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Fundamental limits to radiative heat transfer: theory

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Radiative heat transfer between bodies at the nanoscale can surpass blackbody limits on thermal radiation by orders of magnitude due to contributions from evanescent electromagnetic fields, which carry no energy to the far-field. Thus far, principles guiding explorations of larger heat transfer beyond planar structures have assumed utility in surface nanostructuring, via enhancement of the density of states, and the possibility that such design paradigms can approach Landauer limits, in analogy to conduction. Here, we derive fundamental shape-independent limits to radiative heat transfer, applicable in near- through far-field regimes, that incorporate material and geometric constraints such as intrinsic dissipation and finite object sizes, and show that these preclude reaching the Landauer limits in all but a few restrictive scenarios. Additionally, we show that the interplay of material response and electromagnetic scattering among proximate bodies means that bodies which maximize radiative heat transfer actually maximize *scattering* rather than absorption. Finally, we compare our new bounds to Landauer limits as well as limits that ignore the interplay between material and geometric constraints, and show that these prior limits lead to overly optimistic predictions. Our results have ramifications for the ultimate performance of thermophotovoltaics and nanoscale cooling, as well as incandescent and luminescent devices.

The concept of a blackbody, derived from electromagnetic reciprocity (or detailed balance), has provided a benchmark of the largest emission rates that can be achieved by a heated macroscopic object: through nanoscale texturing, gray objects can be designed in myriad ways to mimic the response of a blackbody at selective wavelengths,[1–6] with implications for a variety of technologies, including high-efficiency solar cells, selective emitters, and thermal sensors.[7] Over the past few decades, motivated by potential applications to thermophotovoltaics,[8–11] nanoscale cooling,[12] and thermal microscopy,[13,14] much effort has gone toward understanding analogous limits to enhancements of near-field radiative heat transfer (RHT),[15–18] supported by a rich and growing number of experimental[6,19–21] and theoretical[22–26] investigations. A key principle underlying further near-field RHT enhancements is the use of materials supporting bound (plasmon and phonon) polaritons in the infrared, where the Planck distribution peaks at typical temperatures probed in experiments. This leads to strong subwavelength responses tied to corresponding enhancements in the density of states;[4,27–29] consequently, the amplified near-field RHT spectrum exhibits a narrow lineshape, justifying focus on selective wavelengths. However, while the properties of such polaritons, particularly their resonance frequencies, associated densities of states, and scattering characteristics can be modified through nanoscale texturing, only recently have computational methods[23–25,30–32] arisen to model RHT between bodies of arbitrary shapes beyond high-symmetry cases.[26,33–35] Furthermore, the challenge of gaining simultaneous control over the scattering properties of large numbers of contributing surface waves has generally precluded general upper bounds on RHT.

RHT between two bodies A and B in vacuum is given as

$$P = \int_0^\infty |\Pi(\omega, T_\mathrm{B}) - \Pi(\omega, T_\mathrm{A})|\Phi(\omega)\,\mathrm{d}\omega, \qquad (1)$$

in terms of the Planck function $\Pi(\omega, T) = \hbar\omega/[\exp(\hbar\omega/(k_\mathrm{B}T)) - 1]$ evaluated at the local temperatures T_A and T_B, and the spectral function $\Phi(\omega)$, which can be enhanced by changing material and geometric properties through the creation of resonances and changes in the electromagnetic density of states. In particular, nanostructuring metallic surfaces or polar dielectrics makes it possible to tailor resonances in the infrared, such that peaks of the spectrum Φ may coincide with the peak of the Planck distribution near room temperature. It remains an open question, however, to what extent the peak value of Φ itself at any given frequency ω may be enhanced through appropriate geometric and material choices, as well as what such optimal structures should be.

Previous attempts at deriving bounds on RHT have primarily focused on extended media,[15–17,36] showing that at least for translationally invariant structures, Φ can be expressed as the trace of a "transmission" matrix whose singular values (corresponding to evanescent Fourier modes) each contribute a finite flux, bounded above by a Landauer limit in analogy with conduction.[37,38] Aside from being restricted to planar geometries, these bounds turn out to be either pessimistic,[17] ignoring the large densities of states that can arise in nanostructured and low-loss materials, or too optimistic,[15,16] ignoring any constraints imposed by Maxwell's equations and assuming instead that all such Fourier modes, up to an unrealistic cutoff on the order of the atomic scale, can saturate the flux.[15] From a design perspective, Landauer limits present a hurdle as they rely on ad-hoc estimates of the number and relative contribution of radiative modes/channels, which depend on specific material and geometric features. More recent works have derived complementary material limits on electromagnetic absorption in subwavelength regimes,[39] showing that absorbed power in a medium of susceptibility χ increases in proportion to an "inverse resistivity" material figure of merit, $|\chi|^2/\operatorname{Im}\chi$, in principle diverging with increasing indices of refraction and decreasing dissipation. Saturation of these bounds for a subwavelength absorber in the quasistatic regime can generally be achieved through the strong polarization currents arising in resonant media supporting surface plasmon or phonon polaritons. These arguments have been extended to near-field RHT[18] by exploiting energy conservation and reciprocity, finding the upper bound of Φ at a polariton resonance

to scale quadratically with $|\chi|^2/\operatorname{Im}\chi$. While near-field RHT between dipolar objects can attain these bounds in a dilute limit, such a universal scaling has yet to be observed in large-area structures. This naïvely suggests room for improvement in Φ through nanostructuring via enhancements in the density of states or equivalently, via saturation of modal contributions, yet trial-and-error explorations and optimization procedures[40,41] have failed to produce nanostructured geometries that bridge this gap, leading to the alternative possibility that existing bounds are too loose.

In this paper, we derive new algebraic bounds on RHT, valid in the near-, mid-, and far-field regimes, through analysis of the singular value decompositions of relevant response quantities. In contrast to prior limits, our bounds incorporate the interplay of constraints imposed by material losses and geometric radiative effects between bodies, and are therefore tighter. In particular, every channel of energy transmission is shown to be generally prohibited from saturating its Landauer limit, in contrast to predictions based on modal decompositions[15–17,36] that neglect material properties and are most applicable in the ray optics regime. Furthermore, the growth of RHT with decreasing material dissipation is shown to be strongly limited by radiative losses, in contrast to predictions based on energy-conservation limits to material response[18] that neglect finite-size scattering effects between bodies and are thus tightest in the quasistatic regime. In upcoming papers closely related to this one, we apply these bounds to various scenarios of interest, providing predictions of the maximum RHT achievable in compact and extended geometries,[42] and deriving related bounds on far-field thermal emission from single bodies in isolation.[43]

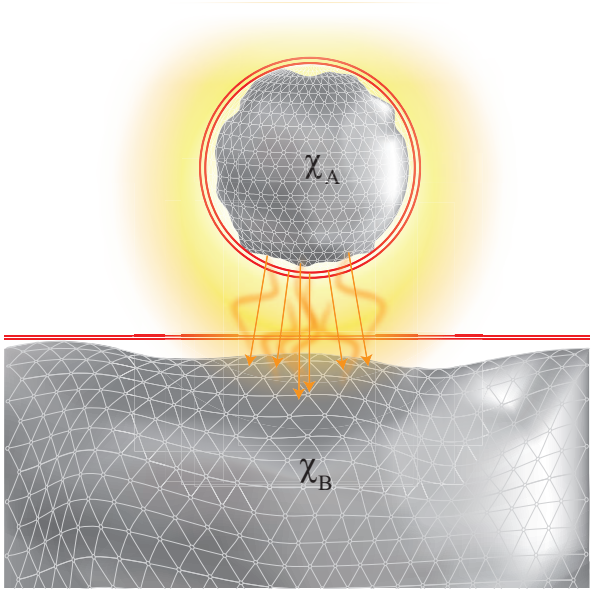

Figure 1. Two bodies labeled A and B exchange heat in vacuum. Each body could be compact or of infinite extent in at least one spatial dimension, and for given susceptibilities χ_p, the optimal structures may be quite complicated, but the upper bounds, which depend on $\zeta_p = |\chi_p|^2/\operatorname{Im}(\chi_p)$, can be evaluated in simpler bounding domains that enclose each object while respecting any other design constraints present.

I. HEAT TRANSFER DEFINITIONS

For two bodies A and B in vacuum [Fig. 1], the spectral function Φ appearing in (1) represents the average power absorbed in B due to fluctuating current sources in A , depicted in Fig. 1, and is reciprocal (invariant under interchange of A and B). Using operator notation, this average absorbed power can be written in terms of the susceptibilities $\mathbb{V}_p$, the vacuum Maxwell Green's function $\mathbb{G}^{\text{vac}}_{pq}$, and scattering T-operators $\mathbb{T}_p$, for $p, q \in \{\text{A}, \text{B}\}$. For local homogeneous isotropic media, each susceptibility is written as $\mathbb{V}_p = \chi_p \mathbb{I}_p$, where $\mathbb{I}_p$ is the projection onto the space of body p. The vacuum Maxwell Green's function $\mathbb{G}^{\text{vac}}$ solves $[(c/\omega)^2\nabla\times(\nabla\times) - \mathbb{I}]\mathbb{G}^{\text{vac}} = \mathbb{I}$ in all space, and its blocks are denoted as $\mathbb{G}^{\text{vac}}_{pq}$ for sources in body q propagating fields to body p. Finally, the T-operators $\mathbb{T}_p = (\mathbb{V}_p^{-1} - \mathbb{G}^{\text{vac}}_{pp})^{-1}$ represent the total induced polarization moment in body p due to a localized field of unit magnitude incident upon it. All of these quantities are reciprocal, so they are equal to their unconjugated transposes in position space: $\mathbb{V}_p = \mathbb{V}_p^{\top}$, $\mathbb{T}_p = \mathbb{T}_p^{\top}$, and $\mathbb{G}^{\text{vac}}_{pq} = (\mathbb{G}^{\text{vac}}_{qp})^{\top}$. This means that Hermitian conjugation is equivalent to complex conjugation: $\mathbb{V}_p^{\dagger} = \mathbb{V}_p^{\star}$, $\mathbb{T}_p^{\dagger} = \mathbb{T}_p^{\star}$, and $(\mathbb{G}^{\text{vac}}_{pq})^{\dagger} = \mathbb{G}^{\text{vac}\star}_{qp}$. Additionally, we may define these operators in the combined space of the two bodies in 2×2 block matrix form as

$$\mathbb{G}^{\text{vac}} = \begin{bmatrix} \mathbb{G}^{\text{vac}}_{\text{AA}} & \mathbb{G}^{\text{vac}}_{\text{AB}} \\ \mathbb{G}^{\text{vac}}_{\text{BA}} & \mathbb{G}^{\text{vac}}_{\text{BB}} \end{bmatrix} \tag{2}$$

$$\mathbb{V}^{-1} = \begin{bmatrix} \mathbb{V}_{\text{A}}^{-1} & 0 \\ 0 & \mathbb{V}_{\text{B}}^{-1} \end{bmatrix} \tag{3}$$

from which the overall T-operator is defined as

$$\mathbb{T}^{-1} = \begin{bmatrix} \mathbb{T}_{\text{A}}^{-1} & -\mathbb{G}^{\text{vac}}_{\text{AB}} \\ -\mathbb{G}^{\text{vac}}_{\text{BA}} & \mathbb{T}_{\text{B}}^{-1} \end{bmatrix} \tag{4}$$

in terms of the individual T-operators. Finally, we note that all of these quantities depend on frequency ω, though this will be suppressed for the sake of notational brevity.

Given these definitions and relations (see Appendix C for details), the RHT spectrum can be written as[26]

$$\begin{aligned}\Phi = \frac{2}{\pi}\operatorname{Tr}[&\mathbb{T}_{\text{B}}^{\star}(\mathbb{I}_{\text{B}} - \mathbb{G}^{\text{vac}\star}_{\text{BA}}\mathbb{T}_{\text{A}}^{\star}\mathbb{G}^{\text{vac}\star}_{\text{AB}}\mathbb{T}_{\text{B}}^{\star})^{-1}\operatorname{Im}(\mathbb{V}_{\text{B}}^{-1\star})\times \\ &\mathbb{T}_{\text{B}}(\mathbb{I}_{\text{B}} - \mathbb{G}^{\text{vac}}_{\text{BA}}\mathbb{T}_{\text{A}}\mathbb{G}^{\text{vac}}_{\text{AB}}\mathbb{T}_{\text{B}})^{-1}\times \\ &\mathbb{G}^{\text{vac}}_{\text{BA}}\mathbb{T}_{\text{A}}\operatorname{Im}(\mathbb{V}_{\text{A}}^{-1\star})\mathbb{T}_{\text{A}}^{\star}\mathbb{G}^{\text{vac}\star}_{\text{AB}}],\end{aligned} \tag{5}$$

where $\operatorname{Im}(\mathbb{A}) = (\mathbb{A} - \mathbb{A}^{\star})/(2\,\text{i})$ and $\operatorname{Asym}(\mathbb{A}) = (\mathbb{A} - \mathbb{A}^{\dagger})/(2\,\text{i})$ for any operator $\mathbb{A}$; if $\mathbb{A}$ is reciprocal, then $\operatorname{Asym}(\mathbb{A}) = \operatorname{Im}(\mathbb{A})$. This expression is manifestly reciprocal in A and B, and treats the T-operators of A and B on an equal footing, linked only by the Green's function $\mathbb{G}^{\text{vac}}_{\text{BA}}$ propagating fields in vacuum from one body to the other. However, it is possible to write this spectrum more suggestively in terms

of operator combinations that hide this reciprocity in order to more strongly link this expression to absorbed and emitted powers. In particular, Φ may be rewritten as

$$\Phi = \frac{2}{\pi}\,\mathrm{Tr}[\mathbb{Y}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})\mathbb{Y}_\mathrm{B}\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\,\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star})\mathbb{T}_\mathrm{A}^\star\mathbb{G}_\mathrm{AB}^{\mathrm{vac}\star}], \tag{6}$$

in terms of the reciprocal operator $\mathbb{Y}_\mathrm{B} = \mathbb{T}_\mathrm{B}\mathbb{S}_\mathrm{B}^\mathrm{A}$, which is in turn written in terms of the scattering operator $\mathbb{S}_\mathrm{B}^\mathrm{A} = (\mathbb{I}_\mathrm{B} - \mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{G}_\mathrm{AB}^\mathrm{vac}\mathbb{T}_\mathrm{B})^{-1}$. Essentially, $\mathbb{Y}_\mathrm{B}$ is a new "dressed T-operator" describing absorption and scattering in B in the presence of A, just as the bare T-operators $\mathbb{T}_p$ describe absorption and scattering from each body $p \in \{\mathrm{A},\mathrm{B}\}$ in isolation.

The assumption that the susceptibility in each body $p \in \{\mathrm{A},\mathrm{B}\}$ is homogeneous, uniform, and isotropic, yields the identity $\mathrm{Im}(\mathbb{V}_p^{-1\star}) = \frac{\mathrm{Im}(\chi_p)}{|\chi_p|^2}\mathbb{I}_p$. For convenience, we denote $\zeta_p = \frac{|\chi_p|^2}{\mathrm{Im}(\chi_p)}$ as the "material response factor". Using this, we write the RHT spectrum as

$$\Phi = \frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\,\|\mathbb{Y}_\mathrm{B}\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\|_\mathrm{F}^2 \tag{7}$$

where $\|\mathbb{A}\|_\mathrm{F}^2 = \mathrm{Tr}(\mathbb{A}^\dagger\mathbb{A})$ denotes the Frobenius norm for any operator $\mathbb{A}$.

As we show in Appendix C, the RHT spectrum may alternatively be written as

$$\Phi = \frac{2}{\pi}\,\|\mathbb{Q}\|_\mathrm{F}^2 \tag{8}$$

in terms of the transmission operator $\mathbb{Q} = \mathrm{Im}(\mathbb{V}_\mathrm{B})^{1/2}\mathbb{G}_\mathrm{BA}\,\mathrm{Im}(\mathbb{V}_\mathrm{A})^{1/2}$, which in turn depends on the total Green's function $\mathbb{G}_\mathrm{BA} = \mathbb{V}_\mathrm{B}^{-1}\mathbb{T}_\mathrm{B}\mathbb{S}_\mathrm{B}^\mathrm{A}\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{V}_\mathrm{A}^{-1}$ connecting dipole sources in body A to total fields in body B and accounting for multiple scattering to all orders within and between both bodies. This obeys a Landauer limit, as the singular values of $\mathbb{Q}^\dagger\mathbb{Q}$ do not exceed $1/4$, so including the prefactor $2/\pi$, the contribution of each mode/channel in the trace expression to Φ does not exceed $\frac{1}{2\pi}$. Our goal is to explain the conditions under which the Landauer bounds for each of these contributions may be saturated.

II. SINGULAR VALUE BOUNDS

We derive upper bounds on Φ starting from (6) by making liberal use of the singular value decomposition for the relevant operators and associated bounds on the trace of products of operators. To start, in Appendix B, we prove the lemma that the largest real positive value for the trace of a product of operators occurs when those operators share singular vectors and when their fixed singular values are each arranged in a consistent order. This will be useful to bound (6), though the connection is more subtle as we wish to vary the singular values of some of the operators involved, and the extent to which those singular values can vary is restricted by physical constraints requiring nonnegative far-field scattered power. Once it is established that the proof in Appendix B is applicable, we vary the singular values of relevant operators to maximize the upper bound on Φ, which we call Φ_opt.

A. Constraints on nonnegative far-field scattering

As we have cast (6) as an absorption quantity that is guaranteed to be nonnegative, the most relevant physical constraints are that far-field scattering from each individual body, and for the system as a whole in turn, must be nonnegative. In general, given a susceptibility $\mathbb{V}$ and an associated T-operator $\mathbb{T}$, the far-field scattered power from a given incident field $|\mathbf{E}^\mathrm{inc}\rangle$ is $\frac{\omega}{2}\langle\mathbf{E}^\mathrm{inc}, (\mathrm{Im}(\mathbb{T}) - \mathbb{T}^\star\,\mathrm{Im}(\mathbb{V}^{-1\star})\mathbb{T})\mathbf{E}^\mathrm{inc}\rangle$, and for this to be nonnegative, the operator $\mathrm{Im}(\mathbb{T}) - \mathbb{T}^\star\,\mathrm{Im}(\mathbb{V}^{-1\star})\mathbb{T}$ must be positive-semidefinite. This must hold true for each body in isolation, meaning $\mathrm{Im}(\mathbb{T}_p) - \mathbb{T}_p^\star\,\mathrm{Im}(\mathbb{V}_p^{-1\star})\mathbb{T}_p$ is likewise positive-semidefinite for each $p \in \{\mathrm{A},\mathrm{B}\}$, and reciprocity means that $\mathrm{Im}(\mathbb{T}_p) - \mathbb{T}_p\,\mathrm{Im}(\mathbb{V}_p^{-1\star})\mathbb{T}_p^\star$ must also be positive-semidefinite. As the following derivations make clear, this condition is most relevant for body A, meaning that

$$\left\langle \mathbf{E}^\mathrm{inc}, \left[\mathrm{Im}(\mathbb{T}_\mathrm{A}) - \frac{1}{\zeta_\mathrm{A}}\mathbb{T}_\mathrm{A}^\star\mathbb{T}_\mathrm{A}^\star\right]\mathbf{E}^\mathrm{inc}\right\rangle \geq 0 \tag{9}$$

for every incident field $|\mathbf{E}^\mathrm{inc}\rangle$, after using $\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star}) = \frac{1}{\zeta_\mathrm{A}}\mathbb{I}_\mathrm{A}$. Further conditions become relevant when the two bodies are proximate to each other.

As a first step, we show that the operator $\mathbb{Y}_\mathrm{B} = (\mathbb{T}_\mathrm{B}^{-1} - \mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{G}_\mathrm{AB}^\mathrm{vac})^{-1} = \mathbb{T}_\mathrm{B}\mathbb{S}_\mathrm{B}^\mathrm{A}$ is an effective T-operator for body B dressed by the proximity of body A. In particular, referring to Appendix C, the definitions (3) and (4) can be plugged into (C5) and rearranged in order to write

$$|\mathbf{P}_\mathrm{B}\rangle = \mathbb{Y}_\mathrm{B}(\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{V}_\mathrm{A}^{-1}|\mathbf{P}_\mathrm{A}^{(0)}\rangle + \mathbb{V}_\mathrm{B}^{-1}|\mathbf{P}_\mathrm{B}^{(0)}\rangle).$$

If only sources in A are relevant, then we may set $|\mathbf{P}_\mathrm{B}^{(0)}\rangle \to 0$ and define an effective incident field $|\mathbf{E}^{\mathrm{inc}(\mathrm{A})}\rangle = \mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{V}_\mathrm{A}^{-1}|\mathbf{P}_\mathrm{A}^{(0)}\rangle$ which depends on multiple scattering within A but not on any properties of B apart from projection onto its volumetric degrees of freedom. This means that $|\mathbf{P}_\mathrm{B}\rangle = \mathbb{Y}_\mathrm{B}|\mathbf{E}^{\mathrm{inc}(\mathrm{A})}\rangle$, which is interpreted to mean that the total induced polarization in B arises from the response of B dressed in the presence of A, namely $\mathbb{Y}_\mathrm{B}$, acting on the effective incident field $|\mathbf{E}^{\mathrm{inc}(\mathrm{A})}\rangle$ accounting only for body A; this is analogous to $\mathbb{T}_\mathrm{B}$ which relates the total polarization induced in B to incident fields in vacuum.

Given this and the fact that $|\mathbf{E}_\mathrm{B}\rangle = \mathbb{V}_\mathrm{B}^{-1}|\mathbf{P}_\mathrm{B}\rangle$ after setting $|\mathbf{P}_\mathrm{B}^{(0)}\rangle \to 0$, the scattered power only from body B (in the presence of body A) may be written as the difference between extinction and absorption powers only from body B (in the presence of body A), namely $\frac{\omega}{2}\left(\mathrm{Im}(\langle\mathbf{E}^\mathrm{inc}, \mathbf{P}_\mathrm{B}\rangle) - \langle\mathbf{E}_\mathrm{B}, \mathbf{P}_\mathrm{B}\rangle\right) = \frac{\omega}{2}\langle\mathbf{E}^\mathrm{inc}, (\mathrm{Im}(\mathbb{Y}_\mathrm{B}) - \mathbb{Y}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})\mathbb{Y}_\mathrm{B})\mathbf{E}^\mathrm{inc}\rangle$. Nonnegativity of this quantity for any $|\mathbf{P}_\mathrm{A}^{(0)}\rangle$, or more generally any $|\mathbf{E}^\mathrm{inc}\rangle$, means that $\mathrm{Im}(\mathbb{Y}_\mathrm{B}) - \mathbb{Y}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})\mathbb{Y}_\mathrm{B}$ must be positive-semidefinite.

Moreover, nonnegativity of far-field scattering from the system in general means that upon evaluating the inverse of (4), the operator $\mathrm{Im}(\mathbb{T}) - \mathbb{T}\,\mathrm{Im}(\mathbb{V}^{-1\star})\mathbb{T}^\star$ must be positive-semidefinite, which means in turn that each of its diagonal blocks must be positive-semidefinite. Manipulating operators allows for showing that the bottom-right block is $\mathrm{Im}(\mathbb{Y}_\mathrm{B}) -$

$\mathbb{Y}_\mathrm{B}\left(\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})+\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\,\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star})\mathbb{T}_\mathrm{A}^\star\mathbb{G}_\mathrm{AB}^{\mathrm{vac}\star}\right)\mathbb{Y}_\mathrm{B}^\star$, which involves another positive-semidefinite operator (as $\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star})$ is positive-semidefinite and is multiplied on its left and right by operators which are Hermitian adjoints of each other) subtracted from the operator $\mathrm{Im}(\mathbb{Y}_\mathrm{B})-\mathbb{Y}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})\mathbb{Y}_\mathrm{B}$. Therefore, $\mathrm{Im}(\mathbb{Y}_\mathrm{B})-\mathbb{Y}_\mathrm{B}\left(\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})+\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\,\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star})\mathbb{T}_\mathrm{A}^\star\mathbb{G}_\mathrm{AB}^{\mathrm{vac}\star}\right)\mathbb{Y}_\mathrm{B}^\star$ is positive-semidefinite, which can be written more explicitly as the condition

$$\left\langle\mathbf{E}^\mathrm{inc},\mathrm{Im}(\mathbb{Y}_\mathrm{B})\mathbf{E}^\mathrm{inc}\right\rangle-\left\langle\mathbf{E}^\mathrm{inc},\mathbb{Y}_\mathrm{B}\left(\frac{1}{\zeta_\mathrm{B}}\mathbb{I}_\mathrm{B}+\frac{1}{\zeta_\mathrm{A}}\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{T}_\mathrm{A}^\star\mathbb{G}_\mathrm{AB}^{\mathrm{vac}\star}\right)\mathbb{Y}_\mathrm{B}^\star\mathbf{E}^\mathrm{inc}\right\rangle\geq 0 \tag{10}$$

for every incident field $|\mathbf{E}^\mathrm{inc}\rangle$, where we have used $\mathrm{Im}(\mathbb{V}_p^{-1\star})=\frac{1}{\zeta_p}\mathbb{I}_p$. Equation 10 yields stronger bounds on the singular values of $\mathbb{Y}_\mathrm{B}$ than the positive-semidefiniteness of $\mathrm{Im}(\mathbb{Y}_\mathrm{B})-\frac{1}{\zeta_\mathrm{B}}\mathbb{Y}_\mathrm{B}^\star\mathbb{Y}_\mathrm{B}$ alone, because the former also subtracts the absorption in body A in the presence of body B, whereas the latter does not.

B. Optimization of singular values

With the constraints in (9) and (10) in mind, we define the operator $\mathbb{A}\equiv\mathbb{G}_\mathrm{BA}^\mathrm{vac}\mathbb{T}_\mathrm{A}\mathbb{T}_\mathrm{A}^\star\mathbb{G}_\mathrm{AB}^{\mathrm{vac}\star}$ so that (6) may be rewritten as $\Phi=\frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\,\mathrm{Tr}[\mathbb{Y}_\mathrm{B}\mathbb{A}\mathbb{Y}_\mathrm{B}^\star]$ and (10) may be rewritten as the condition that the operator $\mathrm{Im}(\mathbb{Y}_\mathrm{B})-\mathbb{Y}_\mathrm{B}\left(\frac{1}{\zeta_\mathrm{B}}\mathbb{I}_\mathrm{B}+\frac{1}{\zeta_\mathrm{A}}\mathbb{A}\right)\mathbb{Y}_\mathrm{B}^\star$ be positive-semidefinite. The reason for this is as follows. The proof in Appendix B depends on the singular values being fixed and independent of the singular vectors, whereas an arbitrary grouping of operators might have variable singular values whose constraints depend on the singular vectors. However, we take the singular values of $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$ as fixed, and while we choose to vary the singular values of $\mathbb{T}_\mathrm{A}$, the constraints on those singular values from (9) are independent of the various singular vectors or values of other operators. In particular, reciprocity of $\mathbb{T}_\mathrm{A}$ allows for writing the singular value decomposition $\mathbb{T}_\mathrm{A}=\sum_i\tau_i|\mathbf{a}_i\rangle\langle\mathbf{a}_i^\star|$ where $\langle\mathbf{a}_i,\mathbf{a}_j\rangle=\delta_{ij}$. Thus, if the singular values τ_j are appropriately set, the assumptions in Appendix B remain valid. We choose to write the singular decomposition $\mathbb{A}=\sum_j\alpha_j|\mathbf{b}_j\rangle\langle\mathbf{b}_j|$, so (10) implies that $\Phi=\frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\,\mathrm{Tr}[\mathbb{Y}_\mathrm{B}\mathbb{A}\mathbb{Y}_\mathrm{B}^\star]\leq\frac{2}{\pi\zeta_\mathrm{B}}\,\mathrm{Tr}\left[\mathrm{Im}(\mathbb{Y}_\mathrm{B})-\frac{1}{\zeta_\mathrm{B}}\mathbb{Y}_\mathrm{B}\mathbb{Y}_\mathrm{B}^\star\right]$. From this, we can immediately see that the right-hand side is maximized if $\mathbb{Y}_\mathrm{B}=\mathrm{i}\,\mathrm{Im}(\mathbb{Y}_\mathrm{B})$ is purely anti-Hermitian, as any nontrivial Hermitian part increases the magnitude of the negative contribution relative to the positive contribution. Moreover, while the right-hand side is basis-independent, it can be evaluated in the singular value basis $\{|\mathbf{b}_i\rangle\}$ of $\mathbb{A}$, so the overall sum (trace) is guaranteed to be maximized when each individual contribution is maximized: the constraint in (9) is evaluated for a particular $|\mathbf{b}_i\rangle$ as $\frac{1}{\zeta_\mathrm{A}}\sum_j\alpha_j|\langle\mathbf{b}_j,\mathbb{Y}_\mathrm{B}\mathbf{b}_i\rangle|^2\leq\langle\mathbf{b}_i,\mathrm{Im}(\mathbb{Y}_\mathrm{B})\mathbf{b}_i\rangle-\frac{1}{\zeta_\mathrm{B}}\sum_j|\langle\mathbf{b}_j,\mathbb{Y}_\mathrm{B}\mathbf{b}_i\rangle|^2$, and so the right-hand side is maximized for each channel i if $\mathbb{Y}_\mathrm{B}$ has $\{|\mathbf{b}_j\rangle\}$ as its right singular vectors. Thus, the proof in Appendix B is indeed applicable, and reciprocity then allows us to write the singular value decomposition $\mathbb{Y}_\mathrm{B}=\sum_j y_j|\mathbf{b}_j^\star\rangle\langle\mathbf{b}_j|$, where $\langle\mathbf{b}_i,\mathbf{b}_j\rangle=\delta_{ij}$ and $\langle\mathbf{b}_j,\mathbf{b}_j^\star\rangle=\mathrm{i}$ for each channel j. In this basis of singular vectors, we may write $\Phi\leq\frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\sum_j\alpha_j y_j^2$, and the constraint in (9) can be written as $y_j-y_j^2(\zeta_\mathrm{B}^{-1}+\zeta_\mathrm{A}^{-1}\alpha_j)\geq 0$, so $y_j\leq\frac{1}{\zeta_\mathrm{B}^{-1}+\zeta_\mathrm{A}^{-1}\alpha_j}$. Saturating the inequality on y_j means we may write the bound $\Phi\leq\frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\sum_j\frac{\alpha_j}{(\zeta_\mathrm{B}^{-1}+\zeta_\mathrm{A}^{-1}\alpha_j)^2}$, and then optimize each α_j to maximize the bound itself.

Returning to the definition of $\mathbb{A}$, the bound on Φ can potentially be largest when the singular values α_j of $\mathbb{A}$ have the largest possible range from which optimal values may be chosen, and this occurs when $\mathbb{T}_\mathrm{A}$ shares its left singular vectors $\{|\mathbf{a}_j\rangle\}$ with $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$. This in turn combines with the structure of $\mathbb{A}$ to ensure that its singular vectors $\{\langle\mathbf{b}_j|\}$ are in fact the left singular vectors of $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$. Thus, we rewrite $\alpha_i=(g_i\tau_i)^2$, where g_i are the singular values of $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$, so that $\Phi\leq\frac{2}{\pi\zeta_\mathrm{A}\zeta_\mathrm{B}}\sum_j\frac{(g_j\tau_j)^2}{(\zeta_\mathrm{B}^{-1}+\zeta_\mathrm{A}^{-1}(g_j\tau_j)^2)^2}$. As the singular values g_j are fixed, we then optimize the singular values τ_j to maximize this upper bound for each channel j. The constraints on the singular values τ_j from (9) are loosest when $\mathbb{T}_\mathrm{A}=\mathrm{i}\,\mathrm{Im}(\mathbb{T}_\mathrm{A})$ (purely anti-Hermitian), implying that its singular vectors satisfy $\langle\mathbf{a}_j^\star,\mathbf{a}_j\rangle=\mathrm{i}$ for each channel j. Hence, Equation 9 yields the constraint $\tau_j\leq\zeta_\mathrm{A}$ for each channel j.

The contribution of each channel j to the upper bound on Φ is maximized at the Landauer limit of $\frac{1}{2\pi}$ if $\tau_j=\frac{1}{\sqrt{\zeta_\mathrm{A}^{-1}\zeta_\mathrm{B}g_i}}$ is chosen, which requires $\sqrt{\zeta_\mathrm{A}\zeta_\mathrm{B}}g_i\geq 1$ in order for $\tau_i\leq\zeta_\mathrm{A}$ to hold; this also corresponds to $y_j=\frac{\zeta_\mathrm{B}}{2}$. We interpret this to mean that to obtain optimal heat transfer, the T-operator of body A in isolation must be engineered in a way that depends on the presence of body B, due to both the presence of the material response factor ζ_B and the dependence on the singular values g_j of $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$ (propagating electromagnetic fields in vacuum from A to B). In turn, the expression $y_j=\frac{\zeta_\mathrm{B}}{2}$ means that the effective T-operator of body B dressed by scattering from body A must actually exhibit maximal *scattering*, and not absorption, in the presence of body A,[39] though it is more difficult to extract information about the implications for the T-operator of body B in isolation. Importantly, maximal scattering includes both far-field scattering from body B in the presence of body A, as well as absorption from body A in the presence of body B. If these two conditions can be met simultaneously for the given channel j, which is effectively a rate-matching condition relating the absorption and scattering rates of each body in the presence of the other, then the per-channel Landauer transmission upper bound $\frac{1}{2\pi}$ is achieved.

Otherwise, if $\sqrt{\zeta_\mathrm{A}\zeta_\mathrm{B}}g_i<1$, then $\tau_i=\zeta_\mathrm{A}$ must be used to maximize the contribution, which yields $\frac{2\zeta_\mathrm{A}\zeta_\mathrm{B}g_i^2}{\pi(1+\zeta_\mathrm{A}\zeta_\mathrm{B}g_i^2)^2}<\frac{1}{2\pi}$, and corresponds to $y_i=\frac{\zeta_\mathrm{B}}{1+\zeta_\mathrm{A}\zeta_\mathrm{B}g_i^2}\geq\frac{\zeta_\mathrm{B}}{2}$. We interpret this to mean that if the singular value g_i of $\mathbb{G}_\mathrm{BA}^\mathrm{vac}$ falls below a threshold involving the two material response factors, then the optimal T-operator of body A in isolation corresponds to maximal absorption, and the optimal effective T-operator of body B dressed by body A evinces the effects of multiple scattering with A. The contribution to Φ similarly shows the effects of

multiple scattering between the two bodies and is unable to saturate the Landauer bound for that channel.

C. Generality of singular value bounds

To summarize, the bound on RHT may be written as

$$\Phi \leq \Phi_{\mathrm{opt}} = \sum_i \Big[\frac{1}{2\pi} \Theta(\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2 - 1) + \frac{2}{\pi} \frac{\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2}{(1+\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2)^2} \Theta(1 - \zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2) \Big] \quad (11)$$

where Θ is the Heaviside step function. This bound depends intimately on the *interplay* between material response factors ζ_p for $p \in \{\mathrm{A}, \mathrm{B}\}$ and the singular values g_i of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$, which we term "radiative efficacies" as they describe the magnitudes of coupling between natural bases of currents in bodies A and B via propagation of EM fields. As the material response (encoded in $\zeta_{\mathrm{A}}\zeta_{\mathrm{B}}$) increases, progressively more channels may saturate the Landauer limit per channel, so that the Landauer limit (summed over all channels) is reached asymptotically as $\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} \to \infty$. However, the rate at which this divergence occurs depends on the general geometry of the problem, as that determines how the radiative efficacies g_i depend on the index i. We use the term "material-limited contributions" to refer to the terms $\frac{2}{\pi} \frac{\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2}{(1+\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2)^2}$, for $\zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2 < 1$, which do not saturate the Landauer limit for those channels.

We emphasize that while the singular values g_i of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$ are technically restricted to the domains of the objects to give the tightest bound on heat transfer, such a restriction is less than ideal given the explicit dependence on the shapes of the objects. However, as we prove in Appendix D, the singular values g_i of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$ are domain monotonic, meaning that they increase monotonically as the volumes of regions A and B increase; consequently, Φ_{opt} is domain monotonic, as it is monotonically nondecreasing with respect to g_i for each i. Separately from this, the regions containing only the material degrees of freedom of each body can be replaced by larger regions that fully enclose each body, as the T-operators of each body will commute with projections into the smaller subspaces corresponding to the actual material degrees of freedom. Thus, these bounds can be slightly loosened to be independent of body shapes, and can then be evaluated subject to constraints on topology and domain volumes as determined by the desired application [Fig. 1], e.g. ellipsoids with prescribed aspect ratios or films of prescribed thicknesses representative of compact or extended object shapes, respectively. Essentially, the effective rank of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$, which determines the number of modes that could participate in RHT, is largely determined by the size and topology of the choice of bounding surface, which represents a general and fundamental geometric constraint on the bounds of RHT analogous to the general material constraints imposed by ζ_p for each body $p \in \{\mathrm{A}, \mathrm{B}\}$; our bounds in turn capture the coupling between both constraints.

III. COMPARISON TO ALTERNATIVE BOUNDS

The bound for the RHT spectrum Φ in (11) may be compared to a number of other bounds. Strictly speaking, Φ_{opt} is not necessarily the tightest general bound that could be formulated. In particular, using the relation $\mathbb{T}_{\mathrm{A}}^{-1} = \mathbb{V}_{\mathrm{A}}^{-1} - \mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}}$ allows for writing (6) in terms of $\mathbb{T}_{\mathrm{A}}$ and $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})$ without reference to $\mathbb{V}_{\mathrm{A}}$. Such a procedure, in analogy with bounds on thermal emission which we detail in an upcoming paper,[43] would more explicitly capture far-field radiative losses from bodies of finite size, which becomes more relevant at large separations where such losses may compete with RHT itself. However, as we show in Appendix E, we find the resulting bound to be intractable, requiring self-consistent solution of systems of nonlinear equations to find the optimal singular values of $\mathbb{T}_{\mathrm{A}}$. Therefore, we do not further consider such a bound, and henceforth refer only to (11).

With respect to prior work, the most obvious point of comparison is the Landauer bound,[15] namely

$$\Phi_{\mathrm{L}} = \sum_i \frac{1}{2\pi}, \quad (12)$$

which simply depends on the number of modes participating in RHT, without any reference to separation, geometric or radiative constraints, or even material constraints, let alone their interplay; consequently, in contrast to our bounds, there is no metric to evaluate how many participating modes can actually saturate the limit $\frac{1}{2\pi}$. Even modal analyses that technically do not necessarily assume saturation of the Landauer limits for every mode[15–17,36] tend to neglect material effects, so the purely geometric arguments are valid only in the ray-optical regime where blackbody limits are reproduced. Thus, it is clear that $\Phi_{\mathrm{opt}} \leq \Phi_{\mathrm{L}}$.

We also compare to the bound found by Miller et al,[18] tight only in the quasistatic regime, written as

$$\Phi_{\mathrm{qs}} = \sum_i \frac{2}{\pi} \zeta_{\mathrm{A}}\zeta_{\mathrm{B}} g_i^2 \quad (13)$$

which we term the "quasistatic bound". This is derived by limiting the singular values of $\mathbb{Y}_{\mathrm{B}}$ such that only the total scattering of body B in the presence of body A needs to be nonnegative, meaning $\mathrm{Im}(\mathbb{Y}_{\mathrm{B}}) - \mathbb{Y}_{\mathrm{B}}^{\star} \mathrm{Im}(\mathbb{V}_{\mathrm{B}}^{-1\star}) \mathbb{Y}_{\mathrm{B}}$ should be positive-semidefinite; this leads to the bound $y_i \leq \zeta_{\mathrm{B}}$, so maximization of Φ subject to that constraint as well as $\tau_i \leq \zeta_{\mathrm{A}}$ simply requires saturation of both of these constraints. The domain monotonicity of this bound trivially follows from that of g_i for each channel i, and the validity of the embedding argument with respect to $\mathbb{T}_p$ still holds, meaning that Φ_{qs} is also a useful bound for the RHT spectrum when considering bounding surfaces of arbitrary size and topology. However, we have shown that positive-semidefiniteness of $\mathrm{Im}(\mathbb{Y}_{\mathrm{B}}) - \mathbb{Y}_{\mathrm{B}}^{\star} \mathrm{Im}(\mathbb{V}_{\mathrm{B}}^{-1\star}) \mathbb{Y}_{\mathrm{B}}$, corresponding to nonnegative total scattering from body B in the presence of body A, is a looser constraint on y_i than nonnegativity of scattering from the system as a whole, corresponding to positive-semidefiniteness of $\mathrm{Im}(\mathbb{Y}_{\mathrm{B}}) - \mathbb{Y}_{\mathrm{B}} \left(\mathrm{Im}(\mathbb{V}_{\mathrm{B}}^{-1\star}) + \mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}} \mathbb{T}_{\mathrm{A}} \mathrm{Im}(\mathbb{V}_{\mathrm{A}}^{-1\star}) \mathbb{T}_{\mathrm{A}}^{\star} \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star} \right) \mathbb{Y}_{\mathrm{B}}^{\star}$: the former constraint says that only the sum of far-field scattering

from B and absorption in A needs to be nonnegative for B in the presence of A, whereas the latter constraint says that far-field scattering from B in the presence of A needs to be nonnegative by itself after discounting absorption in A. In this way, $\Phi_{\rm opt}$ accounts not only for material constraints on each body but also on the interplay with constraints on radiation between the bodies given their geometries and separations, whereas $\Phi_{\rm qs}$ accounts for each constraint separately without considering the interplay; for this reason, the contribution to $\Phi_{\rm opt}$ per channel is bounded from above, whereas the contribution to $\Phi_{\rm qs}$ per channel may be unbounded. The contribution from each channel to $\Phi_{\rm opt}$ is also bounded above by the corresponding contribution to $\Phi_{\rm qs}$ for that channel, which implies $\Phi_{\rm opt} \leq \Phi_{\rm qs}$ overall.

We may write the overall inequalities as

$$\Phi_{\rm opt} \leq \Phi_{\rm qs}, \Phi_{\rm L}. \tag{14}$$

In general, it is not possible to write an inequality relation between $\Phi_{\rm qs}$ and $\Phi_{\rm L}$ in all situations, because $\Phi_{\rm qs}$ may have some contributions $\frac{2}{\pi}\zeta_{\rm A}\zeta_{\rm B}g_i^2$ which fall above or below $\frac{1}{2\pi}$, and the geometry determining g_i would have to be known in order to know how many fall above or below. This is now a moot point though as we now have a bound that is at least as tight as each of those bounds.

IV. CONCLUDING REMARKS

We have determined bounds for the RHT spectrum Φ based purely on algebraic arguments. In particular, we have shown that there is a tension between optimizing transmission channels and material/geometric constraints placed on each object in isolation as well as in in the presence of the other. As a result, a select number of channels can saturate previously derived Landauer bounds, while others are restricted by the aforementioned constraints. By virtue of domain monotonicity, these bounds can be applied in a shape-independent manner, so while they can be evaluated analytically in highly symmetric bounding surfaces, they can just as easily be evaluated numerically in more complicated domains depending on specific design constraints [Fig. 1]. Similarly, the dependence on the material response factor $\zeta = |\chi|^2/\operatorname{Im}\chi$ does not make explicit reference to a particular frequency or material model. In comparison, the Landauer bounds yield overly optimistic predictions, while choosing a scalar response for each object corresponding to maximal absorption of every incident field in isolation yields overly pessimistic predictions. Additionally, we find that previous work by Miller et al[18] also yields overly optimistic predictions compared to our current bounds, because those derivations neglect the interplay between material and geometric radiative constraints between the two bodies and consequently overestimate the optimal response of one body in the presence of the other. We point out that while our bounds are always at least as tight as Landauer and quasistatic limits for any given bounding domain, they say nothing about which domains may yield the tightest per-volume limits given material constraints, or whether they may in fact be attained by physically realizable structures. In summary, while quasistatic and Landauer limits are technically upper bounds on RHT, their neglect of the coupling between radiative geometric constraints and material losses in both cases (and of each constraint itself in the latter case) render them loose compared to the bounds presented here. We further emphasize that in contrast to quasistatic limits,[18] which can become unphysically loose and diverge beyond the near-field for extended geometries, our bounds are valid and could be tighter from the near-field all the way through the far-field for bodies of arbitrary size: no nonretarded or quasistatic approximations are made, and the saturation of contributions per channel at the Landauer limit constitutes a greater promise of a finite bound.

In a complementary paper,[42] we analyze these bounds in the near-field in specific geometries of interest, particularly high-symmetry domains enclosing dipolar as well as extended (infinite area) bodies. There, we find that $\Phi_{\rm opt}$ either saturates or increases very slowly compared to the rapid increase in $\Phi_{\rm qs}$ as a function of $\zeta_{\rm A}\zeta_{\rm B}$, and that the material-limited contributions to $\Phi_{\rm opt}$, representing the feasible energy transfer spectrum for high-symmetry homogeneous isotropic media at a polariton condition, comes very close to $\Phi_{\rm opt}$ for practically achievable material response factors. These findings suggest that the role of nanostructuring in enhancing the near-field RHT spectrum above results achievable in high-symmetry objects made of appropriately-chosen polar dielectric materials will be limited, and this has important implications for the theoretical and experimental design of devices for cooling, heat dissipation, and energy generation. Additionally, we apply similar ideas to upper bounds on thermal emission (see Appendix C) in a separate work.[43]

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Appendix A: Notation

We briefly discuss the notation used through the main text and the appendices. A vector field $\mathbf{v}(\mathbf{x})$ will be denoted as $|\mathbf{v}\rangle$. The conjugated inner product is $\langle \mathbf{u}, \mathbf{v}\rangle = \int \mathrm{d}^3x\ \mathbf{u}^\star(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$. An operator $\mathbb{A}(\mathbf{x}, \mathbf{x}')$ will be denoted as $\mathbb{A}$, with $\int \mathrm{d}^3x'\ \mathbb{A}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{v}(\mathbf{x}')$ denoted as $\mathbb{A}|\mathbf{v}\rangle$. The Hermitian conjugate $\mathbb{A}^\dagger$ is defined such that $\langle \mathbf{u}, \mathbb{A}^\dagger\mathbf{v}\rangle = \langle \mathbb{A}\mathbf{u}, \mathbf{v}\rangle$. The anti-Hermitian part of a square operator (whose domain and range are the same size) is defined as the operator

$\mathrm{Asym}(\mathbb{A}) = (\mathbb{A} - \mathbb{A}^\dagger)/(2\,\mathrm{i})$. Finally, the trace of an operator is $\mathrm{Tr}(\mathbb{A}) = \int \mathrm{d}^3x\ \mathrm{Tr}(\mathbb{A}(\mathbf{x},\mathbf{x}))$. Through this paper, unless stated explicitly otherwise, all quantities implicitly depend on ω, and such dependence will be notationally suppressed for brevity.

Appendix B: Proof of von Neumann trace inequality

In this section, we reproduce the proof of a trace inequality by von Neumann[44] for clarity. The lemma is as follows: if operators $\mathbb{A}_n$ for $n \in \{1,2,\ldots,N\}$ have fixed singular values labeled $\sigma_i^{(n)}$, then the singular vectors that maximize $\mathrm{Tr}[\mathbb{A}_1\mathbb{A}_2\ldots\mathbb{A}_N]$ are common between operators multiplied together. That is, the singular value decomposition of $\mathbb{A}_n$ should follow $\mathbb{A}_n = \sum_i \sigma_i^{(n)} |\mathbf{a}_i^{(n)}\rangle\langle\mathbf{a}_i^{(n+1)}|$ for $n \in \{1,2,\ldots,N-1\}$, with $\mathbb{A}_N = \sum_i \sigma_i^{(N)} |\mathbf{a}_i^{(N)}\rangle\langle\mathbf{a}_i^{(1)}|$, where the vectors $|\mathbf{a}_i^{(n)}\rangle$ are orthonormal for each n such that $\langle\mathbf{a}_i^{(n)},\mathbf{a}_j^{(n)}\rangle = \delta_{ij}$. This lemma will hold even if each $\mathbb{A}_n$ is not square, as long as $\mathbb{A}_n\mathbb{A}_{n+1}$ forms a valid nontrivial operator product, as these can be embedded in larger spaces padded with more vanishing singular values. Thus, we restrict our consideration to square operators. Moreover, associativity means $\mathbb{A}_n\mathbb{A}_{n+1}\mathbb{A}_{n+2} = (\mathbb{A}_n\mathbb{A}_{n+1})\mathbb{A}_{n+2}$, and the trace of a product of operators is invariant under cyclic permutations, so we ultimately only consider maximizing the trace of a product of two operators, as maximization of the trace of products of more than two operators follows inductively from this.

To maximize $\mathrm{Tr}[\mathbb{A}\mathbb{B}]$, assuming it to be real and nonnegative, we start by writing

$$\mathbb{A} = \sum_{i=1}^{N} \sigma_i |\mathbf{u}_i\rangle\langle\mathbf{v}_i| \tag{B1}$$

$$\mathbb{B} = \sum_{j=1}^{N} \tau_j |\mathbf{w}_j\rangle\langle\mathbf{y}_j| \tag{B2}$$

where N is the size of the space; this may be larger than the rank of either $\mathbb{A}$ or $\mathbb{B}$, but the point is moot because the singular values are fixed, whether they vanish or not, and it has already been assumed that $\mathbb{A}$ and $\mathbb{B}$ are square. We also assume that the singular values are ordered such that $\sigma_i \geq \sigma_{i+1}$ for all $i \in \{1,2,\ldots,N-1\}$ and $\tau_j \geq \tau_{j+1}$ for all $j \in \{1,2,\ldots,N-1\}$. This allows for writing

$$\mathrm{Tr}[\mathbb{A}\mathbb{B}] = \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i p_{ij} \tau_j q_{ji} \tag{B3}$$

in terms of $p_{ij} = \langle\mathbf{v}_i,\mathbf{w}_j\rangle$ and $q_{ji} = \langle\mathbf{y}_j,\mathbf{u}_i\rangle$. As the singular vectors are orthonormal, then p_{ij} and q_{ji} are the elements of unitary matrices, satisfying $\sum_{j=1}^N |p_{ij}|^2 = \sum_{i=1}^N |p_{ij}|^2 = \sum_{j=1}^N |q_{ji}|^2 = \sum_{i=1}^N |q_{ji}|^2 = 1$. As $\mathrm{Tr}[\mathbb{A}\mathbb{B}]$ is assumed to be real and nonnegative, it is maximized when $\sigma_i p_{ij} \tau_j q_{ji}$ are all nonnegative; this means the singular vectors can be chosen without loss of generality such that p_{ij} and q_{ji} are real and nonnegative, implying p_{ij} and q_{ji} are the elements of real-valued orthogonal matrices.

We use induction to prove that maximizing the trace requires that $\{|\mathbf{u}_i\rangle\}$ be the duals of $\{\langle\mathbf{y}_j|\}$, and that $\{\langle\mathbf{v}_i|\}$ by the duals of $\{|\mathbf{w}_j\rangle\}$. The case $N = 1$ is trivial, as all quantities are scalars. For $N = 2$, we use orthogonality to note that $p_{1,2} = p_{2,1}$, $q_{1,2} = q_{2,1}$, $p_{1,1} = p_{2,2} = \sqrt{1-p_{1,2}^2}$, and $q_{1,1} = q_{2,2} = \sqrt{1-q_{1,2}^2}$. As a result, we may write $\mathrm{Tr}[\mathbb{A}\mathbb{B}] = \sqrt{(1-p_{1,2}^2)(1-q_{1,2}^2)}(\sigma_1\tau_1 + \sigma_2\tau_2) + p_{1,2}q_{1,2}(\sigma_1\tau_2 + \sigma_2\tau_1)$. As the first term in parentheses is larger than the second term in parentheses by the nonnegative value $(\sigma_1 - \sigma_2)(\tau_1 - \tau_2)$ given the ordering of singular values, having the left singular vectors of one operator not be duals of the right singular vectors of the other and vice versa could only increase the trace if $\sqrt{(1-p_{1,2}^2)(1-q_{1,2}^2)} + p_{1,2}q_{1,2} > 1$, but this leads to the impossible condition $0 > (p_{1,2} - q_{1,2})^2$, so we can only have $p_{1,2} = q_{1,2}$ for the trace to be maximized, implying the duality result must hold.

The inductive step assumes an arbitrary $N-1$ and moves from there to proving the statement for N. Without loss of generality, we consider first the contribution of the largest singular value τ_1 of $\mathbb{B}$, namely $\tau_1|\mathbf{w}_1\rangle\langle\mathbf{y}_1|$, interacting with $\mathbb{A} = \sum_i \sigma_i |\mathbf{u}_i\rangle\langle\mathbf{v}_i|$ in the trace. This yields the contribution $\sum_i (p_{i,1} - q_{1,i})^2 = 2\,(1 - \sum_i p_{i,1}q_{1,i}) \geq 0$ using the fact that $\sum_i p_{i,1}^2 = \sum_i q_{1,i}^2 = 1$, so this in turn gives the condition $\sum_i p_{i,1}q_{1,i} \leq 1$. The trace can be seen to be maximal when the above condition is saturated, so $\sum_i p_{i,1}q_{1,i} = 1$, which implies $p_{i,1} = q_{1,i}$ for every i. As this also holds when the roles of $\mathbb{A}$ and $\mathbb{B}$ are interchanged, and as this can be progressively carried out for each successively smaller singular value given orthonormality of the singular vectors, then the duality condition must hold, completing the proof.

Appendix C: Derivation of radiative heat transfer formulas

In this section, we derive the formula for the radiative heat transfer spectrum between two bodies, without assumptions about retardation, homogeneity, locality, or isotropy. The formula depends on individual T-operators and the vacuum Green's function, and follows a previous derivation[45] which considered energy transfer by fluctuating volume currents. Through further derivation, we also equate this formula to another formula involving the susceptibilities and the full Maxwell Green's function, and use that to recast the heat transfer spectrum in a Landauer form, whence we prove that the singular values of the Landauer transmission operator for RHT do not exceed $1/4$. Finally, we prove that the formula for thermal emission of a single body in isolation can be derived from the formula for RHT between two bodies in vacuum, by taking the second body to fully enclose the first and to be perfectly absorbing, thus taking on the role of a perfectly absorbing medium (vacuum).

1. T-operator formula

Our derivation of the heat transfer spectrum from the fluctuation–dissipation theorem for dipole sources in each body follows Ref.,[45] which we reproduce here for clarity. Consider two bodies A and B in vacuum with general susceptibilities $\mathbb{V}_p$ for $p \in \{\mathrm{A},\mathrm{B}\}$ which may be inhomogeneous, nonlocal, or anisotropic. Maxwell's equations may be written in integral form as

$$|\mathbf{E}\rangle = \mathbb{G}^{\mathrm{vac}}|\mathbf{P}\rangle \tag{C1}$$

$$|\mathbf{P}\rangle = |\mathbf{P}^{(0)}\rangle + \mathbb{V}|\mathbf{E}\rangle \tag{C2}$$

for the fields $|\mathbf{E}\rangle$ and total polarizations $|\mathbf{P}\rangle$ in terms of the polarization sources $|\mathbf{P}^{(0)}\rangle$, after defining

$$|\mathbf{E}\rangle = \begin{bmatrix} |\mathbf{E}_\mathrm{A}\rangle \\ |\mathbf{E}_\mathrm{B}\rangle \end{bmatrix}, \;\; |\mathbf{P}\rangle = \begin{bmatrix} |\mathbf{P}_\mathrm{A}\rangle \\ |\mathbf{P}_\mathrm{B}\rangle \end{bmatrix} \tag{C3}$$

(C4)

in block form for the material degrees of freedom constituting each object. Using these in conjunction with (2), (3), and (4), where as a reminder $\mathbb{T}_p^{-1} = \mathbb{V}_p^{-1} - \mathbb{G}_{pp}^{\mathrm{vac}}$, Maxwell's equations can be formally solved to yield

$$\begin{aligned} |\mathbf{E}\rangle &= \mathbb{G}^{\mathrm{vac}}\mathbb{T}\mathbb{V}^{-1}|\mathbf{P}^{(0)}\rangle \\ |\mathbf{P}\rangle &= \mathbb{T}\mathbb{V}^{-1}|\mathbf{P}^{(0)}\rangle \end{aligned} \tag{C5}$$

obtained by applying formulas for the block matrix inverse to compute $\mathbb{T}$. We also define the projection operators,

$$\mathbb{I}_\mathrm{A} = \begin{bmatrix} \mathbb{I}_\mathrm{A} & 0 \\ 0 & 0 \end{bmatrix}, \;\; \mathbb{I}_\mathrm{B} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_\mathrm{B} \end{bmatrix}, \tag{C6}$$

such that (abusing notation) $\mathbb{I}_p$ is the projection onto the material degrees of freedom of body p.

We consider the energy flow from fluctuating dipole sources only in body A into material degrees of freedom in body B, noting that reciprocity would yield the same heat transfer if the roles of bodies A and B were interchanged. This means $|\mathbf{P}^{(0)}\rangle = \begin{bmatrix} |\mathbf{P}_\mathrm{A}^{(0)}\rangle \\ 0 \end{bmatrix}$ defines the fluctuating sources in body A. The heat transfer spectrum is the ensemble-averaged work, denoted by $\langle\cdots\rangle$, done by the field,

$$\Phi = \frac{1}{2}\,\mathrm{Re}(\langle\langle \mathbb{I}_\mathrm{B}\mathbf{E}, \mathbb{I}_\mathrm{B}\mathbf{J}\rangle\rangle) \tag{C7}$$

where $|\mathbf{J}\rangle = -\,\mathrm{i}\,\omega|\mathbf{P}\rangle$. Using the Hermiticity and idempotence of $\mathbb{I}_p$ yields $\Phi = -\frac{\omega}{4\,\mathrm{i}}(\langle\langle \mathbb{I}_\mathrm{B}\mathbf{P}, \mathbf{E}\rangle\rangle - \langle\langle \mathbf{E}, \mathbb{I}_\mathrm{B}\mathbf{P}\rangle\rangle)$, and using the results of (C5) gives

$$\Phi = -\frac{\omega}{2}\langle\langle \mathbf{P}_\mathrm{A}^{(0)}, \mathbb{I}_\mathrm{A}\mathbb{V}^{-1\dagger}\mathbb{T}^\dagger\;\mathrm{Asym}(\mathbb{I}_\mathrm{B}\mathbb{G}^{\mathrm{vac}})\mathbb{T}\mathbb{V}^{-1}\mathbb{I}_\mathrm{A}\mathbf{P}_\mathrm{A}^{(0)}\rangle\rangle \tag{C8}$$

in terms of the fluctuating sources $|\mathbf{P}_\mathrm{A}^{(0)}\rangle$. As these fluctuations are thermal in nature, their correlations are given by the fluctuation–dissipation theorem

$$\langle|\mathbf{P}_\mathrm{A}^{(0)}\rangle\langle\mathbf{P}_\mathrm{A}^{(0)}|\rangle = \frac{4}{\pi\omega}\,\mathrm{Asym}(\mathbb{V}_\mathrm{A}) \tag{C9}$$

(suppressing the Planck function Π as it has already been factored to be separate from Φ), yielding

$$\Phi = -\frac{2}{\pi}\,\mathrm{Tr}(\mathrm{Asym}(\mathbb{V}_\mathrm{A}^{-1\dagger})\mathbb{I}_\mathrm{A}\mathbb{T}^\dagger\;\mathrm{Asym}(\mathbb{I}_\mathrm{B}\mathbb{G}^{\mathrm{vac}})\mathbb{T}\mathbb{I}_\mathrm{A}) \tag{C10}$$

as the dressed radiative heat transfer spectrum.

To prove equivalence of this expression for Φ to that involving only $\mathbb{G}^{\mathrm{vac}}$ and $\mathbb{T}_p$, it is useful to explicitly invoke reciprocity: $\mathbb{V}_p^\top = \mathbb{V}_p$, $\mathbb{T}_p^\top = \mathbb{T}_p$, and $(\mathbb{G}_{pq}^{\mathrm{vac}})^\top = \mathbb{G}_{qp}^{\mathrm{vac}}$ for $p, q \in \{\mathrm{A},\mathrm{B}\}$, implying that $\mathbb{V}_p^\dagger = \mathbb{V}_p^\star$, $\mathbb{T}_p^\dagger = \mathbb{T}_p^\star$, $(\mathbb{G}_{pq}^{\mathrm{vac}})^\dagger = \mathbb{G}_{qp}^{\mathrm{vac}\star}$, and $\mathrm{Asym}(\mathbb{A}) = \mathrm{Im}(\mathbb{A})$ for $\mathbb{A} \in \{\mathbb{V}_p, \mathbb{T}_p, \mathbb{G}_{pp}^{\mathrm{vac}}\}$. This allows for writing the operators

$$\begin{aligned} \mathrm{Asym}(\mathbb{I}_\mathrm{B}\mathbb{G}^{\mathrm{vac}}) &= \begin{bmatrix} 0 & -\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star}/(2\,\mathrm{i}) \\ \mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}/(2\,\mathrm{i}) & \mathrm{Im}(\mathbb{G}_{\mathrm{BB}}^{\mathrm{vac}}) \end{bmatrix} \\ \mathbb{T}\mathbb{I}_\mathrm{A} &= \begin{bmatrix} (\mathbb{T}_\mathrm{A}^{-1} - \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^{-1} \\ \mathbb{T}_\mathrm{B}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}(\mathbb{T}_\mathrm{A}^{-1} - \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^{-1} \end{bmatrix} \\ \mathbb{I}_\mathrm{A}\mathbb{T}^\dagger &= \begin{bmatrix} (\mathbb{T}_\mathrm{A}^{-1\star} - \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star}\mathbb{T}_\mathrm{B}^\star\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}\star})^{-1} \\ (\mathbb{T}_\mathrm{A}^{-1\star} - \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star}\mathbb{T}_\mathrm{B}^\star\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}\star})^{-1}\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star}\mathbb{T}_\mathrm{B}^\star \end{bmatrix} \end{aligned}$$

in block matrix form, where the projection onto A allows for truncation to the appropriate block column or row for notational convenience; note that $\mathbb{I}_\mathrm{A}\mathbb{T}^\dagger$ should actually be a row vector, but has been written as a column for ease of reading. Multiplying these matrices together, it can be noted that $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}(\mathbb{T}_\mathrm{A}^{-1} - \mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^{-1} = \mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}(\mathbb{I}_\mathrm{A} - \mathbb{T}_\mathrm{A}\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^{-1}\mathbb{T}_\mathrm{A} = (\mathbb{I}_\mathrm{B} - \mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\mathbb{T}_\mathrm{A}\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B})^{-1}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\mathbb{T}_\mathrm{A} = \mathbb{S}_\mathrm{B}^\mathrm{A}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\mathbb{T}_\mathrm{A}$ using the definition of the scattering operator $\mathbb{S}_\mathrm{B}^\mathrm{A} = (\mathbb{I}_\mathrm{B} - \mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\mathbb{T}_\mathrm{A}\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}}\mathbb{T}_\mathrm{B})^{-1}$. Additionally, using the definition $\mathbb{T}_p^{-1} = \mathbb{V}_p^{-1} - \mathbb{G}_{pp}^{\mathrm{vac}}$, it is easy to prove that $\mathrm{Im}(\mathbb{T}_\mathrm{B}) - \mathbb{T}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{G}_{\mathrm{BB}}^{\mathrm{vac}})\mathbb{T}_\mathrm{B} = \mathbb{T}_\mathrm{B}^\star\,\mathrm{Im}(\mathbb{V}_\mathrm{B}^{-1\star})\mathbb{T}_\mathrm{B}$, and likewise $\mathrm{Im}(\mathbb{T}_\mathrm{A}) - \mathbb{T}_\mathrm{A}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})\mathbb{T}_\mathrm{A}^\star = \mathbb{T}_\mathrm{A}\,\mathrm{Im}(\mathbb{V}_\mathrm{A}^{-1\star})\mathbb{T}_\mathrm{A}^\star$. Thus, the result is (5) in the main text, as expected.

2. Derivation of Green's function heat transfer formula

Our derivation of the bounds in the main text relies on the relationship between the heat transfer spectrum Φ written in terms of the vacuum Green's function and the T-operators of individual objects, to the heat transfer formula[1]

$$\Phi = \frac{2}{\pi}\,\mathrm{Tr}[\mathrm{Asym}(\mathbb{V}_\mathrm{A})\mathbb{G}_{\mathrm{BA}}^\dagger\,\mathrm{Asym}(\mathbb{V}_\mathrm{B})\mathbb{G}_{\mathrm{BA}}] \tag{C11}$$

where $\mathbb{G} = (\mathbb{G}^{\mathrm{vac}-1} - \mathbb{V}_\mathrm{A} - \mathbb{V}_\mathrm{B})^{-1}$ is the full Maxwell Green's function in the presence of both bodies, with the block $\mathbb{G}_{\mathrm{BA}}$ representing the fields in body B due to dipole sources in body A. We start with the T-operator form by writing

$$\begin{aligned} \Phi(\omega) = \frac{2}{\pi}\,\mathrm{Tr}[&\mathrm{Asym}(\mathbb{V}_\mathrm{A})\mathbb{V}_\mathrm{A}^{-1\dagger}\mathbb{T}_\mathrm{A}^\dagger(\mathbb{S}_\mathrm{B}^\mathrm{A}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^\dagger\mathbb{T}_\mathrm{B}^\dagger\times \\ &\mathbb{V}_\mathrm{B}^{-1\dagger}\,\mathrm{Asym}(\mathbb{V}_\mathrm{B})\mathbb{V}_\mathrm{B}^{-1}\mathbb{T}_\mathrm{B}\mathbb{S}_\mathrm{B}^\mathrm{A}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\mathbb{T}_\mathrm{A}\mathbb{V}_\mathrm{A}^{-1}] \end{aligned} \tag{C12}$$

where we have used the facts that $\mathrm{Asym}(\mathbb{V}_\mathrm{B}^{-1\dagger}) = \mathbb{V}_\mathrm{B}^{-1\dagger}\,\mathrm{Asym}(\mathbb{V}_\mathrm{B})\mathbb{V}_\mathrm{B}^{-1}$ and $\mathrm{Asym}(\mathbb{V}_\mathrm{A}^{-1\dagger}) =$

$\mathbb{V}_{\rm A}^{-1}\,\mathrm{Asym}(\mathbb{V}_{\rm A})\mathbb{V}_{\rm A}^{-1\dagger}$ along with invariance of the trace under cyclic permutations of operator products. From this, it can be seen that the two expressions for $\Phi(\omega)$ are guaranteed to be the same if the operator $\mathbb{G}_{\rm BA}$ is the same as $\mathbb{V}_{\rm B}^{-1}\mathbb{T}_{\rm B}\mathbb{S}_{\rm B}^{\rm A}\mathbb{G}_{\rm BA}^{\rm vac}\mathbb{T}_{\rm A}\mathbb{V}_{\rm A}^{-1}$. We use the fact that $\mathbb{V}_\sigma^{-1}=\mathbb{T}_\sigma^{-1}+\mathbb{G}_{\sigma\sigma}^{\rm vac}$ to say that

$$\mathbb{V}_{\rm B}^{-1}\mathbb{T}_{\rm B}\mathbb{S}_{\rm B}^{\rm A}\mathbb{G}_{\rm BA}^{\rm vac}\mathbb{T}_{\rm A}\mathbb{V}_{\rm A}^{-1}=(\mathbb{I}_{\rm B}+\mathbb{G}_{\rm BB}^{\rm vac}\mathbb{T}_{\rm B})\mathbb{S}_{\rm B}^{\rm A}\mathbb{G}_{\rm BA}^{\rm vac}(\mathbb{I}_{\rm A}+\mathbb{T}_{\rm A}\mathbb{G}_{\rm AA}^{\rm vac})$$

must hold. To prove that this is equal to $\mathbb{G}_{\rm BA}$, we use the definition

$$\mathbb{G}=\mathbb{G}^{\rm vac}+\mathbb{G}^{\rm vac}\mathbb{T}\mathbb{G}^{\rm vac} \tag{C13}$$

in conjunction with definitions of $\mathbb{G}^{\rm vac}$ and $\mathbb{T}$ as 2×2 block matrices in (4) to write

$$\mathbb{G}_{\rm BA}=\mathbb{G}_{\rm BA}^{\rm vac}+\begin{bmatrix}\mathbb{G}_{\rm BA}^{\rm vac} & \mathbb{G}_{\rm BB}^{\rm vac}\end{bmatrix}\mathbb{T}\begin{bmatrix}\mathbb{G}_{\rm AA}^{\rm vac}\\ \mathbb{G}_{\rm BA}^{\rm vac}\end{bmatrix} \tag{C14}$$

for this system. Performing this matrix multiplication, recognizing that $\mathbb{G}_{\rm BA}^{\rm vac}\mathbb{T}_{\rm A}\mathbb{G}_{\rm AB}^{\rm vac}(\mathbb{T}_{\rm B}^{-1}-\mathbb{G}_{\rm BA}^{\rm vac}\mathbb{T}_{\rm A}\mathbb{G}_{\rm AB}^{\rm vac})^{-1}=\mathbb{S}_{\rm B}^{\rm A}-\mathbb{I}_{\rm B}$, using the fact definition of $\mathbb{S}_{\rm B}^{\rm A}$, and collecting and canceling terms leads to the proof of the equality $\mathbb{G}_{\rm BA}=\mathbb{V}_{\rm B}^{-1}\mathbb{T}_{\rm B}\mathbb{S}_{\rm B}^{\rm A}\mathbb{G}_{\rm BA}^{\rm vac}\mathbb{T}_{\rm A}\mathbb{V}_{\rm A}^{-1}$.

3. Landauer bounds on heat transfer singular values

We now prove that radiative heat transfer between arbitrarily shaped bodies can also be expressed as the trace of a transmission matrix whose singular values can be bounded above, similar to previously derived bounds in planar media. This relation intuitively connects the finite value of the RHT bounds and approximate low rank of $\mathbb{G}_{\rm BA}$, and can be proved as follows. For this, we use the cyclic property of the trace to define

$$\Phi=\frac{2}{\pi}\,\mathrm{Tr}(\mathbb{Q}^\dagger\mathbb{Q}) \tag{C15}$$

where $\mathbb{Q}=\mathrm{Im}(\mathbb{V}_{\rm B})^{1/2}\mathbb{G}_{\rm BA}\,\mathrm{Im}(\mathbb{V}_{\rm A})^{1/2}$ is the heat transmission operator.

The definition $\mathbb{G}^{-1}=\mathbb{G}^{\rm vac-1}-(\mathbb{V}_{\rm A}+\mathbb{V}_{\rm B})$ along with the fact that the vacuum Maxwell operator $\mathbb{G}^{\rm vac-1}$ is real-valued in position space leads to

$$\mathrm{Asym}(\mathbb{G})=\mathbb{G}^\dagger\,\mathrm{Asym}(\mathbb{V}_{\rm A}+\mathbb{V}_{\rm B})\mathbb{G} \tag{C16}$$

which relates dissipation in polarization currents and electromagnetic fields in equilibrium. Additionally, the fact that $\mathrm{Asym}(\mathbb{V}_p)$ is a Hermitian positive-definite operator for each body $p\in\{\mathrm{A},\mathrm{B}\}$ means it has a unique square root $\mathrm{Asym}(\mathbb{V}_{\rm p})^{1/2}$. Rearranging the above equation, multiplying both sides by $2\ \mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}$, and adding $\mathbb{I}_{\rm A}$ to both sides gives

$$\begin{aligned}&4\ \mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}^\dagger\,\mathrm{Asym}(\mathbb{V}_{\rm B})\mathbb{G}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\\ &\quad+4\ \mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}^\dagger\,\mathrm{Asym}(\mathbb{V}_{\rm A})\mathbb{G}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\\ &\quad+2\,\mathrm{i}(\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\\ &\quad-\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}^\dagger\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2})+\mathbb{I}_{\rm A}=\mathbb{I}_{\rm A}\end{aligned}$$

recognizing the equality $\mathbb{Q}^\dagger\mathbb{Q}=\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}^\dagger\,\mathrm{Asym}(\mathbb{V}_{\rm B})\mathbb{G}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}$. Following this substitution, this may be factored as

$$\begin{aligned}&4\,\mathbb{Q}^\dagger\mathbb{Q}+\Big(\mathbb{I}_{\rm A}+2\,\mathrm{i}\ \mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}_{\rm AA}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\Big)^\dagger\times\\ &\Big(\mathbb{I}_{\rm A}+2\,\mathrm{i}\ \mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\mathbb{G}_{\rm AA}\,\mathrm{Asym}(\mathbb{V}_{\rm A})^{1/2}\Big)=\mathbb{I}_{\rm A}\end{aligned} \tag{C17}$$

where $\mathbb{G}$ has been replaced by its blocks $\mathbb{G}_{\rm AA}$ and $\mathbb{G}_{\rm AB}$ due to multiplications on each each side by $\mathrm{Asym}(\mathbb{V}_p)^{1/2}$ for $p\in\{\mathrm{A},\mathrm{B}\}$ (and likewise for $\mathbb{G}^\dagger$). This expression is the sum of two Hermitian positive-semidefinite operators equal to the identity; though this has been done for body A, reciprocity of heat transfer yields a similar expression in terms of the operators for body B. Consequently, the singular values of the operator $\mathbb{Q}^\dagger\mathbb{Q}$ entering the trace expression for $\Phi(\omega)$ must all be less than or equal to $1/4$. We emphasize that this derivation is valid for compact or extended structures of *arbitrary* geometry, without any need to expand heat transfer in terms of incoming and outgoing plane waves specific to translationally symmetric systems.[15]

4. Single-body thermal radiation from two-body radiative heat transfer

In this section, we prove that the formula for thermal emission of a single body in isolation in vacuum can be derived by starting from the formula for heat transfer between two bodies in vacuum under the following conditions. We take body A to be the thermal emitter in question, while body B is taken to fully surround body A as a shell of inner radius $r_{\rm B}$ and outer radius $R_{\rm B}$ with susceptibility $\mathbb{V}_{\rm B}=\chi_{\rm B}\mathbb{I}_{\rm B}$, and take the simultaneous limits $\mathrm{Im}(\chi_{\rm B})\to 0$ and $\omega r_{\rm B}/c\to\infty$ constrained by $\omega(R_{\rm B}-r_{\rm B})/c\to\infty$ and $\omega(R_{\rm B}-r_{\rm B})\,\mathrm{Im}(\chi_{\rm B})/c\to 1$, in which case body B takes on the role of a perfectly absorbing medium.

To start, we note that $\mathbb{T}_{\rm B}^{-1}=\mathbb{V}_{\rm B}^{-1}-\mathbb{G}_{\rm BB}^{\rm vac}$ in conjunction with reciprocity allows for writing (5) as

$$\begin{aligned}\Phi=\frac{2}{\pi}\,\mathrm{Tr}[&\mathbb{T}_{\rm B}\mathbb{S}_{\rm B}^{\rm A}\,\mathrm{Asym}(\mathbb{V}_{\rm B}^{-1\dagger})\mathbb{S}_{\rm B}^{\rm A\dagger}\mathbb{T}_{\rm B}^\dagger\times\\ &(\mathbb{G}_{\rm AB}^{\rm vac})^\dagger(\mathrm{Asym}(\mathbb{T}_{\rm A})-\mathbb{T}_{\rm A}^\dagger\,\mathrm{Asym}(\mathbb{G}_{\rm AA}^{\rm vac})\mathbb{T}_{\rm A})\mathbb{G}_{\rm AB}^{\rm vac}].\end{aligned}$$

In the aforementioned size and susceptibility limits for body B, $\mathbb{T}_{\rm B}\to 0$ so $\mathbb{S}_{\rm B}^{\rm A}\to\mathbb{I}_{\rm B}$, and $\mathbb{T}_{\rm B}\mathbb{S}_{\rm B}^{\rm A}\,\mathrm{Asym}(\mathbb{V}_{\rm B}^{-1\dagger})\mathbb{S}_{\rm B}^{\rm A\dagger}\mathbb{T}_{\rm B}^\dagger\to\mathrm{Asym}(\mathbb{V}_{\rm B})$. Using reciprocity, this yields $\Phi=\frac{2}{\pi}\,\mathrm{Tr}[\mathbb{G}_{\rm AB}^{\rm vac}\,\mathrm{Im}(\mathbb{V}_{\rm B})\mathbb{G}_{\rm BA}^{\rm vac\star}(\mathrm{Im}(\mathbb{T}_{\rm A})-\mathbb{T}_{\rm A}^\star\,\mathrm{Im}(\mathbb{G}_{\rm AA}^{\rm vac})\mathbb{T}_{\rm A})$. For a system with a general susceptibility $\mathbb{V}$ and Maxwell Green's function $\mathbb{G}=(\mathbb{G}^{\rm vac-1}-\mathbb{V})^{-1}$, the relations $\mathbb{G}\,\mathrm{Im}(\mathbb{V})\mathbb{G}^\star=\mathbb{G}^\star\,\mathrm{Im}(\mathbb{V})\mathbb{G}=\mathrm{Im}(\mathbb{G})$ will always hold. Considering B in isolation, the simultaneous constrained limits of infinite size and infinitesimal susceptibility mean that $\mathbb{G}_{\rm AB}^{\rm vac}\,\mathrm{Im}(\mathbb{V}_{\rm B})\mathbb{G}_{\rm BA}^{\rm vac\star}\to\mathrm{Im}(\mathbb{G}_{\rm AA}^{\rm vac})$. Finally, this yields the emission formula

$$\Phi=\frac{2}{\pi}\,\mathrm{Tr}[\mathrm{Im}(\mathbb{G}^{\rm vac})(\mathrm{Im}(\mathbb{T})-\mathbb{T}^\star\,\mathrm{Im}(\mathbb{G}^{\rm vac})\mathbb{T})] \tag{C18}$$

in agreement with the formula derived by Krüger et al,[26] where the subscripts A have been dropped as there is only one material body under consideration given that body B has effectively vanished.

Appendix D: Proof of domain monotonicity of singular values of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$

In this section, we prove that the singular values g_i of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$ are domain monotonic. The singular values of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$ are the eigenvalues of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}(\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^\dagger$. We consider the effects of a perturbative addition of volume only to body A; a perturbative effect on body B can be considered through reciprocity, and the proof will remain the same. Under this condition, we write the block row vector of operators

$$\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}} = \begin{bmatrix}\mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}} & \mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\end{bmatrix} \tag{D1}$$

where $\mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}$ is the operator propagating fields in vacuum from the unperturbed volume A_0 to body B, and $\mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}$ is the operator propagating fields in vacuum from the perturbative volume $\Delta\mathrm{A}$ to body B. Using reciprocity, we may then write

$$\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}(\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}})^\dagger = \mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}\mathbb{G}_{\mathrm{A}_0\mathrm{B}}^{\mathrm{vac}\star} + \mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star} + \mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\mathrm{A}_0\mathrm{B}}^{\mathrm{vac}\star} + \mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star} \tag{D2}$$

for which the first term $\mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}\mathbb{G}_{\mathrm{A}_0\mathrm{B}}^{\mathrm{vac}\star}$ is the Hermitian positive-semidefinite unperturbed operator, the terms $\mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star}$ and $\mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\mathrm{A}_0\mathrm{B}}^{\mathrm{vac}\star}$ vanish because the projections onto the volume A_0 and $\Delta\mathrm{A}$ are orthogonal to each other, and $\mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star}$ is the Hermitian positive-semidefinite perturbation. From Rayleigh-Schrödinger perturbation theory, if ρ_i is an unperturbed singular value of $\mathbb{G}_{\mathrm{BA}_0}^{\mathrm{vac}}$ with $|\mathbf{b}_i\rangle$ being the corresponding normalized right singular vector, then the perturbation to ρ_i is $\langle\mathbf{b}_i, \mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star}\mathbf{b}_i\rangle$, which is nonnegative by virtue of the positive-semidefiniteness of $\mathbb{G}_{\mathrm{B}\Delta\mathrm{A}}^{\mathrm{vac}}\mathbb{G}_{\Delta\mathrm{AB}}^{\mathrm{vac}\star}$. Therefore, any increase in the volume of a body will increase the singular values of $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}$.

Appendix E: Alternative bounds incorporating far-field radiative losses through $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})$

In this section, we derive an alternative bound to (11) that involves $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})$, thus capturing constraints on scattering losses purely from finite object sizes rather than through multiple scattering. Starting from $\mathbb{T}_{\mathrm{A}}^{-1} = \mathbb{V}_{\mathrm{A}}^{-1} - \mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}}$, operator manipulations lead to $\mathbb{T}_{\mathrm{A}}^{-1}\,\mathrm{Im}(\mathbb{V}_{\mathrm{A}}^{-1\star})\mathbb{T}_{\mathrm{A}}^{\star} = \mathrm{Im}(\mathbb{T}_{\mathrm{A}}) - \mathbb{T}_{\mathrm{A}}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})\mathbb{T}_{\mathrm{A}}^{\star}$. Hence, from (6),

$$\Phi = \frac{2}{\pi\zeta_{\mathrm{B}}}\,\mathrm{Tr}\left[\mathbb{G}_{\mathrm{AB}}^{\mathrm{vac}\star}\mathbb{Y}_{\mathrm{B}}^{\star}\mathbb{Y}_{\mathrm{B}}\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}}\left(\mathrm{Im}(\mathbb{T}_{\mathrm{A}}) - \mathbb{T}_{\mathrm{A}}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})\mathbb{T}_{\mathrm{A}}^{\star}\right)\right],$$

which now hides reciprocity, as no similar transformation has been made to eliminate terms giving rise to ζ_{B}. Using the singular value decompositions $\mathbb{G}_{\mathrm{BA}}^{\mathrm{vac}} = \sum_i g_i|\mathbf{b}_i\rangle\langle\mathbf{a}_i|$ and $\mathbb{Y}_{\mathrm{B}} = \sum_i y_i|\mathbf{b}_i^\star\rangle\langle\mathbf{b}_i|$ but leaving $\mathbb{T}_{\mathrm{A}}$ general, we find that the constraint on y_i is saturated when $y_i = \left(\zeta_{\mathrm{B}}^{-1} + \zeta_{\mathrm{A}}^{-1}g_i^2\langle\mathbf{a}_i, \mathbb{T}_{\mathrm{A}}\mathbb{T}_{\mathrm{A}}^{\star}\mathbf{a}_i\rangle\right)^{-1}$. Completing the square, one may write

$$\mathrm{Im}(\mathbb{T}_{\mathrm{A}}) - \mathbb{T}_{\mathrm{A}}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})\mathbb{T}_{\mathrm{A}}^{\star} = \mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{-1/2} \times \left[\frac{1}{4}\mathbb{I}_{\mathrm{A}} - \left(\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{1/2}\mathbb{T}_{\mathrm{A}}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{1/2} - \frac{\mathrm{i}}{2}\mathbb{I}_{\mathrm{A}}\right)\times \left(\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{1/2}\mathbb{T}_{\mathrm{A}}\,\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{1/2} - \frac{\mathrm{i}}{2}\mathbb{I}_{\mathrm{A}}\right)^{\star}\right]\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})^{-1/2}.$$

This strongly suggests that the optimal $\mathbb{T}_{\mathrm{A}}$ should be diagonalized in the same basis as $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})$, so if we write

$$\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}}) = \sum_i \rho_i|\mathbf{q}_i\rangle\langle\mathbf{q}_i|,$$

then one may also write $\mathbb{T}_{\mathrm{A}} = \mathrm{i}\sum_i \tau_i|\mathbf{q}_i\rangle\langle\mathbf{q}_i|$. This implies that the constraint on the singular values of $\mathbb{Y}_{\mathrm{B}}$ becomes

$$y_i = \left(\zeta_{\mathrm{B}}^{-1} + \zeta_{\mathrm{A}}^{-1}g_i^2\sum_j \tau_j^2|\langle\mathbf{a}_i, \mathbf{q}_j\rangle|^2\right)^{-1}$$

so y_i depends on τ_j for every channel j, not just $j = i$. Consequently, we arrive at the following bound:

$$\Phi \leq \frac{2}{\pi}\sum_i \frac{\zeta_{\mathrm{B}}g_i^2\left(\sum_j \tau_j(1-\rho_j\tau_j)|\langle\mathbf{a}_i, \mathbf{q}_j\rangle|^2\right)}{\left[1 + \zeta_{\mathrm{A}}^{-1}\zeta_{\mathrm{B}}g_i^2\left(\sum_j \tau_j^2|\langle\mathbf{a}_i, \mathbf{q}_j\rangle|^2\right)\right]^2} \tag{E1}$$

for which finding the optimal values of τ_i for each channel i requires self-consistently solving a large set of nonlinear equations subject to the constraint $\tau_i \leq \zeta_{\mathrm{A}}$ for each i. While this expression should yield tighter bounds on RHT owing to the incorporation of constraints on scattering losses for both objects in isolation (in addition to multiple scattering), it appears to be analytically intractable and must therefore be evaluated numerically, which we leave to future work; that said, numerical solution of the optimal values of τ_i for evaluating this bound requires solving a large set of nonlinear polynomial equations, which is generally computationally easier than brute-force optimization of the RHT spectrum due to the non-polynomial dependence of Φ on the T-operators in general. We point out that in the nonretarded quasistatic limit, $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}}) \to 0$, so all of its singular values may be taken to vanish as well; this means that its singular vectors become arbitrary, allowing for choosing $|\mathbf{q}_i\rangle = |\mathbf{a}_i\rangle$. Doing so, the above expression simplifies to $\Phi = \frac{2}{\pi}\sum_i \frac{\zeta_{\mathrm{B}}g_i^2\tau_i}{(1+\zeta_{\mathrm{A}}^{-1}\zeta_{\mathrm{B}}g_i^2\tau_i^2)^2}$, from which it can be seen that if the material bound $\tau_i \leq \zeta_{\mathrm{A}}$ is saturated, each contribution is identical to the corresponding contribution from (11); if the material bound is not saturated, then the optimal $\tau_i = \frac{1}{\sqrt{3\zeta_{\mathrm{A}}^{-1}\zeta_{\mathrm{B}}g_i}}$ leads to a larger contribution than what we find in (11), yielding an overall looser bound. This corroborates the notion that our present bounds, which do not include an explicit expression in terms of $\mathrm{Im}(\mathbb{G}_{\mathrm{AA}}^{\mathrm{vac}})$, do not fully account for dissipation via far-field radiative losses. Considering the singular values ρ_i will remedy this, but at the cost of greater complexity.

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[1] W. Jin, S. Molesky, Z. Lin, and A. W. Rodriguez, "Material scaling and frequency-selective enhancement of near-field radiative heat transfer for lossy metals in two dimensions via inverse design," Phys. Rev. B **99**, 041403 (2019).

[2] D. Costantini, A. Lefebvre, A.-L. Coutrot, I. Moldovan-Doyen, J.-P. Hugonin, S. Boutami, F. Marquier, H. Benisty, and J.-J. Greffet, "Plasmonic metasurface for directional and frequency-selective thermal emission," Phys. Rev. Applied **4**, 014023 (2015).

[3] A. W. Rodriguez, O. Ilic, P. Bermel, I. Celanovic, J. D. Joannopoulos, M. Soljačić, and S. G. Johnson, "Frequency-selective near-field radiative heat transfer between photonic crystal slabs: A computational approach for arbitrary geometries and materials," Phys. Rev. Lett. **107**, 114302 (2011).

[4] A. I. Volokitin and B. N. J. Persson, "Near-field radiative heat transfer and noncontact friction," Rev. Mod. Phys. **79**, 1291–1329 (2007).

[5] M. Francoeur, M. P. Mengüç, and R. Vaillon, "Near-field radiative heat transfer enhancement via surface phonon polaritons coupling in thin films," Applied Physics Letters **93**, 043109 (2008), https://doi.org/10.1063/1.2963195.

[6] S. Shen, A. Narayanaswamy, and G. Chen, "Surface phonon polaritons mediated energy transfer between nanoscale gaps," Nano Letters **9**, 2909–2913 (2009), http://dx.doi.org/10.1021/nl901208v.

[7] T. Asano, M. Suemitsu, K. Hashimoto, M. De Zoysa, T. Shibahara, T. Tsutsumi, and S. Noda, "Near-infrared–to–visible highly selective thermal emitters based on an intrinsic semiconductor," Science Advances **2** (2016), 10.1126/sciadv.1600499, http://advances.sciencemag.org/content/2/12/e1600499.full.pdf.

[8] A. Lenert, D. M. Bierman, Y. Nam, W. R. Chan, I. Celanović, M. Soljačić, and E. N. Wang, "A nanophotonic solar thermophotovoltaic device," Nature nanotechnology **9**, 126–130 (2014).

[9] A. Karalis and J. Joannopoulos, "'squeezing' near-field thermal emission for ultra-efficient high-power thermophotovoltaic conversion," Scientific reports **6**, 28472 (2016).

[10] A. Fiorino, L. Zhu, D. Thompson, R. Mittapally, P. Reddy, and E. Meyhofer, "Nanogap near-field thermophotovoltaics," Nature nanotechnology **13**, 806 (2018).

[11] T. Inoue, T. Koyama, D. D. Kang, K. Ikeda, T. Asano, and S. Noda, "One-chip near-field thermophotovoltaic device integrating a thin-film thermal emitter and photovoltaic cell," Nano Letters **19**, 3948–3952 (2019), pMID: 31137936, https://doi.org/10.1021/acs.nanolett.9b01234.

[12] B. Guha, C. Otey, C. B. Poitras, S. Fan, and M. Lipson, "Near-field radiative cooling of nanostructures," Nano Letters **12**, 4546–4550 (2012), pMID: 22891815, https://doi.org/10.1021/nl301708e.

[13] B. D. Boudreau, J. Raja, R. J. Hocken, S. R. Patterson, and J. Patten, "Thermal imaging with near-field microscopy," Review of Scientific Instruments **68**, 3096–3098 (1997), https://doi.org/10.1063/1.1148248.

[14] A. C. Jones and M. B. Raschke, "Thermal infrared near-field spectroscopy," Nano Letters **12**, 1475–1481 (2012), pMID: 22280474, https://doi.org/10.1021/nl204201g.

[15] J. B. Pendry, "Radiative exchange of heat between nanostructures," Journal of Physics: Condensed Matter **11**, 6621–6633 (1999).

[16] S.-A. Biehs, E. Rousseau, and J.-J. Greffet, "Mesoscopic description of radiative heat transfer at the nanoscale," Phys. Rev. Lett. **105**, 234301 (2010).

[17] P. Ben-Abdallah and K. Joulain, "Fundamental limits for non-contact transfers between two bodies," Phys. Rev. B **82**, 121419 (2010).

[18] O. D. Miller, S. G. Johnson, and A. W. Rodriguez, "Shape-independent limits to near-field radiative heat transfer," Phys. Rev. Lett. **115**, 204302 (2015).

[19] R. St-Gelais, B. Guha, L. Zhu, S. Fan, and M. Lipson, "Demonstration of strong near-field radiative heat transfer between integrated nanostructures," Nano Letters **14**, 6971–6975 (2014), http://dx.doi.org/10.1021/nl503236k.

[20] L. Cui, W. Jeong, V. Fernández-Hurtado, J. Feist, F. J. García-Vidal, J. C. Cuevas, E. Meyhofer, and P. Reddy, "Study of radiative heat transfer in ångström-and nanometre-sized gaps," Nature Communications **8** (2017).

[21] K. Kloppstech, N. Könne, S.-A. Biehs, A. W. Rodriguez, L. Worbes, D. Hellmann, and A. Kittel, "Giant heat transfer in the crossover regime between conduction and radiation," Nature Communications **8** (2017).

[22] C. Luo, A. Narayanaswamy, G. Chen, and J. D. Joannopoulos, "Thermal radiation from photonic crystals: A direct calculation," Phys. Rev. Lett. **93**, 213905 (2004).

[23] C. R. Otey, L. Zhu, S. Sandhu, and S. Fan, "Fluctuational electrodynamics calculations of near-field heat transfer in non-planar geometries: A brief overview," Journal of Quantitative Spectroscopy and Radiative Transfer **132**, 3–11 (2014).

[24] A. G. Polimeridis, M. T. H. Reid, W. Jin, S. G. Johnson, J. K. White, and A. W. Rodriguez, "Fluctuating volume-current formulation of electromagnetic fluctuations in inhomogeneous media: Incandescence and luminescence in arbitrary geometries," Phys. Rev. B **92**, 134202 (2015).

[25] A. W. Rodriguez, M. T. H. Reid, and S. G. Johnson, "Fluctuating-surface-current formulation of radiative heat transfer: Theory and applications," Phys. Rev. B **88**, 054305 (2013).

[26] M. Krüger, G. Bimonte, T. Emig, and M. Kardar, "Trace formulas for nonequilibrium casimir interactions, heat radiation, and heat transfer for arbitrary objects," Phys. Rev. B **86**, 115423 (2012).

[27] A. I. Volokitin and B. N. J. Persson, "Radiative heat transfer between nanostructures," Phys. Rev. B **63**, 205404 (2001).

[28] G. Domingues, S. Volz, K. Joulain, and J.-J. Greffet, "Heat transfer between two nanoparticles through near field interaction," Phys. Rev. Lett. **94**, 085901 (2005).

[29] B. Song, Y. Ganjeh, S. Sadat, D. Thompson, A. Fiorino, V. Fernández-Hurtado, J. Feist, F. J. Garcia-Vidal, J. C. Cuevas, P. Reddy, *et al.*, "Enhancement of near-field radiative heat transfer using polar dielectric thin films," Nature nanotechnology **10**, 253–258 (2015).

[30] M. T. H. Reid, A. W. Rodriguez, and S. G. Johnson, "Fluctuation-induced phenomena in nanoscale systems: Harnessing the power of noise," Proceedings of the IEEE **101**, 531–545 (2013).

[31] S. Edalatpour, J. DeSutter, and M. Francoeur, "Near-field thermal electromagnetic transport: An overview," Journal of Quantitative Spectroscopy and Radiative Transfer **178**, 14–21 (2016).

[32] S. Edalatpour and M. Francoeur, "Near-field radiative heat transfer between arbitrarily shaped objects and a surface," Phys. Rev. B **94**, 045406 (2016).

[33] A. Pérez-Madrid, J. M. Rubí, and L. C. Lapas, "Heat transfer between nanoparticles: Thermal conductance for near-field interactions," Phys. Rev. B **77**, 155417 (2008).

[34] R. Messina, M. Tschikin, S.-A. Biehs, and P. Ben-Abdallah, "Fluctuation-electrodynamic theory and dynamics of heat transfer in systems of multiple dipoles," Phys. Rev. B **88**, 104307 (2013).

[35] R. Messina, A. Noto, B. Guizal, and M. Antezza, "Radiative heat transfer between metallic gratings using fourier modal method with adaptive spatial resolution," Phys. Rev. B **95**, 125404 (2017).
[36] G. Bimonte, "Scattering approach to casimir forces and radiative heat transfer for nanostructured surfaces out of thermal equilibrium," Phys. Rev. A **80**, 042102 (2009).
[37] S. Datta, *Electronic Transport in Mesoscopic Systems*, Cambridge Studies in Semiconductor Physics and Microelectronic Engineering (Cambridge University Press, 1995).
[38] J. C. Klöckner, M. Bürkle, J. C. Cuevas, and F. Pauly, "Length dependence of the thermal conductance of alkane-based single-molecule junctions: An ab initio study," Phys. Rev. B **94**, 205425 (2016).
[39] O. D. Miller, A. G. Polimeridis, M. T. H. Reid, C. W. Hsu, B. G. DeLacy, J. D. Joannopoulos, M. Soljačić, and S. G. Johnson, "Fundamental limits to optical response in absorptive systems," Opt. Express **24**, 3329–3364 (2016).
[40] W. Jin, R. Messina, and A. W. Rodriguez, "Overcoming limits to near-field radiative heat transfer in uniform planar media through multilayer optimization," Opt. Express **25**, 14746–14759 (2017).
[41] V. Fernández-Hurtado, F. J. García-Vidal, S. Fan, and J. C. Cuevas, "Enhancing near-field radiative heat transfer with si-based metasurfaces," Phys. Rev. Lett. **118**, 203901 (2017).
[42] P. S. Venkataram, S. Molesky, W. Jin, and A. W. Rodriguez, "Fundamental limits to radiative heat transfer: the limited role of nanostructuring in the near-field," (2019), arXiv:1903.07968.
[43] S. Molesky, W. Jin, P. S. Venkataram, and A. W. Rodriguez, "Bounds on absorption and thermal radiation from arbitrary objects," (2019), arXiv:1907.04418.
[44] L. Mirsky, "A trace inequality of john von neumann," Monatshefte für Mathematik **79**, 303–306 (1975).
[45] P. S. Venkataram, J. Hermann, A. Tkatchenko, and A. W. Rodriguez, "Phonon-polariton mediated thermal radiation and heat transfer among molecules and macroscopic bodies: Nonlocal electromagnetic response at mesoscopic scales," Phys. Rev. Lett. **121**, 045901 (2018).