly via the symmetry-extension [4].
- ⎧ a 3+1d symmetric anomalous gapped TQFT with the $\nu \in \mathbb{Z}_{16}$ 't Hooft anomaly.

The 3+1d symmetric anomalous gapped TQFT with the $\nu \in \mathbb{Z}_{16}$ 't Hooft anomaly may only require a \mathbb{Z}_2 gauge group if the symmetry extension only requires a \mathbb{Z}_2 -extension.

If there is no 't Hooft anomaly, namely $\nu = 0 \pmod{16}$, then we can in principle have a 3+1d symmetric gap boundary preserving the $\text{Spin} \times_{\mathbb{Z}_2} \mathbb{Z}_4$ symmetry with a trivial order (no topological order, no TQFT, no symmetry-breaking order, etc.). See the related work on the 3+1d physics along this research direction [62–67].

7.5 5+1d

For $d = 6$ or $5 + 1d$: We have $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ SPTs classified by $\Omega_6^{\text{Pin}^-} = \mathbb{Z}_{16}$, which involves 6d SPT invariant with $\nu \in \mathbb{Z}_{16}$ [53]:

$$\exp\left(\frac{2\pi i \nu}{16} \int_{M^6} \eta(\text{PD}(w_1(TM)^2))\right). \quad (7.10)$$

- When $\nu = 8$, the $8\eta(\text{PD}(w_1(TM)^2))$ is related to $w_1(TM)^6$ up to trivial gapped fermions, namely

$$\exp\left(\frac{2\pi i 8}{16} \int_{M^6} \eta(\text{PD}(w_1(TM)^2))\right) = \exp\left(\pi i \int_{M^6} w_1(TM)^6\right).$$

- When $\nu = 4$, the $4\eta(\text{PD}(w_1(TM)^2))$ is related to $w_1(TM)^5 \eta' \equiv \eta'(\text{PD}(w_1(TM)^6))$, namely

$$\exp\left(\frac{2\pi i 4}{16} \int_{M^6} \eta(\text{PD}(w_1(TM)^2))\right) = \exp\left(\frac{\pi i}{2} \int_{M^6} \eta'(\text{PD}(w_1(TM)^5))\right).$$

- When $\nu = 2$, the $2\eta(\text{PD}(w_1(TM)^2))$ is related to $w_1(TM)^4 \text{ABK} \equiv \text{ABK}(\text{PD}(w_1(TM)^4))$, namely

$$\exp\left(\frac{2\pi i 2}{16} \int_{M^6} \eta(\text{PD}(w_1(TM)^2))\right) = \exp\left(\frac{\pi i}{4} \int_{M^6} \text{ABK}(\text{PD}(w_1(TM)^4))\right).$$

- When $\nu = 1$, we have the $\eta(\text{PD}(w_1(TM)^2))$. However, by pulling back Pin^- to EPin , we can trivialize $w_1(TM)^2$ by a \mathbb{Z}_2 extension, thus we also trivialize all these classes $\nu = 8, 4, 2, 1$ and any $\nu \in \mathbb{Z}_{16}$ at once [48, 53].

7.6 6+1d

For $d = 7$ or $6 + 1d$: We have $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ SPTs classified by $\Omega_7^{\text{Spin} \times \mathbb{Z}_2} = \mathbb{Z}_{16}$, which involves 7d SPT invariant with $\nu \in \mathbb{Z}_{16}$ [53]:

$$\exp\left(\frac{2\pi i \nu}{16} \int_{M^7} \eta(\text{PD}(A^3))\right). \quad (7.11)$$

- When $\nu = 8$, the $8\eta(\text{PD}(A^3))$ is related to A^7 up to trivial gapped fermions, namely

$$\exp\left(\frac{2\pi i 8}{16} \int_{M^7} \eta(\text{PD}(A^3))\right) = \exp\left(\pi i \int_{M^7} A^7\right).$$

- When $\nu = 4$, the $4\eta(\text{PD}(A^3))$ is related to $A^6\tilde{\eta} \equiv \tilde{\eta}(\text{PD}(A^6))$, namely

$$\exp\left(\frac{2\pi i 4}{16} \int_{M^7} \eta(\text{PD}(A^3))\right) = \exp\left(\frac{\pi i}{2} \int_{M^7} \eta'(\text{PD}(A^6))\right).$$

- When $\nu = 2$, the $2\eta(\text{PD}(A^3))$ is related to $A^5\text{ABK} \equiv \text{ABK}(\text{PD}(A^5))$, namely

$$\exp\left(\frac{2\pi i 2}{16} \int_{M^7} \eta(\text{PD}(A^3))\right) = \exp\left(\frac{\pi i}{4} \int_{M^7} \text{ABK}(\text{PD}(A^5))\right).$$

- When $\nu = 1$, we have the $\eta(\text{PD}(A^3))$. However, by pulling back $\text{Spin} \times \mathbb{Z}_2$ to $\text{Spin} \times \mathbb{Z}_4$, we can trivialize A^3 by a \mathbb{Z}_2 extension, thus we also trivialize all these classes $\nu = 8, 4, 2, 1$ and any $\nu \in \mathbb{Z}_{16}$ at once [48, 53].

7.7 Smooth (DIFF) manifolds to topological (TOP) manifolds

Sec. 7.3 shows that the following extension can trivialize the cobordism invariant of $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$ when ν is even $(2, 4, 6, 8, 10, 12, 14, 16) \in \mathbb{Z}_{16}$:

$$\mathbb{Z}_2 \rightarrow \text{EPin} \rightarrow \text{Pin}^+, \quad (7.12)$$

because we can trivialize the $w_1^2(TM)$ in EPin . The EPin is introduced in Ref. [48] and reviewed in Sec. 6.1. Is there any way we can trivialize when the ν is odd $(1, 3, 5, 7, 9, 11, 13, 15) \in \mathbb{Z}_{16}$?

- One known way is by the construction of Ref. [68], which shows that the 3d $\text{SO}(3)_{3,\pm}$ Chern-Simons (CS) theory can saturate the 't Hooft anomaly of $\nu = \pm 3 \in \mathbb{Z}_{16}$. The \pm in 3d $\text{SO}(3)_{3,\pm}$ CS means two version of time-reversal symmetry assignment.

- Here we provide another possibility by looking into more general categories of manifolds. The bordism group $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$ classifies the 4d smooth and differentiable (DIFF) manifolds with Pin^+ structures. However, Ref. [69] looks at the bordism group

$$\Omega_4^{\text{TopPin}^+} = \mathbb{Z}_8 \times \mathbb{Z}_2 \quad (7.13)$$

which classifies the 4d non-triangulable topological (Top) manifolds also with Pin^+ structures. It turns out that what survives as \mathbb{Z}_8 in $\Omega_4^{\text{TopPin}^+}$ is the even ν class in $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$. Which means all the odd ν class in $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$ is “trivialized” in the more general category of topological manifolds in $\Omega_4^{\text{TopPin}^+}$.²⁵

Since EPin trivializes the even $\nu \in \mathbb{Z}_{16}$ of $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$, while the TopPin^+ trivializes odd $\nu \in \mathbb{Z}_{16}$ of $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$, therefore we expect that the combined categories of topological manifolds with the extended EPin structure, which we name TopEPin , can trivialize all $\nu \in \mathbb{Z}_{16}$ of $\Omega_4^{\text{Pin}^+} = \mathbb{Z}_{16}$. For future work, it will be illuminating to study the bordism group $\Omega_4^{\text{TopEPin}}$ further.

²⁵The remained \mathbb{Z}_2 in $\Omega_4^{\text{TopPin}^+} = \mathbb{Z}_8 \times \mathbb{Z}_2$ is given by the nontrivial Kirby-Siebenmann class $\kappa \in H^4(M, \mathbb{Z}_2)$, whose manifold generator is the Freedman’s E8 manifold. The E8 manifold is a unique compact, simply connected 4d topological manifold whose intersection form is the positive-definite even unimodular rank-8 matrix of the E8 lattice. Th E8 manifold does not have a smooth DIFF or piecewise linear PL structure.

7.8 Intrinsic fermionic gapless topological phase protected by supersymmetry

Recent work [70] points out the relation between (1) the boundary of $D + 1$ dimensional SPTs whose 't Hooft anomaly can be trivialized by symmetry extension, and (2) the intrinsic gapless D dimensional SPTs living in one lower dimension.

Ref. [70] focus on the bosonic SPT example, here we can propose a novel intrinsically fermionic “gapless” SPTs protected by supersymmetry.

Take the 0+1d boundary of the 1+1d 2-layer of Kitaev chains, we had found that

$$\begin{cases} \text{the bulk } \mathbb{Z}_2^T \times \mathbb{Z}_2^F \text{ symmetry can be extended to the boundary } \mathbb{D}_8^{F,T} \text{ with 't Hooft anomaly trivialized.} \\ \text{the bulk } \mathbb{Z}_4^{TF} \text{ symmetry can be extended to the boundary } \mathbb{M}_{16}^{F,T} \text{ with 't Hooft anomaly trivialized.} \end{cases}$$

So there should be a 0+1d intrinsic fermionic “gapless” SPTs protected by supersymmetry $\mathbb{D}_8^{F,T}$ or $\mathbb{M}_{16}^{F,T}$, living in 0+1d (without the requirement of an extra dimension 1+1d). “Gapless” in 0+1d only means degenerate zero energy modes. How to find such a 0+1d intrinsic fermionic “gapless” SPTs with degenerate zero energy modes? We can consider $N = 0 \pmod 8$ -layer of Kitaev chains, such as the 8-layer of 1+1d Kitaev chains. The 8-layer of 1+1d Kitaev chains can be gapped in the bulk with a trivial order without any SPTs. Thus, the 0+1d boundary of 1+1d 8-layer Kitaev chains must be intrinsically 0+1d.

Say we consider the Hilbert space of 8 Majorana fermion operators: $\gamma_1, \gamma_{1'}, \gamma_2, \gamma_{2'}, \gamma_3, \gamma_{3'}, \gamma_4, \gamma_{4'}$ (on the boundary of 8-layer of Kitaev chains). The 8 Majorana fermions have at most $\text{GSD} = 2^4 = 16$ ground states. But we can turn on the 0+1d time-reversal symmetric Hamiltonian:

(1). Consider the 0+1d Hamiltonian

$$\begin{aligned} H'_{BB} &= (\gamma_1 \gamma_{1'} \gamma_2 \gamma_{2'}) + (\gamma_3 \gamma_{3'} \gamma_4 \gamma_{4'}) + (\vec{S}_B^\alpha \cdot \vec{S}_B^\beta) \\ &= (-P_{F,1} P_{F,2}) + (-P_{F,3} P_{F,4}) + (\vec{S}_B^\alpha \cdot \vec{S}_B^\beta) \equiv H'_{B,11'22'} + H'_{B,33'44'} + (\vec{S}_B^\alpha \cdot \vec{S}_B^\beta). \end{aligned} \quad (7.14)$$

The $P_{F,a} = -i\gamma_a \gamma_{a'}$ is the fermion parity from the a -th Majorana pairs, for $a = 1, 2, 3, 4$. The H'_{BB} leaves us a single bosonic ground state ($\text{GSD} = 1$) with a total fermion parity $P_F = P_{F,1} P_{F,2} P_{F,3} P_{F,4} = +1$. The reasoning is the following.

- The first interaction term $H'_{B,11'22'} \equiv (\gamma_1 \gamma_{1'} \gamma_2 \gamma_{2'})$ in H'_{BB} energetically selects the bosonic ground state sector in $P_{F,1} P_{F,2} = +1$ out of the $\text{GSD} = 2^2 = 4$ from the first four Majorana $\gamma_1, \gamma_{1'}, \gamma_2, \gamma_{2'}$. There are two bosonic sectors: $P_{F,1} = P_{F,2} = +1$ and $P_{F,1} = P_{F,2} = -1$. In short, we gap the other two fermionic sectors out of $\text{GSD} = 4$ to leave with only bosonic $\text{GSD} = 2$ remained for the first four Majorana modes.
- The second interaction term $H'_{B,33'44'} \equiv (\gamma_3 \gamma_{3'} \gamma_4 \gamma_{4'})$ in H'_{BB} also selects the bosonic ground state sector in $P_{F,3} P_{F,4} = +1$ out of the $\text{GSD} = 2^2 = 4$ from the second four Majorana $\gamma_3, \gamma_{3'}, \gamma_4, \gamma_{4'}$. We gap the other two fermionic sectors out of $\text{GSD} = 4$ to leave with only bosonic $\text{GSD} = 2$ remained for the last four Majorana modes.
- So far the first two interaction terms $H'_{B,11'22'} + H'_{B,33'44'}$ in H'_{BB} energetically select 4 bosonic ground states out of the original 16 states. Out of the bosonic $\text{GSD} = 4$, the first four Majorana modes form a bosonic doublet, while the last Majorana modes form another bosonic doublet. Each bosonic doublet is the same degree of freedom as a $J = \frac{1}{2}$ spin (as a representation of $\text{SU}(2)$) at the one end of Haldane spin-1 chain. We can write these two $\text{SU}(2)$ bosonic spin- $\frac{1}{2}$ operators as, \vec{S}_B^α and \vec{S}_B^β , respectively. Since the two bosonic doublets can form a spin-0 singlet and a spin-1

triplet (via $\frac{1}{2}_B \otimes \frac{1}{2}_B = 0_B \oplus 1_B$), we can gap the spin-1 triplet via the third interaction term, the Heisenberg-type interaction ($\vec{S}_B^\alpha \cdot \vec{S}_B^\beta$). Thus turning on the full H'_{BB} , we end up with only a bosonic sector of spin-0 singlet with GSD = 1. Let us call this single bosonic ground state sector $|B_0\rangle$.

- In fact the H'_B of eq (7.14) can be rewritten as 14 terms of four-Majorana interactions similar to Ref. [32], firstly known by Fidkowski-Kitaev.

- (2). We can also tune the 0+1d Hamiltonian via different interactions:

$$\begin{aligned} H'_{FF} &= (-\gamma_1\gamma_{1'}\gamma_2\gamma_{2'}) + (-\gamma_3\gamma_{3'}\gamma_4\gamma_{4'}) + (\vec{S}_F^\alpha \cdot \vec{S}_F^\beta) \\ &= (P_{F,1} P_{F,2}) + (P_{F,3} P_{F,4}) + (\vec{S}_F^\alpha \cdot \vec{S}_F^\beta) \equiv H'_{F,11'22'} + H'_{F,33'44'} + (\vec{S}_F^\alpha \cdot \vec{S}_F^\beta). \end{aligned} \quad (7.15)$$

The $H'_{F,11'22'}$ gives two fermionic ground states with $(P_{F,1} P_{F,2}) = -1$, and the $H'_{F,33'44'}$ gives another two fermionic ground states with $(P_{F,3} P_{F,4}) = -1$. Here we write these two SU(2) fermionic spin- $\frac{1}{2}$ operators as, \vec{S}_F^α and \vec{S}_F^β , respectively. However, the full fermion parity of each of these GSD = 4 is still $P_F = P_{F,1}P_{F,2}P_{F,3}P_{F,4} = +1$ thus bosonic. We can further gap the spin-1 triplet (3 ground states) via the spin-1 interaction in $\frac{1}{2}_F \otimes \frac{1}{2}_F = 0_B \oplus 1_B$, which can be constructed out of two fermionic spin- $\frac{1}{2}$ doublets as $(\vec{S}_F^\alpha \cdot \vec{S}_F^\beta)$. Thus turning on the full H'_{FF} , we still end up with only a bosonic sector of spin-0 singlet with GSD = 1.

- (3). An interesting question is whether we can tune Hamiltonian ending up with only a fermionic sector with GSD = 1? Consider the 0+1d Hamiltonian

$$\begin{aligned} H'_{BF} &= (\gamma_1\gamma_{1'}\gamma_2\gamma_{2'}) + (-\gamma_3\gamma_{3'}\gamma_4\gamma_{4'}) + (\vec{S}_B^\alpha \cdot \vec{S}_F^\beta) \\ &= (-P_{F,1} P_{F,2}) + (+P_{F,3} P_{F,4}) + (\vec{S}_B^\alpha \cdot \vec{S}_F^\beta) \equiv H'_{B,11'22'} + H'_{F,33'44'} + (\vec{S}_B^\alpha \cdot \vec{S}_F^\beta). \end{aligned} \quad (7.16)$$

The $H'_{B,11'22'}$ gives two bosonic ground states with $(P_{F,1} P_{F,2}) = +1$, and the $H'_{F,33'44'}$ gives another two fermionic ground states with $(P_{F,3} P_{F,4}) = -1$. However, the full fermion parity of each of these GSD = 4 is $P_F = P_{F,1}P_{F,2}P_{F,3}P_{F,4} = -1$ fermionic. We further gap the spin-1 triplet (3 ground states) via the spin-1 interaction $\frac{1}{2}_B \otimes \frac{1}{2}_F = 0_F \oplus 1_F$ constructed out of one bosonic and one fermionic spin- $\frac{1}{2}$ doublet as $(\vec{S}_B^\alpha \cdot \vec{S}_F^\beta)$. Thus turning on the full H'_{BF} , we end up with only a fermionic sector of spin-0 singlet with GSD = 1. Let us call this single fermionic ground state sector $|F_0\rangle$.

Similarly, we can also define H'_{FB} analogous to eq (7.16), which also ends up with only a fermionic sector of spin-0 singlet with GSD = 1.

The combined Hamiltonian above, by tuning the coupling appropriately

$$H' = g_{BB} H'_{BB} + g_{FF} H'_{FF} + g_{BF} H'_{BF} + g_{FB} H'_{FB} + \dots \quad (7.17)$$

can energetically select ground states out of the original 16 ground states. With certain appropriate couplings, when the bosonic and fermionic states, $|B_0\rangle$ and $|F_0\rangle$, are both favored as ground states, we have GSD = 2. We can imagine a process when the $|B_0\rangle$ is favored on one side of coupling choice, while the $|F_0\rangle$ is favored on the other side of coupling choice (both have GSD = 1); while at some critical coupling, we have the GSD = 2 formed by a 2-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F = \{|B_0\rangle, |F_0\rangle\}$. Although we are studying the boundary of 8-layer Kitaev chains, we find with appropriate interactions, we land on the same 2-dimensional Hilbert space $\mathcal{H} = \{|B_0\rangle, |F_0\rangle\}$ for 2-layer of Kitaev chain studied in Sec. 3 and Sec. 5!

As mentioned, the boundary of 8-layer Kitaev chains is intrinsically 0+1d. Thus, this theory at the critical coupling with a Hilbert space $\mathcal{H} = \{|B_0\rangle, |F_0\rangle\}$, can be regarded as a *critical “gapless”*

intrinsically 0+1d fermionic SPTs protected by supersymmetry (here gapless only means the degenerate ground states in 0+1d)! Depends on the assignment of time-reversal \mathcal{T} symmetry shown in Sec. 3, the supersymmetry here can be $\mathbb{D}_8^{F,T}$ or $\mathbb{M}_8^{F,T}$.

We leave the generalization of this understanding on a critical “gapless” *intrinsically fermionic SPTs protected by supersymmetry* in higher-dimensions for future work.

7.9 Final comments and future directions

We conclude by enlisting some additional comments and future directions:

1. *All intrinsically fermionic 1+1d SPT phases have supersymmetric 0+1d boundaries:*

- In the present work, especially in Sec. 3 and Sec. 5, we have shown by examples that when the number of layers of 1+1d Kitaev chains is $8N+2$ or $8N+6$ (namely the layer number is 2 mod 4), then the 0+1d boundary theory exhibits at least an $\mathcal{N} = 2$ supersymmetric quantum mechanics. In the sense that the boundary energy spectrum always have bosonic energy eigenstate $|\mu, +\rangle$ and fermionic energy eigenstate $|\mu, -\rangle$ paired together for every eigenenergy E_μ .

- In a companion work [36], we prove that “all intrinsically fermionic 1+1d SPT phases have supersymmetric 0+1d boundaries.” Another way to rephrase this result in a mathematically precise statement is the following:

Consider a 2d fermionic bordism group $\Omega_2^{G_{f,\text{total}}}$ with a total group $G_{f,\text{total}}$ obtained from the extension of a normal subgroup of fermionic internal G_f global symmetry, and the d -dimensional spacetime rotational symmetry (say $O(d)/SO(d)$, etc.)²⁶:

$$\begin{aligned} & 1 \rightarrow G_f \rightarrow G_{f,\text{total}} \rightarrow O(d) \rightarrow 1, \quad \text{with time-reversal symmetry.} \\ \text{or} \quad & 1 \rightarrow G_f \rightarrow G_{f,\text{total}} \rightarrow SO(d) \rightarrow 1, \quad \text{without time-reversal symmetry.} \end{aligned} \quad (7.18)$$

The fermionic internal G_f contains the fermion parity \mathbb{Z}_2^F and the bosonic internal G_b , via a central extension:

$$1 \rightarrow \mathbb{Z}_2^F \rightarrow G_f \rightarrow G_b \rightarrow 1. \quad (7.19)$$

The intrinsically fermionic SPTs, corresponds to the cobordism invariants (representing some invertible TQFTs in field theory) of $\Omega_2^{G_{f,\text{total}}}$, but those do *not overlap* with cobordism invariants of fermionic invertible topological order’s bordism group $\Omega_2^{\text{Spin}}(pt)$, *nor* overlap with those of bosonic SPT’s bordism group $\Omega_2^{G_{b,\text{total}}}$. Here the $G_{b,\text{total}}$ is obtained from:

$$\begin{aligned} & 1 \rightarrow G_b \rightarrow G_{b,\text{total}} \rightarrow O(d) \rightarrow 1, \quad \text{with time-reversal symmetry.} \\ \text{or} \quad & 1 \rightarrow G_b \rightarrow G_{b,\text{total}} \rightarrow SO(d) \rightarrow 1, \quad \text{without time-reversal symmetry.} \end{aligned} \quad (7.20)$$

We can prove that the 0+1d boundary of these 1+1d cobordism invariants (namely, the invertible TQFTs) of intrinsically fermionic SPTs must be supersymmetric [36].

2. *Boundary supersymmetry and the symmetry extension:*

Various previous works pointed out the fractionalized boundary symmetry of fermionic SPTs may be related to supersymmetry [34, 71–75].²⁷

²⁶The Spin/Pin group can be obtained via the \mathbb{Z}_2^F -extension of O and SO groups (see Appendix B.1).

²⁷Naively Ref. [34] results look similar to ours, such as Gu’s superalgebra mentioned in his equation (39). However, we can tell the *differences* of our result and Gu’s Ref. [34] by comparing the commutativity of time reversal \mathcal{T} and fermion parity P_f . Ref. [34] shows the commutative $\mathcal{T}P_f = +P_f\mathcal{T}$ (which has no SUSY, in our definition). Our work shows the anti-commutative $\mathcal{T}P_f = -P_f\mathcal{T}$ which implies the SUSY quantum mechanics.

In the present work, we point out that on the boundary of intrinsic fermionic SPTs, the non-commutative properties between fermion parity P_f and other symmetry generator (such as time reversal \mathcal{T}) implies the boundary supersymmetry. (In a companion work [36], we present a precise proof for this statement.) We also provide a framework based on a generalization of symmetry extension in terms of group extension [4], to the case of an extended total group involving supersymmetry — which we name it a supersymmetry extension.

3. *Quantum criticality and emergent supersymmetry*: In Sec. 7.8, we point out a potential relation between the 0+1d boundary supersymmetric degenerate ground states can be regarded as a 0+1d intrinsically gapless fermionic SPTs (here gapless means degenerate in 0+1d), indeed also as a 0+1d critical theory on the boundary of the 1+1d trivial class of SPTs (by tuning the 0+1d boundary Hamiltonian). This result may be generalizable to other higher dimensions, thus connecting to the literature [71–75] of quantum criticality and supersymmetry on the boundary of fermionic SPTs. Hopefully our approach can provide a more systematic and mathematically precise framework.
4. *'t Hooft anomalies on the boundary of worldsheet or worldvolume in string theory*: In a different language setting, we can also apply our results to the fermionic SPTs living on the 1+1d worldsheet of string theory. The 0+1d boundary of worldsheet may have various 't Hooft anomalies related to the reflection, time-reversal \mathcal{T} , and other symmetries. These 't Hooft anomalies impose the constraints on the consistency of worldsheet and string theories [5, 49, 50, 76, 77]. The constraints are tightly connected to the string landscape vs swampland program, where the *cobordism conjecture* plays an important role to restrict the consistency of quantum gravity [78].
5. *Higher-symmetry extension and fermionic symmetry extension construction of symmetric gapped boundaries of SPTs*: The present work focus on the ordinary 0-form global symmetry acting on bosonic and fermionic Hilbert spaces. We can also attempt to include higher-form generalized global symmetries [79]. In fact the symmetry extension method had been generalized to a higher-symmetry extension [6, 80–82], also include fermionic symmetry extension [26, 83, 84] — these methods are helpful to construct symmetric gapped boundaries of SPTs, as explained in [4]. By including the supersymmetry extension, we hope to also construct more general boundary theories of fermionic SPTs in higher dimensions.

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A Tantivasadakarn-Vishwanath model and the basis change

Here, we give details of how the model in section 4 was arrived at. We start by writing down the model constructed by Tantivasadakarn and Vishwanath in [35]. They start with the Hilbert space of one fermion on odd sites $(\gamma, \bar{\gamma})$, one fermion on odd sites $(\eta, \bar{\eta})$ and one qubit on even sites and the following trivial Hamiltonian

$$H_0 = - \sum_{\text{even } j} \sigma_j^x - i \sum_{\text{even } j} \bar{\eta}_j \eta_j - i \sum_{\text{odd } j} \bar{\gamma}_j \gamma_j. \quad (\text{A.1})$$

The SPT Hamiltonian is obtained from H_0 by a finite depth quantum circuit W . Up to an inconsequential difference in minus signs, we can write W considered by the authors of [35] as a two-layer FDUC:

$$W = W_2 W_1. \quad (\text{A.2})$$

$$W_1 = \prod_{\text{even } j} [|0\rangle\langle 0|_j + |1\rangle\langle 1|_j i \bar{\eta}_j \xi_{jj+1} \gamma_{j+1}]. \quad (\text{A.3})$$

$$W_2 = \prod_{\text{even } j} [|0\rangle\langle 0|_j + |1\rangle\langle 1|_j i \gamma_{j-1} \bar{\eta}_j]. \quad (\text{A.4})$$

Where $|0/1\rangle$ are the eigenstates of σ^z with eigenvalues $(-1)^{0/1}$. The $\xi_{jj+1} = \pm 1$ are \mathbb{Z}_2 valued background fields that denote the spin structure to be Ramond (periodic) or Neveu-Schwarz (anti-periodic) depending on whether $\prod_j \xi_{jj+1} = +1$ or $\prod_j \xi_{jj+1} = -1$ respectively. The time reversal symmetry \mathcal{T} is

$$\mathcal{T} = \prod_{\text{even } j} \sigma_j^x. \quad (\text{A.5})$$

It can easily be checked that \mathcal{T} does not commute with W_1 or W_2 . However, depending on the spin structure, \mathcal{T} commutes (Neveu-Schwarz) or anti-commutes (Ramond) with W

$$\mathcal{T} W \mathcal{T}^{-1} = - \left(\prod_j \xi_{jj+1} \right) W. \quad (\text{A.6})$$

Regardless of spin structure, the SPT Hamiltonian $H = W H_0 W^\dagger$ shown below commutes with \mathcal{T} .

$$H = -i \sum_{\text{odd } j} \sigma_{j-1}^z \bar{\gamma}_j \gamma_j \sigma_{j+1}^z + \sum_{\text{odd } j} \xi_{j+1,j+2} \gamma_j (\sigma_{j+1}^z \sigma_{j+1}^x) \gamma_{j+2} \sigma_{j+3}^z - i \sum_{\text{even } j} \bar{\eta}_j \eta_j. \quad (\text{A.7})$$

Notice that the term $\sum_{\text{even } j} \bar{\eta}_j \eta_j$ remains unchanged from its trivial form and only plays an intermediary role in defining the FDUC, W . It is ignored in the final form by the authors of [35]. However, we shall retain it. To get the form of the Hamiltonian and symmetry presented in the main text, we redefine the Hilbert space by considering both the fermions to live on the sites and qubits to live on the links with the following relabeling

$$(\bar{\gamma}_{2k+1}, \gamma_{2k+1}) \mapsto (\bar{\gamma}_{\downarrow,k}, \gamma_{\downarrow,k}), \quad (\bar{\eta}_{2k+2}, \eta_{2k+2}) \mapsto (\bar{\gamma}_{\uparrow,k}, \gamma_{\uparrow,k}), \quad \vec{\sigma}_{2k+2} \mapsto \vec{\sigma}_{k,k+1}. \quad (\text{A.8})$$

Finally, by performing a change-of-basis using W_2 , we get the form of the Hamiltonian and symmetry operators used in the main text.

$$W_2 H W_2^\dagger = -i \sum_j [\sigma_{j,j+1}^z (\bar{\gamma}_{\uparrow,j} \gamma_{\uparrow,j} + \bar{\gamma}_{\downarrow,j+1} \gamma_{\downarrow,j+1}) + \sigma_{j,j+1}^x \bar{\gamma}_{\uparrow,j} \gamma_{\downarrow,j+1}]. \quad (\text{A.9})$$

$$W_2 \mathcal{T} W_2^\dagger = \prod_j (i \gamma_{\downarrow,j} \bar{\gamma}_{\uparrow,j}) \sigma_{j,j+1}^x \mathcal{K}. \quad (\text{A.10})$$

B Appendices for Section 6

B.1 Group extension: Formal setup

Below we illuminate further the group extension used throughout our work:

$$1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1. \quad (\text{B.1})$$

We will later include also the continuous spacetime symmetry $\text{SO}(d)$ in the continuum limit²⁸ in terms of

$$1 \rightarrow N \rightarrow \tilde{G}_{\text{Tot}} \rightarrow G_{\text{Tot}} \rightarrow 1, \quad (\text{B.2})$$

where \tilde{G}_{Tot} and G_{Tot} can be written in terms of the spacetime-internal symmetry group form of $\frac{G_{\text{spacetime}} \times G_{\text{internal}}}{N_{\text{shared}}}$. In Table 4, Table 5, and discussions below, we shorthand the short exact sequences in eq (B.1) and eq (B.2) as $N \rightarrow \tilde{G} \rightarrow G$ and $N \rightarrow \tilde{G}_{\text{Tot}} \rightarrow G_{\text{Tot}}$ for simplicity.

To precisely specify an extended group

$$\tilde{G} = N \rtimes_{\rho, \varphi} G \quad (\text{B.3})$$

in of eq (B.1), we need the following data

- The normal subgroup N and the quotient group Q .
- $\rho : G \rightarrow \text{Aut}(N)$. If and only if a nontrivial ρ implies a non-central extension eq (B.1).
- A 2-cocycle $\varphi \in \text{H}^2(\text{BG}, N)$ in the cohomology group with the classifying space BG mapping to values in N (usually defined for an Abelian group) is solved from the cocycle condition, in the additive form,

$$0 = \delta\varphi(g_1, g_2, g_3) = (\rho(g_1) \cdot \varphi(g_2, g_3)) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2)$$

$$\text{or in the multiplicative form } \delta\varphi(g_1, g_2, g_3) = \frac{(\rho(g_1) \cdot \varphi(g_2, g_3)) \varphi(g_1, g_2 g_3)}{\varphi(g_1 g_2, g_3) \varphi(g_1, g_2)}.$$

With two group elements in \tilde{G} specified by a doublet (n_1, g_1) and $(n_2, g_2) \in (N, G)$, we have the group law

$$(n_1, g_1)(n_2, g_2) = (n_1 + \rho(g_1) \cdot n_2 + \varphi(g_1, g_2), g_1 \cdot g_2).$$

Here we use the addition $+$ for the group operation in N , and the product \cdot for the group operation in G . Here the 2-cocycle $\varphi(g_1, g_2)$ specifies how much the group extension is twisted. We illustrate the precise $\tilde{G} = N \rtimes_{\rho, \varphi} G$ for the examples of Table 4 in a footnote.²⁹

²⁸By the continuous spacetime symmetry, we mean the Lorentz symmetry $\text{SO}(d-1, 1)$ or Euclidean rotational symmetry $\text{SO}(d)$ in the continuum limit. In the condensed matter system with a discretized lattice, the spacetime symmetry, such as continuous rotations, is not usually manifested. We may regard this continuous spacetime symmetry as an emergent symmetry at an intermediate energy scale of some effective field theories, *below* the ultraviolet (UV) high-energy lattice cutoff, but *above* the infrared (IR) low-energy field theory. In this work, we mainly do $d = 2$ or $1+1\text{d}$, however we will also discuss general d in Sec. 7.

²⁹Here let us illustrate the mathematically precise $\tilde{G} = N \rtimes_{\rho, \varphi} G$ for the examples in Table 4:

1. $\mathbb{Z}_4^T \times \mathbb{Z}_2^F = \mathbb{Z}_2 \rtimes_{0, \varphi} (\mathbb{Z}_2^T \times \mathbb{Z}_2^F)$ has no ρ but a nontrivial $\varphi \in \text{H}^2(\text{B}\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2$. We may denote this generator

B.2 Bosonic, Fermionic, Time Reversal, Spacetime Symmetries and Group Extensions

We start from elaborating the familiar groups: the continuous rotation $SO(d)$,³⁰ the reflection \mathbb{Z}_2^{TE} or time-reversal \mathbb{Z}_2^T , and the fermion parity \mathbb{Z}_2^F (See useful discussions in [25, 85]). Then we can write down the short exact sequences relating the familiar groups to $O(d)$, $Spin(d)$, and $Pin^\pm(d)$, by symmetry group extension along the horizontal and vertical directions of:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{Z}_2^F & & \mathbb{Z}_2^F & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & Spin(d) & \longrightarrow & Pin^\pm(d) & \longrightarrow & \mathbb{Z}_2^{TE} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & SO(d) & \longrightarrow & O(d) & \xrightarrow{\det=\pm 1} & \mathbb{Z}_2^{TE} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array} \tag{B.4}$$

Some remarks about these symmetry groups:

1. The $SO(d)$ is a spatial rotation group. The general linear group $GL(d, \mathbb{R})$ is the group of the rank- d invertible matrix with all the entry with \mathbb{R} coefficients as group elements, together with the operation of ordinary matrix multiplication. The special linear group $SL(d, \mathbb{R})$ is the subgroup of $GL(d, \mathbb{R})$, then

$$SO(d) = \{g_{SO(d)} \in SL(d, \mathbb{R}) \mid g_{SO(d)} g_{SO(d)}^T = g_{SO(d)}^T g_{SO(d)} = 1\}, \tag{B.5}$$

with the transpose operation T and its determinant $\det(g_{SO(d)}) = 1$.

2. The $Spin(d) = \mathbb{Z}_2^F \rtimes SO(d)$ is from the spatial rotation $SO(d)$ extended/graded by the fermion parity \mathbb{Z}_2^F . For $d > 2$, $Spin(d)$ is double cover of $SO(d)$ and simply-connected, thus the (unique) universal cover of $SO(d)$. So its first homotopy group is $\pi_1(Spin(d)) = 0$. The centers of $Spin(d)$

$\varphi = w_T^2$ with w_T can be regarded as a time-reversal background field as a generator from $H^1(B\mathbb{Z}_2^T, \mathbb{Z}_2) = \mathbb{Z}_2$.

2. $\mathbb{D}_8^{F,T} = \mathbb{Z}_2 \rtimes_{0,\varphi} (\mathbb{Z}_2^T \times \mathbb{Z}_2^F)$ has no ρ but a nontrivial $\varphi = w_T A_F \in H^2(B(\mathbb{Z}_2^T \times \mathbb{Z}_2^F), \mathbb{Z}_2) = \mathbb{Z}_2^3$ with A_F can be regarded as a fermion-parity background field as a generator from $H^1(B\mathbb{Z}_2^F, \mathbb{Z}_2) = \mathbb{Z}_2$. But for $\mathbb{D}_8^{F,T} = \mathbb{Z}_4^T \rtimes_{\rho,0} \mathbb{Z}_2^F$, it has a nontrivial $\rho : \mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$ thus not a central extension, but no ϕ .

3. $\mathbb{M}_{16}^{F,T} = \mathbb{Z}_4 \rtimes_{0,\varphi} \mathbb{Z}_4^{TF}$ has no $\rho : \mathbb{Z}_4 \rightarrow \text{Aut}(\mathbb{Z}_4) = \mathbb{Z}_2$ and a nontrivial odd class $\varphi \in H^2(B\mathbb{Z}_4^{TF}, \mathbb{Z}_4) = \mathbb{Z}_4$.

4. Other groups in Table 4 obtained by adding extra $U(1)$ and $SU(2)$ in group extension twisted by (ρ, φ) are similar in the spirit as the previous examples.

We can also use the information below to specify the group laws for $\tilde{g}_1 = (n_1, g_1) \in \tilde{G}$ and $\tilde{g}_2 = (n_2, g_2) \in \tilde{G}$, with $n_i \in N$ and $g_i \in G$ as $\tilde{g}_1 \tilde{g}_2 = (n_1, g_1)(n_2, g_2) = (n_1 + \rho(g_1).n_2 + \varphi(g_1, g_2), g_1 + g_2)$.

³⁰The major part of our paper focus on $d = 2$ or $1+1d$, but now we keep the general d notation in order to generalize our approach to higher dimensions in Sec. 7.

and $\text{SO}(d)$ for $d \geq 2$ are:

$$Z(\text{Spin}(d)) = \begin{cases} \mathbb{Z}_2^2, & d = 0 \pmod{4} \\ \mathbb{Z}_2, & d = 1 \pmod{4} \\ \mathbb{Z}_4, & d = 2 \pmod{4} \\ \mathbb{Z}_2, & d = 3 \pmod{4} \end{cases} \cdot \quad Z(\text{SO}(d)) = \begin{cases} \mathbb{Z}_2, & d = 0 \pmod{2} \\ 1, & d = 1 \pmod{2} \end{cases} \cdot \quad (\text{B.6})$$

3. The $\text{O}(d) = \text{SO}(d) \rtimes \mathbb{Z}_2^{T_E}$ is

$$\text{O}(d) = \{g_{\text{O}(d)} \in \text{GL}(d, \mathbb{R}) \mid g_{\text{O}(d)} g_{\text{O}(d)}^T = g_{\text{O}(d)}^T g_{\text{O}(d)} = 1\} \quad (\text{B.7})$$

from the time reversal $\mathbb{Z}_2^{T_E}$ extended by the spatial rotation $\text{SO}(d)$:

$$\text{O}(d) \cong \begin{cases} \text{SO}(d) \rtimes \mathbb{Z}_2^{T_E}, & d \in \text{even} \\ \text{SO}(d) \times \mathbb{Z}_2^{T_E}, & d \in \text{odd} \end{cases} \cdot \quad (\text{B.8})$$

When d is odd, the $\mathbb{Z}_2^{T_E} = \{+1, -1\}$ along the diagonal element, exactly matches the determinant map $\det(g_{\text{O}(d)}) = \pm 1$. The center of $\text{O}(d)$ for $d \geq 2$ is: $Z(\text{O}(d)) = \mathbb{Z}_2^{T_E}$.

4. The $\text{Pin}^\pm(d) = \mathbb{Z}_2^F \rtimes \text{O}(d)$ are double covers of $\text{O}(d)$. For physics purpose, we need to interpret the spacetime symmetry in the Lorentzian signature (i.e., Minkowski signature). Let us clarify the notations of groups in Euclidean signature (time-reversal \mathcal{T}_E) and Lorentzian signature (time-reversal \mathcal{T}):³¹

$$\begin{aligned} \text{Pin}^+(d) &\cong \begin{matrix} \text{Spin}(d) \rtimes \mathbb{Z}_2^{T_E} \\ \mathbb{Z}_2^{T_E} \times \mathbb{Z}_2^F \end{matrix} \quad \text{or} \quad \mathbb{Z}_4^{TF} \text{ for discrete subgroups.} \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} \text{Pin}^-(d) &\cong \frac{\text{Spin}(d) \rtimes \mathbb{Z}_4^{T_E F}}{\mathbb{Z}_2^F} \quad \mathcal{T}_E^2 = (-1)^F \text{ or } \mathcal{T}^2 = +1. \\ &\quad \mathbb{Z}_4^{T_E F} \quad \text{or} \quad \mathbb{Z}_2^T \times \mathbb{Z}_2^F \text{ for discrete subgroups.} \end{aligned} \quad (\text{B.10})$$

In Euclidean signature, the $\mathbb{Z}_2^{T_E}$ is generated by a reflection \mathcal{T}_E . The reflection $\mathcal{T}_E \in \text{O}(d)$ has $\mathcal{T}_E^2 = +1$. The pullback from $\mathcal{T}_E \in \text{O}(d)$ to $\mathcal{T}_E \in \text{Pin}^\pm(d)$ via the \mathbb{Z}_2^F extension shows that $\mathcal{T}_E^2 = +1$ for $\text{Pin}^+(d)$ and $\mathcal{T}_E^2 = (-1)^F$ for $\text{Pin}^-(d)$.

In Lorentzian signature, the reflection \mathcal{T}_E becomes the time-reversal \mathcal{T} by a Wick rotation [23]. The Wick rotation introduces the imaginary i sign to the Dirac gamma matrix $\gamma_E \rightarrow \gamma_0 = i\gamma_E$ of the temporal component in order to fit the Lorentzian version of Clifford algebra $(\gamma_0)^2 = -(\gamma_E)^2$, so $\mathcal{T}^2 = (i\mathcal{T}_E)^2 = -\mathcal{T}_E^2$, which suggests that $\mathcal{T}^2 = (-1)^F$ for $\text{Pin}^+(d)$ and $\mathcal{T}^2 = 1$ for $\text{Pin}^-(d)$.

The centers of $\text{Pin}^+(d)$ and $\text{Pin}^-(d)$ for $d \geq 2$ are:

$$Z(\text{Pin}^+(d)) = \begin{cases} \mathbb{Z}_2, & d = 0 \pmod{4} \\ \mathbb{Z}_2^2, & d = 1 \pmod{4} \\ \mathbb{Z}_2, & d = 2 \pmod{4} \\ \mathbb{Z}_4, & d = 3 \pmod{4} \end{cases} \cdot \quad Z(\text{Pin}^-(d)) = \begin{cases} \mathbb{Z}_2, & d = 0 \pmod{4} \\ \mathbb{Z}_4, & d = 1 \pmod{4} \\ \mathbb{Z}_2, & d = 2 \pmod{4} \\ \mathbb{Z}_2^2, & d = 3 \pmod{4} \end{cases} \cdot \quad (\text{B.11})$$

³¹Follow eq (3.31), for $1 \rightarrow \mathbb{Z}_2^F \rightarrow \tilde{G} \rightarrow \mathbb{Z}_2^T \rightarrow 1$, we have two possible extended $\tilde{G} = \mathbb{Z}_4^{TF}$ or $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$. We can also replace T to T_E so $\tilde{G} = \mathbb{Z}_4^{T_E F}$ or $\mathbb{Z}_2^{T_E} \times \mathbb{Z}_2^F$.

5. When d is an odd integer, because of $O(d) = SO(d) \times \mathbb{Z}_2^{T_E}$, we can simplify eq (B.9) and eq (B.10), in the Euclidean or Lorentzian signatures (with time coordinates T_E and T respectively), to:

$$\text{Pin}^+(d) \cong \begin{cases} \text{Spin}(d) \times \mathbb{Z}_2^{T_E}, & d = +1 \pmod{4} \\ (\text{Spin}(d) \times \mathbb{Z}_4^{T_E F})/\mathbb{Z}_2^F, & d = +3 \pmod{4} \end{cases}. \quad (\text{B.12})$$

$$\text{Pin}^-(d) \cong \begin{cases} (\text{Spin}(d) \times \mathbb{Z}_4^{T_E F})/\mathbb{Z}_2^F, & d = +1 \pmod{4} \\ \text{Spin}(d) \times \mathbb{Z}_2^{T_E}, & d = +3 \pmod{4} \end{cases}. \quad (\text{B.13})$$

These forms match the center of Pin^\pm groups in eq (B.11). Furthermore, we can still find in these expressions, $\text{Pin}^+(d)$ and $\text{Pin}^-(d)$ contain the discrete subgroups $\mathbb{Z}_2^{T_E} \times \mathbb{Z}_2^F$ and $\mathbb{Z}_4^{T_E F}$ respectively.

6. The $E(d)$ is defined in [24, 48, 82] which is a subgroup of $O(d) \times \mathbb{Z}_4$, described by

$$E(d) = \{(M, j) \in (O(d), \mathbb{Z}_4) \mid \det M = j^2\} \quad (\text{B.14})$$

in our interpretations, we take $\mathbb{Z}_4 = \mathbb{Z}_4^{TB}$ which is from $1 \rightarrow \mathbb{Z}_2^B \rightarrow \mathbb{Z}_4^{TB} \rightarrow \mathbb{Z}_2^T \rightarrow 1$.³² The $E(d)$ sits at the following extensions: $1 \rightarrow \mathbb{Z}_2^B \rightarrow E(d) \rightarrow O(d) \rightarrow 1$ and $1 \rightarrow SO(d) \rightarrow E(d) \rightarrow \mathbb{Z}_4^{TB} \rightarrow 1$ [82].

For the later convenience, we will specify two species of fermions F_- , F_+ , and their boson bound states $B = F_+ F_-$; their time-reversal \mathcal{T} and spacetime symmetries in the continuum correspond to:

$$\text{Fermion } F_- : \mathcal{T}^2 = +1, \mathcal{T}_E^2 = -1, \quad (\mathbb{Z}_2^{F_-} \rtimes SO(d)) \rtimes \mathbb{Z}_2^{T_E} = \text{Spin}(d) \rtimes \mathbb{Z}_2^{T_E} = \text{Pin}^-(d). \quad (\text{B.15})$$

$$\text{Fermion } F_+ : \mathcal{T}^2 = -1, \mathcal{T}_E^2 = +1, \quad (\mathbb{Z}_2^{F_+} \rtimes SO(d)) \rtimes \mathbb{Z}_2^{T_E} = \frac{\text{Spin}(d) \rtimes \mathbb{Z}_4^{T_E F}}{\mathbb{Z}_2^F} = \text{Pin}^+(d). \quad (\text{B.16})$$

$$\text{Boson } B : \mathcal{T}^2 = -1, \mathcal{T}_E^2 = -1, \quad (SO(d) \times \mathbb{Z}_2^B) \rtimes \mathbb{Z}_2^{T_E} = SO(d) \rtimes \mathbb{Z}_4^{T_E B} = E(d). \quad (\text{B.17})$$

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³²For bosonic case, we can identify the Lorentz and Euclidean time reversal symmetries directly (without extra -1 sign), thus $\mathbb{Z}_4^{TB} = \mathbb{Z}_4^{T_E B}$ and $\mathbb{Z}_2^T = \mathbb{Z}_2^{T_E}$.

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