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Bosonic Crystalline Symmetry Protected Topological Phases Beyond the Group Cohomology Proposal

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It is demonstrated by explicit construction that three-dimensional bosonic crystalline symmetry protected topological (cSPT) phases are classified by $H^5_\phi(G;\mathbb{Z})\oplus H^1_\phi(G;\mathbb{Z})$ for all 230 space groups G, where $H^n_\phi(G;\mathbb{Z})$ denotes the nth twisted group cohomology of G with $\mathbb{Z}$ coefficients, and ϕ indicates that $g\in G$ acts non-trivially on coefficients by sending them to their inverses if g reverses spacetime orientation and acts trivially otherwise. The previously known summand $H^5_\phi(G;\mathbb{Z})$ corresponds only to crystalline phases built without the E_8 state or its multiples on 2-cells of space. It is the crystalline analogue of the "group cohomology proposal" for classifying bosonic symmetry protected topological (SPT) phases, which takes the form $H^{d+2}_\phi(G;\mathbb{Z}) \cong H^{d+1}_\phi(G;U(1))$ for finite internal symmetry groups in d spatial dimensions. The new summand $H^1_\phi(G;\mathbb{Z})$ classifies possible configurations of E_8 states on 2-cells that can be used to build crystalline phases beyond the group cohomology proposal. The completeness of our classification and the physical meaning of $H^1_\phi(G;\mathbb{Z})$ are established through a combination of dimensional reduction, surface topological order, and explicit cellular construction. The value of $H^1_\phi(G;\mathbb{Z})$ can be easily read off from the international symbol for G. Our classification agrees with the prediction of the "generalized cohomology hypothesis," which concerns the general structure of the classification of SPT phases, and therefore provides strong evidence for the validity of the said hypothesis in the realm of crystalline symmetries.

I. INTRODUCTION

As the interacting generalization of topological insulators and superconductors [1–8], symmetry protected topological (SPT) phases [9] have garnered considerable interest in the past decade [10–46]. The early studies of SPT phases focused on phases with *internal* symmetries (*i.e.*, symmetries that do not change the position of local degrees of freedom, such as Ising symmetry, $U(1)$ symmetry, and time reversal symmetry). Now it is slowly being recognized [24, 26, 27, 47, 48] that the classification of internal SPT phases naturally satisfies certain axioms which happen to define a well-known structure in mathematics called generalized cohomology [49–52]. In particular, different existing proposals for the classification of internal SPT phases are simply different examples of generalized cohomology theories.

Ref. [47] distilled the above observations regarding the general structure of the classification of SPT phases into a "generalized cohomology hypothesis." It maintained that (a) there exists a generalized cohomology theory h that correctly classifies internal SPT phases in all dimensions for all symmetry groups, and that (b) even though we may not know exactly what h is, meaningful physical results can still be derived from the fact that h is a generalized cohomology theory alone. Indeed, it can be shown, on the basis of the generalized cohomology hypothesis, that three-dimensional bosonic SPT phases with internal symmetry G are classified by $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ [47, 48],[1] where $H^n_\phi(G;\mathbb{Z})$ denotes the nth twisted group cohomology of G with $\mathbb{Z}$ coefficients, and ϕ emphasizes that $g\in G$ acts non-trivially on coefficients by sending them to their inverses if g reverses spacetime orientation and acts trivially otherwise. The first summand, $H^5_\phi(G;\mathbb{Z})$,[2] corresponds to the "group cohomology proposal" for the classification of SPT phases [18]. The second summand, $H^1_\phi(G;\mathbb{Z})$, corresponds to phases beyond the group cohomology proposal, and are precisely the phases constructed in Ref. [54] using decorated domain walls. Specifically, in Ref. [54], the domain walls were decorated with multiples of the E_8 state, which is a 2D bosonic state with quantized thermal Hall coefficient [17, 29, 55, 56].

However, physical systems tend to crystallize. What is the classification of SPT phases if G is a *space-group* symmetry rather than internal symmetry? In the fermionic case, one can incorporate crystalline symmetries by imposing point-group actions on the Brillouin zone before activating interactions [57]. This method is not applicable to bosonic systems, because nontrivial bosonic SPT phases cannot be realized by Hamiltonians quadratic in bosonic creation and annihilation operators. As a get-around, Refs. [45, 58] proposed to build bosonic crys-

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[1] The direct sum of two abelian groups is the same as their direct product, but the direct sum notation $\oplus$ is more common for abelian groups in the mathematical literature.

[2] For $n = 0,1,2,\cdots$ and a compact group G, $H^{n+1}_\phi(G;\mathbb{Z})$ is isomorphic to the "Borel group cohomology" $H^n_{\mathrm{Borel},\phi}(G;U(1))$ considered in Ref. [53], which is simply $H^n_\phi(G;U(1))$ if G is finite.

talline SPT (cSPT) phases by focusing on high-symmetry points in the real space rather than momentum space. Concretely, on every high-symmetry line, plane, etc. of a d-dimensional space with space-group action by G, one can put an SPT phase of the appropriate dimensions with an internal symmetry equal to the stabilizer subgroup of (any point on) the line, plane, etc. In particular, it was shown, for every element of $H^5_\phi(G;\mathbb{Z})$, that there is a 3D bosonic crystalline SPT phase with space group symmetry G that one can construct. Curiously, the same mathematical object, $H^5_\phi(G;\mathbb{Z})$, is also the group cohomology proposal for the classification of 3D bosonic SPT phases with *internal* symmetry G. A heuristic insight into this apparent correspondence between crystalline and internal SPT phases was provided in Ref. [42], which drew an analogy between internal gauge fields and a certain notation of crystalline gauge fields. The correspondence was referred to therein as the "crystalline equivalence principle."

Just like for internal symmetries, the group cohomology proposal $H^5_\phi(G;\mathbb{Z})$ does not give the complete classification for crystalline symmetries either. In fact, in the block-state construction of Ref. [45], E_8 state was excluded from being used as a building block for simplicity. Appealing to the crystalline equivalence principle, one might guess that the complete classification of 3D bosonic crystalline SPT phases with space group symmetry G would be $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$, since that is the classification when G is internal. The recent work [59] gives us added confidence in this conjecture. In that work, the authors justified the extension of generalized cohomology theory to crystalline symmetries by systematically interpreting terms of a spectral sequence of a generalized cohomology theory as building blocks of crystalline phases. Related discussions along this direction can also be found in Ref. [60, 61]. (To be precise, Ref. [59] focused on generalized *homology* theories, but it is highly plausible that that is equivalent to a generalized cohomology formulation via a Poincaré duality.)

In this paper, we will conduct a thorough investigation into 3D bosonic cSPT phases protected by any space group symmetry G, dubbed G-SPT phases for short, and establish that their classification is indeed given by

$$H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z}). \tag{1}$$

We will see that distinct embedding copies of the E_8 state in the Euclidean space $\mathbb{E}^3$ produce G-SPT phases with different $H^1_\phi(G;\mathbb{Z})$ labels. To obtain the classification of G-SPT phases and to understand its physical meaning, we will invoke three techniques: dimensional reduction, surface topological order, and explicit cellular construction. Technically, we will establish

(a) every 3D bosonic G-SPT phase can be mapped, via a homomorphism, to an element of $H^1_\phi(G;\mathbb{Z})$,

(b) every element of $H^1_\phi(G;\mathbb{Z})$ can be mapped, via a homomorphism, to a 3D bosonic G-SPT phase,

(c) the first map is a left inverse of the second, and

(d) a 3D bosonic G-SPT phase comes from $H^5_\phi(G;\mathbb{Z})$ if and only if it maps to the trivial element of $H^1_\phi(G;\mathbb{Z})$.

Therefore, with minor caveats such as the correctness of $H^5_\phi(G;\mathbb{Z})$ in classifying non-E_8-based phases and certain assumptions about the correlation length of the states representing SPT (including cSPT) phases, we will be providing an essentially rigorous proof that 3D bosonic cSPT phases are classified by $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$. In addition, we will show that the value of $H^1_\phi(G;\mathbb{Z})$ can be easily read off from the international symbol for G, following this formula:

$$H^1_\phi(G;\mathbb{Z}) = \begin{cases} \mathbb{Z}^k, & G \text{ preserves orientation,} \\ \mathbb{Z}^k \times \mathbb{Z}_2, & \text{otherwise,} \end{cases} \tag{2}$$

where $k = 0$ if there is more than one symmetry direction listed in the international symbol, $k = 3$ if the international symbol has one symmetry direction listed and it is 1 or $\bar{1}$, and $k = 1$ if the international symbol has one symmetry direction listed and it is not 1 or $\bar{1}$.

Naturally, we expect that $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ also works for more general crystalline symmetries with $H^1_\phi(G;\mathbb{Z})$ computed in a similar way. In particular, for magnetic space groups G (which include space groups as a subclass called type I), we still have $H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}^k \times \mathbb{Z}_2^\ell$ with $k \in \mathbb{Z}$ and $\ell \in \{0,1\}$ only depending on the associated magnetic point group. For magnetic group G of type II or type IV, $H^1_\phi(G;\mathbb{Z})$ is simply $\mathbb{Z}_2$, while k and ℓ can be read off from the associated magnetic point group type according to Table III for G of type III.

Throughout this paper, we will use the term SPT phases with symmetry G to mean all invertible topological phases with that symmetry [19, 22, 23, 25, 26, 47, 62, 63]. As there is a more restrictive alternative convention in the literature [9] requiring SPT phases to be trivializable by breaking symmetry, let us clarify the basic notions used in this paper here. First, the notion of topological phases under discussion is about ground states of gapped quantum systems. A state $|\psi\rangle$, which is the ground state of some gapped system, is called *invertible* if there exists a ground state $|\eta\rangle$ (of another gapped system) such that $|\psi\rangle \otimes |\eta\rangle$ is trivializable, *i.e.*, can be deformed continuously (along a path of ground states of gapped Hamiltonians with finite range interactions) into the trivial state (which is a tensor product state for spin systems and the vacuum for fermionic systems). Trivializable states represent the trivial phase (without symmetry protection) and they are all invertible. However, not all invertible states are trivializable; integer quantum hall states are examples of invertible but not trvializable states for fermionic systems. We will follow Kitaev's terminology as well, using *short-range entangled* (SRE) states as a synonym of invertible states; this terminology is motivated by the fact that the invertibility property tells whether a state can be completely specified by local measurements [26]. Two G-symmetric SRE states $|\Psi_a\rangle$ and $|\Psi_b\rangle$ (probably of different systems a and b) are said to be of the same G-SPT order, if there exists a continuous path of G-symmetric SRE states between $|\Psi_a\rangle \otimes |T_b\rangle$ and $|T_a\rangle \otimes |\Psi_b\rangle$, where $|T_a\rangle$ and $|T_b\rangle$ denotes the trivial states for systems a and b respectively. As discussed in Appendix B, the set of G-SPT orders forms an Abelian group under the stacking operation in general. Our classification Eq. (1) is about the group structure of *bosonic* G-SPT orders (*i.e.*, G-SPT orders for systems involving no fermionic degrees of freedom), which we denote as $\mathrm{SPT}^d(G)$. We emphasize that we do not require G-SPT orders trivializable by breaking symmetry: the corresponding SRE states may be invertible but not trivializable (even without symmetry protection).

The alternate convention used in the literature [9] focuses only on SPT orders which are trivializable by symmetry breaking. Let $\mathrm{SPT}^d_*(G)$ be the set of such bosonic G-SPT orders in d spatial dimensions. Clearly, $\mathrm{SPT}^d_*(G)$ is a subgroup of $\mathrm{SPT}^d(G)$ in general. For any space group or magnetic group G in $d = 3$ spatial dimensions, it will be clear from our discussion that $\mathrm{SPT}^3_*(G)$ is the torsion subgroup of $\mathrm{SPT}^3(G)$ [*i.e.*, the subgroup of $\mathrm{SPT}^3(G)$ containing all elements of finite order]. Explicitly, the group structure of $\mathrm{SPT}^3_*(G)$ is $H^5_\phi(G;\mathbb{Z})$ if G preserves spacetime orientation and $H^5_\phi(G;\mathbb{Z}) \times \mathbb{Z}_2$ (which contains an extra $\mathbb{Z}_2$ factor beyond the group cohomology proposal) if G contains any elements reversing spacetime orientation.

This paper is organized as follows. In Sec. II, we will explain how $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ arises from the generalized cohomology hypothesis. In Sec. III, we will present examples of cSPT phases described by $H^1_\phi(G;\mathbb{Z})$ for select space groups. In Sec. IV, we will establish that 3D bosonic cSPT phases are classified by $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ in full generality. In Sec. V, we will discuss possible generalizations and conclude the paper. There are three appendices to the paper. In Appendix A, we will prove formula (2) and tabulate $H^1_\phi(G;\mathbb{Z})$ for all 230 space groups. Moreover, we generalize the calculation to magnetic space groups as well. In Appendix B, we review the stacking operation on SPT phases. In Appendix C, we review the generalized cohomology hypothesis.

II. PREDICTION BY GENERALIZED COHOMOLOGY HYPOTHESIS

To pave the way for us to generally construct and completely classify 3D E_8-based cSPT phases, let us make an prediction for what the classification of these phases might be using the generalized cohomology hypothesis [47, 63]. The generalized cohomology hypothesis was based on Kitaev's proposal [24, 26, 27] that the classification of SPT phases must carry the structure of generalized cohomology theories [49–52]. This proposal was further developed in Refs. [47, 48, 63].

The key idea here is that the classification of SPT phases can be encoded by a sequence $F_\bullet = \{F_d\}$ of topological spaces,

$$F_0, F_1, F_2, F_3, F_4, \ldots \tag{3}$$

where F_d is the space made up of all d-dimensional short-range entangled (SRE) states. It can be argued [24, 26, 27, 47, 48] that the spaces F_d are related to each other: the d-th space is homotopy equivalent to the loop space [49] of the $(d+1)$st space,

$$F_d \simeq \Omega F_{d+1}. \tag{4}$$

Physically, this says that there is a correspondence between d-dimensional SRE states and one-parameter families of $(d+1)$-dimensional SRE states.

To state how the sequence $F_\bullet$ determines the classification of SPT phases, let us introduce a homomorphism,

$$\phi : G \to \{\pm 1\}\,, \tag{5}$$

that tracks which elements of the symmetry group G preserve the orientation of spacetime (mapped to $+1$) and which elements do not (mapped to -1). As discussed in Appendix B, the collections of G-SPT phases (*i.e.*, topological phase protected by symmetry G) in d dimensions are classified by the Abelian group of G-SPT orders

$$\mathrm{SPT}^d(G), \tag{6}$$

whose addition operation is defined by stacking. It is conjectured that the group structure of $\mathrm{SPT}^d(G)$ can be obtained by computing the mathematical object

$$h^d_\phi(G;F_\bullet) := [EG, \Omega F_{d+1}]_G\,. \tag{7}$$

Here, EG is the total space of the universal principal G-bundle [64], and $[EG, \Omega F_{d+1}]_G$ denotes the set of deformation classes of G-equivariant maps from the space EG to the space ΩF_{d+1} (see Appendix C for detail). Explicitly, the generalized cohomology hypothesis states that we have an isomorphism

$$\mathrm{SPT}^d(G) \cong h^d_\phi(G;F_\bullet). \tag{8}$$

To compute (7), we note by definition that the 0th homotopy group of F_d,

$$\pi_0(F_d), \tag{9}$$

(*i.e.*, the set of connected components of F_d) classifies d-dimensional invertible topological orders (*i.e.*, SPT phases without symmetry). In 0, 1, 2, and 3 dimensions, the classification of invertible topological orders is believed to be [24, 26, 27, 47, 48]

$$\pi_0(F_0) = 0,\ \pi_0(F_1) = 0,\ \pi_0(F_2) = \mathbb{Z},\ \pi_0(F_3) = 0, \tag{10}$$

respectively, where the $\mathbb{Z}$ in 2 dimensions is generated by the E_8 phase [17, 29, 55, 56]. Next, we note that 0-dimensional SRE states are nothing but rays in Hilbert

Table I. Homotopy groups of the space F_d of d-dimensional SRE states, for $0 \le d \le 3$.

	F_0	F_1	F_2	F_3
$\pi_{>5}$	0	0	0	0
π_5	0	0	0	$\mathbb{Z}$
π_4	0	0	$\mathbb{Z}$	0
π_3	0	$\mathbb{Z}$	0	0
π_2	$\mathbb{Z}$	0	0	0
π_1	0	0	0	$\mathbb{Z}$
π_0	0	0	$\mathbb{Z}$	0

spaces. These rays form the infinite-dimensional complex projective space [49], so

$$F_0 = \mathbb{C}P^\infty. \tag{11}$$

Finally, we note, as a consequence of Eq. (4), that the $(k+1)$st homotopy group of F_{d+1} is the same as the k-th homotopy group of F_d for all k and d:

$$\pi_k(F_d) \cong \pi_{k+1}(F_{d+1}). \tag{12}$$

This allows us to determine all homotopy groups of F_1, F_2, F_3 from the classification of invertible topological orders (10) and the known space of 0D SRE states (11). The results are shown in Table I.

It turns out the homotopy groups in Table I completely determine the space F_1, F_2, F_3 themselves [47]:

$$F_1 = K(\mathbb{Z},3), \tag{13}$$
$$F_2 = K(\mathbb{Z},4) \times \mathbb{Z}, \tag{14}$$
$$F_3 = K(\mathbb{Z},5) \times \mathbf{S}^1, \tag{15}$$

where $K(\mathbb{Z},n)$ is the n-th Eilenberg-MacLane space of $\mathbb{Z}$ [defined by the property $\pi_k(K(\mathbb{Z},n)) = \mathbb{Z}$ for $k = n$ and 0 otherwise] [49]. Plugging Eqs. (11)(13)(14)(15) into Eqs. (7)(8), we arrive at the prediction

$$\mathrm{SPT}^0(G) \cong H^2_\phi(G;\mathbb{Z}), \tag{16}$$
$$\mathrm{SPT}^1(G) \cong H^3_\phi(G;\mathbb{Z}), \tag{17}$$
$$\mathrm{SPT}^2(G) \cong H^4_\phi(G;\mathbb{Z}) \oplus H^0_\phi(G;\mathbb{Z}), \tag{18}$$
$$\mathrm{SPT}^3(G) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z}), \tag{19}$$

where $H^n_\phi(G;\mathbb{Z})$ denotes the n-th twisted group cohomology of G with coefficient $\mathbb{Z}$ and twist ϕ [64]. For finite or compact groups, we have $H^5_\phi(G;\mathbb{Z}) \cong H^4_{\mathrm{Borel},\phi}(G;U(1))$; we identify this as the contribution from the group cohomology proposal [53] to the 3D classification (19). The existence of $H^1_\phi(G;\mathbb{Z})$ in Eq. (19), on the other hand, can be traced back to the fact that $\pi_0(F_2) = \mathbb{Z}$ in Eq. (10); we identify it as the contribution of E_8-based phases [17, 29, 55, 56] to the 3D classification. Therefore, we predict that 3D bosonic cSPT phases built from E_8 states are classified by

$$H^1_\phi(G;\mathbb{Z}) \tag{20}$$

(up to some non-E_8-based phases), where G is the space group and $\phi : G \to \{\pm 1\}$ keeps track of which elements of G preserve/reverse the orientation [see Eq. (5)].

$H^1_\phi(G;\mathbb{Z})$ can be intuitively thought of as the set of "representations of G in the integers." Explicitly, an element of $H^1_\phi(G;\mathbb{Z})$ is represented by a map (called a group 1-cocycle)

$$\nu^1 : G \to \mathbb{Z} \tag{21}$$

satisfying the cocycle condition

$$\nu^1(g_1 g_2) = \nu^1(g_1) + \phi(g_1)\nu^1(g_2) \tag{22}$$

for all $g_1, g_2 \in G$. Suppose ϕ was trivial for the moment (mapping all elements to $+1$). Then we would have $\nu^1(g_1 g_2) = \nu^1(g_1) + \nu^1(g_2)$, which is precisely the axiom $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for a representation ρ of G, written additively as opposed to multiplicatively. Now if we allowed ρ to be antilinear, then we would have the modified condition $\rho(g_1 g_2) = \rho(g_1)\overline{\rho(g_2)}$ (overline denoting complex conjugation) for all g_1's that are represented antilinearly. The analogue of this for ν^1 is precisely (22). The cohomology group $H^1_\phi(G;\mathbb{Z})$ itself is defined to be the quotient of group 1-cocycles [as in Eqs. (21)(22)] by what is called group 1-coboundaries. In Appendix A, we make the definition of group 1-coboundary explicit and show how one can easily read off $H^1_\phi(G;\mathbb{Z})$ from the international symbol of G.

III. EXAMPLES OF 3D CRYSTALLINE PHASES

Now let us focus on cSPT phases in three spatial dimensions (*i.e.*, $d = 3$). For each fixed space group G, they (or more precisely G-SPT orders) form an Abelian group $\mathrm{SPT}^3(G)$ equipped with the stacking operation as explained in Appendix B. To examine whether G-SPT phases are classified by $h^d_\phi(G;F_\bullet) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$, we notice that the first summand $H^5_\phi(G;\mathbb{Z})$ has been already computed in Ref. [42] and matches with the phases built by lower dimensional group cohomology states explicitly constructed and classified in Ref. [45]. It was also noticed that distinct cSPT phases can be built with E_8 states [58]. In the following sections, we are going to construct such phases systematically for all 230 space groups and show that they are classified by the second summand $H^1_\phi(G;\mathbb{Z})$.

To clarify the notion of cSPT phases and to motivate the general construction, let us present explicit examples of cSPT phases for the space groups $P1\,(1)$, $P\bar{1}\,(2)$, $Pm\,(6)$, $Pc\,(7)$, $Pmm2\,(25)$, and $Pmmm\,(47)$. Here, for the reader's convenience, the sequential number of a space group (as in the *International Tables for Crystallography* [65]) are provided in the parentheses after its Hermann-Mauguin symbol.

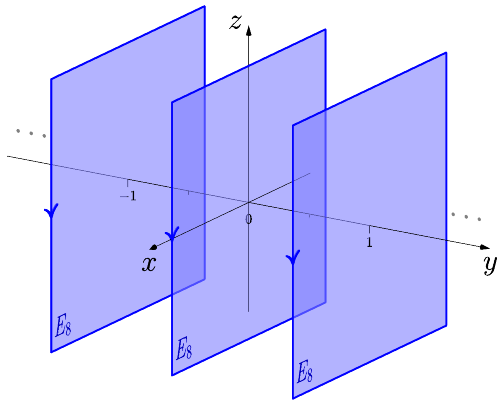

Figure 1. Layered E_8 states in the t_y direction.

A. Space group $P1\,(1)$

The space group $P1$ contains only translation symmetries. In proper coordinates, $P1$ is generated by

$$t_x : (x, y, z) \mapsto (x+1, y, z)\,, \tag{23}$$
$$t_y : (x, y, z) \mapsto (x, y+1, z)\,, \tag{24}$$
$$t_z : (x, y, z) \mapsto (x, y, z+1)\,. \tag{25}$$

As an abstract group, it is isomorphic to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \equiv \mathbb{Z}^3$. Thus, we have

$$H^5_\phi(P1, \mathbb{Z}) = 0, \tag{26}$$
$$H^1_\phi(P1, \mathbb{Z}) = \mathbb{Z}^3. \tag{27}$$

The latter can be identified with layered E_8 states.

For instance, we can put a copy of E_8 state on each of the planes $y = \cdots, -2, -1, 0, 1, 2, \cdots$ as in Fig. 1. Since each E_8 state can be realized with translation symmetries t_x and t_z respected, the layered E_8 states can be made to respect all three translations and thus realize a 3D cSPT phase with $P1$ symmetry.

To characterize this cSPT phase, we take a periodic boundary condition in t_y direction requiring $t_y^{L_y} = 1$. Such a procedure is called a compactification in the t_y direction, which is well-defined for any integer $L_y \gg 1$ in general for a gapped system with a translation symmetry t_y. The resulting model has a finite thickness in the t_y direction and thus can be viewed as a 2D system extending in the t_x, t_z directions. Further, neglecting the translation symmetries t_x and t_z, we take an open boundary condition of the 2D system. Then its edge supports $8L_y$ co-propagating chiral boson modes (chiral central charge $c_-^y = 8L_y$). The resulting quantized thermal Hall effect is proportional to L_y and shows the nontriviality of the layered E_8 states as a cSPT phase with the $P1$ symmetry.

Actually, even without translation symmetries, we cannot trivialize such a system into a tensor product state with local unitary gates of a universal finite depth homogeneously in space. However, such a state is trivializable in a weaker sense: if the system has correlation length less than $\xi > 0$, then any ball region of size much larger than ξ can be trivialized with correlation length kept smaller than ξ. We call such a state *weakly trivialiable*. In this paper, the notion of cSPT phase includes all invertible gapped quantum phases and in particular these weakly trivializable states.

A bosonic SPT system has $c_-^y = \gamma_y L_y$ with γ_y a multiple of 8 (*i.e.*, $\gamma_y \in 8\mathbb{Z}$) in general. To see $\gamma_y \in 8\mathbb{Z}$, we notice that the net number of chiral boson modes along the interface between compactified systems of thicknesses L_y and L_y+1 is γ_y. Then the absence of anyons in both sides implies $\gamma_y \in 8\mathbb{Z}$ [66]. It is obvious that γ_y's are added during a stacking operation.

Clearly, this cSPT phase of layered E_8 states is invertible; its inverse is made of layered $\overline{E_8}$ states, where $\overline{E_8}$ denotes the chiral twin of E_8. The edge modes of $\overline{E_8}$ propagate in the direction opposite to those of E_8 and hence we have $\gamma_y = -8$ for the cSPT phase of the layered $\overline{E_8}$ states. Via the stacking operation explained in Appendix B, all possible $\gamma_y \in 8\mathbb{Z}$ is generated by the cSPT phase of layered E_8 states and its inverse.

Analogously, we can define γ_x (resp. γ_z) by compactifying a system in the t_x (resp. t_z) direction. Thus, the cSPT phases with $P1$ symmetry are classified by $\frac{1}{8}(\gamma_x, \gamma_y, \gamma_z) \in \mathbb{Z}^3 = H^1_\phi(P1, \mathbb{Z})$. Via the stacking operation, they can be generated by the three cSPT phases made of layered E_8 states in the t_x, t_y and t_y directions respectively and their inverses.

B. Space groups $Pm\,(6)$, $Pmm2\,(25)$, and $Pmmm\,(47)$

To explore possible cSPT phases in the presence of reflection symmetries (*i.e.*, mirror planes), let us look at the space groups Pm, $Pmm2$ and $Pmmm$ as examples.

1. *Space group* $Pm\,(6)$

The space group Pm is generated by t_x, t_y, t_y and a reflection

$$m_y : (x, y, z) \mapsto (x, -y, z)\,. \tag{28}$$

Thus, the mirror planes are $y = \cdots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \cdots$ (*i.e.*, integer y planes and half-integer y planes). For Pm, the group structure of G and its subgroup G_0 of orientation preserving symmetries are

$$G = \mathbb{Z}\times\mathbb{Z}\times\left(\mathbb{Z}\rtimes\mathbb{Z}_2^\phi\right), \tag{29}$$
$$G_0 = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}, \tag{30}$$

where $\mathbb{Z}_2^\phi$ is the group generated by m_y with the superscript ϕ emphasizing $\phi(m_y) = -1$.

Theorem 1 in Appendix A computes the second part of $\mathrm{SPT}^3(G)$ shown in Eq. (19); the result is

$$H^1_\phi(G; \mathbb{Z}) = \mathbb{Z}\times\mathbb{Z}_2. \tag{31}$$

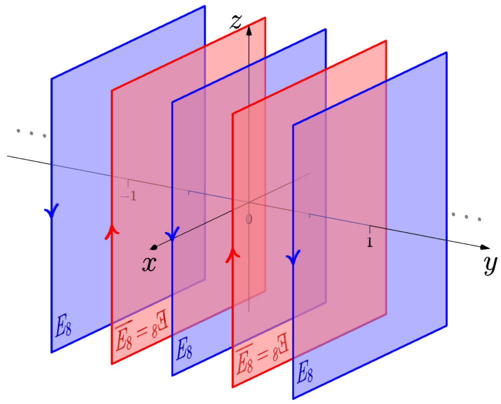

Figure 2. Alternately layered E_8 states in the t_y direction.

The $\mathbb{Z}$ factor specifies $\frac{1}{8}\gamma_y$ of the cSPT phases compatible with Pm symmetry; the reflection symmetry m_y requires that $\gamma_x = \gamma_z = 0$. Explicitly, putting an E_8 state on each integer y plane produces a phase with $\frac{1}{8}\gamma_y = 1$.

The factor $\mathbb{Z}_2$ in Eq. (31) is generated by the cSPT phase built by putting an E_8 state at each integer y plane and its inverse $\overline{E_8}$ at each half-integer y plane, as shown in Fig. 2. This construction was proposed and the resulting phases were studied in Ref. [58]. In particular, the order of this cSPT phase is 2, meaning that stacking two copies of such a phase produces a trivial phase. In fact, we will soon see that this generating phase can be realized by a model with higher symmetry like $Pmmm$ and is protected nontrivial by any single orientation-reversing symmetry.

In addition, the other part of $\mathrm{SPT}^3(G)$ is

$$H^5_\phi(G;\mathbb{Z}) = H^5_\phi\left(\mathbb{Z}\rtimes\mathbb{Z}_2^\phi;\mathbb{Z}\right)\oplus\left[H^4_\phi\left(\mathbb{Z}\rtimes\mathbb{Z}_2^\phi;\mathbb{Z}\right)\right]^2 \oplus H^3_\phi\left(\mathbb{Z}\rtimes\mathbb{Z}_2^\phi;\mathbb{Z}\right) = \mathbb{Z}_2^2\oplus 0\oplus\mathbb{Z}_2^2. \quad (32)$$

The three summands correspond to the phases built from group cohomology SPT phases in 2, 1, and 0 spatial dimensions respectively in Ref. [45].

For the reader's convenience, let us review the construction briefly here. The four cSPT phases corresponding to first summand $\mathbb{Z}_2^2$ in Eq. (32) can be built by putting 2D SPT states with Ising symmetry on mirror planes. We notice that each reflection acts as an Ising symmetry (*i.e.*, a unitary internal symmetry of order 2) on its mirror plane. Moreover, the space group Pm contains two families of inequivalent mirrors (*i.e.*, integer y planes and half-integer y planes). For each family, there are two choices of 2D SPT phases with Ising symmetry, classified by $H^4(\mathbb{Z}_2,\mathbb{Z}) \cong H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$. Thus, we realize four cSPT phases with group structure $\mathbb{Z}_2^2$. Further, noticing the translation symmetries within each mirror, we can put, to every unit cell of the mirror, a 0D state carrying eigenvalue ± 1 of the corresponding reflection; 0D states with Ising symmetry are classified by their symmetry charges and are formally labeled by $H^2(\mathbb{Z}_2,\mathbb{Z}) \cong H^1(\mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. They produce another $\mathbb{Z}_2$ factor of cSPT phases associated with each inequivalent family of mirrors. Thus, we get another four cSPT phases labeled by the last summand $\mathbb{Z}_2^2$ in Eq. (32).

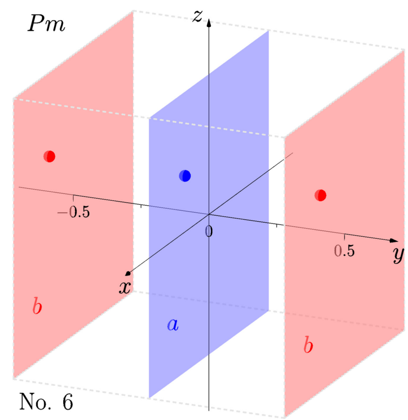

Figure 3. For space group No. 6 (*i.e.*, $G = Pm$), there are two inequivalent family of mirrors (blue and red online), whose Wyckoff position labels are a and b respectively in Ref. [65]. The cSPT phases corresponding to $H^5_\phi(G;\mathbb{Z})$ are built by assigning to the mirrors 2D SPT phases protected by Ising symmetry and $\mathbb{Z}_2$ point charges (blue and red dots online).

To summarize, this construction of $H^5_\phi(G;\mathbb{Z})$ phases for the space group Pm can be presented by Fig. 3. It can be generalized to all the other space groups: to each Wyckoff position, we assign group cohomology SPT phases protected by its site symmetry. Further technical details can be found in Ref. [45].

Thus, we have constructed the invertible topological phases with $G = Pm$ symmetry; they are classified by $\mathrm{SPT}^3(G) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$. It is clear that all the cSPT phases labeled by $H^5_\phi(G;\mathbb{Z})$ are trivializable by breaking symmetry. The cSPT phase constructed in Fig. 3, which corresponds to the $\mathbb{Z}_2$ factor of $H^1_\phi(G;\mathbb{Z})$, can also be trivialized by breaking the crystalline symmetry and annihilating the E_8, $\overline{E}_8$ states in pairs. However, the phases with nonzero $\frac{1}{8}\gamma_y \in \mathbb{Z}$ cannot be completely trivialized by local unitary gates of finite depth even without any symmetry. Therefore, if we only focus on the Pm-SPT phases which can be trivialized by breaking symmetry, they are classified by $\mathrm{SPT}^3_*(G) \cong H^5_\phi(G;\mathbb{Z}) \times \mathbb{Z}_2$, which is a subgroup of $\mathrm{SPT}^3(G)$.

2. *Space groups $Pmm2$ (25)*

In the following, we will focus on developing a universal construction of $H^1_\phi(G;\mathbb{Z})$ phases. To get motivated, let us look at more examples.

The space group $G = Pmm2$ is generated by t_x, t_y, t_y, and two reflections

$$m_x : (x, y, z) \mapsto (-x, y, z)\,, \quad (33)$$

$$m_y : (x, y, z) \mapsto (x, -y, z)\,. \quad (34)$$

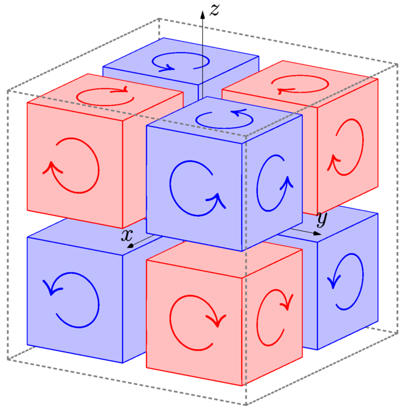

Figure 4. A unit cell for the space group $Pmmm\,(47)$ is partitioned into eight cuboids (red and blue online), each of which works as a fundamental domain. A chiral $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ state is put on the surface of each fundamental domain; its chirality is indicated by the arrowed arcs such that all symmetries are respected. On each interface between two fundamental domains, there are two copies of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ states. Let $(\mathfrak{e}_\mathfrak{f},\mathfrak{e}_\mathfrak{f})$ (resp. $(\mathfrak{m}_\mathfrak{f},\mathfrak{m}_\mathfrak{f})$) denote the anyon formed by pairing $\mathfrak{e}_\mathfrak{f}$ (resp. $\mathfrak{m}_\mathfrak{f}$) from each copy. Then the anyons $(\mathfrak{e}_\mathfrak{f},\mathfrak{e}_\mathfrak{f})$ and $(\mathfrak{m}_\mathfrak{f},\mathfrak{m}_\mathfrak{f})$ can be condensed, leading to a model that generates (via the stacking operation) cSPT phases corresponding to $H^1_\phi(G;\mathbb{Z})$ for $G = Pmmm$.

This time, Theorem 1 in Appendix A tells us that

$$H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}_2. \tag{35}$$

There is no $\mathbb{Z}$ factor any more as expected, because m_x and m_y together require $\gamma_x = \gamma_y = \gamma_z = 0$.

The cSPT phase with $Pmm2$ symmetry generating $H^1_\phi(G;\mathbb{Z})$ is actually compatible with a higher symmetry $Pmmm$. Hence let us combine the study on this phase with the discussion of the space group $Pmmm$ below.

3. *Space group Pmmm (47)*

The space group $Pmmm$ is generated by t_x, t_y, t_y, and three reflections

$$m_x : (x,y,z) \mapsto (-x,y,z)\,, \tag{36}$$
$$m_y : (x,y,z) \mapsto (x,-y,z)\,, \tag{37}$$
$$m_z : (x,y,z) \mapsto (x,y,-z)\,. \tag{38}$$

For $G = Pmmm$, Theorem 1 in Appendix A gives

$$H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}_2. \tag{39}$$

There is no $\mathbb{Z}$ factor as in the case of $Pmm2\,(25)$ above.

The cSPT phase generating $H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}_2$ can be constructed as in Fig. 4. Step 1: we partition the 3D space into cuboids of size $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$. Each such cuboid works as a *fundamental domain*, also know as an *asymmetric unit* in crystallography [65]; it is a smallest simply connected closed part of space from which, by application of all symmetry operations of the space group, the whole of space is filled. Every orientation-reversing symmetry relates half of these cuboids (blue online) to the other half (red online). Step 2: we attach an $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological state to the surface of each cuboid from inside with all symmetries in G respected. An $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological state hosts three anyon species, denoted $\mathfrak{e}_\mathfrak{f}$, $\mathfrak{m}_\mathfrak{f}$, and ε, all with fermionic self-statistics. Such a topological order can be realized by starting with a $\nu = 4$ integer quantum Hall state and then coupling the fermion parity to a $\mathbb{Z}_2$ gauge field in its deconfined phase [66]. This topological phase exhibits net chiral edge modes under an open boundary condition. Step 3: there are two copies of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ states at the interface of neighbor cuboids and we condense $(\mathfrak{e}_\mathfrak{f},\mathfrak{e}_\mathfrak{f})$ and $(\mathfrak{m}_\mathfrak{f},\mathfrak{m}_\mathfrak{f})$ simultaneously without breaking any symmetries in G, where $(\mathfrak{e}_\mathfrak{f},\mathfrak{e}_\mathfrak{f})$ (resp. $(\mathfrak{m}_\mathfrak{f},\mathfrak{m}_\mathfrak{f})$) denotes the anyon formed by pairing $\mathfrak{e}_\mathfrak{f}$ (resp. $\mathfrak{m}_\mathfrak{f}$) from each copy. After condensation, all the other anyons are confined, resulting in the desired cSPT phase, denoted $\mathcal{E}$.

To see that $\mathcal{E}$ generates $H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}_2$, we need to check that $\mathcal{E}$ is nontrivial and that $2\mathcal{E}$ (*i.e.*, two copies of $\mathcal{E}$ stacking together) is trivial. First, we notice that any orientation-reversing symmetry (*i.e.*, a reflection, a glide reflection, an inversion, or a rotoinversion) $g \in G$ is enough to protect $\mathcal{E}$ nontrivial. Let us consider an open boundary condition of the model, keeping only the fundamental domains enclosed by the surface shown in 5. Then the construction in Fig. 4 leaves a surface of the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order respecting g. If the bulk cSPT phase is trivial, then the surface topological order can be disentangled in a symmetric way from the bulk by local unitary gates with a finite depth. However, this strictly 2D system of the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order is chiral, wherein the orientation-reversing symmetry g has to be violated. This contradiction shows that the bulk is nontrivial by the protection of g.

To better understand the incompatibility of a strictly 2D system of the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order with any orientation-reversing symmetry g, we view the 2D system as a gluing result of two regions related by g as in Fig. 5. The $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order implies net chiral edge modes for each region. Further, g requires that edge modes from the two region propagate in the same direction at their 1D interface. Thus, a gapped gluing is impossible, which shows the non-existence of the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order compatible with g in a strictly 2D system. Therefore, the surface $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order respecting g proves the non-triviality of the bulk cSPT phase $\mathcal{E}$.

On the other hand, $2\mathcal{E}$ is equivalent to the model obtained by attaching an E_8 state to the surface of each fundamental domain from inside in a symmetric way. To check this, let us trace back the construction of $2\mathcal{E}$: we start with attaching two copies of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ states on the surface of each fundamental domain from inside. Then let us focus a single rectangle interface between two cuboids. There are four copies of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ states along it, labeled by $i = 1, 2$ for one side and $i = 3, 4$ for the other side. The symmetries relate $i = 1 \leftrightarrow i = 4$ and $i = 2 \leftrightarrow i = 3$ sep-

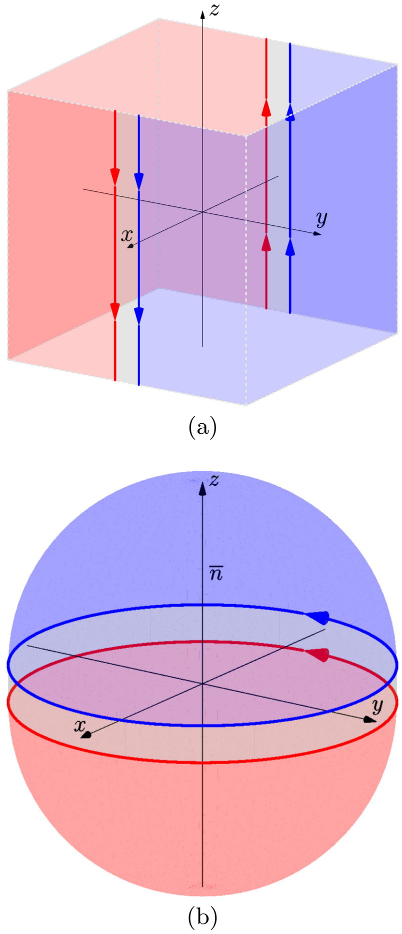

Figure 5. Two 2D systems (red and blue online) of the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological order cannot be glued together into a *gapped* state respecting any orientation-reversing symmetry; their chiral edge modes must propagate in the same direction because of the symmetry. (a) illustrates cases with an rotoinversion axis $\overline{n}$ (including the spacial cases with an inversion center or a mirror plane). (b) illustrates cases with a glide reflection plane such as $c : (x, y, z) \mapsto (x, -y, z + 1/2)$.

arately. If we further condense $(\mathfrak{e}_\mathfrak{f}^1, \mathfrak{e}_\mathfrak{f}^4)$, $(\mathfrak{m}_\mathfrak{f}^1, \mathfrak{m}_\mathfrak{f}^4)$, $(\mathfrak{e}_\mathfrak{f}^2, \mathfrak{e}_\mathfrak{f}^3)$, and $(\mathfrak{m}_\mathfrak{f}^2, \mathfrak{m}_\mathfrak{f}^3)$, then we get the local state of $2\mathcal{E}$ near the rectangle. Here $\mathfrak{e}_\mathfrak{f}^i$ (resp. $\mathfrak{m}_\mathfrak{f}^i$) denotes the $\mathfrak{e}_\mathfrak{f}$ (resp. $\mathfrak{m}_\mathfrak{f}$) particle from the i^{th} copy of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ state and $(\mathfrak{e}_\mathfrak{f}^i, \mathfrak{e}_\mathfrak{f}^j)$ (resp. $(\mathfrak{m}_\mathfrak{f}^i, \mathfrak{m}_\mathfrak{f}^j)$) is the anyon obtained by pairing $\mathfrak{e}_\mathfrak{f}^i$, $\mathfrak{e}_\mathfrak{f}^j$ (resp. $\mathfrak{m}_\mathfrak{f}^i$, $\mathfrak{m}_\mathfrak{f}^j$). However, we may alternately condense $(\mathfrak{e}_\mathfrak{f}^1, \mathfrak{e}_\mathfrak{f}^2)$, $(\mathfrak{m}_\mathfrak{f}^1, \mathfrak{m}_\mathfrak{f}^2)$, $(\mathfrak{e}_\mathfrak{f}^3, \mathfrak{e}_\mathfrak{f}^4)$, and $(\mathfrak{m}_\mathfrak{f}^3, \mathfrak{m}_\mathfrak{f}^4)$. The resulting local state is two copies of E_8 states connecting the same environment in a symmetric gapped way. Thus, the two local states produced by different condensation procedures have the same edge modes with the same symmetry behavior and hence are equivalent. Therefore, $2\mathcal{E}$ is equivalent to the model constructed by attaching an E_8 state to the surface of each fundamental domain from inside. Since the later can be obtained by blowing an E_8 state bubble inside each fundamental domain, it (and hence $2\mathcal{E}$) clearly presents a trivial cSPT phase.

Thus, we have shown that $\mathcal{E}$ is nontrivial (by the protection of any orientation-reversing symmetry) and that $2\mathcal{E}$ is trivial. Therefore, $\mathcal{E}$ contributes a $\mathbb{Z}_2$ factor to the classification of cSPT phases for $G = Pmmm$ and any of its non-orientation-preserving subgroups such as $Pm\,(6)$ and $Pmm2\,(25)$. Clearly, $\mathcal{E}$ should be counted as a cSPT phase even in the more restrictive convention; we can trivialize the state constructed in Fig. 4 by blowing an E_8 state bubble inside each blue cube if orientation-reversing symmetries are allowed to be broken.

For Pm, the model constructed in Fig. 2 actually presents the same cSPT phase as $\mathcal{E}$ constructed in Fig. 4. To see this, we could blow an E_8 state bubble inside cuboids centered at $\frac{1}{4}(1, \pm 1, -1) + \mathbb{Z}^3$ and $\frac{1}{4}(-1, \pm 1, 1) + \mathbb{Z}^3$, with chirality opposite to those indicated by the arrowed arcs shown on the corresponding cuboids in Fig. 4. This relates $\mathcal{E}$ to alternately layered E_8 states; however, each reflection does not acts trivially on the resulting E_8 layer on its mirror as the model in Fig. 2. To further show their equivalence, we look at a single E_8 layer at $y = 0$ for instance. Since it may be obtained by condensing $(\mathfrak{e}_\mathfrak{f}, \mathfrak{e}_\mathfrak{f})$ and $(\mathfrak{m}_\mathfrak{f}, \mathfrak{m}_\mathfrak{f})$ in a pair of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological states attached to the mirror, it hence can connect the $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ surface shown in Fig. 5 in a gapped way with the reflection m_y respected. On the other hand, we know that an E_8 state put at $y = 0$ with trivial m_y action can also connect to this surface in gapped m_y-symmetric way [58] and is thus equivalent to the corresponding E_8 layer just mentioned with nontrivial m_y action. As a result, the models constructed in Figs. 2 and 4 realize the same cSPT phase with Pm symmetry.

C. Space group $P\overline{1}\,(2)$

Let us explain the role of inversion symmetry by the example of space group $P\overline{1}$, which is generated by t_x, t_y, t_y, and an inversion

$$\overline{1} : (x, y, z) \mapsto (-x, -y, -z)\,. \tag{40}$$

For $G = P\overline{1}$, Theorem 1 in Appendix A tells us that

$$H^1_\phi\,(G; \mathbb{Z}) = \mathbb{Z}^3 \times \mathbb{Z}_2. \tag{41}$$

The factor $\mathbb{Z}^3$ specifies $\frac{1}{8}(\gamma_x, \gamma_y, \gamma_z)$ as in the case of $P1$. For instance, the phase labeled by $\frac{1}{8}(\gamma_x, \gamma_y, \gamma_z) = (0, 1, 0)$ can be constructed by putting a copy of $E8$ state on each of the planes $y = \cdots, -2, -1, 0, 1, 2, \cdots$ with the symmetries t_x, t_y, t_z and $\overline{1}$ respected.

On the other hand, we can generate the factor $\mathbb{Z}_2$ in Eq. (41) by the phase constructed in Fig. 4. In particular, we have shown that this phase is nontrivial under the protection any orientation-reversing symmetry, like the inversion symmetry here, in Sec. III B 3.

D. Space group $Pc\,(7)$

Finally, we explain the role of glide reflection symmetry by the example of space group Pc, which is generated by t_x, t_y, t_y, and a glide reflection

$$c:(x,y,z)\mapsto(x,-y,z+1/2)\,. \tag{42}$$

For $G=Pc$, Theorem 1 in Appendix A tells us that

$$H^1_\phi\,(G;\mathbb{Z})=\mathbb{Z}\times\mathbb{Z}_2. \tag{43}$$

The factor $\mathbb{Z}$ specifies $\frac{1}{8}\,(0,\gamma_y,0)$; the glide reflection symmetry c requires $\gamma_x=\gamma_z=0$.

To construct the cSPT phase that generates the summand $\mathbb{Z}_2$, we consider the space group G' generated by translations t_x, t_y, $t'_z:(x,y,z)\mapsto\left(x,y,z+\frac{1}{2}\right)$ together with reflections m_x, m_y, m_z in Eqs. (36-38). Obviously, $G\subset G'$ and G' is a space group of type $Pmmm$. Then the model constructed as in Fig. 4 with respect to G' generates this $\mathbb{Z}_2$ factor of cSPT phases protected by space group G. Particularly, the glide reflection $c\in G$ is enough to protect the corresponding cSPT phase nontrivial by the argument in Sec. III B 3.

IV. DIMENSIONAL REDUCTION AND GENERAL CONSTRUCTION

In the above examples, we have presented cSPT phases by lower dimensional short-range entangled (SRE) states. Such a representation for a generic cSPT phase can be obtained by a dimensional reduction procedure; it is quite useful for constructing, analyzing, and classifying cSPT phases [45, 58]. Below, let us review the idea of dimension reduction and illustrate how to build cSPT phases with lower dimensional SRE states in general. More emphasis will be put on the construction of cSPT phases involving E_8 states, which have not been systematically studied for all space groups in the literature.

Given a space group G, we first partition the 3D euclidean space $\mathbb{E}^3$ into fundamental domains accordingly. A *fundamental domain*, also know as an *asymmetric unit* in crystallography [65], is a smallest simply connected closed part of space from which, by application of all symmetry operations of the space group, the whole of space is filled. Formally, the partition is written as

$$\mathbb{E}^3=\bigcup_{g\in G} g\mathcal{F}, \tag{44}$$

where $\mathcal{F}$ is a fundamental domain and $g\mathcal{F}$ its image under the action of $g\in G$. If g is not the identity of space group G, then by definition $\mathcal{F}$ and $g\mathcal{F}$ only intersect in their surfaces at most. In general, $\mathcal{F}$ can be chosen to be a convex polyhedron: the Dirichlet-Voronoi cell of a point $\mathsf{P}\in\mathbb{E}^3$ to its G-orbit, with P chosen to have a trivial stabilizer subgroup (this is always possible by discreteness of space groups) [67].

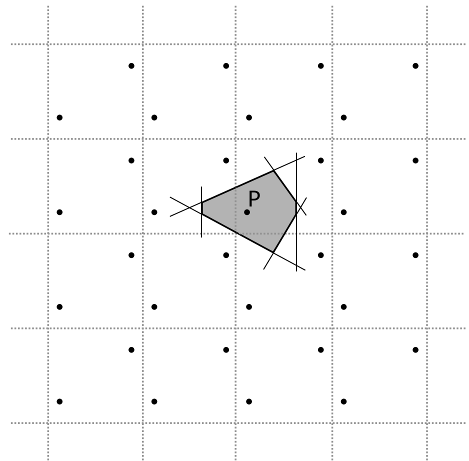

Figure 6. The Dirichlet-Voronoi cell (dark region) of a point $\mathsf{P}\in\mathbb{E}^2$ to its G-orbit (black dots), where G is generated by translations and an in-plane two-fold rotation.

The above definition and general construction of fundamental domain works for space groups in any dimensions. Let us take a lower dimensional case for a simple illustration: Fig. 6 shows a fundamental domain given by the Dirichlet-Voronoi cell construction for wallpaper group No. 2, which is generated by translations and a two-fold rotation. Clearly, the choice of fundamental domain is often not unique as in this case; a regular choice of fundamental domain for each wallpaper group and 3D space group is available in the International Tables for Crystallography [65].

To use terminology from simplicial homology [68], we further partition the fundamental domain $\mathcal{F}$ (resp. $g\mathcal{F}$) into tetrahedrons $\{\varsigma_\alpha\}_{\alpha=1,2,\cdots,N}$ (resp. $\{g\varsigma_\alpha\}_{\alpha=1,2,\cdots,N}$) such that $X=\mathbb{E}^3$ becomes a G-*simplicial* complex. A simplicial complex is a special kind of cell complex whose n-cells are n-simplices (*i.e.*, vertices for $n=0$, edges for $n=1$, triangles for $n=2$, and tetrahedrons for $n=3$). In a G-simplicial complex, each of its simplices is either completely fixed or mapped onto another simplex by g, $\forall g\in G$. Clearly, all internal points of every simplex share the same site symmetry[3]. Let $\Delta^n\,(X)$ denote the set of n-simplices and X_n the n-skeleton of X (*i.e.*, the subspace made of all k-simplices of X for $k\leq n$).

A. Dimensional reduction of cSPT phases

Topological phases of matter should admit a topological quantum field theory (TQFT) description, whose correlation functions do not depend on the metric of spacetime. Thus, it is natural to conjecture the following two basic properties of SPT (including cSPT) phases.

[3] The *site symmetry* of a point p is the group of symmetry operations under which p is not moved, *i.e.*, $G_p := \{g\in G|gp=p\}$.

(1) Each SPT phase can be represented by a symmetric SRE state $|\psi\rangle$ with arbitrary short correlation length. (2) Within the same SPT phase defined on d-dimensional Euclidean space $\mathbb{E}^d$, any two symmetric SRE states $|\psi_0\rangle$ and $|\psi_1\rangle$ with correlation length shorter than $r>0$ can be connected a path of symmetric SRE states $|\psi_\tau\rangle$ (parameterized by $\tau\in[0,1]$) whose correlation length is shorter than r for all τ. The conjecture is satisfied by all the topological states investigated in this paper, allowing the dimensional reduction procedure described below. Its rigorous proof in a reasonable setting, however, remains an interesting question and goes beyond the scope of this paper. If the conjecture holds in general, our classification in this paper will be complete. Otherwise, we would miss the cSPT phases where the two properties fail.

Given any cSPT phase for space group G, let us now describe the dimensional reduction procedure explaining why it can be built by lower dimensional states in general. To start, as conjectured, we can present this cSPT phase by a state $|\Psi\rangle$ with correlation length ξ much smaller than the linear size of the fundamental domain $\mathcal{F}$. The short-range correlation nature implies that $|\Psi\rangle$ is the ground state of some gapped Hamiltonian H whose interaction range is ξ as well. The local part of H inside $\mathcal{F}$ thus describes a 3D SRE state. It is believed that all 3D SRE states are trivializable or weakly trivializable (e.g. layered E_8 states). Thus, we can continuously change H into a trivial Hamiltonian inside $\mathcal{F}$ (except within a thin region near the boundary of $\mathcal{F}$) keeping the correlation length of its ground state smaller than ξ all the time. Removing the trivial degrees of freedom, we are left with a system on the 2-skeleton X_2 of $\mathbb{E}^3$.

Still, the reduced system hosts no nontrivial excitations and is SRE. Thus, there is an SRE state on each 2-simplex (*i.e.*, triangle) τ_α, indexed by α, of X_2. In particular, it could be $q_2(\alpha)$ copies of E_8 states (without specifying symmetry), where $q_2(\alpha)\in\mathbb{Z}$ with sign specifying the chirality. These data may be written collectively as a formal sum $q_2=\sum_\alpha q_2(\alpha)\tau_\alpha$, which may contain infinitely many terms as $X=\mathbb{E}^3$ is noncompact. On each edge ℓ, the chiral modes from all triangles connecting to ℓ have to cancel in order for the system to be gapped. In terms of the simplicial boundary map ∂, we thus have that $\partial q_2 \coloneqq \sum q_2(\alpha)\partial\tau_\alpha$ equals 0. In general, let $C_k(X)$ the set of such formal sums of k-simplices of X. Naturally, $C_k(X)$ has an Abelian group structure; $C_{-1}(X)$ is taken to be the trivial group. For any integer $k\geq 0$, the boundary map ∂_k (or simply ∂): $C_k(X)\to C_{k-1}(X)$ is a group homomorphism and let $B_k(X)$ (resp. $Z_{k-1}(X)$) denote its kernel (resp. image). Thus, the E_8 state configuration on X_2 is encoded by $q_2\in B_2(X)$.

It clear that $\partial_{k+1}\circ\partial_k=0$ and hence $Z_k(X)\subseteq B_k(X)$. As $C_k(X)$ contains *infinite* sums of simplices, it is not a standard group of k-chains. Instead, it can be viewed as $(3-k)$-cochains on the *dual polyhedral decomposition* (also called *dual block decomposition* [68]) of X. Thus, $B_k(X)/Z_k(X)$ should be understood as the $(3-k)^{th}$ cohomology $H^{3-k}(X)$ rather than the k^{th} homology of X. Since $X=\mathbb{E}^3$ is contractible, its cohomology groups are the same as a point: $H^n(X)$ is $\mathbb{Z}$ for $n=0$ and trivial for $n>0$. Thus, $B_2(X)=Z_2(X)$ and hence any gapped E_8 state configuration $q_2\in B_2(X)$ can be expressed as $q_2=\partial q_3$ for some $q_3\in C_3(X)$. Also, it is clear that $B_3(X)$ is generated by the sum of all 3-simplices with the right-handed orientation, which is simply denoted by X as well.

As $q_2=\partial q_3$ is symmetric under G, we have $\partial(gq_3)=g\partial q_3=\partial q_3$ and hence

$$gq_3=q_3+\nu^1(g)\,X, \tag{45}$$

for some $\nu^1(g)\in\mathbb{Z}$. Clearly, $\nu^1(e)=0$ for the identity element $e\in G$. The consistent condition $(gh)q_3=g(hq_3)$ requires that

$$\nu^1(gh)=\phi(g)\nu^1(h)+\nu^1(g), \tag{46}$$

i.e., the cocycle condition for $Z^1_\phi(G;\mathbb{Z})$. Definitions of group cocycles $Z^1_\phi(G;\mathbb{Z})$ as well as coboundaries $B^1_\phi(G;\mathbb{Z})$ and cohomologies $H^1_\phi(G;\mathbb{Z})$ are given in Appendix A 1. Thus, ν^1 is a normalized 1-cocycle. Moreover, we notice that q_3 is not uniquely determined by the E_8 state configuration q_2; solutions to $\partial q_3=q_2$ may differ by a multiple of X. According to Eq. (45), the choice change $q_3\to q_3+\nu^0X$ leads to $\nu^1(g)\to\nu^1(g)+\left(d\nu^0\right)(g)$, where $\nu^0\in\mathbb{Z}$ and $\left(d\nu^0\right)(g)\coloneqq\phi(g)\nu^0-\nu^0$. Thus, ν^1 is only specified up to a 1-coboundary. Therefore, any G-symmetric model (with correlation length much shorter than simplex size) on X_2 defines a cohomology group element $[\nu^1]\in H^1_\phi(G;\mathbb{Z})$.

By Lemma 1, each $[\nu^1]\in H^1_\phi(G;\mathbb{Z})$ can be parameterized by $\nu^1(t_{v_1}),\nu^1(t_{v_2}),\nu^1(t_{v_3})\in\mathbb{Z}$ (together with $\nu^1(r) \pmod 2\in\mathbb{Z}_2$ if G is non-orientation-preserving), where t_{v_1}, t_{v_2}, t_{v_3} are three elementary translations that generate the translation subgroup and r is an orientation-reversing symmetry. Let us explain their physical meaning by examples. For the model in Fig. 1, we could pick

$$q_3=-\sum_{i,j,k\in\mathbb{Z}} j\left(t_x^i t_y^j t_z^k\mathcal{F}\right) \tag{47}$$

with fundamental domain $\mathcal{F}=[0,1]\times[0,1]\times[0,1]$. To use the terminology of simplicial homology, $\mathcal{F}$ can be partitioned into tetrahedrons and be viewed as a formal sum of them with the right-handed orientation. Moreover, $t_x^i t_y^j t_z^k\mathcal{F}$ denote the translation result of $\mathcal{F}$. It is straightforward to check that ∂q_3 equals the sum of 2-simplices (oriented toward the positive y direction according to the right-hand rule) on integer y planes; thus, ∂q_3 corresponds to the model in Fig. 1. It is also clear that $t_vq_3-q_3=v^yX$ for a translation by $v=(v^x,v^y,v^z)$; hence $\nu^1(t_x)=0$, $\nu^1(t_y)=1$, and $\nu^1(t_z)=0$. Thus, we get a physical interpretation of $\nu^1(t_v)$: if the model is compactified in the v direction such that $t_v^L=1$, then it is equivalent to $L\nu^1(t_v)$ copies of E_8 states as a 2D system. Clearly, models with different $\nu^1(t_v)$ on any translation symmetry t_v must present distinct cSPT phases.

As another example, for $G=Pmmm$, $H_{\phi}^{1}(G;\mathbb{Z})=\mathbb{Z}_2$ with element $[\nu^1]$ parameterized by $\nu^1(r) \pmod 2$ on any orientation-reversing element r of G; here $\nu^1(t_v)$ on any translation $t_v\in G$ is required to be zero by symmetry. The E_8 state configuration of the model in Fig. 4 can be encoded by $q_3=\sum_{g\in G_0} g\mathcal{F}$ (*i.e.*, the sum of cuboids colored blue online), where $\mathcal{F}=[0,\frac{1}{2}]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]$ is a fundamental domain [4] and G_0 is the orientation-preserving subgroup of G. For any orientation-reversing symmetry $r\in G$ (e.g. m_x, m_y, and m_z in Eqs. (36-38)), it is clear that $rq_3=q_3-\gamma_X$ and hence $\nu^1(r)=-1$.

For any space group G, let $\mathrm{SPT}^3(G)$ be the set of G-SPT phases, which forms a group under the stacking operation as shown in Appendix B. we will see in Sec. IV B that the dimensional reduction procedure actually gives a well-define group homorphism $\mathfrak{D}:\mathrm{SPT}^3(G)\to H_{\phi}^{1}(G;\mathbb{Z})$. In particular, $[\nu^1]\in H_{\phi}^{1}(G;\mathbb{Z})$ is a well-defined invariant, independent of the dimensional reduction details, for each generic G-SPT phase. Conversely, we will show in Sec. IV C that a G-symmetric SRE state, denoted $\left|[\nu^1]\right\rangle$, can always be constructed to present a G-SPT phase corresponding to each $[\nu^1]\in H_{\phi}^{1}(G;\mathbb{Z})$, resulting in a group homomorphism $\mathfrak{C}:H_{\phi}^{1}(G;\mathbb{Z})\to\mathrm{SPT}^3(G)$.

Then a generic G-symmetric SRE state $|\Psi\rangle$ is equivalent (as a G-SPT phase) to $\left|[\nu^1]\right\rangle\otimes|\Psi_2\rangle$ with $[\nu^1]\in H_{\phi}^{1}(G;\mathbb{Z})$ specified by $|\Psi\rangle$ via the dimensional reduction and $|\Psi_2\rangle=\left|-[\nu^1]\right\rangle\otimes|\Psi\rangle$, where $\left|-[\nu^1]\right\rangle$ denotes the inverse of $\left|[\nu^1]\right\rangle$. Further, by dimensional reduction, we deform $|\Psi_2\rangle$ into a G-symmetric SRE state (with arbitrarily short correlation length) on X_2, whose E_8 state configuration specifies $0\in H_{\phi}^{1}(G;\mathbb{Z})$. Actually, $|\Psi_2\rangle$ can be presented on X_2 without using E_8 states: since E_8 state configuration of $|\Psi_2\rangle$ is given by q_3 satisfying $gq_3=q_3$, we can reduce $q_2=\partial q_3$ to $q_2=0$ by blowing $-q_3(\alpha)$ copies of E_8 state bubbles in a process respecting G symmetry. Explicitly, as in the example of $Pm(6)$ in Sec. III B 1, such phases equipped with the stacking operation form the summand $H_{\phi}^{5}(G;\mathbb{Z})$ in Eq. (19) and can be built with lower dimensional group cohomology phases [42, 45]. Let us briefly describe how to decompose $|\Psi_2\rangle$ (which has been deformed into an SRE state on X_2 with $q_2=0$) into these lower dimensional components.

Without E_8 states, $|\Psi_2\rangle$ (an SRE state on X_2) can only show nontrivial 2D phases on 2-simplices in mirror planes, where each point has a $\mathbb{Z}_2$ site symmetry effectively working as an Ising symmetry protecting 2D phases classified by $H^3(\mathbb{Z}_2,U(1))=\mathbb{Z}_2$. As the system is symmetric and gapped, the 2D phases associated with all 2-simplices in the same mirror plane have to be either trivial or nontrivial simultaneously. Thus, each inequivalent mirror plane contributes a $\mathbb{Z}_2$ factor to cSPT phase classification. Conversely, a reference model of the 2D nontrivial phase protected by the $\mathbb{Z}_2$ site symmetry on a mirror plane can be constructed by sewing 2-simplices together with all symmetries respected. By adding such reference models to mirror planes where $|\Psi_2\rangle$ shows the nontrivial 2D phase, we can symmetrically trivialize all the 2-simplices and obtain a state $|\Psi_1\rangle$ which is only nontrivial on the 1-skeleton X_1. Actually, only 1D phases along the intersection of inequivalent mirror planes can be protected nontrivial by the site symmetry C_{nv} for $n=2,4,6$. A reference model, which generates the 1D phases protected by the C_{nv} site symmetry and classified by $H^2(C_{nv};U(1))=\mathbb{Z}_2$ for n even, can be constructed by connecting states on 1-simplices in a gapped and symmetric way. By adding such reference models to the C_{nv} axes where $|\Psi_1\rangle$ presents a nontrivial 1D phase, we can trivialize all the 1-simplices and get a state $|\Psi_0\rangle$ which may be nontrivial on the 0-skeleton X_0. Explicitly, $|\Psi_0\rangle$ is a tensor product of trivial degrees of freedom and some isolated 0D states centered at the 0-simplices (*i.e.*, vertices) of X carrying nontrivial site symmetry charges (*i.e.*, one-dimensional representations of site symmetry). However, some distinct site symmetry charge configurations may be deformed into each other by charge splitting and fusion; they are not in 1-1 correspondence with cSPT phases, whose classification and characterization are studied in Ref. [45].

Putting all the above ingredient together, we see that a generic cSPT state $|\Psi\rangle$ can be reduced to the stacking of $\left|[\nu^1]\right\rangle$, 2D nontrivial states protected by $\mathbb{Z}_2$ site symmetry on some mirror planes, 1D nontrivial states protected by C_{nv} site symmetry on some C_{nv} axes with n even, and 0D site symmetry charges. As we have mentioned, the cSPT phases built without using E_8 states have a group structure $H_{\phi}^{5}(G;\mathbb{Z})$ [*i.e.*, the first summand in Eq. (19)] under stacking operation [42, 45]. Next, we will focus on the models with ground states $\left|[\nu^1]\right\rangle$ for $[\nu^1]\in H_{\phi}^{1}(G;\mathbb{Z})$.

B. $H_{\phi}^{1}(G;\mathbb{Z})$ as a cSPT phase invariant

Let $\mathrm{SPT}^3(G)$ be the set of cSPT phases with space group G symmetry in $d=3$ spatial dimension. As shown in Appendix B, the stacking operation equips $\mathrm{SPT}^3(G)$ with an Abelian group structure. Below, let us prove that the dimensional reduction procedure defines a group homomorphism

$$\mathfrak{D}:\mathrm{SPT}^3(G)\to H_{\phi}^{1}(G;\mathbb{Z})\,. \tag{48}$$

To check the well-definedness of $\mathfrak{D}$, we notice the following two facts.

Proposition 1. *Let $\mathcal{M}$ be a model on X_2 with $\nu^1(t_{v_1})$ specified by its E_8 configuration, where t_{v_1} is the translation symmetry along a vector $v_1\in\mathbb{R}^3$. Compactifying $\mathcal{M}$ such that $t_{v_1}^{L}=1$ results in a 2D system with the invertible topologicial order of $L\nu^1(t_{v_1})$ copies of E_8 states.*

[4] To use the terminology of simplicial homology, $\mathcal{F}$ can be partitioned into tetrahedrons and be viewed as a formal sum of them with the right-handed orientation.

Proof. Let us only keep the subgroup $H \subseteq G$ of symmetries generated by three translations t_{v_1}, t_{v_2}, and t_{v_3} along linearly independent vectors v_1, v_2, and v_3 respectively. For convenience, we now use the coordinate system such that t_{v_1}, t_{v_2}, t_{v_3} work as t_x, t_y, t_z in Eqs. (23-25). Let $\mathcal{F}' = [0,1] \times [0,1] \times [0,1]$. Then $\mathcal{F}'$ is both a fundamental domain and a unit cell for H. The original triangulation X partitions $\mathcal{F}'$ into convex polyhedrons, which can be further triangulated resulting in a finer H-simplex complex X'. Clearly, q_3 can be viewed as an element of $C_3(X')$ as well. Moreover, putting $\nu^1(t_{v_1})$ (resp. $\nu^1(t_{v_2})$, $\nu^1(t_{v_3})$) copies of $\overline{E_8}$ states on each of integer x (resp. y, z) planes produces a H-symmetric model $\mathcal{M}'$ with E_8 configuration encoded by q_3' satisfying $t_{v_j} q_3' = q_3' + \nu^1(t_{v_j}) X, \forall j = 1, 2, 3$. Then $q_3 - q_3'$ is invariant under the action of H. Thus, the original model $\mathcal{M}$ can be continuously changed into $\mathcal{M}'$ through a path of H-symmetric SRE states. In particular, it reduces to a 2D system with the invertible topological order of $L\nu^1(t_{v_1})$ copies of E_8 states, when the system is compactified such that $t_{v_1}^L = 1$. □

Remark 1. This compactification procedure provides an alternate interpretation of $\nu^1(t_{v_1})$, which is now clearly independent of the dimensional reduction details and invariant under a continuous change of cSPT states. Thus, $\nu^1(t_{v_1})$ is well-defined for a cSPT phase.

Proposition 2. *Suppose that G contains an orientation-reversing symmetry r. Given two G-symmetric models $\mathcal{M}_a$ and $\mathcal{M}_b$ on G-simplicial complex structures X_a and X_b of $\mathbb{E}^3$ respectively, let $\nu_a^1(r) \pmod 2$ and $\nu_b^1(r) \pmod 2$ be specified by their E_8 configurations. If $\mathcal{M}_a$ and $\mathcal{M}_b$ are in the same G-SPT phase, then $\nu_a^1(r) = \nu_b^1(r) \pmod 2$.*

Proof. The triangulation of X_a partitions each tetrahedron of X_b into convex polyhedrons, which can be further triangulated resulting a simplicial complex X finer than both X_a and X_b. Naturally, both $\mathcal{M}_a$ and $\mathcal{M}_b$ can be viewed as models on X for the convenience of making a comparison; the values of $[\nu_a^1]$ and $[\nu_b^1]$ are clearly invariant when computed on a finer triangulation.

To make a proof by contradiction, suppose $\nu_a^1(r) \neq \nu_b^1(r) \pmod 2$. Without loss of generality, we may assume $\nu_a^1(r) \pmod 2 = 1$ and $\nu_b^1(r) \pmod 2 = 0$. To compare the difference between $\mathcal{M}_a$ and $\mathcal{M}_b$, let us construct an inverse of $\mathcal{M}_b$ with symmetry r respected. Since $\nu_b^1(r) \pmod 2 = 0$, the E_8 configuration of $\mathcal{M}_b$ can be described by ∂q_3^b with $q_3^b \in C_3(X)$ satisfying $rq_3^b = q_3^b$. Let $\overline{\mathcal{M}_b'}$ be a model obtained by attaching a bubble of $-q_3^b(\varsigma)$ copies of E_8 states to $\partial\varsigma, \forall \varsigma \in \Delta_3(X)$ from its inside. Then adding $\overline{\mathcal{M}_b'}$ to $\mathcal{M}_b$ cancels its E_8 configuration; the resulting state may only have group cohomology SPT phases left on simplices and hence has an inverse, denoted $\overline{\mathcal{M}_b''}$. Let $\overline{\mathcal{M}_b} = \overline{\mathcal{M}_b'} \oplus \overline{\mathcal{M}_b''}$ (*i.e.*, the stacking of $\overline{\mathcal{M}_b'}$ and $\overline{\mathcal{M}_b''}$). Then $\overline{\mathcal{M}_b}$ clearly is an inverse of $\mathcal{M}_b$ with symmetry r respected. Since $\mathcal{M}_a$ and $\mathcal{M}_b$ are in the same G-SPT phase, $\overline{\mathcal{M}_b}$ is also an inverse of $\mathcal{M}_a$ with symmetry r respected. Thus, $\mathcal{M}_a \oplus \overline{\mathcal{M}_b}$ admits a r-symmetric surface. Moreover, since $\nu_a^1(r) \pmod 2 = 1$, the E_8 configuration of $\mathcal{M}_a \oplus \overline{\mathcal{M}_b}$ can be described by ∂q_3 for some $q_3 \in C_3(X)$ satisfying $rq_3 = q_3 - X$.

To proceed, we pick an r-symmetric region Y with surface shown in Fig. 5. For convenience, Y is chosen to be a subcomplex of X (*i.e.*, union of simplices in X). In addition, r can be an inversion, a rotoinversion, a reflection, and a glide reflection in three dimensions. Let Π be a plane passing the inversion/rotoinversion center, the mirror plane, and the glide reflection plane respectively. Clearly, $r\Pi = \Pi$. Adding Π to the triangulation of Y, some tetrahedrons are divided in convex polyhedrons, which can be further triangulated resulting in a finer triangulation of Y. Below, we treat Y as a simplicial complex specified by the new triangulation. By restriction, q_3 can be viewed as an element of $C_3(Y)$ satisfying $rq_3 = q_3 - Y$. Let $\mathcal{M}$ be a model on Y with an r-symmetric SRE surface and a bulk identical to $\mathcal{M}_a \oplus \overline{\mathcal{M}_b}$. Thus, ∂q_3 describes the bulk E_8 configuration of $\mathcal{M}$.

On the other hand, putting an r-symmetric E_8 state on Π gives a configuration described by $\partial q_3'$ with $q_3'(\varsigma)$ equal to 1 for all 3-simplex ς on side of Π and 0 on the other side; accordingly, $rq_3' = q_3' - Y$ and hence $r(q_3' - q_3) = q_3' - q_3$. Thus, adding a bubble of $q_3'(\varsigma) - q_3(\varsigma)$ copies of E_8 states to $\partial\varsigma, \forall\varsigma \in \Delta_3(Y)$ from inside gives an r-symmetric SRE model, denoted $\mathcal{M}'$. By construction, the bulk E_8 configuration of $\mathcal{M}'$ get concentrated on Π; more precisely, $\mathcal{M}'$ can be viewed as a gluing result of the two half surfaces (red and blue) in Fig. 5 and an E_8 state on Π. As ∂Y hosts no anyons, each half surface has $8n$ co-propagating chiral boson modes along its boundary, where $n \in \mathbb{Z}$. Due to the symmetry r, they add up to $16n$ co-propagating chiral boson modes, which cannot be canceled by boundary modes of the E_8 state on Π. This implies that $\mathcal{M}'$ cannot be gapped, which contradicts that E_8 is SRE and hence disproves our initial assumption. Therefore, $\nu_a^1(r) = \nu_b^1(r) \pmod 2$ for any two models $\mathcal{M}_a$ and $\mathcal{M}_b$ in the same G-symmetric cSPT phase. □

By Lemma 1, $[\nu^1] \in H_\phi^1(G;\mathbb{Z})$ is specified by $\nu^1(t_{v_1})$, $\nu^1(t_{v_2})$, $\nu^1(t_{v_3})$ together with $\nu^1(r) \pmod 2 \in \mathbb{Z}_2$ if G is non-orientation-preserving, where t_{v_1}, t_{v_2}, t_{v_3} are three linearly independent translation symmetries and r is an orientation-reversing symmetry. Combining the above two facts, we get that any two models (probably on different simplicial complex structures of $\mathbb{E}^3$) in the same G-symmetric cSPT phase must determine a unique $[\nu^1] \in H_\phi^1(G;\mathbb{Z})$. Thus, $\mathfrak{D}$ in Eq. (48) is well-defined, independent of the details of the dimensional reduction. Moreover, it clearly respects the group structure.

C. Construction of $H_\phi^1(G;\mathbb{Z})$ cSPT phases

To show that the group structure of $\mathrm{SPT}^3(G)$ (*i.e.*, G-SPT phases in d spatial dimensions) is $H_\phi^5(G;\mathbb{Z}) \oplus$

$H^1_\phi(G;\mathbb{Z})$, we analyze three group homomorphisms $\mathfrak{I}$, $\mathfrak{C}$, and $\mathfrak{D}$, which can be organized as

$$\begin{tikzcd} H^5_\phi(G;\mathbb{Z}) \arrow[r, hook, "\mathfrak{I}"] & \mathrm{SPT}^3(G) \arrow[r, two heads, "\mathfrak{D}"] & H^1_\phi(G;\mathbb{Z}). \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd} \tag{49}$$

A hooked (resp. two-head) arrow is used to indicate that $\mathfrak{I}$ is injective (resp. $\mathfrak{D}$ is surjective). The map $\mathfrak{I}$ is an inclusion identifying $H^5_\phi(G;\mathbb{Z})$ as a subgroup of $\mathrm{SPT}^3(G)$; Refs. [42, 45] have shown that $H^5_\phi(G;\mathbb{Z})$ classifies the G-symmetric cSPT phases built with lower dimensional group cohomology phases protected site symmetry. We have defined $\mathfrak{D}$ via dimensional reduction. Clearly, $\mathfrak{D}$ maps all G-SPT phases labeled by $H^5_\phi(G;\mathbb{Z})$ to $0 \in H^1_\phi(G;\mathbb{Z})$. Conversely, the E_8 configuration of any model on X_2 with $[\nu^1] = 0$ can be described by ∂q_3 with $q_3 \in C_3(X)$ satisfying $gq_3 = q_3, \forall g \in G$. Thus, it is possible to grow a bubble of $-q_3(\varsigma)$ of E_8 states inside ς to its boundary for all $\varsigma \in \Delta_3(X)$ in a G-symmetric way, canceling all E_8 states on 2-simplicies. Thus, any phase with $[\nu^1] = 0$ can be represented by a model built with group cohomology phases only. Formally, the image of $\mathfrak{I}$ equals the kernel of $\mathfrak{D}$. Below, we will define the group homomophism $\mathfrak{C}$ by constructing a G-symmetric SRE state representing a G-SPT phase with each $[\nu^1] \in H^1_\phi(G;\mathbb{Z})$ and show that $\mathfrak{D}\circ\mathfrak{C}$ equals the identity map on $H^1_\phi(G;\mathbb{Z})$, which implies the surjectivity of $\mathfrak{D}$ and further

$$\mathrm{SPT}^3(G) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z}) \tag{50}$$

by the splitting lemma in homological algebra.

Suppose that t_{v_1}, t_{v_2}, and t_{v_3} generate the translation subgroup of G. Let G_0 be the orientation-preserving subgroup of G, *i.e.*, $G_0 := \{g \in G \mid \phi(g) = 1\}$. By Lemma 1, each $[\nu^1] \in H^1_\phi(G;\mathbb{Z})$ can be parameterized by $\nu^1(t_{v_1}), \nu^1(t_{v_2}), \nu^1(t_{v_3}) \in \mathbb{Z}$ together with $\nu^1(r) \pmod 2 \in \mathbb{Z}_2$ if G is non-orientation-preserving (*i.e.*, $G_0 \neq G$), where r is an orientation-reversing symmetry. Below, we construct models for generators of $H^1_\phi(G;\mathbb{Z})$ and then the model corresponding to a generic $[\nu^1]$ will be obtained by stacking.

For a non-orientation-preserving space group G, let ν^1_r be a 1-cocycle satisfying $\nu^1_r(g) = 0, \forall g \in G_0$ and $\nu^1_r(r) \pmod 2 = 1$; the corresponding $[\nu^1_r] \in H^1_\phi(G;\mathbb{Z})$ clearly has order 2. Let us first construct a G-symmetric SRE state with $[\nu^1_r] \in H^1_\phi(G;\mathbb{Z})$, which is done by recasting the illustrative construction in Fig. 4 in a general setting. We attach an bubble of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological state to $\partial\mathcal{F}$ from inside and duplicate it at $\partial(g\mathcal{F})$ by all symmetries $g \in G$. Then there are two copies of $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ topological states on each 2-simplex, which is always an interface between two fundamental domains $g_1\mathcal{F}$ and $g_2\mathcal{F}$ for some $g_1, g_2 \in G$. Let $(\mathfrak{e}_\mathfrak{f}, \mathfrak{e}_\mathfrak{f})$ (resp. $(\mathfrak{m}_\mathfrak{f}, \mathfrak{m}_\mathfrak{f})$) denote the anyon formed by pairing $\mathfrak{e}_\mathfrak{f}$ (resp. $\mathfrak{m}_\mathfrak{f}$) from each copy. Condensing $(\mathfrak{e}_\mathfrak{f}, \mathfrak{e}_\mathfrak{f})$ and $(\mathfrak{m}_\mathfrak{f}, \mathfrak{m}_\mathfrak{f})$ on all 2-simplices results in a G-symmetric SRE state, denoted $|\Psi_r\rangle$. Its E_8 configuration can be described by ∂q_3 with $q_3 = \sum_{g\in G_0} g\mathcal{F}$; there is a (resp. no) E_8 state on the interface between $g_1\mathcal{F}$ and $g_2\mathcal{F}$ with $\phi(g_1) \neq \phi(g_2)$ (resp. $\phi(g_1) = \phi(g_2)$). Clearly, $rq_3 = q_3 - X$ and hence $\nu^1(r) \pmod 2 = 1$. In addition, with only G_0 respected, the E_8 configuration can be canceled by growing a bubble of an $\overline{E_8}$ state inside each $g\mathcal{F}$ for $g \in G_0$. Thus, as a G_0-symmetric cSPT phase, $|\Psi_r\rangle$ corresponds to $0 \in H^1_\phi(G_0;\mathbb{Z})$; in particular, $\nu^1(t_v) = 0$ for any translation t_v. Therefore, $|\Psi_r\rangle$ is a G-symmetric SRE state mapped to $[\nu^1_r] \in H^1_\phi(G;\mathbb{Z})$ by $\mathfrak{D}$. Moreover, the order of $|\Psi_r\rangle$ is also 2 in $\mathrm{SPT}^3(G)$ by the same argument as the one in Sec. III B 3 showing that stacking two copies of the models in Fig. 4 gives a trivial cSPT phase.

Below, the construction of $\mathfrak{C}$ will be completed case by case based on the number of symmetry directions in the international (Hermann-Mauguin) symbol of G.

1. *G with more than one symmetry direction*

For G with more than one symmetry direction, if G is orientation-preserving, then $H^1_\phi(G;\mathbb{Z}) = 0$ and hence the definition of $\mathfrak{C}$ is obvious: $\mathfrak{C}$ maps $0 \in H^1_\phi(G;\mathbb{Z})$ to the trivial phase in G-SPT. Clearly, $\mathfrak{D}\circ\mathfrak{C}$ is the identity.

On the other hand, if G does not preserve the orientation of $\mathbb{E}^3$, then $H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}_2$ with $[\nu^1_r]$ the only nontrivial element. Since the order of $|\Psi_r\rangle$ is 2 in $\mathrm{SPT}^d(G)$, the group homomorphism $\mathfrak{C}$ can be specified by mapping $[\nu^1_r]$ to the cSPT phase represented by $|\Psi_r\rangle$. By construction, $\mathfrak{D}\circ\mathfrak{C}$ equals the identity on $H^1_\phi(G;\mathbb{Z})$.

2. *$G = P1$ and $P\bar{1}$*

For $G = P1$ and $P\bar{1}$, we pick a coordinate system such that t_{v_1}, t_{v_2}, t_{v_3} work as t_x, t_y, t_z in Eqs. (23-25) (and such that the origin is an inversion center[5] of r for $P\bar{1}$). Let $[\nu^1_i]$ be an element of $H^1_\phi(G;\mathbb{Z})$ represented by a 1-cocycle satisfying $\nu^1_i(t_{v_j}) = \delta_{ij}$ and $\nu^1_i(r) \pmod 2 = 1$ for $i = 1, 2, 3$. As an example, a G-symmetric SRE state $|\Psi_2\rangle$ for $[\nu^1_2]$ can be constructed by putting an E_8 state $|E^y_8\rangle$ on the plane $y = 0$ with the symmetries t_{v_1}, t_{v_3} (as well as the inversion $r : (x, y, z) \mapsto -(x, y, z)$ for $P\bar{1}$) respected and its translation image $t^n_{v_2}|E^y_8\rangle$ on planes $y = n$ for $n \in \mathbb{Z}$. In particular, since there is a single E_8 layer passing the inversion center of r in the case $G = P\bar{1}$, the E_8 configuration of $|\Psi_2\rangle$ determines $\nu^1_2(r) \pmod 2 = 1$. A G-symmetric SRE state $|\overline{\Psi}_2\rangle$, inverse to $|\Psi_2\rangle$ in $\mathrm{SPT}^3(G)$, can be obtained by replacing each E_8 layer by its inverse. Clearly, $|\overline{\Psi}_2\rangle$ is mapped to $-[\nu^1_y]$ by $\mathfrak{D}$. Analogously, we construct a G-symmetric SRE state $|\Psi_i\rangle$ for $[\nu^1_i]$ and its inverse $|\overline{\Psi}_i\rangle$ for $i = 1, 3$ as well.

[5] Every orientation-reversing symmetry of $P\bar{1}$ is an inversion.

Noticing that each $[\nu^1] \in H^1_\phi(G;\mathbb{Z})$ can uniquely be expressed as $\sum_{i=1}^3 n_i[\nu_i^1]$ for $G = P1$ and $n_r[\nu_r^1] + \sum_{i=1}^3 n_i[\nu_i^1]$ for $G = P\overline{1}$ with $n_r = 0, 1$ and $n_1, n_2, n_3 \in \mathbb{Z}$, we can define $\mathfrak{C}([\nu^1])$ as the phase represented by the G-symmetric SRE state $|\Psi_r\rangle^{\otimes n_r} \otimes |\Psi_1\rangle^{\otimes n_1} \otimes |\Psi_2\rangle^{\otimes n_2} \otimes |\Psi_3\rangle^{\otimes n_3}$, where $|\Psi_r\rangle^{\otimes n_r}$ is needed only for $G = P\overline{1}$ and $|\Psi\rangle^{\otimes n}$ denotes the stacking of $|n|$ copies of $|\Psi\rangle$ (resp. its inverse $|\overline{\Psi}\rangle$) for $n \geq 0$ (resp. $n < 0$). In particular, $|\Psi\rangle^{\otimes 0}$ denotes any trivial state. By construction, $\mathfrak{C}$ is a group homomorphism and $\mathfrak{D} \circ \mathfrak{C}$ is the identity on $H^1_\phi(G;\mathbb{Z})$.

3. *G with exactly one symmetry diretion other than* 1 *or* $\overline{1}$

For G with exactly one symmetry direction other than 1 or $\overline{1}$ (e.g. $I\overline{4}$, $P2$, $P2_1/c$, Pm), we pick a Cartesian coordinate system whose z axis lies along the symmetry direction. With each point represented by a column vector of its coordinates $u = (x, y, z)^\mathsf{T} \in \mathbb{R}^3$, each $g \in G$ is represented as $u \mapsto \phi(g) R_z(\varphi) u + w$ with $w = (w^x, w^y, w^z)^\mathsf{T} \in \mathbb{R}^3$ and an orthogonal matrix

$$R_z(\varphi) := \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{51}$$

describing a rotation about the z axis. Clearly, w^z is independent of the origin position for $g \in G_0$, where G_0 is the orientation-preserving subgroup of G. Let $\mathbb{A}$ be the collection of w^z of all orientation-preserving symmetries. Then $\mathbb{A} = \kappa\mathbb{Z}$ for some positive real number κ. Pick $h \in G_0$ with $w^z = \kappa$.

If $G_0 \neq G$, then $r \in G - G_0$ may be an inversion, a rotoinversion, a reflection, or a glide reflection. Clearly, there is a plane Π perpendicular to the z direction satisfying $r\Pi = \Pi$. If $G_0 = G$, we just pick Π to be any plane perpendicular to the z direction. For convenience, we may choose the coordinate origin on Π such that $w^z = 0$ for r. In such a coordinate system, $w^z \in \kappa\mathbb{Z}$ for all $g \in G$ even if $\phi(g) = -1$. Moreover, $G_\Pi := \{g \in G | g\Pi = \Pi\}$ contains symmetries with $w^z = 0$. It is a wallpaper group for Π, which preserves the orientation of Π.

Putting a G_Π-symmetric E_8 layer $|E_8^z\rangle$ on Π and its duplicate $h^n|E_8^z\rangle$ on $h^n\Pi$, we get an h-symmetric SRE state $|\Psi_z\rangle := \cdots \otimes h^{-1}|E_8^z\rangle \otimes |E_8^z\rangle \otimes h|E_8^z\rangle \otimes \cdots$. For all $f \in G_\Pi$, $fh^n|E_8^z\rangle = h^{\phi(f)n} f_n|E_8^z\rangle = h^{\phi(f)n}|E_8^z\rangle$, where $f_n = h^{-\phi(f)n} g h^n$ is an element of G_Π and hence leaves $|E_8^z\rangle$ invariant. Thus, $|\Psi_z\rangle$ is G_Π-symmetric and hence G-symmetric, as each $g = G$ can be expressed as $h^m f$ with $m \in \mathbb{Z}$ and $f \in G_\Pi$. A G-symmetric SRE state $|\overline{\Psi}_z\rangle$, inverse to $|\Psi_z\rangle$, can be obtained by replacing each E_8 by its inverse.

Let $[\nu_z^1] \in H^1_\phi(G;\mathbb{Z})$ (represented by a 1-cocyle ν_z^1) be the image of $|\Psi_z\rangle$ under $\mathfrak{D}$. By construction, $\nu_z^1(g) = w^z/\kappa$ on each orientation-preserving symmetry $g : u \mapsto R_z(\varphi) u + (w^x, w^y, w^z)$. When $G_0 \neq G$, a single E_8 layer on Π implies the nonexistence of r-symmetric q_3 with ∂q_3 describing the E_8 configuration of $|E_8^z\rangle$ and hence $\nu_z^1(r)$ (mod 2) $= 1$. Clearly, $|\overline{\Psi}_z\rangle$ is mapped to $-[\nu_z^1]$ by $\mathfrak{D}$.

Further, we notice that every $[\nu^1] \in H^1_\phi(G;\mathbb{Z})$ can be uniquely expressed as $n_h[\nu_z^1]$ (resp. $n_r[\nu_r^1] + n_h[\nu_z^1]$) when $G_0 = G$ (resp. $G_0 \neq G$); comparing values on r and h gives $n_h = \nu^1(h) \in \mathbb{Z}$ and $n_r = \nu^1(r) + \nu^1(h)$ (mod 2) $\in \{0, 1\}$. Thus, a map $\mathfrak{C}$ can be defined by $n_h[\nu_z^1] \mapsto |\Psi_z\rangle^{\otimes n_h}$ (resp. $n_r[\nu_r^1] + n_h[\nu_z^1] \mapsto |\Psi_r\rangle^{\otimes n_r} \otimes |\Psi_z\rangle^{\otimes n_h}$), where $|\Psi\rangle^{\otimes n}$ denotes the stacking of $|n|$ copies of $|\Psi\rangle$ (resp. its inverse $|\overline{\Psi}\rangle$) for $n \geq 0$ (resp. $n < 0$). In particular, $|\Psi\rangle^{\otimes 0}$ denotes any trivial state. Since $|\Psi_r\rangle$ has order 2 in $\mathrm{SPT}^3(G)$, $\mathfrak{C}$ is a group homomorphism. By construction, $\mathfrak{D} \circ \mathfrak{C}$ is clearly the identity on $H^1_\phi(G;\mathbb{Z})$.

V. DISCUSSION

Dimensional reduction procedure suggests the cSPT phases can be represented by lower dimensional SRE states in general. Earlier works systematically showed that the restricted bosonic cases ignoring possible presence of 2D nontrivial invertible topological orders (*i.e.*, E_8 states or its multiples) in $d \leq 3$ spatial dimensions are classified by the group cohomology $H^{d+2}_\phi(G;\mathbb{Z})$.

In this paper, we studied the ignored cSPT states and found that the complete classification of cSPT phases is given by the twisted generalized cohomology $h^d_\phi(G; F_\bullet) := [EG, \Omega F_{d+1}]_G$ with Ω-spectrum $F_\bullet = \{F_d\}$ given by Eq. (13-15) in low dimensions for bosonic systems. Since $h^d_\phi(G; F_\bullet)$ was previously conjectured to completely classify SPT phases with *internal* symmetry (*i.e.*, symmetry keeping the location of degrees of freedom) G; thus, our result supports the Crystalline Equivalent Principle at the level beyond group cohomology phases.

In particular, the abelian group $\mathrm{SPT}^d(G)$ of cSPT phases (equipped with the stacking operation) for space group G in $d = 3$ spatial dimensions is isomorphic to $h^3_\phi(G; F_\bullet) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$, where ϕ indicates that $g \in G$ acting as multiplication by $\phi(g) = \pm 1$ depending on whether space orientation is preserved by g or not. The summand $H^1_\phi(G;\mathbb{Z})$, missed in the group cohomology proposal, is related to E_8 state configurations on the 2-skeleton of the space $\mathbb{E}^3$ which is triangulated into a G-simplicial complex. According to Theorem 1, it is isomorphic to $\mathbb{Z}^k$ if G preserves orientation and $\mathbb{Z}^k \times \mathbb{Z}_2$ otherwise, where $k = 0, 1, 3$ can be easily ready off from the international (Hermann-Mauguin) symbol of G. The results of $H^5_\phi(G;\mathbb{Z})$, $H^1_\phi(G;\mathbb{Z})$, and $\mathrm{SPT}^3(G)$ is tabulated in Table II.

To show $H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z}) \cong \mathrm{SPT}^3(G)$ and to explain its physical meaning, we defined a group homorphism $\mathfrak{C} : H^1_\phi(G;\mathbb{Z}) \to \mathrm{SPT}^3(G)$ by constructing a model for each $[\nu^1] \in H^1_\phi(G;\mathbb{Z})$. Explicitly, if G does not preserve space orientation, then a model corresponding to the nontrivial element of the $\mathbb{Z}_2$ factor in $H^1_\phi(G;\mathbb{Z})$ can be obtained through anyon condensation from $\mathfrak{e}_\mathfrak{f}\mathfrak{m}_\mathfrak{f}$ state bubbles on boundary of fundamental domains. Such a model gives an example how lower dimensional invertible topological orders naturally appear and is responsible for the existence of bosonic cSPT phases beyond group cohomology. Thus, a twisted generalized cohomology $h^3_\phi(G; F_\bullet) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ is needed for classifying cSPT phases completely; it naturally includes crystalline invertible topological phases generated by layered E_8 states, which corresponds to the $\mathbb{Z}^k$ factor in $H^1_\phi(G;\mathbb{Z})$.

We now discuss some natural generalizations and the outlook for further developments motivated by the results presented here. First, the isomorphism $h^3_\phi(G; F_\bullet) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ holds and our computation strategy of $H^1(G;\mathbb{Z})$ works in general, predicting a universal $\mathbb{Z}_2$ factor [from $H^1_\phi(G;\mathbb{Z})$] lying beyond the group cohomology classification [*i.e.*, $H^5_\phi(G;\mathbb{Z})$] in the presence of any spacetime-orientation-reversing symmetry. There may be an extra $\mathbb{Z}^k$ factor labeling the phases of infinite order (e.g. layered E_8 states with the same chirality), which are invertible but not completely trivial even without symmetry. On the other hand, phases of finite order would become trivial when symmetry protection is removed.

In particular, the result of $H^1_\phi(G;\mathbb{Z})$ for any magnetic space group G is given in Appendix A 2. It only depends on the magnetic point group $\wp G$ associated with G and hence can be read off directly from the Opechowski-Guccione or Belov-Neronova-Smirnova symbol of G. Type I magnetic space group is just pure space group and the corresponding $H^\phi_\phi(G;\mathbb{Z})$ is already tabulated in Table II. If G is of type II or type IV, then $H^1_\phi(G;\mathbb{Z})$ is always $\mathbb{Z}_2$. If G is of type III, then $H^1_\phi(G;\mathbb{Z}) = \mathbb{Z}^k \times \mathbb{Z}_2^\ell$ with k and ℓ listed in Table III for all the 58 possibilities of the associated magnetic point group.

Secondly, we have carefully checked that $h^d_\phi(BG; F_\bullet)$ gives a complete classification of cSPT phases, *i.e.*, $h^d_\phi(BG; F_\bullet) \cong \mathrm{SPT}^d(G)$, for any space group G (at least for low dimensions $d \leq 3$), provided that every cSPT phase can be represented by a state with arbitrarily short correlation length. It is still an open question to see weather a generic crystalline gapped quantum phase hosting no nontrivial fractional excitation always has such a representation. If it is true, then the dimensional reduction procedure applies to cSPT phases in general and we will be more confident that our classification is complete without further technical assumption.

Another remaining important problem is to complete a generalized cohomology theory for fermionic SPT phases. Along the same line of thinking, fermionic SPT (including cSPT) phases should also be classified by some generalized cohomology. Obviously, the corresponding Ω-spectrum $F_\bullet = \{F_n\}$ must be different the one describing Bosonic SPT phases. However, the fermionic situation is much richer: a symmetry group G may fractionalize on fermions in many different ways. For instance, a mirror reflection may square to $(-1)^F$ and translations may only commute we may have $t_x t_y = (-1)^F t_y t_x$ and $m^2 = (-1)^F$, where t_x, t_y are two translations, m denotes a mirror reflection, and $(-1)^F$ is the fermion parity which assigns distinct signs to states with even and odd number of fermions. It is clear that different symmetry fractionalizations on fermions are labeled by $H^2(G;\mathbb{Z}_2)$. If possible, we would desire a generalized cohomology that classifies fermionic SPT phases in a universal way, which produces a classification with a quadruple input (d, G, ϕ, ω) and apply it for various physical systems, where d is the dimension of space, G is the symmetry group, $\phi : G \to \{\pm 1\}$ determines whether spacetime orientation is preserved or not, and $\omega \in H^2(G;\mathbb{Z}_2)$ describing symmetry fractionalization on fermions.

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Appendix A: Computation of $H^1_\phi(G;\mathbb{Z})$

1. 230 space groups (3D)

Here we compute $H^1_\phi(G;\mathbb{Z})$, which classifies the E_8 state configuration for bosonic SPT phases, for all 230 space groups in three spatial dimensions. We will first prove a formula for $H^1_\phi(G;\mathbb{Z})$, in Theorem 1, which will then be used to obtain the classification, in Table II.

Theorem 1. *Let G be a 3D space group and $\phi : G \to \{\pm 1\}$ be the homomorphism that sends orientation-preserving elements to 1 and the rest to -1. Then*

$$H^1_\phi(G;\mathbb{Z}) = \begin{cases} \mathbb{Z}^k, & G \text{ preserves orientation}, \\ \mathbb{Z}^k \times \mathbb{Z}_2, & \text{otherwise}, \end{cases} \tag{A1}$$

where $k = 0, 1, 3$ respectively if, in the international (Hermann-Mauguin) symbol, there are more than one symmetry direction listed, exactly one symmetry direction listed and it is not 1 or $\bar{1}$, and exactly one symmetry direction listed and it is 1 or $\bar{1}$.

We recall that $H^1_\phi(G;\mathbb{Z})$ is the quotient of group 1-cocycles by group 1-coboundaries [64]. That is,

$$H^1_\phi(G;\mathbb{Z}) := Z^1_\phi(G;\mathbb{Z})/B^1_\phi(G;\mathbb{Z}), \tag{A2}$$

where

$$Z^1_\phi(G;\mathbb{Z}) := \left\{\nu^1 : G \to \mathbb{Z} \mid d\nu^1 = 0\right\}, \tag{A3}$$
$$B^1_\phi(G;\mathbb{Z}) := \left\{d\nu^0 \mid \nu^0 \in C^0_\phi(G;\mathbb{Z})\right\}. \tag{A4}$$

The condition $d\nu^1 = 0$ (cocycle condition) reads

$$\nu^1(g_1g_2) = \nu^1(g_1) + \phi(g_1)\nu^1(g_2), \quad \forall g_1, g_2 \in G, \tag{A5}$$

whereas the elements $d\nu^0$ (group 1-coboundaries) are of the form

$$\begin{aligned} d\nu^0 &: G \to \mathbb{Z} \\ g &\mapsto \begin{cases} 2m, & \phi(g) = -1, \\ 0, & \phi(g) = 1, \end{cases} \end{aligned} \tag{A6}$$

for some $m \in \mathbb{Z}$. To proceed, we fix three linearly independent translations $t_{v_1}, t_{v_2}, t_{v_3} \in G$ as well as an orientation-reversing element $r \in G$, if any, where t_v denotes a translation by vector $v \in \mathbb{R}^3$. We have the following lemma.

Lemma 1. *An element $\nu^1 \in Z^1_\phi(G;\mathbb{Z})$ is completely determined by $\nu^1(t_{v_1})$, $\nu^1(t_{v_2})$, and $\nu^1(t_{v_3})$, together with $\nu^1(r)$ if G is non-orientation-preserving.*

Proof. Setting $g_1 = g_2$ to be the identity element $1 \in G$ in Eq. (A5), we see that $\nu^1(1) = 2\nu^1(1)$, so $\nu^1(1) = 0$. If an element $g \in G$ is orientation-preserving and has finite order n, then Eq. (A5) implies $n\nu^1(g) = \nu^1(g^n) = \nu^1(1) = 0$, so $\nu^1(g) = 0$. If an element $g \in G$ is orientation-preserving and has infinite order, then it is either a translation or a screw rotation; either way, there exists an integer n such that $g^n = t^{n_1}_{v_1}t^{n_2}_{v_2}t^{n_3}_{v_3}$ for some integers n_1, n_2, n_3, and Eq. (A5) implies

$$\nu^1(g) = \frac{1}{n}\left[n_1\nu^1(t_{v_1}) + n_2\nu^1(t_{v_2}) + n_3\nu^1(t_{v_3})\right]. \tag{A7}$$

Thus the value of ν^1 on the orientation-preserving subgroup of G is completely determined. The orientation-reversing elements of G, if any, can all be written gr for some orientation-preserving g, and Eq. (A5) implies

$$\nu^1(gr) = \nu^1(g) + \nu^1(r). \tag{A8}$$

Thus the value of ν^1 on the orientation-reversing subset of G is completely determined as well. $\square$

Let us show that there are constraints on the integers $\nu^1(t_{v_1})$, $\nu^1(t_{v_2})$, and $\nu^1(t_{v_3})$. It is convenient to represent the triple $\left(\nu^1(t_{v_1}), \nu^1(t_{v_2}), \nu^1(t_{v_3})\right)$ by the unique vector $\mu \in \mathbb{R}^3$ such that $\nu^1(t_v) = \mu\cdot v$. In Cartesian coordinates, each point of the Euclidean space $\mathbb{E}^3$ can be identified with a column vector $u = (x, y, z)^\intercal \in \mathbb{R}^3$ and each $g \in G$ can be represented as $u \mapsto Wu+w$, where the superscript $\intercal$ denotes matrix transpose, $W \in O(3)$ (*i.e.*, W is an orthogonal matrix) and $w \in \mathbb{R}^3$ is a column vector. Thus, any $g \in G$ can be denoted by a pair $(W, w) \in O(3) \times \mathbb{R}^3$. Clearly, ϕ is also well-defined for $W \in O(3)$ based on whether W preserves orientation and we have $\phi(g) = \phi(W) = \det(W)$. Noticing

$$gt_vg^{-1} = t_{Wv}, \quad \forall t_v \in G, \tag{A9}$$

and applying ν^1 to both sides and utilizing the cocycle condition (A5), we obtain

$$\phi(g)\nu^1(t_v) = \nu^1(t_{Wv}), \tag{A10}$$

which can be recast as

$$\phi(W)\mu\cdot v = \mu\cdot Wv. \tag{A11}$$

Since this holds for all $v \in \mathbb{R}^3$ for which $t_v \in G$, which span all three dimensions, we must have

$$\left[\phi(W)I - W^\intercal\right]\mu = 0, \tag{A12}$$

or equivalently $[\phi(W)W - I]\mu = 0$ with I denoting the identity matrix. Thus we see that

$$\mu \in \bigcap_{W\in\wp G} \ker\left[\phi(W)W - I\right], \tag{A13}$$

where $\wp$ denotes the group homomorphism $G \to O(3)$ that maps (W, w) to W and hence $\wp G$ is the point group of G.

The dimensionality of $\bigcap_{W\in\wp G}\ker\left[\phi(W)W - I\right] \subseteq \mathbb{R}^3$ can be determined as follows.

Lemma 2. *The vector space $\bigcap_{W\in\wp G}\ker\left[\phi(W)W - I\right]$ is 0-, 1-, or 3-dimensional if, respectively, in the international (Hermann-Mauguin) symbol of space group G, there are more than one symmetry directions listed, exactly one symmetry direction listed and it is not 1 or $\bar{1}$, or exactly one symmetry direction listed and it is 1 or $\bar{1}$.*

Proof. We can focus on international symbol of the point group, which has the same number of symmetry directions as the space group. A symbol n (resp. $\bar{n}$) represents an n-fold rotation (resp. rotoinversion) about a given axis. It is easy to see, if W represents such a rotation or rotoinversion, that $\ker\left[\phi(W)W - I\right]$ is nothing but the symmetry axis, unless $n = 1$, in which case $\ker\left[\phi(W)W - I\right] = \mathbb{R}^3$. Now we simply take the intersection for all symmetry generators listed in the international (Hermann-Mauguin) symbol. $\square$

For short, let V denote $\bigcap_{W\in\wp G}\ker\left[\phi(W)W - I\right]$. Due to the constraint that ν^1 is integer-valued, μ in fact lives in a discrete subset of V. Below, let us express this subset explicitly and show that it is isomorphic to $\mathbb{Z}^k$ with $k := \dim V$. To label the translations in G along V, let $\mathbb{L}(G) := \left\{v \in \mathbb{R}^3 | t_v \in G\right\}$ and $\mathbb{L}(G, V) := \mathbb{L}(G) \cap V$. We have $\mathbb{L}(G, V) \cong \mathbb{Z}^k$ with generators denoted a_j for $j = 1, 2, \cdots, k$. There is a group homomorphism $\beta : G_0 \to V$ mapping $g = (W, w' + w'')$ to $w' \in V$, where $w'' \perp V$ and G_0 denotes the orientation-preserving subgroup of G. Then we have $\beta\left(G_0\right) \subseteq \frac{1}{|\wp G_0|}\mathbb{L}\left(G, V\right)$ based on a case-by-case discussion: (1) This is obvious if $\wp G_0$ is trivial;

(2) If $\wp G_0$ is nontrivial and $W=1$, then $w'+w''\in\mathbb{L}(G)$ and $|\wp G_0|\,w'=\sum_{i=1}^{|\wp G_0|}W_*^i(w'+w'')\in\mathbb{L}(G,V)$ for any nontrivial $W_*\in\wp G_0$; (3) If $W\neq 1$, then $g^{|\wp G_0|}$ is the translation by $|\wp G_0|\,w'\in\mathbb{L}(G,V)$. Thus, $\mathbb{L}\left(G,V\right)\subseteq\beta\left(G_0\right)\subseteq\frac{1}{|\wp G_0|}\mathbb{L}\left(G,V\right)$ and hence $\beta(G_0)\cong\mathbb{Z}^k$. Pick $\{a_j\}_{j=1,2,\cdots,k}$ that generates $\beta(G_0)$ and let $\{\alpha_i\}_{i=1,2,\cdots,k}$ be the dual basis satisfying $\alpha_i\cdot a_j=\delta_{ij}$. Pick g_j such that $\beta(g_j)=a_j$ for $j=1,2,\cdots,k$. It is clear that μ corresponding to $\nu^1\in Z_\phi^1(G;\mathbb{Z})$ has the form $\sum_{i=1}^k\nu^1(g_i)\alpha_i$.

Conversely, each $\mu\in\langle\{\alpha_i\}_{i=1,2,\cdots,k}\rangle$ determines a cocycle $\nu^1\in Z_\phi^1(G_0;\mathbb{Z})$ mapping $(W,w)\in G_0$ to $\mu\cdot w\in\mathbb{Z}$, where $\langle\{\alpha_i\}_{i=1,2,\cdots,k}\rangle\cong\mathbb{Z}^k$ is the lattice generated by $\{\alpha_i\}_{i=1,2,\cdots,k}$. If $G_0=G$, this completes a one-to-one correspondence between $Z_\phi^1(G;\mathbb{Z})$ and $\langle\{\alpha_i\}_{i=1,2,\cdots,k}\rangle$. If $G_0\neq G$, then it is straightforward to check that $\nu^1\left(ghg^{-1}\right)=\phi\left(g\right)\nu^1\left(h\right),\forall g\in G,\forall h\in G_0$ from construction and hence that Eq. (A8) always extends to a well-defined cocycle $\nu^1\in Z_\phi^1(G;\mathbb{Z})$ regardless of the value of $\nu^1(r)\in\mathbb{Z}$ with $r\in G-G_0$. Thus, the cocycles in $Z_\phi^1(G;\mathbb{Z})$ can be labeled by $\mu\in\langle\{\alpha_i\}_{i=1,2,\cdots,k}\rangle$ and $\nu^1(r)\in\mathbb{Z}$.

From definitions (A2-A4) and Eq. (A6), we further see that $\nu^1(r)$ is defined modulo 2 when the cohomology class $[\nu^1]\in H_\phi^1(G;\mathbb{Z})$ is concerned. Thus, the proof of Theorem 1 is completed.

Now we can use Theorem 1 to obtained the missed factor $H_\phi^1(G;\mathbb{Z})$ (due to nontrivial E_8 state configurations) in the classification of 3D bosonic cSPT phases for all 230 space groups. The results have been tabulated in Table II. We have juxtaposed these results with the classification of non-E_8-based cSPT phases previously obtained in Ref. [45]. The product of the two gives the complete classification of 3D bosonic cSPT phases.

2. Point groups and magnetic groups (3D)

The above computation will becomes much simpler when we only considers the 32 point groups (3D). Due to the absence of translation symmetry, the factor $\mathbb{Z}^k$ is not needed. The final result is $H_\phi^1(G;\mathbb{Z})$ is trivial if the point group G preserves orientation and equals $\mathbb{Z}_2$ otherwise; it is also gives the extra factor describing the E_8 state configuration missed in the classification of 3D bosonic SPT phases protected point group symmetry in Ref. [45]. Actually, this result of $H_\phi^1(G;\mathbb{Z})$ holds even for a magnetic point group G.

Furthermore, $H_\phi^1(G;\mathbb{Z})$ can also be computed as in Appendix A 1 for the 1651 magnetic space groups; only $k=\dim\bigcap_{W\in\wp G}\left(\phi\left(W\right)W-I\right)$ is computed in a slight different way, where $\wp G$ is the magnetic point group associated with G. To proceed, let us first review the basic structure of magnetic space groups.

Let $\mathbb{E}^d$ be the d-dimensional Euclidean space and $\mathcal{I}\left(\mathbb{E}^d\right)$ be the group of its isometries. A subgroup G of $\mathcal{I}\left(\mathbb{E}^d\right)\cup\mathcal{I}\left(\mathbb{E}^d\right)\mathcal{T}$ is called a magnetic space group if its translation subgroup is generated by translations along d linearly independent vectors v_j, $j=1,2,\cdots,d$, where $\mathcal{T}$ is the time reversion and $\mathcal{I}\left(\mathbb{E}^d\right)\mathcal{T}$ collects the operations of an space isometry combined with $\mathcal{T}$. Elements of $\mathcal{I}\left(\mathbb{E}^d\right)$ (resp. $\mathcal{I}\left(\mathbb{E}^d\right)\mathcal{T}$) are called *unitary* (resp. *anti-unitary*); they can be represented by (W,w) with $W=Q$ (resp. $W=Q'$) in a cartesian coordinate system of $\mathbb{E}^d$, where $Q\in O\left(d\right)$ (*i.e.*, is a $d\times d$ orthogonal matrix), $w\in\mathbb{R}^d$, and $Q':=Q\mathcal{T}$ denotes the combined operation of rotation Q and time reversion $\mathcal{T}$. Clearly, $(Q',w)=(Q,w)\,\mathcal{T}$ and a point labeled by $u\in\mathbb{R}^3$ is moved to $Wu+w$ by (W,w) with Wu denoting Qu for both $W=Q,Q'$. Let G^+ (resp. G^-) collect all the unitary (resp. anti-unitary) operations in G; it is clear that G^+ is a normal subgroup of G. There is a well-defined group homomorphism $\wp:G\to O\left(d\right)\cup O\left(d\right)\mathcal{T}$ sending (W,w) to W. The image of $\wp$ (*i.e.*, $\wp G=\wp G^+\cup\wp G^-$) is the magnetic point group associated with G. In addition, $\beta^{-1}\left(I\right)\subseteq G^+$ is the translation subgroup of G and $\beta^{-1}\left(I'\right)\subseteq G^-$ contains operations of the form (I',w), where I is the identity matrix.

A generic magnetic space group G must be one of four basic types. Type I: all elements of G are unitary, *i.e.*, $G^-=\emptyset$; in other words, G is just a space group. Type II: $G^-=G^+\mathcal{T}$; equivalently, G contains the time reversion $\mathcal{T}$. Type III: $G^-\neq\emptyset$ but $I'\notin\wp G$. Type IV: $I'\in\wp G$ but $\mathcal{T}\notin G$. In three ($d=3$) spatial dimensions, there are 230 type I, 230 type II, 674 type III, and 517 type IV magnetic space groups respectively; thus, we have $230+230+674+517=1651$ in total [69].

The calculation process for $H_\phi^1\left(G;\mathbb{Z}\right)$ in Appendix A 1 can be repeated for magnetic space groups and we get the following result.

Theorem 2. *Let G be a 3D magnetic space group and $\phi:G\to\{\pm1\}$ be the homomorphism sending elements that preserve the orientation of spacetime to 1 and the rest to −1. Then*

$$H_\phi^1\left(G;\mathbb{Z}\right)=\mathbb{Z}^k\times\mathbb{Z}_2^\ell,\tag{A14}$$

where $k=\dim\bigcap_{W\in\wp G}\ker\left(\phi\left(W\right)W-I\right)$ with $\wp G$ denoting the magnetic point group associated with G and $\ell=0$ (resp. 1) if there exists no (resp. some) element of G that reverses spacetime orientation.

For G of type I, $k=0,1,3$ respectively if, in the international (Hermann-Mauguin) symbol, there are more than one symmetry direction listed, exactly one symmetry direction listed and it is not 1 or $\bar{1}$, and exactly one symmetry direction listed and it is 1 or $\bar{1}$.

For G of type II or type IV, we always have $k=0$ and $\ell=1$, i.e., $H_\phi^1\left(G;\mathbb{Z}\right)=\mathbb{Z}_2$.

For G of type III, k and ℓ only depend on the magnetic point group $\wp G$ as given in Table III.

Theorem 1 already gives $H_\phi^1\left(G;\mathbb{Z}\right)$ for type I magnetic space group (*i.e.*, space group) G and the results are tabulated in Table II. For type II and type IV magnetic space group G, we have $I'\in\wp G$, $\ker(\phi(I')I'-I)=0$ and hence $H_\phi^1\left(G;\mathbb{Z}\right)=\mathbb{Z}_2$, where both I' and I act as the identity map on $\mathbb{R}^3$ but $\phi(I') = -1$ because of the time reversion.

For any magnetic group G of type III, $\wp G$ must be one of the 58 magnetic point groups listed in Table III, which are called type III as well. For $W = Q \in O(d)$, $\ker(\phi(W)W - I) = \ker(\det(Q)Q - I)$ is the symmetry direction if Q is n (*i.e.*, an n-fold rotation) or $\overline{n}$ (*i.e.*, an n-fold rotoinvertion) with $n > 1$. For $W = Q' \in O(d)\mathcal{T}$, $\ker(\phi(W)W - I) = \ker(\det(Q)Q + I)$ is the plane perpendicular to the symmetry direction if Q is 2 or $\overline{2} = m$ (*i.e.*, a mirror reflection); otherwise, $\ker(\phi(W)W - I) = 0$. Thus, it is easy to obtain k as well as ℓ tabulated in Table III from the international symbols, where primed symmetries are those combined with time reversion.

Appendix B: Stacking of crystalline states

In this appendix, we discuss the notation of stacking operation and establish the Abelian group structure of $\mathrm{SPT}^d(G)$ (*i.e.*, the collection of SPT orders protected by symmetry G in d spatial dimensions) with an emphasis on space group symmetries.

Given a symmetry group (probably a space group) G, let a be a system made of spins (or more generic bosonic degrees of freedom) σ_u located at discrete positions $u \in A \subseteq \mathbb{E}^d$ in d spatial dimensions. Let $\mathcal{H}_a$ denote its Hilbert space with G represented by $\rho_a : G \to \mathrm{Aut}(\mathcal{H}_a)$, where $\mathrm{Aut}(\mathcal{H}_a)$ is the group of unitary and antiunitary[6] operators on $\mathcal{H}_a$. Clearly, we need $\rho_a(g)\sigma_u\rho_a^{-1}(g) = \sigma_{g.u}$, *i.e.*, g mapping the spin at u to the one at $g.u$. With spins symmetrically arranged in space, such a system has a G-symmetric tensor product state $|T_a\rangle$.

In addition, let $|\Psi_a\rangle$ be the ground state of some gapped Hamiltonian $\hat{H}_a$ with finite range interactions. We call $|\Psi_a\rangle$ *short range entangled* (SRE) if it can be completely specified by local measurements. In order for $|\Psi_a\rangle$ to be SRE, the system cannot be intrinsically topologically ordered, *i.e.*, $\hat{H}_a$ allows no fractional excitations like anyons; otherwise, nontrivial global logical operators are needed to specified a topologically ordered state. More precisely, $|\Psi_a\rangle$ is SRE if and only if there exists a gapped system with ground state $|\eta\rangle$ such that the combined system $|\Psi_a\rangle \otimes |\eta\rangle$ is trivializable, *i.e.*, can be adiabatically deformed into a tensor product state without closing the energy gap [26]. Clearly, $|T_a\rangle$ is SRE. Theory of SPT phases presented in this paper classifies G-symmetric SRE states.

Given another system b of spins τ_u at $u \in B \subseteq \mathbb{E}^d$ with symmetry G represented by ρ_b on its Hilbert space $\mathcal{H}_b$, we can stack a and b together as illustrated in Fig. 7 to obtain a new system, whose local degrees of freedom are given by $\sigma_u \otimes \tau_u$ for $u \in A \cup B$. If $u \notin A$ (resp. $u \notin B$), then σ_u (resp. τ_u) is trivial. The resulting Hilbert space is $\mathcal{H}_a \otimes \mathcal{H}_b$ with G represented as $\rho_a \otimes \rho_b$. Clearly, if both $|\Psi_a\rangle$ and $|\Psi_b\rangle$ are SRE (resp. G-symmetric), then $|\Psi_a\rangle \otimes |\Psi_b\rangle$ is also SRE (resp. G-symmetric).

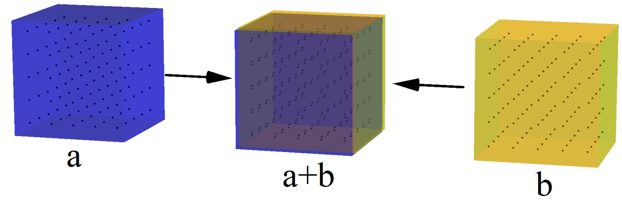

Figure 7. The stacking operation combines two crystalline states with symmetry G in dimension d into a new crystalline state with symmetry G in dimension d.

To see that the stacking operation defines an Abelian group structure of $\mathrm{SPT}^d(G)$, let us clarify the notion of SPT (including cSPT) orders first. For a specific system like a, two G-symmetric SRE states $|\Psi_a\rangle, |\Psi'_a\rangle \in \mathcal{H}_a$ are considered to lie in the same G-SPT phase and we write $|\Psi_a\rangle \simeq_G |\Psi'_a\rangle$ if they are connected by a continuous path of G-symmetric SRE states. Further, to establish a universal notion of SPT orders regardless of specific systems, we would like to compare G-symmetric SRE states $|\Psi_a\rangle \in \mathcal{H}_a$ and $|\Psi_b\rangle \in \mathcal{H}_b$ of different systems. For this, we need to choose[7] a G-symmetric tensor product state $|T_a\rangle$ (resp. $|T_b\rangle$) as the reference for system a (resp. b) and call it the trivial state. We say that $|\Psi_a\rangle$ and $|\Psi_b\rangle$ have the same G-SPT order (compared to $|T_a\rangle$ and $|T_a\rangle$ respectively) and write $(\Psi_a, T_a) \simeq_G (\Psi_b, T_b)$ if $|\Psi_a\rangle \otimes |T_b\rangle, |T_a\rangle \otimes |\Psi_b\rangle \in \mathcal{H}_a \otimes \mathcal{H}_b$ lie in the same G-SPT phase. This is an equivalence relation: it is clearly reflective ($(\Psi, T) \simeq_G (\Psi, T)$) and symmetric ($(\Psi_a, T_a) \simeq_G (\Psi_b, T_b) \Longleftrightarrow (\Psi_b, T_b) \simeq_G (\Psi_a, T_a)$) and its transitive property ($(\Psi_a, T_a) \simeq_G (\Psi_b, T_b)$ together with $(\Psi_b, T_b) \simeq_G (\Psi_c, T_c)$ implies $(\Psi_a, T_a) \simeq_G (\Psi_c, T_c)$) is checked by noticing that G-SPT phases are stable against extra trivial degrees of freedom (*i.e.*, for any G-symmetric state $|T\rangle$, $|\Psi\rangle \simeq_G |\Psi'\rangle \Longleftrightarrow |\Psi\rangle \otimes |T\rangle \simeq_G |\Psi'\rangle \otimes |T\rangle$). As an equivalence class, an G-SPT order represented by G-symmetric SRE states (Ψ, T) of some system is denoted $[\Psi, T]_G$ (or simply $[\Psi]_G$ if there is a unique G-symmetric tensor product state). We denote the collection of all possible G-SPT orders in d spatial dimensions by $\mathrm{SPT}^d(G)$. For $d = 0$, $\mathrm{SPT}^d(G)$ is actually the set of symmetries charges: $[\Psi_a, T_a]_G$ corresponds to the symmetry charge of $|\Psi_a\rangle$ relative to $|T_a\rangle$.

Any two $[\Psi_a, T_a]_G, [\Psi_b, T_b]_G \in \mathrm{SPT}^d(G)$ can be added by stacking: their sum $[\Psi_a, T_a]_G + [\Psi_b, T_b]_G$ is defined as $[\Psi_a \otimes \Psi_b, T_a \otimes T_b]_G$, where a and b may or may not be the same system. It is straightforward to check that

[6] Antiunitary operators are needed when time reversion is involved.

[7] The choice may not be unique; as we have seen in Fig. 3, distinct cSPT phases may be constructed with only 0D symmetry charges and hence can be presented by a tensor product state.

this binary operation is associative and commutative. Clearly, $[T_a, T_a]_G = [T_b, T_b]_G$ is the identity and denoted 0 with respect to this addition. Moreover, as in the plenty of examples we have seen in this paper, there is an inverse $|\Psi_b\rangle$ for each G-symmetric SRE state $|\Psi_a\rangle$ satisfying $|\Psi_a\rangle \otimes |\Psi_b\rangle \simeq_G |T_a\rangle \otimes |T_b\rangle$, which implies $[\Psi_a, T_a]_G + [\Psi_b, T_b]_G = 0$. Thus, G-SPT is an Abelian group.

In practice, we could always pick a tensor product of spin states that transform trivially under site symmetry as the preferred trivial state. Explicitly, a G-SPT order is simply represented by a G-symmetric model $\mathcal{M}_a$, which can be encoded by the distribution of spins (*i.e.*, bosonic local degrees of freedom) $\sigma_a(u)$ in space, a representation ρ_a of G on its Hilbert space $\mathcal{H}_a$, and a Hamiltonian $\hat{H}_a$ fixing a G-symmetric SRE ground state $|\Psi_a\rangle$. The trivial state $|T_a\rangle$ is chosen by default to be a tensor product of spin states that transform trivially under the action ρ_a of their site symmetry. For any two G-SPT models $\mathcal{M}_a = (\sigma_a(u), \mathcal{H}_a, \rho_a, \hat{H}_a, |\Psi_a\rangle)$ and $\mathcal{M}_b = (\sigma_b(u), \mathcal{H}_b, \rho_b, \hat{H}_b, |\Psi_b\rangle)$, we simply write $\mathcal{M}_a \oplus \mathcal{M}_b$ for the G-SPT order represented by the model $(\sigma_a(u) \otimes \sigma_b(u), \mathcal{H}_a \otimes \mathcal{H}_b, \rho_a \otimes \rho_b, \hat{H}_a \otimes \hat{\mathbb{I}}_b + \hat{\mathbb{I}}_a \otimes \hat{H}_b, |\Psi_a\rangle) \otimes |\Psi_b\rangle)$ (with respect to $|T_a\rangle \otimes |T_b\rangle$).

Appendix C: The generalized cohomology hypothesis

A generalized cohomology theory [49–52] h can be represented by an Ω-spectrum $F_\bullet = \{F_n\}$, which by definition is a sequence

$$\ldots, F_{-2}, F_{-1}, F_0, F_1, F_2, \ldots \tag{C1}$$

of pointed topological spaces together with pointed homotopy equivalences

$$F_n \simeq \Omega F_{n+1}, \tag{C2}$$

where Ω is the loop space functor. In the non-twisted case, the generalized cohomology theory h outputs an abelian group $h^n(X)$ for each given topological space X and integer n, according to

$$h^n(X) := [X, \Omega F_{n+1}]. \tag{C3}$$

Here $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. One can define abelian group structure on $h^n(X)$. Namely, given representatives $a, b : X \to \Omega F_{n+1}$ of two homotopy classes $[a]$ and $[b]$, one defines $[a] + [b]$ to be the homotopy class represented by the map $a + b : X \to \Omega F_{n+1}$, where $(a+b)(x)$ is the concatenation of the loops $a(x)$ and $b(x)$ for any x.

In the twisted case, one is given an integer n, a pointed topological space X, and an action ϕ_X of the fundamental group $\pi_1(X)$ on the Ω-spectrum F. The generalized cohomology theory h then outputs an abelian group according to

$$h^n(X, \phi_X) := \left[\widetilde{X}, \Omega F_{n+1}\right]_{\pi_1(X)}, \tag{C4}$$

where $\widetilde{X}$ denotes the universal cover of X, and $[X, Y]_G$ denotes the set of homotopy classes of G-equivariant maps from X to Y. As in the non-twisted case, $h^n(X, \phi_X)$ can be given an abelian group structure by concatenating loops. Here, recall that a G-equivariant map $f : X \to Y$ is a map that commutes with the action of G, *i.e.*, $g.(f(x)) = f(g.x)$ $\forall g \in G$ and $x \in X$. It is a simple exercise to show that if $\pi_1(X)$ acts trivially on the Ω-spectrum, then $h^n(X, \phi_X) = h^n(X)$.

In Appendix B, we noted that for given dimension d, symmetry G, and homomorphism $\phi : G \to \{\pm 1\}$ indicating whether a group element reverses the spacetime orientation, the set of bosonic SPT phases

$$\mathrm{SPT}^d(G) \tag{C5}$$

has a natural abelian group structure defined by stacking. Based on Kitaev's argument [24, 26, 27] that the classification of SPT phases should carry the structure of a generalized cohomology theory, Refs. [47, 63] formulated the following "generalized cohomology hypothesis."

Generalized cohomology hypothesis. *There exists a generalized cohomology theory h such that for any $d \in \mathbb{N}$, group G, and continuous homomorphism $\phi : G \to \{\pm 1\}$, we have an abelian group isomorphism*

$$\mathrm{SPT}^d(G) \cong h^d(BG, \phi) \tag{C6}$$

between the classification of bosonic SPT phases and the generalized cohomology group.

(The formulation of the generalized cohomology hypothesis in Refs. [47, 63] was actually stronger than as stated here and incorporated an additional structure called functoriality. There was also a fermionic counterpart to the hypothesis [63].)

In the right-hand side of isomorphism (C6), the action of $\pi_1(BG) \cong \pi_0(G)$ on the Ω-spectrum $F_\bullet$ that represents h is as follows. First, note that by continuity $\phi : G \to \{\pm 1\}$ can be viewed as a homomorphism from $\pi_0(G)$ to $\{\pm 1\}$ instead. Then, an element x of $\pi_1(BG) \cong \pi_0(G)$ acts on the ΩF_{n+1} in Eq. (C4) by reversing the loop if $\phi(x) = -1$ and by the identity if $\phi(x) = +1$. In other words, we have the following

Addendum to hypothesis. *A symmetry element behaves antiunitarily in the generalized cohomology hypothesis if it reverses the orientation of spacetime, and unitarily otherwise.*

In the main text, $h^d(BG, \phi)$ is explicitly defined by Eq. (7) and denoted as $h^d_\phi(G; F_\bullet)$ in a way analogous to $H^n_\phi(G; \mathbb{Z})$.

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Table II. The classification of 3D bosonic crystalline SPT (cSPT) phases for all 230 space groups. The first and second columns list the numbers and international (Hermann-Mauguin) symbols of space groups. The third through fifth columns list the classification of non-E_8-based phases (given by $H^5_\phi(G;\mathbb{Z})$ and cited from Ref. [45]), E_8 state configurations (given by $H^1_\phi(G;\mathbb{Z})$ and computed using Theorem 1), and all 3D bosonic cSPT phases $\mathrm{SPT}^3(G) \cong H^5_\phi(G;\mathbb{Z}) \oplus H^1_\phi(G;\mathbb{Z})$ respectively. Entries left blank have trivial classifications.

No.	Symbol	Classification of 3D bosonic cSPT phases		
		$H^5_\phi(G;\mathbb{Z})$	$H^1_\phi(G;\mathbb{Z})$	$\mathrm{SPT}^3(G)$
1	$P1$		$\mathbb{Z}^3$	$\mathbb{Z}^3$
2	$P\bar{1}$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2\times\mathbb{Z}^3$	$\mathbb{Z}_2^9\times\mathbb{Z}^3$
3	$P2$	$\mathbb{Z}_2^4$	$\mathbb{Z}$	$\mathbb{Z}_2^4\times\mathbb{Z}$
4	$P2_1$		$\mathbb{Z}$	$\mathbb{Z}$
5	$C2$	$\mathbb{Z}_2^2$	$\mathbb{Z}$	$\mathbb{Z}_2^2\times\mathbb{Z}$
6	Pm	$\mathbb{Z}_2^4$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^5\times\mathbb{Z}$
7	Pc		$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}$
8	Cm	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^3\times\mathbb{Z}$
9	Cc		$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}$
10	$P2/m$	$\mathbb{Z}_2^{18}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^{19}\times\mathbb{Z}$
11	$P2_1/m$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^7\times\mathbb{Z}$
12	$C2/m$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^{12}\times\mathbb{Z}$
13	$P2/c$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^7\times\mathbb{Z}$
14	$P2_1/c$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^5\times\mathbb{Z}$
15	$C2/c$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^6\times\mathbb{Z}$
16	$P222$	$\mathbb{Z}_2^{16}$		$\mathbb{Z}_2^{16}$
17	$P222_1$	$\mathbb{Z}_2^4$		$\mathbb{Z}_2^4$
18	$P2_12_12$	$\mathbb{Z}_2^2$		$\mathbb{Z}_2^2$
19	$P2_12_12_1$			
20	$C222_1$	$\mathbb{Z}_2^2$		$\mathbb{Z}_2^2$
21	$C222$	$\mathbb{Z}_2^9$		$\mathbb{Z}_2^9$
22	$F222$	$\mathbb{Z}_2^8$		$\mathbb{Z}_2^8$
23	$I222$	$\mathbb{Z}_2^8$		$\mathbb{Z}_2^8$
24	$I2_12_12_1$	$\mathbb{Z}_2^3$		$\mathbb{Z}_2^3$
25	$Pmm2$	$\mathbb{Z}_2^{16}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{17}$
26	$Pmc2_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
27	$Pcc2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
28	$Pma2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
29	$Pca2_1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
30	$Pnc2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
31	$Pmn2_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
32	$Pba2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
33	$Pna2_1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
34	$Pnn2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
35	$Cmm2$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
36	$Cmc2_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
37	$Ccc2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
38	$Amm2$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
39	$Aem2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
40	$Ama2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
41	$Aea2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
42	$Fmm2$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2$	$\mathbb{Z}_2^7$
43	$Fdd2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
44	$Imm2$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^9$
45	$Iba2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
46	$Ima2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
47	$Pmmm$	$\mathbb{Z}_2^{42}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{43}$
48	$Pnnn$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
49	$Pccm$	$\mathbb{Z}_2^{17}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{18}$
50	$Pban$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
51	$Pmma$	$\mathbb{Z}_2^{17}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{18}$
52	$Pnna$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
53	$Pmna$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
54	$Pcca$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
55	$Pbam$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
56	$Pccn$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
57	$Pbcm$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
58	$Pnnm$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
59	$Pmmn$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
60	$Pbcn$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
61	$Pbca$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
62	$Pnma$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
63	$Cmcm$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
64	$Cmca$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2^8$
65	$Cmmm$	$\mathbb{Z}_2^{26}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{27}$
66	$Cccm$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
67	$Cmme$	$\mathbb{Z}_2^{17}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{18}$
68	$Ccce$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2^8$
69	$Fmmm$	$\mathbb{Z}_2^{20}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{21}$
70	$Fddd$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2$	$\mathbb{Z}_2^7$
71	$Immm$	$\mathbb{Z}_2^{22}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{23}$
72	$Ibam$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
73	$Ibca$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
74	$Imma$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
75	$P4$	$\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2\times\mathbb{Z}$
76	$P4_1$		$\mathbb{Z}$	$\mathbb{Z}$
77	$P4_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}$	$\mathbb{Z}_2^3\times\mathbb{Z}$
78	$P4_3$		$\mathbb{Z}$	$\mathbb{Z}$
79	$I4$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}$
80	$I4_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}$
81	$P\bar{4}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^4\times\mathbb{Z}$
82	$I\bar{4}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^3\times\mathbb{Z}$
83	$P4/m$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^{12}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^{13}\times\mathbb{Z}$
84	$P4_2/m$	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_2^{12}\times\mathbb{Z}$
85	$P4/n$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4^2\times\mathbb{Z}_2^4\times\mathbb{Z}$
86	$P4_2/n$	$\mathbb{Z}_4\times\mathbb{Z}_2^4$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4\times\mathbb{Z}_2^5\times\mathbb{Z}$
87	$I4/m$	$\mathbb{Z}_4\times\mathbb{Z}_2^8$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4\times\mathbb{Z}_2^9\times\mathbb{Z}$
88	$I4_1/a$	$\mathbb{Z}_4\times\mathbb{Z}_2^3$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_4\times\mathbb{Z}_2^4\times\mathbb{Z}$
89	$P422$	$\mathbb{Z}_2^{12}$		$\mathbb{Z}_2^{12}$
90	$P42_12$	$\mathbb{Z}_4\times\mathbb{Z}_2^4$		$\mathbb{Z}_4\times\mathbb{Z}_2^4$
91	$P4_122$	$\mathbb{Z}_2^3$		$\mathbb{Z}_2^3$
92	$P4_12_12$	$\mathbb{Z}_2$		$\mathbb{Z}_2$

No.	Symbol	Classification of 3D bosonic cSPT phases		
		$H^5_\phi(G;\mathbb{Z})$	$H^1_\phi(G;\mathbb{Z})$	$\mathrm{SPT}^3(G)$
93	$P4_222$	$\mathbb{Z}_2^{12}$		$\mathbb{Z}_2^{12}$
94	$P4_22_12$	$\mathbb{Z}_2^{5}$		$\mathbb{Z}_2^{5}$
95	$P4_322$	$\mathbb{Z}_2^{3}$		$\mathbb{Z}_2^{3}$
96	$P4_32_12$	$\mathbb{Z}_2$		$\mathbb{Z}_2$
97	$I422$	$\mathbb{Z}_2^{8}$		$\mathbb{Z}_2^{8}$
98	$I4_122$	$\mathbb{Z}_2^{5}$		$\mathbb{Z}_2^{5}$
99	$P4mm$	$\mathbb{Z}_2^{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{13}$
100	$P4bm$	$\mathbb{Z}_4\times\mathbb{Z}_2^{4}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{5}$
101	$P4_2cm$	$\mathbb{Z}_2^{6}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{7}$
102	$P4_2nm$	$\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{6}$
103	$P4cc$	$\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{4}$
104	$P4nc$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{2}$
105	$P4_2mc$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
106	$P4_2bc$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
107	$I4mm$	$\mathbb{Z}_2^{7}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{8}$
108	$I4cm$	$\mathbb{Z}_2^{4}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{5}$
109	$I4_1md$	$\mathbb{Z}_2^{4}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{5}$
110	$I4_1cd$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{2}$
111	$P\bar{4}2m$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
112	$P\bar{4}2c$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
113	$P\bar{4}2_1m$	$\mathbb{Z}_4\times\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{6}$
114	$P\bar{4}2_1c$	$\mathbb{Z}_4\times\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{3}$
115	$P\bar{4}m2$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
116	$P\bar{4}c2$	$\mathbb{Z}_2^{7}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{8}$
117	$P\bar{4}b2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{6}$
118	$P\bar{4}n2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{6}$
119	$I\bar{4}m2$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
120	$I\bar{4}c2$	$\mathbb{Z}_2^{6}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{7}$
121	$I\bar{4}2m$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
122	$I\bar{4}2d$	$\mathbb{Z}_4\times\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{3}$
123	$P4/mmm$	$\mathbb{Z}_2^{32}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{33}$
124	$P4/mcc$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
125	$P4/nbm$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
126	$P4/nnc$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
127	$P4/mbm$	$\mathbb{Z}_4\times\mathbb{Z}_2^{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{16}$
128	$P4/mnc$	$\mathbb{Z}_4\times\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^{9}$
129	$P4/nmm$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
130	$P4/ncc$	$\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{6}$
131	$P4_2/mmc$	$\mathbb{Z}_2^{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{25}$
132	$P4_2/mcm$	$\mathbb{Z}_2^{18}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{19}$
133	$P4_2/nbc$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
134	$P4_2/nnm$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
135	$P4_2/mbc$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
136	$P4_2/mnm$	$\mathbb{Z}_2^{14}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{15}$
137	$P4_2/nmc$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
138	$P4_2/ncm$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
139	$I4/mmm$	$\mathbb{Z}_2^{20}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{21}$
140	$I4/mcm$	$\mathbb{Z}_2^{14}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{15}$
141	$I4_1/amd$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
142	$I4_1/acd$	$\mathbb{Z}_2^{5}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{6}$

No.	Symbol	Classification of 3D bosonic cSPT phases		
		$H^5_\phi(G;\mathbb{Z})$	$H^1_\phi(G;\mathbb{Z})$	$\mathrm{SPT}^3(G)$
143	$P3$	$\mathbb{Z}_3^{3}$	$\mathbb{Z}$	$\mathbb{Z}_3^{3}\times\mathbb{Z}$
144	$P3_1$		$\mathbb{Z}$	$\mathbb{Z}$
145	$P3_2$		$\mathbb{Z}$	$\mathbb{Z}$
146	$R3$	$\mathbb{Z}_3$	$\mathbb{Z}$	$\mathbb{Z}_3\times\mathbb{Z}$
147	$P\bar{3}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{4}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{5}\times\mathbb{Z}$
148	$R\bar{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^{4}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_3\times\mathbb{Z}_2^{5}\times\mathbb{Z}$
149	$P312$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
150	$P321$	$\mathbb{Z}_3\times\mathbb{Z}_2^{2}$		$\mathbb{Z}_3\times\mathbb{Z}_2^{2}$
151	$P3_112$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
152	$P3_121$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
153	$P3_212$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
154	$P3_221$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
155	$R32$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
156	$P3m1$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
157	$P31m$	$\mathbb{Z}_3\times\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^{3}$
158	$P3c1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
159	$P31c$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2$
160	$R3m$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
161	$R3c$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
162	$P\bar{3}1m$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
163	$P\bar{3}1c$	$\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{4}$
164	$P\bar{3}m1$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
165	$P\bar{3}c1$	$\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{4}$
166	$R\bar{3}m$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
167	$R\bar{3}c$	$\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{4}$
168	$P6$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{2}$	$\mathbb{Z}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{2}\times\mathbb{Z}$
169	$P6_1$		$\mathbb{Z}$	$\mathbb{Z}$
170	$P6_5$		$\mathbb{Z}$	$\mathbb{Z}$
171	$P6_2$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}$	$\mathbb{Z}_2^{2}\times\mathbb{Z}$
172	$P6_4$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}$	$\mathbb{Z}_2^{2}\times\mathbb{Z}$
173	$P6_3$	$\mathbb{Z}_3^{2}$	$\mathbb{Z}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}$
174	$P\bar{6}$	$\mathbb{Z}_3^{3}\times\mathbb{Z}_2^{4}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_3^{3}\times\mathbb{Z}_2^{5}\times\mathbb{Z}$
175	$P6/m$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{10}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{11}\times\mathbb{Z}$
176	$P6_3/m$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{4}$	$\mathbb{Z}_2\times\mathbb{Z}$	$\mathbb{Z}_3^{2}\times\mathbb{Z}_2^{5}\times\mathbb{Z}$
177	$P622$	$\mathbb{Z}_2^{8}$		$\mathbb{Z}_2^{8}$
178	$P6_122$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
179	$P6_522$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
180	$P6_222$	$\mathbb{Z}_2^{8}$		$\mathbb{Z}_2^{8}$
181	$P6_422$	$\mathbb{Z}_2^{8}$		$\mathbb{Z}_2^{8}$
182	$P6_322$	$\mathbb{Z}_2^{2}$		$\mathbb{Z}_2^{2}$
183	$P6mm$	$\mathbb{Z}_2^{8}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{9}$
184	$P6cc$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
185	$P6_3cm$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
186	$P6_3mc$	$\mathbb{Z}_2^{2}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{3}$
187	$P\bar{6}m2$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
188	$P\bar{6}c2$	$\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{4}$
189	$P\bar{6}2m$	$\mathbb{Z}_3\times\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^{10}$
190	$P\bar{6}2c$	$\mathbb{Z}_3\times\mathbb{Z}_2^{3}$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^{4}$
191	$P6/mmm$	$\mathbb{Z}_2^{22}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{23}$
192	$P6/mcc$	$\mathbb{Z}_2^{9}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$

No.	Symbol	Classification of 3D bosonic cSPT phases		
		$H^5_\phi(G;\mathbb{Z})$	$H^1_\phi(G;\mathbb{Z})$	$\mathrm{SPT}^3(G)$
193	$P6_3/mcm$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
194	$P6_3/mmc$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{10}$
195	$P23$	$\mathbb{Z}_3\times\mathbb{Z}_2^4$		$\mathbb{Z}_3\times\mathbb{Z}_2^4$
196	$F23$	$\mathbb{Z}_3$		$\mathbb{Z}_3$
197	$I23$	$\mathbb{Z}_3\times\mathbb{Z}_2^2$		$\mathbb{Z}_3\times\mathbb{Z}_2^2$
198	$P2_13$	$\mathbb{Z}_3$		$\mathbb{Z}_3$
199	$I2_13$	$\mathbb{Z}_3\times\mathbb{Z}_2$		$\mathbb{Z}_3\times\mathbb{Z}_2$
200	$Pm\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^{14}$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^{15}$
201	$Pn\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^5$
202	$Fm\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^6$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^7$
203	$Fd\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^3$
204	$Im\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^8$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^9$
205	$Pa\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^3$
206	$Ia\overline{3}$	$\mathbb{Z}_3\times\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_3\times\mathbb{Z}_2^4$
207	$P432$	$\mathbb{Z}_2^6$		$\mathbb{Z}_2^6$
208	$P4_232$	$\mathbb{Z}_2^6$		$\mathbb{Z}_2^6$
209	$F432$	$\mathbb{Z}_2^4$		$\mathbb{Z}_2^4$
210	$F4_132$	$\mathbb{Z}_2$		$\mathbb{Z}_2$
211	$I432$	$\mathbb{Z}_2^5$		$\mathbb{Z}_2^5$
212	$P4_332$	$\mathbb{Z}_2$		$\mathbb{Z}_2$
213	$P4_132$	$\mathbb{Z}_2$		$\mathbb{Z}_2$
214	$I4_132$	$\mathbb{Z}_2^4$		$\mathbb{Z}_2^4$
215	$P\overline{4}3m$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2^8$
216	$F\overline{4}3m$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
217	$I\overline{4}3m$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
218	$P\overline{4}3n$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$
219	$F\overline{4}3c$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
220	$I\overline{4}3d$	$\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^2$
221	$Pm\overline{3}m$	$\mathbb{Z}_2^{18}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{19}$
222	$Pn\overline{3}n$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2$	$\mathbb{Z}_2^6$
223	$Pm\overline{3}n$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
224	$Pn\overline{3}m$	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{11}$
225	$Fm\overline{3}m$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
226	$Fm\overline{3}c$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2^8$
227	$Fd\overline{3}m$	$\mathbb{Z}_2^7$	$\mathbb{Z}_2$	$\mathbb{Z}_2^8$
228	$Fd\overline{3}c$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$
229	$Im\overline{3}m$	$\mathbb{Z}_2^{13}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{14}$
230	$Ia\overline{3}d$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^5$

Table III. List of 58 magnetic point groups of type III with its symbol in the second column. For each magnetic group G, the third column gives $k = \dim \bigcap_{W\in G}(\phi(W)W - I)$. The forth column ($\ell$) shows 0 if G preserves the orientation of spacetime and 1 otherwise.

No.	Symbol	k	ℓ
1	$\overline{1}'$	0	0
2	$2'$	2	1
3	m'	2	0
4	$\frac{2'}{m}$	0	1
5	$\frac{2}{m'}$	0	0
6	$\frac{2'}{m'}$	2	1
7	$2'2'2$	1	1
8	$m'm2'$	1	1
9	$m'm'2$	1	0
10	$m'mm$	0	1
11	$m'm'm$	1	1
12	$m'm'm'$	0	0
13	$4'$	0	1
14	$\overline{4}'$	0	0
15	$\frac{4'}{m}$	0	1
16	$\frac{4}{m'}$	0	0
17	$\frac{4'}{m'}$	0	1
18	$4'22'$	0	1
19	$42'2'$	1	1
20	$4'm'm$	0	1
21	$4m'm'$	1	0
22	$\overline{4}'2'm$	0	1
23	$\overline{4}'2m'$	0	0
24	$\overline{4}2'm'$	1	1
25	$\frac{4}{m'}mm$	0	1
26	$\frac{4'}{m}m'm$	0	1
27	$\frac{4'}{m'}m'm$	0	1
28	$\frac{4}{m}m'm'$	1	1
29	$\frac{4}{m'}m'm'$	0	0

No.	Symbol	k	ℓ
30	$\overline{3}'$	0	0
31	$32'$	1	1
32	$3m'$	1	0
33	$\overline{3}'m$	0	1
34	$\overline{3}'m'$	0	0
35	$\overline{3}m'$	1	1
36	$6'$	0	1
37	$\overline{6}'$	0	0
38	$\frac{6'}{m}$	0	1
39	$\frac{6}{m'}$	0	0
40	$\frac{6'}{m'}$	0	1
41	$6'22'$	0	1
42	$62'2'$	1	1
43	$6'mm'$	0	1
44	$6m'm'$	1	0
45	$\overline{6}'m'2$	0	0
46	$\overline{6}'m2'$	0	1
47	$\overline{6}m'2'$	1	1
48	$\frac{6}{m'}mm$	0	1
49	$\frac{6'}{m}mm'$	0	1
50	$\frac{6'}{m'}mm'$	0	1
51	$\frac{6}{m}m'm'$	1	1
52	$\frac{6}{m'}m'm'$	0	0
53	$m'\overline{3}'$	0	0
54	$4'32'$	0	1
55	$\overline{4}'3m'$	0	0
56	$m'\overline{3}'m$	0	1
57	$m\overline{3}m'$	0	1
58	$m'\overline{3}'m'$	0	0

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