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Fractons emerge as charges with reduced mobility in a new class of gauge theories. Here, we generalise fractonic theories of $U(1)$ type to what we call (k,n)-fractonic Maxwell theory, which employs symmetric rank-n tensors of k-forms (rank-k antisymmetric tensors) as "vector potentials". The generalisation, valid in any spatial dimension d, has two key manifestations. First, the objects with mobility restrictions extend beyond simple charges to higher order multipoles (dipoles, quadrupoles, ...) all the way to n^{th}-order multipoles, which we call the order-n fracton condition. Second, these fractonic charges themselves are characterized by tensorial densities of $(k-1)$-dimensional extended objects. For any (k,n), the theory can be constructed to have a gapless "photon modes" with dispersion $\omega \sim |q|^z$, where the integer z can range from 1 to n.

I. INTRODUCTION

Understanding and classifying the phases of systems with many interacting microscopic degrees of freedom is gaining renewed attention. Ideas of topology and entanglement are key to obtaining deeper insight and clarity in the classification of phases of many body systems[1]. Apart from the success in classification of short ranged entangled phases of free fermions[2,3], significant progress has been made in the understanding of symmetry protected phases with interactions[4]. Long range entangled phases with topological order have also been elaborated[1]. Systems with topological order provide opportunities for producing a plethora of new physics with exotic excitations and properties that can, for example, be useful for quantum computation[5].

Quite fascinatingly, recent research[6–15], broadly following the above cues, has uncovered an intriguing new kind of phase of matter – the fracton phase (see[16] for a review). Fractonic phases derive their name from the peculiar properties of their excitations. These have "fractional mobility" in that there are excitations whose motion is restricted. For instance, the X-cube model, which provides a popular example of fracton physics (see, for example,[13]), has point like charge excitations which are immobile, while dipole excitations can move along specific lines. Similar physics, found in all fracton phases, arises not from the energetics, but from the constraint structure encoded in the theory itself that restricts local changes in microscopic degrees of freedom. In addition, a typically sub-extensive ground state 'entropy' of these systems is sensitively dependent on the microscopic lattice on which the theory is defined, leading to the definition of a novel concept of "geometric order"[17].

The obvious question of where the fracton phases fit in the classification has motivated further work. A natural starting point is to ask for a field theoretical description of such phases that obtains both the excitation physics and the ground state degeneracy. Ref.[18], starting from the X-cube model, uses the BF-formulation to obtain a field theory that contains the key fractonic physics. Other approaches[19] combine the Chern-Simons/BF formulation with symmetric tensor (see below) gauge formulations to obtain the fracton characterization. All this clearly points to a rich panorama of possibilities.

Concurrently with these developments, fractons have appeared in an apparently different context in the study[20] of spin liquids described by symmetric tensor gauge theories. Such $U(1)$ symmetric-tensor gauge theories support a gapless "photon" excitation much like Maxwell electrodynamics, which is described by a vector gauge field. Sources(charges) of such a symmetric-tensor gauge theory have a crucial additional character – an isolated charge of a symmetric-tensor gauge theory is immobile[21,22]. This arises from the fact that charge conservation in such theories also forces the conservation of the dipole moment. As a consequence, motion of an isolated charge, as it changes the dipole moment, is forbidden, endowing the charge with a fracton character. An isolated dipole (total charge zero) does not suffer such constraints: dipoles are free to move in an unconstrained fashion (to be compared with the dipole excitations in the X-cube model discussed above). Charge conservation is enforced by the continuity equation

$$\partial_t \rho + \partial_i J^Q_i = 0, \tag{1}$$

where ρ is the charge density, t is time, ∂_t is time, and ∂_i the spatial derivative along the i-th Cartesian coordinate direction, and J^Q_i is the charge current vector. The fractonic nature of charge in symmetric-tensor gauge theories follows from the fact that the charge current vector is itself the divergence of the dipole current tensor J_{ij} , i. e.,

$$J^Q_i = \partial_j J_{ji}. \tag{2}$$

Several interesting aspects of such symmetric-tensor gauge theories, including their connection to gravity, have been explored[23,24]. Field theories where fractonic charges are line like excitations, i. e., extended objects, have also been constructed[25]. While these developments are interesting and encouraging, future research will have to reveal if such symmetric-tensor gauge theories form a generic framework to describe gapless fracton phases (gapped variants with symmetric-tensor descriptions are discussed in ref.[19]).

It is therefore compelling to look for a general framework to describe fractonic theories. Such a general theory should describe order-n fractons that are $(k-1)$-dimensional extended objects in arbitrary spatial dimension d. The order-n fracton condition means that conservation of charge implies conservation of all multipole-moments of the charges, up to some

n-th moment (i.e., 2^n-pole). Physically, this would imply mobility restrictions not only for individual charges, but also on all higher-moment objects such as dipoles, quadrupoles, ..., 2^n-pole.

In this paper, we develop such a framework by formulating what we call (k, n)-fracton theory. These have tensor charge densities of $k-1$-dimensional extended objects with generalized mobility restrictions on all multipoles up to n-th order. This is accomplished by constructing a new gauge structure – gauge fields are rank-n symmetric tensors of k-forms (anti-symmetric tensors) in arbitrary spatial dimension d. This leads naturally to a sequence of theories under the heading of (k, n)-fractonic Maxwell theory, which for $(k = 1, n = 1)$ reduces to standard electromagnetism. An additional feature is that for each (k, n), the framework allows construction of theories with "dynamic exponent" z, with gapless "photon modes" that disperse as $\omega \sim |q|^z$ (ω–frequency, q–wave vector), which we dub $(k, n)_z$-theory, where z is an integer in range 1 to n.

We first present a generalization of the symmetric tensor gauge theories to make $(1, n)$-fracton theories where scalar charges satisfy the order-n fracton condition (section III), followed by the development of the (k, n)-fracton theory (section V) with a dynamical exponent $z = n$. The general $(k, n)_z$-theory is discussed in Appendix A. We close with an extended outlook in section VI.

II. NOTATION

We use a notation that brings out the physical character of our theory that is *not* designed for "relativistic covariance", with x denoting a point in d-dimensional Cartesian space, and t, time. This simple mathematical notation uses symmetric and anti-symmetric tensors. A symmetric tensor of rank n, with n-indices $i^1, i^2, \ldots, i^n$ is denoted by $S_{(i^1 i^2 \ldots i^n)}$ where the ( ) used to explicitly indicate its symmetric nature. On the other hand an anti-symmetric tensor of rank k with indices $i_1, \ldots, i_k$ is denoted by $T_{[i_1,\ldots,i_k]}$ where [] makes the anti-symmetric character of the tensor T explicit. Note that $S_{(i)} = S_i$ and $T_{[i]} = T_i$ is to be understood. A collection of s ordered indices $i_1 i_2 \ldots i_s$ is abbreviated into a composite index I_s

$$I_s \equiv i_1 i_2 \ldots i_s. \tag{3}$$

Further, P_s (whose sign is denoted $(-1)^{P_s}$) acting on I_s permutes $1 \ldots s$,

$$P_s I_s = i_{P_s(1)} \ldots i_{P_s(s)}\ . \tag{4}$$

III. $(1, n)$-FRACTON THEORIES

We first generalize the symmetric tensor gauge theories of ref.[21] to order-n fracton theories, which impose the generalized higher multipole mobility constraints. Consider a gauge field described by $(\phi(x, t), A_{(i^1 i^2 \ldots i^n)}(x, t))$ where the vector potential of the usual Maxwell theory is generalized to a symmetric tensor of rank n. We define the electric field tensor (symmetric tensor of rank n) as

$$E_{(i^1 i^2 \ldots i^n)} = -\partial_{i^1} \partial_{i^2} \ldots \partial_{i^n} \phi - \partial_t A_{(i^1 i^2 \ldots i^n)}. \tag{5}$$

The magnetic field tensor is defined as

$$B_{([I_2^1][I_2^2]\ldots[I_2^n])} = \sum_{\{P_2^r\}} \left(\prod_{r=1}^{n} (-1)^{P_2^r} \partial_{i^r_{P_2^r(1)}} \right) A_{(i^1_{P_2^1(2)} \ldots i^n_{P_2^n(2)})} \tag{6}$$

where $I_2^r \equiv i_1^r i_2^r$, is a composite index and P_2^r permutes I_2^r in the notation introduced above. Note that the magnetic field tensor, with a rather unconventional structure, is symmetric in exchange of I_2^r with $I_2^{r'}$ composite indices. It is anti-symmetric upon exchange of the two indices within any I_2^r.

These fields can be sourced by suitable charges and currents. The charge density ρ and a symmetric n-tensor current density $J_{(i^1 i^2 \ldots i^n)}$ couple minimally to the gauge fields. The system is described by a Lagrangian density

$$L = L_E - L_B + (-1)^n \rho\phi + J_{(i^1 i^2 \ldots i^n)} A_{(i^1 i^2 \ldots i^n)} \tag{7}$$

where repeated indices are contracted ignoring the parentheses (which simply remind us of the symmetric nature of the tensors involved). Here L_E and L_B are electric and magnetic energy densities defined by

$$L_E = \frac{1}{2} E_{(i^1 i^2 \ldots i^n)} \epsilon_{((i^1 i^2 \ldots i^n)(j^1 j^2 \ldots j^n))} E_{(j^1 j^2 \ldots j^n)} \tag{8}$$

where ϵ is a suitable positive definite dielectric symmetric (in interchange of the set i with j) tensor and

$$L_B = \frac{1}{2} B_{([I_2^1][I_2^2]\ldots[I_2^n])} \kappa_{(([I_2^1][I_2^2]\ldots[I_2^n])([J_2^1][J_2^2]\ldots[J_2^n]))} B_{([J_2^1][J_2^2]\ldots[J_2^n])} \tag{9}$$

with κ an inverse permeability tensor, again symmetric positive definite. The electric and magnetic fields are invariant under the gauge transformation involving the function $\psi(x, t)$:

$$\begin{aligned} \phi &\to \phi + \partial_t \psi \\ A_{(i^1 i^2 \ldots i^n)} &\to A_{(i^1 i^2 \ldots i^n)} - \partial_{i^1} \partial_{i^2} \ldots \partial_{i^n} \psi. \end{aligned} \tag{10}$$

The gauge invariance of the action (integral of the Lagrangian density) produces the continuity equation

$$\partial_t \rho + \partial_{i^1} \partial_{i^2} \ldots \partial_{i^n} J_{(i^1 i^2 \ldots i^n)} = 0. \tag{11}$$

This theory thus encodes mobility restrictions not only on charges, but also on any 2^p-pole (a collection of "closeby" charges with leading non vanishing p-th multipole moment), $p < n$, as eqn. (11) implies that

$$\partial_t \int_V \mathrm{d}^d x \left(\prod_{l=1}^{p} x_{i^l} \right) \rho = 0, \quad 0 \le p < n \tag{12}$$

all multipoles up to order n are conserved in this theory (V is the d-volume of the system). The current $J_{(i^1 \ldots i^n)}$ thus has a natural meaning of the 2^n-pole current. While charge current is the divergence of the dipole current, the dipole current is in turn that of the quadrupole current, and so on. In this sense, the theory describes order-n fractonic physics of point charges whose density is ρ. For $n = 2$, the above theory has $z = 2$ (see below); the $n = 2$ theory with $z = 1$ like that in[26], are constructed in Appendix V.

IV. THEORIES WITH EXTENDED SOURCES

To explore the generalizations of the theory where the charges are more complex objects, we first review known Maxwellian theories where charges are extended objects, a well developed subject in itself (cf.[27,28]), casting this theory in our notation. Such gauge theories are described by a gauge field $(\phi_{[i_1 i_2 \ldots i_{k-1}]}(x,t), A_{[i_1 i_2 \ldots i_k]}(x,t))$ where, following our convention, ϕ and A are fully anti-symmetric tensors ($A_{[i_1 i_2 \ldots i_k]}$ is usually called a k-form). The electric field is also an anti-symmetric tensor defined by

$$E_{[i_i i_1 \ldots i_k]} = -\frac{1}{(k-1)!}\sum_{P_k}(-1)^{P_k}\partial_{i_{P_k(1)}}\phi_{[i_{P_k(2)}\ldots i_{P_k(k)}]} - \partial_t A_{[i_1 i_2 \ldots i_k]}. \tag{13}$$

The magnetic field here is obtained as

$$B_{[i_1 i_2 \ldots i_{k+1}]} = \frac{1}{k!}\sum_{P_{k+1}}(-1)^{P_{k+1}}\partial_{i_{P_{k+1}(1)}}A_{[i_{P_{k+1}(2)}\ldots i_{P_{k+1}(k+1)}]}. \tag{14}$$

The extended charged sources[29] are described by densities $\rho_{[i_1 i_2 \ldots i_{k-1}]}$ and currents $J_{[i_1 i_2 \ldots i_k]}$. One can now define a Lagrangian density for this theory as

$$L = L_E - L_B - \rho_{[i_1 i_2 \ldots i_{k-1}]}\phi_{[i_1 i_2 \ldots i_{k-1}]} + \frac{1}{k}J_{[i_1 i_2 \ldots i_k]}A_{[i_1 i_2 \ldots i_k]}, \tag{15}$$

where repeated indices are summed over. The energy densities

$$L_E = \frac{1}{2k}E_{[i_1 i_2 \ldots i_k]}\epsilon_{([i_1 i_2 \ldots i_k][i'_1 i'_2 \ldots i'_k])}E_{[i'_1 i'_2 \ldots i'_k]} \tag{16}$$

and

$$L_B = \frac{1}{2(k+1)}B_{[i_1 i_2 \ldots i_{k+1}]}\kappa_{([i_1 i_2 \ldots i_{k+1}][i'_1 i'_2 \ldots i'_{k+1}])}B_{[i'_1 i'_2 \ldots i'_{k+1}]}, \tag{17}$$

have suitably defined positive definite dielectric, ϵ, and inverse permeability, κ, tensors which are both symmetric under the swapping of primed and unprimed indices (with their internal order kept fixed) – this is the meaning of their subscripts $([i\ldots][i'\ldots])$. Again, the electric and magnetic fields are gauge invariant under

$$\begin{aligned}\phi_{[i_1 i_2 \ldots i_{k-1}]} &= \phi_{[i_1 i_2 \ldots i_{k-1}]} + \partial_t \psi_{[i_1 i_2 \ldots i_{k-1}]}\\ A_{[i_1 i_2 \ldots i_k]} &= A_{[i_1 i_2 \ldots i_k]} - \frac{1}{(k-1)!}\sum_{P_k}(-1)^{P_k}\partial_{i_{P_k(1)}}\psi_{[i_{P_k(2)}\ldots i_{P_k(k)}]}\end{aligned} \tag{18}$$

where $\psi_{[i_1 \ldots i_{k-1}]}(x,t)$ denotes the field that characterizes the gauge transformation. Further, gauge invariance leads to (summing repeated indices):

$$\partial_t \rho_{[i_1 i_2 \ldots i_{k-1}]} + \partial_{i_0}J_{[i_0 i_1 \ldots i_{k-1}]} = 0. \tag{19}$$

Physically, this implies that the rate of change of the "density of the extended charge" is the divergence of its current. Note that this does not yet have any fractonic character; the extended object has no mobility constraints within this theory. In our nomenclature, this is a $(k,1)$-fracton theory. We now turn to the question how to endow such extended objects with an order-n fracton constraint. In any spatial dimension d, we can let $1 \le k \le (d-1)$.

V. (k,n)-FRACTON THEORY

The desideratum of endowing charges with a $(k-1)$-dimensional structure with an order-n fractonic character is achieved by constructing a suitable gauge structure. We here develop a natural extension of $U(1)$ Maxwell electrodynamics towards this end. The gauge field for (k,n)-fracton theory that we propose is of the form

$$(\phi_{([I^1_{k-1}][I^2_{k-1}]\ldots[I^n_{k-1}])}, A_{([I^1_k][I^2_k]\ldots[I^n_k])}). \tag{20}$$

A quantity $T_{([I^1_k][I^2_k]\ldots[I^n_k])}$, called a (k,n)-tensor, is fully anti-symmetric under the action of P_k on any I^r_k, i.e.,

$$T_{([I^1_k][I^2_k]\ldots[P_k I^r_k]\ldots[I^n_k])} = (-1)^{P_k}T_{([I^1_k][I^2_k]\ldots[I^r_k]\ldots[I^n_k])}. \tag{21}$$

Further, the meaning of the subscript $([I^1_k]\ldots[I^n_k])$ i.e., $[I^r_k]$s enclosed in () is that it is symmetric under the exchange of any two Is. In other words,

$$T_{([I^{P_n(1)}_k][I^{P_n(2)}_k]\ldots[I^{P_n(n)}_k])} = T_{([I^1_k][I^2_k]\ldots[I^n_k])} \tag{22}$$

for any permutation P_n of numbers $1\ldots n$. Note that the notation *does not* imply any symmetry in interchanging i^r_l with $i^{r'}_{l'}$, i. e., two particular indices of $[I^r_k]$ and $[I^{r'}_k]$ $(r \neq r')$. In other words, the gauge fields that we posit are "symmetric tensors of anti-symmetric tensors (forms)".

We now define fields in a natural fashion as

$$\begin{aligned}E_{([I^1_k][I^2_k]\ldots[I^n_k])} =& -\frac{1}{[(k-1)!]^n}\sum_{\{P^r_k\}}\left(\prod_{r=1}^{n}(-1)^{P^r_k}\partial_{i^r_{P^r_k(1)}}\right)\phi_{([P^1_k\tilde{I}^1_k]\ldots[P^n_k\tilde{I}^n_k])}\\ &-\partial_t A_{([I^1_k][I^2_k]\ldots[I^n_k]])}\end{aligned} \tag{23}$$

for the electric field. We have used the notation that

$$P^r_k I^r_k \equiv i^r_{P^r_k(1)}\underbrace{i^r_{P^r_k(2)}\ldots i^r_{P^r_k(k)}}_{\equiv P^r_k \tilde{I}^r_k} \tag{24}$$

where $\tilde{I}^r_k \equiv i^r_2 \ldots i^r_k$, i. e., a composite index with $k-1$ members. The magnetic field is

$$B_{([I^1_{k+1}]\ldots[I^n_{k+1}])} = \frac{1}{[k!]^n}\sum_{\{P^r_{k+1}\}}\left(\prod_{r=1}^{n}(-1)^{P^r_{k+1}}\partial_{i^r_{P^r_{k+1}(1)}}\right)A_{([P^1_{k+1}\tilde{I}^1_{k+1}]\ldots[P^n_{k+1}\tilde{I}^n_{k+1}])}. \tag{25}$$

The theory has the appealing feature that all fields are (k,n)-tensor type entities; ϕ is $(k-1,n)$-tensor, A, E are (k,n)-tensors and B, a $(k+1,n)$-tensor. A more general definition of a magnetic field with a more intricate structure is possible in any dimension d and leads to the $(k,n)_z$ theory as discussed in Appendix A.

The charges of this theory are $\rho_{([I^1_{k-1}]\ldots[I^n_{k-1}])}$ and the currents are defined as $J_{([I^1_k]\ldots[I^n_k]])}$ with similar structure as ϕ and A respectively. The Lagrangian density is

$$\begin{aligned}L =& L_E - L_B + (-1)^n \rho_{([I^1_{k-1}]\ldots[I^n_{k-1}])}\phi_{([I^1_{k-1}]\ldots[I^n_{k-1}])}\\ &+\frac{1}{k^n}J_{([I^1_k]\ldots[I^n_k])}A_{([I^1_k]\ldots[I^n_k])}\end{aligned} \tag{26}$$

with

$$L_E = \frac{1}{2k^n} E_\alpha \epsilon_{(\alpha\beta)} E_\beta, \quad L_B = \frac{1}{2(k+1)^n} B_\gamma \kappa_{(\gamma\delta)} B_\delta \tag{27}$$

where ϵ and κ are, again, suitable positive definite dielectric and permeability tensors, α, β are indices of the type $([I^1_k]\dots[I^n_k])$ and γ, δ are indices of the type $([I^1_{k+1}]\dots[I^n_{k+1}])$. The gauge transformation, described by $\psi_{([I^1_{k-1}][I^2_{k-1}]\dots[I^n_{k-1}])}$,

$$\begin{aligned}\phi_{([I^1_{k-1}][I^2_{k-1}]\dots[I^n_{k-1}])} &\to \phi_{([I^1_{k-1}][I^2_{k-1}]\dots[I^n_{k-1}])} + \partial_t \psi_{([I^1_{k-1}][I^2_{k-1}]\dots[I^n_{k-1}])} \\ A_{([I^1_k][I^2_k]\dots[I^n_k])} &\to A_{([I^1_k][I^2_k]\dots[I^n_k])} \\ &- \frac{1}{[(k-1)!]^n} \sum_{\{P^r_k\}} \left(\prod_{r=1}^{n} (-1)^{P^r_k} \partial_{i^r_{P^r_k(1)}} \right) \psi_{([P^1_k \tilde{I}^1_k]\dots[P^n_k \tilde{I}^n_k])},\end{aligned} \tag{28}$$

leaves the electric (eqn. (23)) and magnetic (eqn. (25)) fields unchanged. An illustration of this construction for the $(2,2)$-fracton theory is provided in Appendix B.

From eqn. (26), we obtain the equations of motion

$$\begin{aligned}\partial_{i^1_0}\partial_{i^2_0}\dots\partial_{i^n_0}(\epsilon E)_{([I^1_k][I^2_k]\dots[I^n_k])} &= \rho_{([I^1_{k-1}]\dots[I^n_{k-1}])}, \\ (-1)^n \partial_{i^1_0}\partial_{i^2_0}\dots\partial_{i^n_0}(\kappa B)_{([I^1_{k+1}][I^2_{k+1}]\dots[I^n_{k+1}])} & \\ = \partial_t(\epsilon E)_{([I^1_k][I^2_k]\dots[I^n_k])} &+ \frac{1}{k^n} J_{([I^1_k][I^2_k]\dots[I^n_k])}\end{aligned} \tag{29}$$

where $I^r_k \equiv i^r_0 i^r_1 \dots i^r_{k-1}$; also $i^r_1 \dots i^r_{k-1}$ is identified with I^r_{k-1} (similar convention is used in the second equation of eqn. (29), and in eqn. (31) below). We further have, from eqn. (23) and eqn. (25), that

$$\begin{aligned}\frac{1}{[k!]^n} \sum_{\{P^r_{k+1}\}} \left(\prod_{r=1}^{n} (-1)^{P^r_{k+1}} \partial_{i^r_{P^r_{k+1}(1)}} \right) E_{([P^1_{k+1}\tilde{I}^1_{k+1}]\dots[P^n_{k+1}\tilde{I}^n_{k+1}])} & \\ + \partial_t B_{([I^1_{k+1}]\dots[I^n_{k+1}])} &= 0, \quad (30) \\ \sum_{\{P^r_{k+2}\}} \left(\prod_{r=1}^{n} (-1)^{P^r_{k+2}} \partial_{i^r_{P^r_{k+2}(1)}} \right) B_{([P^1_{k+2}\tilde{I}^1_{k+2}]\dots[P^n_{k+2}\tilde{I}^n_{k+2}])} &= 0.\end{aligned}$$

The relations eqn. (29) and eqn. (30) are the Maxwell equations of our generalized (k,n)-fracton theory. Gauge invariance of the action gives

$$\partial_t \rho_{([I^1_{k-1}]\dots[I^n_{k-1}])} + \partial_{i^1_0}\partial_{i^2_0}\dots\partial_{i^n_0} J_{([I^1_k][I^2_k]\dots[I^n_k])} = 0. \tag{31}$$

This achieves the description of the (k,n)-fracton density via a tensor $\rho_{([I^1_{k-1}]\dots[I^n_{k-1}])}$. These "extended charges" are subject to the order-n fracton condition, i. e., relations similar to eqn. (12) imply mobility constraints on multipoles up to order n of such charges. Finally, we find that there are gapless "photonic" excitations in the source free case, which disperse as $\omega \sim |q|^n$.

The theory presented above is a special case of the more general $(k,n)_z$ theory (Appendix A) with gapless spectrum $\omega \sim |q|^z$ when z is set to n. Known examples(see, for example,[25,26]) of fractonic gauge theories possess $z = 1$, i. e., linearly dispersing gapless excitations. The nonlinear dispersions of the gapless modes of the more general theory can be traced directly to the definition of the magnetic field eqn. (25) (or, more generally, eqn. (A.2) with $z > 1$). It will be particularly interesting to explore physical systems where constraints or symmetries force such a "higher dervative" definition of the magnetic field to be the solely allowed term.

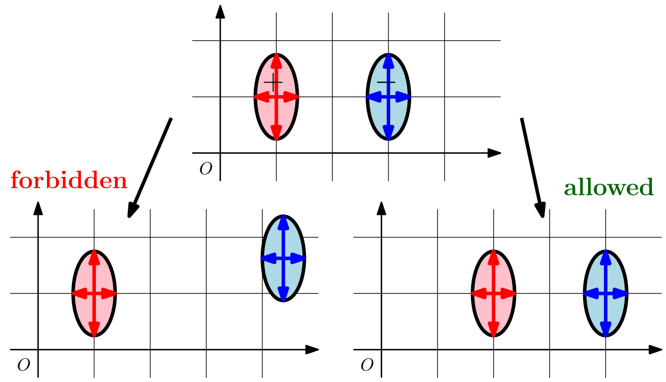

FIG. 1. Illustration of tensor charge mobility constraint of (2,2)-fracton theory (see Appendix B). Charges are 2-nd rank symmetric tensors, represented by an ellipse indicating principal axes. On the top panel the total charge vanishes (red is positive charge, blue is negative charge), but the system has a net dipole moment. A rearrangement of the charges which changes the dipole moment is forbidden (bottom left), while a dipole-preserving "rigid" translations of both charges is allowed (bottom right).

An interesting aspect of our construction is the nature of the (k,n)-fractonic charges (which does not depend on z). The character can be better understood by considering the $(2,2)$-fracton example. Here the charges are of the type $\rho_{(ij)}$, i. e., a symmetric tensor of rank 2. In this theory, individual charges are immobile, but two charges with opposite charge tensors separated in space can move *together* as they preserve the net dipole moment. This is pictorially illustrated in fig. 1.

VI. SUMMARY AND OUTLOOK

In this paper, we have aimed to develop a unified framework that describes extended charges/sources ($k-1$-dimensional) with mobility constraints of order n. We have concluded that a gauge theory with gauge fields that are "symmetric tensors of anti-symmetric tensors" provides the desired framework and also fixes the structure of the corresponding (k,n)-fracton density, and paves way to study theories with different dynamic exponents z.

Our theory, building on[20,21,25], can be viewed as a natural generalization of $U(1)$ Maxwell electromagnetism with freely mobile charges (the $(1,1)$-fracton theory) and constructing a sequence of theories with (k,n)-fractons as sources applicable in arbitrary spatial dimension d and with an appropriate dynamic exponent z. A number of interesting questions follow immediately from these considerations.

First, our formulation may not be a unique generalization of Maxwell electromagnetism, and other routes – e.g. using different forms of Gauss law – leading to yet different kinds of fracton excitations may be worth exploring.

Second, we have not mandated relativistic invariance for our construction. This is rather natural in condensed matter settings, where such an invariance can effectively emerge, but generically does not, see e.g.[30–32]. Nonetheless, identifying possible relativistically covariant generalised fracton theories is surely a worthwhile aim.

Third, a quantum mechanical extension of generalised fracton theories could unearth their novel properties,e.g. concerning nature, quantum numbers and statistics of their excitations. This would be embedded in the broader quest for understanding of the topological properties of quantum locally constrained models such as quantum dimer models[33].

Fourth, what are some microscopic models having generalised fracton theories as effective low-energy description? Particularly noteworthy here is the realization that the $(1,2)_1$ symmetric tensor theory[26] in $d = 2$ is dual to elasticity theory (see also[34,35]). The $(2,2)_1$-fracton theory has also been explored in the context of fractonic dislocation lines[25] in $d = 3$. Following this, it will be interesting to explore theories dual to (k,n)-fracton theories to look for possible physical realizations. The freedom of choosing a desired z for the (k,n)-theory can help broaden the class of such dualities.

Fifth, a related direction is the exploration of the phases of the (k,n)-fracton theory and/or their microscopic models of origin. A case in point is, again, the $(1,2)_1$-theory[36,37] in $d = 2$, which has provided a fresh perspective into defect driven phase transitions. Higher n ($n \geq 3$) in turn hold interesting physics in higher dimensions. This owes to the fact that the electric field, for example an isolated $(1,n)$-fracton at the origin, goes as $E(x) \sim |x|^{n-d}$ (to be interpreted as $\log|x|$ when $n = d$, and constant when $n = d = 1$) making them "weakly confined" (see[21] for a discussion) whenever $n < d$; $n = 3, d = 4$ will be an interesting case to explore.

Finally, it is interesting explore the connection of (k,n)-fracton theory to some of the discrete fracton phases (as reviewed in the introductory section) with subextensive ground-state degeneracies (see, for example,[38]), and possible generalization of lattice gerbe theories[39–42] to lattice tensor-gerbe theories with fractonic physics (see very recent work[43]). A Chern-Simons/BF formulation[19] of (k,n)-fracton theories could also prove fruitful in obtaining field theories of generalized discrete fracton models. Another attractive research direction would be to explore the structure of (k,n)-fracton theories to include compact and non-Abelian gauge structures as in the recent concurrent work[44].

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1 X.-G. Wen, Rev. Mod. Phys. **89**, 041004 (2017).
2 A. Kitaev, AIP Conference Proceedings **1134**, 22 (2009).
3 S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics **12**, 065010 (2010).
4 T. Senthil, Annual Review of Condensed Matter Physics **6**, 299 (2015).
5 A. Kitaev, Annals of Physics **303**, 2 (2003).
6 C. Chamon, Phys. Rev. Lett. **94**, 040402 (2005).
7 S. Bravyi, B. Leemhuis, and B. M. Terhal, Annals of Physics **326**, 839 (2011).
8 C. Castelnovo and C. Chamon, Philosophical Magazine **92**, 304 (2012).
9 J. Haah, Phys. Rev. A **83**, 042330 (2011).
10 B. Yoshida, Phys. Rev. B **88**, 125122 (2013).
11 S. Bravyi and J. Haah, Phys. Rev. Lett. **111**, 200501 (2013).
12 S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **92**, 235136 (2015).
13 S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **94**, 235157 (2016).
14 D. J. Williamson, Phys. Rev. B **94**, 155128 (2016).
15 T. H. Hsieh and G. B. Halász, Phys. Rev. B **96**, 165105 (2017).
16 R. M. Nandkishore and M. Hermele, Annual Review of Condensed Matter Physics **10**, 295 (2019).
17 K. Slagle and Y. B. Kim, Phys. Rev. B **97**, 165106 (2018).
18 K. Slagle and Y. B. Kim, Phys. Rev. B **96**, 195139 (2017).
19 Y. You, T. Devakul, S. L. Sondhi, and F. J. Burnell, arXiv e-prints , arXiv:1904.11530 (2019), arXiv:1904.11530 [cond-mat.str-el].
20 A. Rasmussen, Y.-Z. You, and C. Xu, arXiv e-prints , arXiv:1601.08235 (2016), arXiv:1601.08235 [cond-mat.str-el].
21 M. Pretko, Phys. Rev. B **95**, 115139 (2017).
22 M. Pretko, Phys. Rev. B **96**, 035119 (2017).
23 M. Pretko, Phys. Rev. B **96**, 125151 (2017).
24 M. Pretko, Phys. Rev. D **96**, 024051 (2017).
25 S. Pai and M. Pretko, Phys. Rev. B **97**, 235102 (2018).
26 M. Pretko and L. Radzihovsky, Phys. Rev. Lett. **120**, 195301 (2018).
27 M. Kalb and P. Ramond, Phys. Rev. D **9**, 2273 (1974).
28 M. Henneaux and C. Teitelboim, Foundations of Physics **16**, 593 (1986).
29 In a discrete lattice, these can be viewed as objects defined using links, plaquettes etc., emanating from a lattice point.[40].
30 R. Moessner, S. L. Sondhi, and E. Fradkin, Phys. Rev. B **65**, 024504 (2001).
31 C. Xu, Phys. Rev. B **74**, 224433 (2006).
32 O. Benton, L. D. C. Jaubert, H. Yan, and N. Shannon, Nature Communications **7**, 11572 (2016).
33 R. Moessner and S. L. Sondhi, Phys. Rev. Lett. **86**, 1881 (2001).
34 A. Gromov, ArXiv e-prints (2017), arXiv:1712.06600 [cond-mat.str-el].
35 A. Gromov and P. Surówka, arXiv e-prints , arXiv:1908.06984 (2019), arXiv:1908.06984 [cond-mat.str-el].
36 A. Kumar and A. C. Potter, Phys. Rev. B **100**, 045119 (2019).
37 M. Pretko, Z. Zhai, and L. Radzihovsky, arXiv e-prints , arXiv:1907.12577 (2019), arXiv:1907.12577 [cond-mat.str-el].
38 H. Ma, M. Hermele, and X. Chen, Phys. Rev. B **98**, 035111 (2018).
39 F. J. Wegner, Journal of Mathematical Physics **12**, 2259 (1971).
40 R. Savit, Rev. Mod. Phys. **52**, 453 (1980).
41 A. E. Lipstein and R. A. Reid-Edwards, Journal of High Energy Physics **2014**, 34 (2014).

[42] D. A. Johnston, Phys. Rev. D **90**, 107701 (2014).
[43] M.-Y. Li and P. Ye, arXiv e-prints , arXiv:1909.02814 (2019), arXiv:1909.02814 [cond-mat.str-el].
[44] J. Wang and K. Xu, arXiv e-prints , arXiv:1909.13879 (2019), arXiv:1909.13879 [hep-th].

Appendix A: $(k,n)_z$-fracton theory

In this section we present a more general version (valid in arbitrary spatial dimension d) of the (k,n)-theory, dubbed $(k,n)_z$ fracton theory, that obtains an arbitrary a dynamical exponent z ranging from 1 to n. Such a theory will have gapless modes with dispersion $\omega \sim |q|^z$. The theory presented in the main text is the $(k,n)_n$-fracton theory.

The gauge field for $(k,n)_z$-fracton theory is same as that presented in eqn. (20) of Section V and is reproduced here:

$$(\phi_{([I^1_{k-1}][I^2_{k-1}]\dots[I^n_{k-1}])}, A_{([I^1_k][I^2_k]\dots[I^n_k])}). \tag{20}$$

The electric field, is again, identical to eqn. (23):

$$E_{([I^1_k][I^2_k]\dots[I^n_k])} = -\frac{1}{[(k-1)!]^n}\sum_{\{P^r_k\}}\left(\prod_{r=1}^{n}(-1)^{P^r_k}\partial_{i^r_{P^r_k(1)}}\right)\phi_{([P^1_k\tilde{I}^1_k]\dots[P^n_k\tilde{I}^n_k])} - \partial_t A_{([I^1_k][I^2_k]\dots[I^n_k])}. \tag{23}$$

The key to obtaining $(k,n)_z$-theory is in the suitable definition the magnetic field. This requires an additional mathematical object beyond those introduced in the main text:

$$B_{([I^1_{k+1}][I^2_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k][I^{z+2}_k]\dots[I^n_k])} \tag{A.1}$$

is a tensor that has two sets of composite anti-symmetric indices; there are z composite indices $[I^1_{k+1}]\dots[I^z_{k+1}]$ which are can be exchanged in a symmetric fashion, while the $(n-z)$ composite indices $[I^{z+1}_k]\dots[I^n_k]$ are in turn symmetric among themselves – this is the meaning of the notation $([I^1_{k+1}][I^2_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k][I^{z+2}_k]\dots[I^n_k])$. The magnetic field is a tensor of this kind:

$$B_{([I^1_{k+1}][I^2_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k][I^{z+2}_k]\dots[I^n_k])} = \frac{1}{[k!]^z}\sum_{\{P^r_{k+1}\}}\left(\prod_{r=1}^{z}(-1)^{P^r_{k+1}}\partial_{i^r_{P^r_{k+1}(1)}}\right)A_{([P^1_{k+1}\tilde{I}^1_{k+1}]\dots[P^z_{k+1}\tilde{I}^z_{k+1}][I^{z+1}_k]\dots[I^n_k])}\,. \tag{A.2}$$

The charges and currents of this theory are, respectively, $\rho_{([I^1_{k-1}]\dots[I^n_{k-1}])}$ and $J_{([I^1_k]\dots[I^n_k]))}$ as in the main text. The Lagrangian density is also same as eqn. (26), with the key difference that the magnetic energy is

$$L_B = \frac{1}{2(k+1)^z}B_\gamma\kappa_{(\gamma\delta)}B_\delta \tag{A.3}$$

γ,δ are now indices of the type $([I^1_{k+1}][I^2_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k][I^{z+2}_k]\dots[I^n_k])$. The gauge transformation eqn. (28) leaves eqn. (A.2) also unchanged, along with eqn. (23).

The equations of motion of the $(k,n)_z$ theory are

$$\begin{aligned}\partial_{i^1_0}\partial_{i^2_0}\dots\partial_{i^n_0}(\epsilon E)_{([I^1_k][I^2_k]\dots[I^n_k])} &= \rho_{([I^1_{k-1}]\dots[I^n_{k-1}])},\\ (-1)^z\mathrm{Sym}_{(k,n)}\left[\partial_{i^1_0}\partial_{i^2_0}\dots\partial_{i^z_0}(\kappa B)_{([I^1_{k+1}][I^2_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k]\dots[I^n_k])}\right] &= \partial_t(\epsilon E)_{([I^1_k][I^2_k]\dots[I^n_k])} + \frac{1}{k^n}J_{([I^1_k][I^2_k]\dots[I^n_k])}\end{aligned} \tag{A.4}$$

where the $\mathrm{Sym}_{(k,n)}[\,]$ picks out the (k,n)-tensor part of its argument. Finally, from eqn. (23) and eqn. (A.2) we get

$$\begin{aligned}\frac{1}{[k!]^z}\sum_{\{P^r_{k+1}\}}\left(\prod_{r=1}^{z}(-1)^{P^r_{k+1}}\partial_{i^r_{P^r_{k+1}(1)}}\right)E_{([P^1_{k+1}\tilde{I}^1_{k+1}]\dots[P^n_{k+1}\tilde{I}^z_{k+1}][I^{z+1}_k]\dots[I^n_k])} + \partial_t B_{([I^1_{k+1}]\dots[I^z_{k+1}])|([I^{z+1}_k]\dots[I^n_k])} &= 0,\\ \sum_{\{P^r_{k+2}\}}\left(\prod_{r=1}^{z}(-1)^{P^r_{k+2}}\partial_{i^r_{P^r_{k+2}(1)}}\right)B_{([P^1_{k+2}\tilde{I}^1_{k+2}]\dots[P^n_{k+2}\tilde{I}^z_{k+2}])|([I^{z+1}_k]\dots[I^n_k])} &= 0.\end{aligned} \tag{A.5}$$

The Maxwell equations of the $(k,n)_z$-fracton theory are given by eqn. (A.4) and eqn. (A.5), and imply a gapless spectrum with dispersion $\omega \sim |q|^z$. Again, the theory presented in the main text is the special case $z = n$.

The $(1,2)_1$-fracton theory in $d = 2$ was shown to be dual to elasticity[26], and $(2,2)_1$-theory was shown to describe fractonic dislocation lines[25] in a $d = 3$ elastic medium.

Appendix B: (2,2)-fracton theory

We illustrate the results for the (2,2)-fracton (more precisely $(2,2)_2$-theory in the notation of the previous section) theory. Gauge fields are

$$(\phi_{(ij)}, A_{([ij][kl])}). \tag{B.1}$$

The electic field (eqn. (23))

$$E_{([ij][kl])} = -(\partial_i\partial_k\phi_{(jl)} - \partial_j\partial_k\phi_{(il)} - \partial_i\partial_l\phi_{(jk)} + \partial_j\partial_l\phi_{(ik)}) - \partial_t A_{([ij][kl])}. \tag{B.2}$$

The magnetic field (eqn. (25))

$$\begin{aligned} B_{([ijk][lmn])} = &\ \partial_i\partial_l A_{([jk][mn])} + \partial_j\partial_l A_{([ki][mn])} + \partial_k\partial_l A_{([ij][mn])} \\ &+ \partial_i\partial_m A_{([jk][nl])} + \partial_j\partial_m A_{([ki][nl])} + \partial_k\partial_m A_{([ij][nl])} \\ &+ \partial_i\partial_n A_{([jk][lm])} + \partial_j\partial_n A_{([ki][lm])} + \partial_k\partial_n A_{([ij][lm])}. \end{aligned} \tag{B.3}$$

The gauge transformation is

$$\begin{aligned} \phi_{(ij)} &\to \phi_{(ij)} + \partial_t\psi_{(ij)} \\ A_{([ij][kl])} &\to A_{([ij][kl])} \\ &\quad - \left(\partial_i\partial_k\psi_{(jl)} - \partial_j\partial_k\psi_{(il)} - \partial_i\partial_l\psi_{(jk)} + \partial_j\partial_l\psi_{(ik)}\right) \end{aligned} \tag{B.4}$$

Under the gauge transformation

$$\begin{aligned} E_{([ij][kl])} \to &E_{([ij][kl])} \\ &+ \Big(-\left[\partial_i\partial_k\partial_t\psi_{(jl)} - \partial_j\partial_k\partial_t\psi_{(il)} - \partial_i\partial_l\partial_t\psi_{(jk)} + \partial_j\partial_l\partial_t\psi_{(ik)}\right] \\ &+\partial_t\left[\partial_i\partial_k\psi_{(jl)} - \partial_j\partial_k\psi_{(il)} - \partial_i\partial_l\psi_{(jk)} + \partial_j\partial_l\psi_{(ik)}\right]\Big) \\ B_{([ijk][lmn])} \to &B_{([ijk][lmn])} \\ &-\Big(\partial_i\partial_l\left[\partial_j\partial_m\psi_{(kn)} - \partial_k\partial_m\psi_{(jn)} - \partial_j\partial_n\psi_{(km)} + \partial_k\partial_n\psi_{(jm)}\right] \\ &+\partial_j\partial_l\left[\partial_k\partial_m\psi_{(in)} - \partial_i\partial_m\psi_{(kn)} - \partial_k\partial_n\psi_{(im)} + \partial_i\partial_n\psi_{(km)}\right] \\ &+\partial_k\partial_l\left[\partial_i\partial_m\psi_{(jn)} - \partial_j\partial_m\psi_{(in)} - \partial_i\partial_n\psi_{(jm)} + \partial_j\partial_n\psi_{(im)}\right] \\ &+\partial_i\partial_m\left[\partial_j\partial_n\psi_{(kl)} - \partial_k\partial_n\psi_{(jl)} - \partial_j\partial_l\psi_{(kn)} + \partial_k\partial_l\psi_{(jn)}\right] \\ &+\partial_j\partial_m\left[\partial_k\partial_n\psi_{(il)} - \partial_i\partial_n\psi_{(kl)} - \partial_k\partial_l\psi_{(in)} + \partial_i\partial_l\psi_{(kn)}\right] \\ &+\partial_k\partial_m\left[\partial_i\partial_n\psi_{(jl)} - \partial_j\partial_n\psi_{(il)} - \partial_i\partial_l\psi_{(jn)} + \partial_j\partial_l\psi_{(in)}\right] \\ &+\partial_i\partial_n\left[\partial_j\partial_l\psi_{(km)} - \partial_k\partial_l\psi_{(jm)} - \partial_j\partial_m\psi_{(kl)} + \partial_k\partial_m\psi_{(jl)}\right] \\ &+\partial_j\partial_n\left[\partial_k\partial_l\psi_{(im)} - \partial_i\partial_l\psi_{(km)} - \partial_k\partial_m\psi_{(il)} + \partial_i\partial_m\psi_{(kl)}\right] \\ &+\ \partial_k\partial_n\left[\partial_i\partial_l\psi_{(jm)} - \partial_j\partial_l\psi_{(im)} - \partial_i\partial_m\psi_{(jl)} + \partial_j\partial_m\psi_{(il)}\right]\Big) \end{aligned} \tag{B.5}$$

where the terms in the () vanish ensuring gauge invariant electric and magnetic fields. Gauge invariance of the action will enforce

$$\partial_t\rho_{(ik)} + \partial_j\partial_l J_{([ji][lk])} = 0. \tag{B.6}$$

A similar theory was outlined in[25] for $d = 3$. Our description is applicable in all spatial dimensions $d \geq 3$.