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Fermion decoration construction of symmetry protected trivial orders for fermion systems with any symmetries and in any dimensions

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We use higher dimensional bosonization and fermion decoration to construct exactly soluble interacting fermion models to realize fermionic symmetry protected trivial (SPT) orders (which are also known as symmetry protected topological orders) in any dimensions and for generic fermion symmetries G_f, which can be a non-trivial Z_2^f extension $Z_2^f \leftthreetimes G_b$ (where Z_2^f is the fermion-number-parity symmetry and G_b is the bosonic symmetry). This generalizes the previous results from group super-cohomology of Gu and Wen (arXiv:1201.2648), where G_f is assumed to be $Z_2^f \times G_b$. We find that the $(d+1)$D fermionic SPT phases with bosonic symmetry G_b and from fermion decoration construction can be described in a compact way using higher group homomorphism: $\mathcal{B}G_b \xrightarrow{\varphi} \mathcal{B}(Z_2, 2; Z_2, d)$. In fact, the fermion symmetry is more precisely described by the structure $Z_2^f \leftthreetimes G_b \leftthreetimes SO_\infty$ (or $Z_2^f \leftthreetimes G_b \leftthreetimes O_\infty$ with time reversal symmetry). In this case the $(d+1)$D fermionic SPT phases are better described by $\mathcal{B}(Z_2^f \leftthreetimes G_b \leftthreetimes SO_\infty) \xrightarrow{\varphi} \mathcal{B}(SO_\infty, 1; Z_2, d)$ [or $\mathcal{B}(Z_2^f \leftthreetimes G_b \leftthreetimes O_\infty) \xrightarrow{\varphi} \mathcal{B}(O_\infty, 1; Z_2, d)$].

I. INTRODUCTION

We used to think that different phases of matter all come from spontaneous symmetry breaking[1,2]. In last 30 years, we started to realize that even without symmetry and without symmetry breaking, we can still have different phases of matter, due to a new type of order – topological order[3,4] (*i.e.* patterns of long range entanglement[5–7]).

If there is no symmetry breaking nor topological order, it appears that systems must be in the same trivial phase. So it was a surprise to find that even without symmetry breaking and without topological order, systems can still have distinct phases, which are called Symmetry-Protected Trivial (SPT), or synonymously, Symmetry-Protected Topological (SPT) phases[8,9]. The realization of the existence of SPT orders and the fact that there is no topological order in 1+1D[10,11] lead to a classification of all 1+1D gapped phases of bosonic and fermionic systems with any symmetries[12–15], in terms of projective representations[16]. It is the first time, after Landau symmetry breaking, that a large class of interacting phases are completely classified.

In higher dimensions, the SPT orders, or more generally symmetric invertible topological (SIT) orders,[27] in bosonic systems can be systematically described by group cohomology theory[28–30], cobordism theory[19,31], or generalized cohomology theory[19,32]. The SPT and SIT orders in fermionic systems can be systematically described by group super-cohomology theory[17,33–36], or spin cobordism theory[18–20]. In 2+1D, the SPT orders in bosonic or fermionic systems can also be systematically classified by the modular extensions of $\mathcal{R}\text{ep}(G_b)$ or $\text{s}\mathcal{R}\text{ep}(G_f)$[37]. Here $\mathcal{R}\text{ep}(G_b)$ is the symmetric fusion category formed by representations of the boson symmetry G_b where all representations are bosonic, and $\text{s}\mathcal{R}\text{ep}(G_f)$ is the symmetric fusion category formed by representations of the fermion symmetry $G_f = Z_2^f \leftthreetimes G_b$ where the representations with non-zero Z_2^f charge are fermionic. ($Z_2^f \leftthreetimes G_b$ denote an extension of G_b by the fermion-number-parity symmetry Z_2^f.)

For SPT orders in fermionic systems, the modular extension approach in 2+1D can handle generic fermion symmetry $G_f = Z_2^f \leftthreetimes G_b$. However, in higher dimensions, the group super-cohomology theory can only handle a special form of fermion symmetry $G_f = Z_2^f \times G_b$. In this paper, we will develop a more general group super-cohomology theory for SPT orders of fermion systems based on the decoration construction[38] by fermions,[17] which covers generic fermion symmetry G_f beyond $Z_2^f \times G_b$. The symmetry group G_f can also include time reversal symmetry, and in this case, the fermions can be time-reversal singlet or Kramers doublet. Our approach works in any dimensions. But our theory does not covers the fermionic SPT orders obtained by decorating symmetry line-defects[33,35,36] with Majorana chains (*i.e.* the p-wave topological superconducting chains[39]).

Our theory is constructive in nature. We have constructed exactly soluble local fermionic path integrals (in the bosonized form) to realize the fermionic SPT orders systematically. The simple physical results of this paper is summarized in Table I. A breif mathematical summary of the results is represented Sections III and XVII (and in Section VIII C where more details are given). However, one needs to use mathematical language of cohomology or higher group to state the results precisely.

We note that there are seven non-trivial fermionic $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$-SPT phases in 3+1D, while non-interacting fermions only realize one of them. Other SPT phases are obtained by stacking the bosonic $(U_1 \rtimes_\phi Z_2^T)$-SPT phases formed by electron-hole pairs.

G_f	$1+1$D	$2+1$D	$3+1$D	Realization
$Z_2\times Z_2^f$	$\mathbb{Z}_2$ [$\mathbb{Z}_2$] ($\mathbb{Z}_2$)	$\mathbb{Z}_4$ [$\mathbb{Z}_8$] ($\mathbb{Z}$)	1 [1] (1)	Double-layer superconductors with layer symmetry
Z_4^f	1 [?] (1)	1 [?] (1)	1 [?] (1)	Charge-$2e$ superconductors with a 180° spin rotation symmetry or charge-$4e$ superconductors
$Z_2^T\times Z_2^f$	$\mathbb{Z}_4$ [$\mathbb{Z}_4$] ($\mathbb{Z}$)	1 [1] (1)	1 [1] (1)	Charge-$2e$ superconductors with coplanar spin order
$Z_4^{T,f}$	$\mathbb{Z}_2$ [$\mathbb{Z}_2$] ($\mathbb{Z}_2$)	1 [$\mathbb{Z}_2$] ($\mathbb{Z}_2$)	$\mathbb{Z}_4$ [$\mathbb{Z}_{16}$] ($\mathbb{Z}$)	Charge-$2e$ superconductors with spin-orbital coupling
$(U_1^f\rtimes_\phi Z_4^{T,f})/Z_2$	1 [1] (1)	1 [$\mathbb{Z}_2$] ($\mathbb{Z}_2$)	$\mathbb{Z}_2^3$ [$\mathbb{Z}_2^3$] ($\mathbb{Z}_2$)	Insulator with spin-orbital coupling
SU_2^f	1 [1] (1)	$\mathbb{Z}$ [$\mathbb{Z}$] ($\mathbb{Z}$)	1 [1] (1)	Charge-$2e$ spin-singlet superconductor
$Z_2\times Z_4\times Z_2^f$	$\mathbb{Z}_2^3$ [?] ($\mathbb{Z}_2^7$)	$8\cdot 16$ [?] ($\mathbb{Z}^7$)	$2\cdot 4$ [$\mathbb{Z}_2\times\mathbb{Z}_4$] (1)	

TABLE I. A table for G_f-symmetric fermionic SPT orders obtained via the fermion decoration.[17] We either list the group that describes the SPT phases or the number of SPT phases (both including the trivial one). Those fermionic SPT phases have no topological order, *i.e.* they become trivial if we break the symmetry down to Z_2^f. The numbers in [ ] are results from spin-cobordism approach[18–23]. The numbers in ( ) are for non-interacting fermionic SPT phases,[24–26]. Note that the numbers given in Ref. 18, 24–26 are for SIT orders which include both fermionic SPT orders and invertible fermionic topological orders, while the above numbers only include fermionic SPT orders. The number, for example, $8\cdot 16$ means that the 128 SPT phases can be divided into 8 classes with 16 SPT phases in each class. The SPT phases in the same class only differ by stacking bosonic SPT phases from fermion pairs. The last column indicates how to realize those fermionic SPT phases by electronic systems.

II. NOTATIONS AND CONVENTIONS

Let us first explain some notations used in this paper. We will use extensively the mathematical formalism of cochains, coboundaries, and cocycles, as well as their higher cup product $\underset{k}{\smile}$, Steenrod square Sq^k, and the Bockstein homomorphism β_n. A brief introduction can be found in Appendix A. We will abbreviate the cup product $a\smile b$ as ab by dropping $\smile$. We will use a symbol with bar, such as $\bar a$ to denote a cochain on the classifying space $\mathcal{B}$ of a group or higher group. We will use a to denote the corresponding pullback cochain on space-time $\mathcal{M}^{d+1}$: $a=\phi^*\bar a$, where ϕ is a homomorphism of complexes $\phi:\mathcal{M}^{d+1}\to\mathcal{B}$. In this paper, when we say $\mathbb{R}/\mathbb{Z}$-valued cocycle or coboundary we really mean $\mathbb{R}/\mathbb{Z}$-valued almost-cocycle and almost-coboundary (see Appendix B).

We will use $\overset{n}{=}$ to mean equal up to a multiple of n, and use $\overset{\mathrm{d}}{=}$ to mean equal up to $\mathrm{d}f$ (*i.e.* up to a coboundary). We will use $\lfloor x\rfloor$ to denote the largest integer smaller than or equal to x, and $\langle l,m\rangle$ to denote the greatest common divisor of l and m ($\langle 0,m\rangle\equiv m$).

Also, we will use $Z_n=\{1,\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}},\mathrm{e}^{\mathrm{i}\,2\frac{2\pi}{n}},\cdots,\mathrm{e}^{\mathrm{i}\,(n-1)\frac{2\pi}{n}}\}$ to denote an Abelian group, where the group multiplication is "$*$". We use $\mathbb{Z}_n=\{\lfloor-\frac{n}{2}+1\rfloor,\lfloor-\frac{n}{2}+1\rfloor+1,\cdots,\lfloor\frac{n}{2}\rfloor\}$ to denote an integer lifting of Z_n, where "$+$" is done without mod-n. In this sense, $\mathbb{Z}_n$ is not a group under "$+$". But under a modified equality $\overset{n}{=}$, $\mathbb{Z}_n$ is the Z_n group under "$+$". Similarly, we will use $\mathbb{R}/\mathbb{Z}=(-\frac12,\frac12]$ to denote an $\mathbb{R}$-lifting of U_1 group. Under a modified equality $\overset{1}{=}$, $\mathbb{R}/\mathbb{Z}$ is the U_1 group under "$+$". In this paper, there are many expressions containing the addition "$+$" of $\mathbb{Z}_n$-valued or $\mathbb{R}/\mathbb{Z}$-valued, such as $a_1^{\mathbb{Z}_n}+a_2^{\mathbb{Z}_n}$ where $a_1^{\mathbb{Z}_n}$ and $a_2^{\mathbb{Z}_n}$ are $\mathbb{Z}_n$-valued. Those additions "$+$" are done without mod n or mod 1. In this paper, we also have expressions like $\frac1n a_1^{\mathbb{Z}_n}$. Such an expression convert a $\mathbb{Z}_n$-valued $a_1^{\mathbb{Z}_n}$ to a $\mathbb{R}/\mathbb{Z}$-valued $\frac1n a_1^{\mathbb{Z}_n}$, by viewing the $\mathbb{Z}_n$-value as a $\mathbb{Z}$-value. (In fact, $\mathbb{Z}_n$ is a $\mathbb{Z}$ lifting of Z_n.)

We introduced a symbol $\leftthreetimes$ to construct fiber bundle X from the fiber F and the base space B:

$$pt\to F\to X=F\leftthreetimes B\to B\to pt. \tag{1}$$

We will also use $\leftthreetimes$ to construct group extension of H by N[40]:

$$1\to N\to N\leftthreetimes_{e_2,\alpha}H\to H\to 1. \tag{2}$$

Here $e_2\in H^2[H;Z(N)]$ and $Z(N)$ is the center of N. Also H may have a non-trivial action on $Z(N)$ via $\alpha: H\to\mathrm{Aut}(N)$. e_2 and α characterize different group extensions.

We will also use the notion of higher group in some part of the paper. Here we will treat a d-group $\mathcal{B}(\Pi_1,1;\Pi_2,2;\cdots)$ as a special one-vertex triangulation of a manifold K that satisfy $\pi_n(K)=\Pi_n$ (with unlisted Π_n treated as 0, see Ref. 41 and Appendix L). We see that Π_1 is a group and Π_n, $n\geq 2$, are Abelian groups. The 1-group $\mathcal{B}(G,1)$ is nothing but an one-vertex triangulation of the classifying space of G. We will abbreviate $\mathcal{B}(G,1)$ as $\mathcal{B}G$. (More precisely, the so-called one-vertex triangulation is actually a simplicial set.)

III. A BRIEF MATHEMATICAL SUMMARY

In this paper, we use a higher dimensional bosonization[35,42] to describe local fermion systems in $d+1$-dimensional space-time via a path integral on a random space-time lattice (which is called a space-time complex that triangulates the space-time manifold). This allows us to construct exactly soluble path integrals on space-time complexes based on fermion decoration construction[17] to systematically realize a large class of fermionic SPT orders with a generic fermion symmetry $G_f = Z_2^f \leftthreetimes G_b$. This generalizes the previous result of Ref. 17 and 33 that only deal with fermion symmetry of form $G_f = Z_2^f \times G_b$. The constructed models are exactly soluble since the partition functions are invariant under any re-triangulation of the space-time.

The constructed exactly soluble path integrals and the corresponding fermionic SPT phases are labeled by some data. Those data can be described in a compact form using terminology of higher group $\mathcal{B}(\Pi_1, 1; \Pi_2, 2; \cdots)$ (see Appendix L for details). We note that, for a d-group $\mathcal{B}_f(Z_2, 2; Z_2, d)$ (*i.e.* a complex with only one vertex), its triangles are labeled by group elements $\mathbb{Z}_2$. This gives rise to the so called *canonical 2-cochain* $\bar{e}_2$ on the complex $\mathcal{B}_f(Z_2, 2; Z_2, d)$. On each d-simplex in $\mathcal{B}(Z_2, 2; Z_2, d)$ we also have a $\mathbb{Z}_2$ label. This gives us the canonical $\mathbb{Z}_2$-valued d-cochain $\bar{f}_d$ on the complex $\mathcal{B}_f(Z_2, 2; Z_2, d)$. Here $\mathcal{B}_f(Z_2, 2; Z_2, d)$ is a particular higher group characterized by

$$\mathrm{d}\bar{e}_2 \overset{2}{=} 0, \qquad \mathrm{d}\bar{f}_d \overset{2}{=} 0. \tag{3}$$

Such a higher group is uniquely determined by the above conditions. Now, we are ready to state our results:

1. **The data**: For unitary symmetry $G_f = Z_2^f \leftthreetimes_{e_2} G_b$, the fermionic SPT phases obtained via fermion decoration are described by a pair $(\varphi, \bar{\nu}_{d+1})$, where

   (a) $\varphi : \mathcal{B}G_b \to \mathcal{B}_f(Z_2, 2; Z_2, d)$ is a homomorphism between two higher groups and

   (b) $\bar{\nu}_{d+1}$ is a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{B}G_b$ that trivializes the pullback of a $\mathbb{R}/\mathbb{Z}$-valued $d+2$-cocycle $\bar{\omega}_{d+2} = \frac{1}{2}\mathrm{Sq}^2\bar{f}_d + \frac{1}{2}\bar{f}_d\bar{e}_2$ on $\mathcal{B}_f(Z_2, 2; Z_2, d)$, *i.e.* $-\mathrm{d}\bar{\nu}_{d+1} = \varphi^*\bar{\omega}_{d+2}$.

2. **Model construction and SPT invariant**: Using the data $(\varphi, \bar{\nu}_{d+1})$, we can write down the explicit path integral that describes a local fermion model (in bosonized form) that realizes the corresponding SPT phase (see (45)). The path integral can be evaluated exactly, which leads to the SPT invariant[18,30,31,43,44] that characterize the resulting fermionic SPT phase (see (48)). The bosonized fermion path integral, (45), and the corresponding SPT invariants, (48), are the main results of this paper.

3. **Equivalence relation**: Only the pairs $(\varphi, \bar{\nu}_{d+1})$ that give rise to distinct SPT invariants correspond to distinct SPT phases. The pairs (φ, ν_{d+1}) that give rise to the same SPT invariant are regarded as equivalent. In particular, two homotopically connected $(\varphi, \bar{\nu}_{d+1})$'s are equivalent. So our data is really the homotopy classes of the trivializations of $\frac{1}{2}\mathrm{Sq}^2\bar{f}_d + \frac{1}{2}\bar{f}_d\bar{e}_2$ on $\mathcal{B}_f(Z_2, 2; Z_2, d)$ by the homomorphism $\varphi : \mathcal{B}G_b \to \mathcal{B}_f(Z_2, 2; Z_2, d)$.

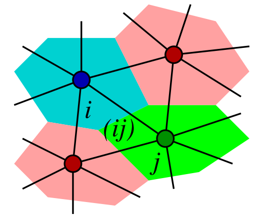

FIG. 1. (Color online) The vertices i and j mark the regions with order parameter g_i and g_j. The link labeled by (ij) connects the vertices i and j.

The data $(\varphi, \bar{\nu}_{d+1})$ give rise to the fermion SPT states obtained via fermion decoration. But they do not include the fermion SPT states obtained via decoration of chains of 1+1D topological p-wave superconducting states.

IV. EXACTLY SOLUBLE MODELS FOR BOSONIC SPT PHASES

Let us first review the construction of exactly soluble models for bosonic SPT orders with on-site symmetry group G_b[28]. The same line of thinking will also be used in our discussions of fermionic SPT phases.

A. Constructing path integral

We start with a phase that breaks the G_b symmetry completely. Then we consider the quantum fluctuations of the G_b-symmetry-breaking domains which restore the symmetry. We mark each domain with a vertex (see Fig. 1), which form a space-time complex $\mathcal{M}^{d+1}$ (see Appendix A for details). So the quantum fluctuations of the domains are described by the degrees of freedom given by $g_i \in G_b$ on each vertex i. In other words, the degrees of freedom is a G_b-valued 0-cochain field $g \in C^0(\mathcal{M}^{d+1}; G_b)$.

In order to probe the SPT orders in a universal way, we can add the symmetry twist to the system (*i.e.* gauging the symmetry)[31,43–45]. This is done by adding a fixed G_b-valued 1-cochain field $A \in C^1(\mathcal{M}^{d+1}; G_b)$ with $A_{ij}^{G_b} \in G_b$ on each link (ij) that connect two vertices i and j. The cochain field A satisfy the flat condition

$$(\delta A^{G_b})_{ijk} \equiv A_{ij}^{G_b} A_{jk}^{G_b} A_{ki}^{G_b} = 1, \tag{4}$$

where we have assumed $A_{ij}^{G_b} = (A_{ji}^{G_b})^{-1}$. Those flat 1-cochain field will be called 1-cocycle field. The collection of those G_b-valued 1-cocycle fields will be denoted by $Z^1(\mathcal{M}^{d+1}, G_b)$. We stress that the 1-cocycle field A is a fixed background field that do not fluctuate.

From the dynamical g and non-dynamical background field A^{G_b}, we can construct an effective dynamic G_b-valued 1-cocycle field a^{G_b} whose values on links are given by

$$a_{ij}^{G_b} = g_i A_{ij}^{G_b} g_j^{-1}. \tag{5}$$

Using such an effective dynamic field, we can construct our model as

$$Z(\mathcal{M}^{d+1}, A^{G_b}) = \sum_{g \in C^0(\mathcal{M}^{d+1};G_b)} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})}, \quad (6)$$

where $\omega_{d+1}(a^{G_b})$ is $\mathbb{R}/\mathbb{Z}$-valued d-cochain: $\omega_{d+1}(a^{G_b}) \in C^{d+1}(\mathcal{M}^{d+1};\mathbb{R}/\mathbb{Z})$, whose value on a $d+2$-simplex $(i_0 i_1 \cdots i_{d+1})$ is a function of $a^{G_b}_{i_0i_1}, a^{G_b}_{i_0i_2}, a^{G_b}_{i_1i_2}, \cdots$: $(\omega_{d+1})_{i_0i_1\cdots i_{d+1}} = \omega_{d+1}(a^{G_b}_{i_0i_1}, a^{G_b}_{i_0i_2}, a^{G_b}_{i_1i_2}, \cdots)$. Note that a^{G_b} also satisfies the flat condition

$$(\delta a^{G_b})_{ijk} \equiv a^{G_b}_{ij} a^{G_b}_{jk} a^{G_b}_{ki} = 1. \quad (7)$$

So we say $a^{G_b} \in Z^1(\mathcal{M}^{d+1};G_b)$. In this case the function $\omega_{d+1}(a^{G_b}_{i_0i_1}, a^{G_b}_{i_0i_2}, a^{G_b}_{i_1i_2}, \cdots)$ only depends on $a^{G_b}_{i_0i_1}, a^{G_b}_{i_1i_2}, a^{G_b}_{i_2i_3}$, *etc* , since other variables are determined from those variables:

$$(\omega_{d+1})_{i_0i_1\cdots i_{d+1}} = \omega_{d+1}(a^{G_b}_{i_0i_1}, a^{G_b}_{i_1i_2}, a^{G_b}_{i_2i_3}, \cdots). \quad (8)$$

We note that the assignment of $a^{G_b}_{ij} \in G_b$ on each link (ij) can be viewed as a map ϕ (a homomorphism of complexes) from space-time complex $\mathcal{M}^{d+1}$ to $\mathcal{B}G_b$ which is a simplicial complex that model the classifying space BG_b of the group G_b. The $d+1$-cochain $\omega_{d+1}(a^{G_b})$ can be viewed as a pull back of a cochain $\bar\omega_{d+1}(\bar a^{G_b})$ in the classifying space $\mathcal{B}G_b$: $\bar\omega_{d+1}(\bar a^{G_b}) \in C^{d+1}(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$. Here $\bar\omega_{d+1}(\bar a^{G_b})$ is a function of the canonical 1-cochain $\bar a^{G_b}$. Note that links in $\mathcal{B}G_b$ are labeled by elements of G_b, which give rise to the canonical 1-cochain $\bar a^{G_b}$ on $\mathcal{B}G_b$ (see Ref. 41 and Appendix L). The above can be written as

$$a^{G_b} = \phi^* \bar a^{G_b},$$
$$\omega_{d+1}(a^{G_b}) = \phi^* \bar\omega_{d+1}(\bar a^{G_b}). \quad (9)$$

B. Making path integral exactly soluble

To make the model exactly soluble, we require ω_{d+1} to be a pullback of a cocycle $\bar\omega_{d+1}$ in the classifying space BG_b:

$$\mathrm{d}\bar\omega_{d+1}(\bar a^{G_b}) \overset{1}{=} 0 \quad \text{or} \quad \bar\omega_{d+1}(\bar a^{G_b}) \in Z^{d+1}(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z}). \quad (10)$$

Why the above condition make our model exactly soluble? Let us compare two action amplitudes $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})}$ and $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b\prime})}$ for two different field values a^{G_b} and $a^{G_b\prime}$. We like to show that if a^{G_b} and $a^{G_b\prime}$ can homotopically deform into each other, then $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})} = \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b\prime})}$. But a^{G_b} and $a^{G_b\prime}$ are discrete fields on discrete lattice. It seems that they can never homotopically deform into each other, in the usual sense.

To define the homotopical deformation for discrete fields on discrete lattice, we try to find a flat connection $\tilde a^{G_b}$ on a complex $\mathcal{M}^{d+1} \times I$ in one higher dimension, such that $\tilde a^{G_b} = a^{G_b}$ on one boundary of $\mathcal{M}^{d+1} \times I$, and $\tilde a^{G_b} = a^{G_b\prime}$ on the other boundary of $\mathcal{M}^{d+1} \times I$. If such a field $\tilde a^{G_b}$ exists, then we say a^{G_b} and $a^{G_b\prime}$ can homotopically deform into each other. In this case, we find that

$$\frac{\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b\prime})}}{\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})}} = \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}\times I} \mathrm{d}\omega_{d+1}(\tilde a^{G_b})}$$
$$= \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\phi(\mathcal{M}^{d+1}\times I)} \mathrm{d}\bar\omega_{d+1}(\bar a^{G_b})}, \quad (11)$$

where ϕ is a homomorphism from $\mathcal{M}^{d+1} \times I$ to $\mathcal{B}G_b$. Therefore $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})} = \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b\prime})}$ if $\mathrm{d}\bar\omega_{d+1}(\bar a^{G_b}) \overset{1}{=} 0$ and if a^{G_b} can homotopically deform into $a^{G_b\prime}$ without breaking the flat condition (7).

Let us define a^{G_b} and $a^{G_b\prime}$ to be equivalent, if they are related by

$$a^{G_b\prime}_{ij} = h_i a^{G_b}_{ij} h_j^{-1}, \quad h_i \in G_b. \quad (12)$$

Clearly, gauge equivalent configurations can always homotopically deform into each other. We believe the reverse is also true: two configurations that can homotopically deform into each other are always gauge equivalent.

Clearly, the action amplitude $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})}$ only depends on the gauge equivalent classes of the field configurations a^{G_b}. In fact

$$\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_b})} = \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(A^{G_b})} \quad (13)$$

which is independent of g_i in eqn. (5).

Since the gauge fluctuations represent all the fluctuations of a^{G_b} which is described by g_i in eqn. (5), there is only one equivalent classe for a fixed A^{G_b}. Thus path integral eqn. (6) is given by

$$Z(\mathcal{M}^{d+1}, A^{G_b}) = |G_b|^{N_v} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(A^{G_b})}, \quad (14)$$

and is thus exactly soluble.

The space $\mathcal{Z}^{d+1}(G_b;\mathbb{R}/\mathbb{Z})$ of the cocylces $\omega_{d+1}(a^{G_b})$ is not connected. Each connected piece describes models in the same phase. Thus the different phases of the models are given by $\pi_0[\mathcal{Z}^{d+1}(G_b;\mathbb{R}/\mathbb{Z})] = H^{d+1}(\mathcal{B}G_b,\mathbb{R}/\mathbb{Z})$. Those models do not spontaneously break the G_b-symmetry. Without symmetry twist $A^{G_b}_{ij} = 1$, the volume-independent partition function $Z^{\text{top}}(\mathcal{M}^{d+1})$[46,47] of the above models is $Z^{\text{top}}(\mathcal{M}^{d+1}) = 1$ for any closed oriented manifolds. So the models have no topological orders, and realize only SPT orders. We see that the SPT orders described by those models are labeled by $H^{d+1}(\mathcal{B}G_b,\mathbb{R}/\mathbb{Z})$.

We believe that different classes $[\omega_{d+1}] \in \mathcal{H}^{d+1}(G_b;\mathbb{R}/\mathbb{Z})$ can give rise to different topological invariant for space-time $\mathcal{M}^{d+1}$ decorated with a flat G_b connection A^{G_b}, *i.e.* for two different $[\omega_{d+1}]$ and $[\omega'_{d+1}]$, we can fine the pairs $(\mathcal{M}^{d+1}, A^{G_b})$ such that $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(A^{G_b})}$ and $\mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega'_{d+1}(A^{G_b})}$ are different. Thus different classes, $[\omega_{d+1}]$'s, characterize different exactly soluble models.

C. Including time reversal symmetry Z_2^T

In the above, we did not include time reversal symmetry described by group Z_2^T. In the presence of time reversal symmetry the full symmetry group G_b is an extension of Z_2^T: $G_b = G_b^0 \rightthreetimes Z_2^T$, where G_b^0 is the unitary on-site symmetry. In this case, the dynamical variable is still $g_i \in G_b$, the symmetry twist is still describe by $A_{ij}^{G_b} \in G_b$. But the symmetry twist satisfy a constraint: The natural projection $G_b \to Z_2^T$ reduce the G_b connection $A_{ij}^{G_b}$ to a Z_2^T connection $A_{ij}^T \in Z_2^T$. A_{ij}^T describes a Z_2 bundle over the space-time $\mathcal{M}^{d+1}$. The tangent bundle of space-time $\mathcal{M}^{d+1}$ give rise to a O_{d+1} bundle over $\mathcal{M}^{d+1}$. From the O_{d+1} bundle we can get its determinant bundle, which is also a Z_2 bundle over $\mathcal{M}^{d+1}$. Such a Z_2 bundle must be the homotopic equivalent to the Z_2 bundle described by the Z_2^T connection A_{ij}^T. In other words,

$$A^T \overset{2,\mathrm{d}}{=} \mathrm{w}_1, \tag{15}$$

where w_n is the n^{th} Stiefel-Whitney class of $\mathcal{M}^{d+1}$. In addition to the above constraint, we also require the Lagrangian $2\pi\omega_{d+1}(a^{G_b})$ to have a time reversal symmetry. As pointed out in Ref. 28, this can be achieved by requiring $\omega_{d+1}(a^{G_b})$ to be a $\mathbb{R}/\mathbb{Z}$-valued cocycle where Z_2^T has a non-trivial action on the value $\mathbb{R}/\mathbb{Z} \to -\mathbb{R}/\mathbb{Z}$.

We see that, to include time reversal symmetry, we need to extend a space-time symmetry, space-time refection Z_2^T, by the internal symmetry G_b^0 to obtain the full symmetry group G_b. As shown in Ref. 28 and 31, in this case, the SPT states are labeled by the elements in $H^{d+1}(\mathcal{B}G_b, (\mathbb{R}/\mathbb{Z})_T)$ where time reversal in G_b has a non-trivial action on the value $T : \mathbb{R}/\mathbb{Z} \to -\mathbb{R}/\mathbb{Z}$. Also, the differential operator d should be understood as the one with this non-trivial action.

D. Classification of bosonic SPT phases

However, $H^{d+1}(\mathcal{B}G_b, (\mathbb{R}/\mathbb{Z})_T)$ fail to cover all bosonic SPT orders[29]. It misses the SPT orders obtained by decorating[38] symmetry defects with the invertible bosonic topological orders[19,31,46]. This problem can be fixed if we replace G_b by $G_{bO} = G_b^0 \rightthreetimes O_\infty$, where O_n is the n-dimensional orthogonal group. This generalizes the extension $G_b = G_b^0 \rightthreetimes Z_2^T$ discussed in the last section to include time reversal symmetry. The time reversal symmetry is included in O_∞ as the disconnected component, which is denoted by Z_2^T.

After replacing G_b by G_{bO}, we can obtain local bosonic models that realize more general SPT states, as well as the bosonic SIT orders[30]:

$$Z(\mathcal{M}^{d+1}) = \sum_{g \in C^0(\mathcal{M}^{d+1};G_{bO})} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_{bO}})}, \tag{16}$$

where $(a^{G_{bO}})_{ij} = g_i A_{ij}^{bO} g_j^{-1}$ and $A_{ij}^{bO} \in G_{bO}$. Also $\omega_{d+1}(a^{G_{bO}})$ is the pullback of $\bar\omega_{d+1}(\bar a^{G_{bO}}) \in \mathcal{Z}^{d+1}[\mathcal{B}G_{bO}; (\mathbb{R}/\mathbb{Z})_T]$.

We like to stress that the A_{ij}^{bO} is not the most general connection of a G_{bO} bundle on $\mathcal{M}^{d+1}$. We may project $A_{ij}^{bO} \in G_{bO}$ to $A_{ij}^{O} \in O_\infty$ via the natural projection $G_{bO} \to O_\infty$. The resulting A_{ij}^{O} describes a O_∞ bundle on $\mathcal{M}^{d+1}$, and such a O_∞ bundle must be the tangent bundle of $\mathcal{M}^{d+1}$ (extended by a trivial bundle).

The different exactly soluble models are labeled by the elements in $H^{d+1}[\mathcal{B}G_{bO}; (\mathbb{R}/\mathbb{Z})_T]$, where G_{bO} has a non-trivial action on the value $\mathbb{R}/\mathbb{Z}$, to ensure the time-reversal symmetry of the Lagrangian[28]. However, since the O_∞ part of A_{ij}^{bO} is only the connection of tangent bundle of $\mathcal{M}^{d+1}$, different cohomology classes may give rise to the same volume-independent partition function[46,47] (*i.e.* topological invariant) for those limited choices of A_{ij}^{bO}. As a result, different cohomology classes in $H^{d+1}[\mathcal{B}G_{bO}; \mathbb{R}/\mathbb{Z}]$ may give rise to the same G_b-SPT order. Thus, $H^{d+1}[\mathcal{B}G_{bO}; \mathbb{R}/\mathbb{Z}]$ provides a many-to-one label of bosonic SPT orders in any dimensions and for any on-site symmetries[30].

As pointed out in Ref. 30, adding O_∞ has the same effect as decorating symmetry membrane-defects with E_8^3 quantum Hall states[29,38]. In other words, our models not only contain the fluctuating field g_i, they also contain the fluctuations of membrane object formed by E_8^3 quantum Hall states. The E_8 quantum Hall state is described by the following 8-layer K-matrix wave function[48] in 2D space (with coordinate $z = x + \mathrm{i}y$)

$$\Psi(\{z_i^I\}) = \prod_{i<j,I} (z_i^I - z_j^I)^{K_{II}} \prod_{i,j,I<J} (z_i^I - z_j^J)^{K_{IJ}}\, \mathrm{e}^{-\sum \frac{|z_i^I|^2}{4}} \tag{17}$$

$$K = \begin{pmatrix} \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{2} \end{pmatrix}, \tag{18}$$

It has gapless chiral edge excitation with chiral central charge $c = 8$. The E_8^3 membrane is formed by stacking three E_8 quantum Hall state together.

Another important question to ask is that whether all the G_b-SPT states can be obtained this way. It is known that G_{bO} cocycle models (or $H^{d+1}(\mathcal{B}G_{bO}; \mathbb{R}/\mathbb{Z})$) do not produce all the H-type[46] bosonic invertible topological orders in 2+1D. But $H^{d+1}(\mathcal{B}G_{bO}; \mathbb{R}/\mathbb{Z})$ may classify all the L-type[46] SPT phases with G_b symmetry in a many-to-one way.

Similarly, if we do not have time reversal symmetry, more general bosonic SPT states can be constructed by choosing the dynamical variables on each vertex to be $g_i \in G_{bSO} = G_b \rightthreetimes SO_\infty$. In this case, the effective variable in the link will be $a^{G_{bSO}} \in G_{bSO}$. The corresponding local bosonic models that can produce more general G_b-SPT phases for bosons are given by

$$Z(\mathcal{M}^{d+1}) = \sum_{g\in C^0(\mathcal{M}^{d+1};G_{bSO})} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \omega_{d+1}(a^{G_{bSO}})}, \quad (19)$$

where $\omega_{d+1}(a^{G_{bSO}})$ is the pullback of $\bar\omega_{d+1}(\bar a^{G_{bSO}}) \in H^{d+1}(\mathcal{B}G_{bSO};\mathbb{R}/\mathbb{Z})$. The cohomology classes in $H^{d+1}(\mathcal{B}G_{bSO};\mathbb{R}/\mathbb{Z})$ give rise to a many-to-one classification of bosonic SPT orders which contain decoration of bosonic invertible topological orders, such as the E_8^3 states.

V. BOSONIZATION OF FERMIONS IN ANY DIMENSIONS WITH ANY SYMMETRY $G_f = Z_2^f \rtimes G_b$

In this section, we will construct exactly soluble models to realize fermionic SPT phases. We will use high dimensional bosonization[35,42] and use the approach in the last section to construct exactly soluble path integrals. Our discussion in this section is similar to that in Ref. 34, but with a generalization at one point, so that the formalism can be applied to study fermionic SPT phases with generic fermionic symmetry $G_f = Z_2^f \rtimes G_b$ beyond $G_f = Z_2^f \times G_b$.

A. 3+1D cochain models for fermion

Let us first construct a cochain model that describes fermion system in 3+1D. A world line of the fermions is dual to a $\mathbb{Z}_2$-valued 3-cocycle f_3. So we will use f_3 as the field to describe the dynamics of the fermions.

There are several ways to make f_3 describe fermionic particles. The first way only works when f_3 is a coboundary. In this case, we can do a 3+1D statistics transmutation[35,42], *i.e.* by adding a term $\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4} b^2 + b \underset{1}{\smile} \mathrm{d}b}$, $\mathrm{d}b = f_3$ in the action amplitude. So our model has the form

$$\begin{aligned} Z(\mathcal{M}^4) &= \sum_{f_3\in B^3(\mathcal{M}^4;\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4} \mathcal{L}(f_3) + \frac12 [b^2 + b\underset{1}{\smile}\mathrm{d}b]} \\ &= \sum_{f_3\in B^3(\mathcal{M}^4;\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4} \mathcal{L}(f_3) + \frac12 \mathbb{Sq}^2 b}, \end{aligned} \quad (20)$$

where the path integral is a summation of the coboundaries $f_3 \in B^3(\mathcal{M}^4;\mathbb{Z}_2)$. Here $\mathcal{L}(f_3)$ is a $\mathbb{R}/\mathbb{Z}$-valued 4-cochain that depends on the field f_3, which can be viewed as the Lagrangian density of our model. Different choices of $\mathcal{L}(f_3)$ will give rise to different models. In the above, $\mathbb{Sq}^2$ defined in (A24) is a square operation acting on cochains . It coincides with the Steenrod square when acting on cohomology classes

$$\mathbb{Sq}^k x = \mathrm{Sq}^k x, \qquad \text{if } \mathrm{d}x = 0. \quad (21)$$

The term $\frac12 \mathbb{Sq}^2 b$ is included to make f_3 to describe fermions. Here b is a 2-cochain that is a function of f_3 as determined by $\mathrm{d}b = f_3$. However, there are many different b's that satisfy $\mathrm{d}b = f_3$. We hope those different b all give rise to the same action amplitude[34]. To see this, let us change b by a 2-cocycle $b_0 \in Z^2(\mathcal{M}^4;\mathbb{Z}_2)$. We find that (using (A28))

$$\mathbb{Sq}^2(b+b_0) - \mathbb{Sq}^2 b \overset{2,\mathrm{d}}{=} \mathrm{Sq}^2 b_0 \overset{2,\mathrm{d}}{=} (\mathrm{w}_2 + \mathrm{w}_1^2) b_0, \quad (22)$$

where we have used $\mathrm{Sq}^2 b_0 \overset{2,\mathrm{d}}{=} (\mathrm{w}_2+\mathrm{w}_1^2)b_0$ on $\mathcal{M}^4$ (see Appendix I). We see that different solutions of b will all give rise to the same action amplitude, provided that $\mathcal{M}^4$ is a Pin^- manifold satisfying $\mathrm{w}_2+\mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} 0$ (see Appendix K). Without time reversal symmetry, the space-time must orientable $\mathrm{w}_1 \overset{2,\mathrm{d}}{=} 0$. So the first way also requires the space-time to be spin manifold $\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$.

In the second way to make f_3 to describe fermion world lines, we do not require f_3 to be a coboundary, but we require the pair $(\mathcal{M}^4, f_3)$ can be extended to one higher dimensions. In other words, there is a 5-dimensional complex $\mathcal{N}^5$ and a cocycle $\tilde f_3$ on $\mathcal{N}^5$, such that $\mathcal{M}^4$ is the boundary of $\mathcal{N}^5$: $\mathcal{M}^4 = \partial\mathcal{N}^5$, and $\tilde f_3 = f_3$ when restricted on the boundary $\mathcal{M}^4$. In fact, we have a stronger requirement on $\mathcal{M}^4$:

> For all $\mathbb{Z}_2$-valued 3-cocycles f_3 on $\mathcal{M}^4$, the pair $(\mathcal{M}^4, f_3)$ can be extended to a pair $(\mathcal{N}^5, \tilde f_3)$ in one higher dimension.

We note that (see (A26))[34].

$$\mathrm{d}\mathbb{Sq}^2 b \overset{2}{=} \mathbb{Sq}^2\, \mathrm{d}b \overset{2}{=} \mathrm{Sq}^2 f_3. \quad (23)$$

So (20) can be rewritten as

$$Z(\mathcal{M}^4) = \sum_{f_3\in Z^3(\mathcal{M}^4;\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4}\mathcal{L}(f_3) + \mathrm{i}\,\pi\int_{\mathcal{N}^5} \mathrm{Sq}^2 \tilde f_3}. \quad (24)$$

The above expression directly depends on $\tilde f_3$. We do not need to solve b to define the action amplitude. This is the better way to write 3+1D statistics transmutation.

However, in order for (26) to define a path integral in 3+1D, the action amplitude $\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4}\nu_4(f_3) + \mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2\tilde f_3}$ must not depend[34] on how we extend f_3 on $\mathcal{M}^4$ to $\tilde f_3$ on $\mathcal{N}^5$. This requires that

$$\int_{\mathcal{N}^5} \mathrm{Sq}^2 \tilde f_3 \overset{2}{=} 0 \quad (25)$$

for any $\tilde f_3 \in Z^3(\mathcal{N}^5;\mathbb{Z}_2)$ and for any closed $\mathcal{N}^5$. We note that on $\mathcal{N}^5$, $\mathrm{Sq}^2 \tilde f_3 = (\mathrm{w}_1^2 + \mathrm{w}_2)\tilde f_3$ (see Appendix I). So the condition (25) cannot be satisfied.

To fix this problem, we rewrite the above partition function as

$$Z(\mathcal{M}^4, A^{\mathbb{Z}_2^f})$$

$$= \sum_{f_3 \in Z^3(\mathcal{M}^4;\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^4} \mathcal{L}(f_3) + A^{\mathbb{Z}_2^f} f_3 + \mathrm{i}\,\pi \int_{\mathcal{N}^5} \mathrm{Sq}^2 \tilde f_3 + (\mathrm{w}_2+\mathrm{w}_1^2)\tilde f_3}$$
$$\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 \text{ on } \mathcal{M}^4, \tag{26}$$

and restrict $\mathcal{M}^4$ to be Pin$^-$ manifold where $\mathrm{w}_1^2 + \mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$ (see Appendix K). When $\mathcal{N}^5$ is also a Pin$^-$ manifold, then eqn. (26) reduces to eqn. (24). In general,

$$\int_{\mathcal{N}^5} \mathrm{Sq}^2 \tilde f_3 + \tilde f_3(\mathrm{w}_2 + \mathrm{w}_1^2) \overset{2}{=} 0 \tag{27}$$

for any $\tilde f_3 \in Z^3(\mathcal{N}^5;\mathbb{Z}_2)$ and for any closed $\mathcal{N}^5$. So indeed, the action amplitude in eqn. (26) does not depend on how we extend f_3 on $\mathcal{M}^4$ to $\tilde f_3$ on $\mathcal{N}^5$.

We note that corrected partition depends on the Pin$^-$ structure described by $A^{\mathbb{Z}_2^f}$ that satisfy $\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2$ on $\mathcal{M}^4$. So the fermionic path integral can be defined on Pin$^-$ manifold[34] $\mathcal{M}^4$, that is a boundary of 5-dimensional complex $\mathcal{N}^5$ (which may not be Pin$^-$), and for f_3's that can be extended to $\mathcal{N}^5$.

When $\mathcal{L}(f_3)$ respect the time reversal symmetry, we may ask if the fermion is a time-reversal singlet or a Kramers doublet? The fact that the path integral can be defined on Pin$^-$ manifold implies that the fermions are time-reversal singlet[18,42].

We also like to remark that the two path integrals (20) and (26) are not exactly the same. In (20) the summation is over the coboundaries $f_3 \in B^3(\mathcal{M}^4;\mathbb{Z}_2)$, while in (26) the summation is over the cocycles $f_3 \in Z^3(\mathcal{M}^4;\mathbb{Z}_2)$, which is what we really want for a fermion path integral.

B. Bosonization of fermion models in any dimensions

The above bosonization of fermion models also works in other dimensions. In $d+1$D space-time, the fermion world line is described by d-cocycles $f_d \in Z^d(\mathcal{M}^{d+1}, \mathbb{Z}_2)$, after the Poincaré duality. The bosonized fermion model is given by

$$Z(\mathcal{M}^{d+1}, A^{\mathbb{Z}_2^f}) = \sum_{f_d \in Z^d(\mathcal{M}^{d+1};\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \mathcal{L}(f_d, e_2) + \frac{1}{2} f_d A^{\mathbb{Z}_2^f}} \mathrm{e}^{\,\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)},$$
$$\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + (\mathrm{w}_1)^2 + e_2 \text{ on } \mathcal{M}^{d+1}. \tag{28}$$

Here we have generalized the discussion in Ref. 34 by including an extra term e_2 in $\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + (\mathrm{w}_1)^2 + e_2$ where e_2 is a fixed $\mathbb{Z}_2$-valued 2-cocycle background field on $\mathcal{M}^{d+1}$. As we will see later that such a generalization allow us to study fermionic SPT phases with generic fermionic symmetry $G_f = Z_2^f \leftthreetimes G_b$.

At the moment, we only mention that, in the presence of time reversal symmetry, when $e_2 = 0$, the model is well defined on space-time with a Pin$^-$ structure, *i.e.* $\mathrm{w}_1^2 + \mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$ (see Appendix K), which means that the fermions are time-reversal singlet[42]. When $e_2 = \mathrm{w}_1^2$, the model is well defined on space-time with a Pin$^+$ structure, *i.e.* $\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$, which means that the fermions are Kramers doublet[42].

We see that to make the fermion model well defined, the path integral will depend on the twisted spin structure of the space-time $\mathcal{M}^{d+1}$. The above expression generalizes the one in Ref. 34 by including an extra term e_2 in $\mathrm{d}A^{\mathbb{Z}_2^f}$ which defines a twisted spin structure. As we will see later that such a generalization allows us to study fermionic SPT phases with generic fermionic symmetry $G_f = Z_2^f \leftthreetimes G_b$.

Let us end this section by describing in more detail how do we compute the partition function eqn. (28). We first start with a space-time $\mathcal{M}^{d+1}$, with a $\mathbb{Z}_2$ valued 1-cochain, and a $\mathbb{Z}_2$-valued 2-cocycle e_2 that satisfy $\mathrm{w}_2 + \mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} e_2$. Then for energy $\mathbb{Z}_2$-valued d-cocycle f_d, we can find an $\mathcal{N}^{d+2}$ and $\mathbb{Z}_2$-valued d-cocycle $\tilde f_d$ on $\mathcal{N}^{d+2}$ such that $\mathcal{M}^{d+1} = \partial \mathcal{N}^{d+2}$ and $f_d \overset{2}{=} \tilde f_d$ on $\mathcal{M}^{d+1}$. (We require $\mathcal{M}^{d+1}$ to have this property.) We also choose $\mathrm{w}_2 + \mathrm{w}_1^2$ on $\mathcal{N}^{d+2}$ such that $\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + (\mathrm{w}_1)^2 + e_2$ on $\mathcal{M}^{d+1}$. This choice is always possible. This allows us to compute the action amplitude and the partition function in eqn. (28), which are independent of the choices of $\mathcal{N}^{d+2}$, as well as $\tilde f_d$ and $\mathrm{w}_2 + (\mathrm{w}_1)^2$ on $\mathcal{N}^{d+2}$. This corresponds to a higher dimensional bosonization of a fermion system.

C. Bosonized fermion models with $G_f = Z_2^f \leftthreetimes G_b$ symmetry in $(d+1)$-dimensions

In this section, we try to construct models that describe fermionic SPT orders in d+1D with G_f symmetry. The fermion symmetry group $G_f = Z_2^f \leftthreetimes G_b$ is a central extension of G_b by fermion-number-parity symmetry Z_2^f. Such a central extension is characterized by a 2-cocycle $e_2 \in H^2(\mathcal{B}G_b;\mathbb{Z}_2)$ (see Appendix N). So we write, more precisely, $G_f = Z_2^f \leftthreetimes_{e_2} G_b$.

To construct fermionic models to realize G_f symmetric SPT phases, we can first break the boson symmetry G_b completely. We then consider the domain fluctuations of the symmetry breaking state to restore the symmetry. Just like the bosonic model discussed in the Section IV, such domain fluctuations are described by $g_i \in G_b$ on each vertex. The fermion world-lines are described by d-cocycle f_d as in the last subsection. After bosonization, such a fermion system is described by

$$Z(\mathcal{M}^{d+1}, A^{\mathbb{Z}_2}) = \sum_{g \in C^0(\mathcal{M}^{d+1};G_b);\, f_d \in Z^d(\mathcal{M}^{d+1};\mathbb{Z}_2)} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \mathcal{L}(g, f_d) + \frac{1}{2} f_d A^{\mathbb{Z}_2^f}} \mathrm{e}^{\,\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \tag{29}$$

where $\mathcal{N}^{d+2}$ is an extension of $\mathcal{M}^{d+1}$: $\mathcal{M}^{d+1} = \partial \mathcal{N}^{d+2}$, and $A^{\mathbb{Z}_2^f}$ is a twisted spin structure

$$\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2. \tag{30}$$

Now let us try to couple the above model to the G_b-symmetry twist described by G_b-valued 1-cocycle $A^{G_b} \in Z^1(\mathcal{M}^4, G_b)$, which satisfies the flat condition

$$(\delta A^{G_b})_{ijk} \equiv A^{G_b}_{ij} A^{G_b}_{jk} A^{G_b}_{ki} = 1 \tag{31}$$

In the presence of the background G_b-connection, the fermion model becomes

$$Z(\mathcal{M}^{d+1}, A^{G_b}, A^{\mathbb{Z}_2^f}) = \sum_{g\in C^0(\mathcal{M}^{d+1};G_b); f_d\in Z^d(\mathcal{M}^{d+1};\mathbb{Z}_2)} \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \mathcal{L}(g,f_d,A^{G_b})+\frac{1}{2} f_d A^{\mathbb{Z}_2^f}} \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)},$$
$$\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}). \tag{32}$$

where e_2 is a $\mathbb{Z}_2$-valued cocycle $e_2 \in H^2(\mathcal{B}G_b, \mathbb{Z}_2)$. More precisely the evaluation of e_2 on a triangle (ijk) is a function of A^{G_b} on link (ij) and (jk):

$$(e_2)_{ijk}(A^{G_b}_{ij}, A^{G_b}_{jk}). \tag{33}$$

This is the meaning of $e_2(A^{G_b})$. The Lagrangian $\mathcal{L}$ is a $\mathbb{R}/\mathbb{Z}$-valued $(d+1)$-cochain in $C^{d+1}(\mathcal{M}^{d+1}, \mathbb{R}/\mathbb{Z})$.

Equation (32) is one of the main results of this paper. It is a bosonization of fermions with an arbitrary finite symmetry $G_f = Z_2^f \leftthreetimes_{e_2} G_b$ in any dimensions. In fact, when $\mathrm{w}_2 + \mathrm{w}_1^2 = 0$, from $\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} e_2(A^{G_b})$ we see that the pair $(A^{G_b}, A^{\mathbb{Z}_2^f})$ actually describes the G_f symmetry twist A^{G_f}, which a flat G_f connection:

$$A^{G_f}_{ij} A^{G_f}_{jk} = A^{G_f}_{ik}, \quad A^{G_f}_{ij}, A^{G_f}_{jk}, A^{G_f}_{ik} \in G_f. \tag{34}$$

To see the above result, we rewrite the above using $A^{G_f}_{ij} \sim (A^{G_b}_{ij}, A^{\mathbb{Z}_2^f}_{ij})$ (see Appendix N)

$$\begin{aligned}(A^{G_b}_{ik}, A^{\mathbb{Z}_2^f}_{ik}) &= (A^{G_b}_{ij}, A^{\mathbb{Z}_2^f}_{ij})(A^{G_b}_{jk}, A^{\mathbb{Z}_2^f}_{jk}) \\ &= \Big(A^{G_b}_{ij} A^{G_b}_{jk}, A^{\mathbb{Z}_2^f}_{ij} + A^{\mathbb{Z}_2^f}_{jk} + e(A^{G_b}_{ij}, A^{G_b}_{jk})\Big)\end{aligned} \tag{35}$$

which reduces to

$$A^{G_b}_{ij} A^{G_b}_{jk} = A^{G_b}_{ik}, \tag{36}$$
$$A^{\mathbb{Z}_2^f}_{ij} + A^{\mathbb{Z}_2^f}_{jk} + e(A^{G_b}_{ij}, A^{G_b}_{jk}) \overset{2}{=} A^{\mathbb{Z}_2^f}_{ik}, \quad \text{or } \mathrm{d}A^{\mathbb{Z}_2} \overset{2}{=} e(A^{G_b}).$$

VI. EXACTLY SOLUBLE MODELS FOR FERMIONIC SPT PHASES: FERMION DECORATION

Now, we are going to choose the Lagrangian $\mathcal{L}$ such that the fermion model is exactly soluble. We first wrote

$$\int_{\mathcal{M}^{d+1}} \mathcal{L}(g, f_d, A^{G_b}) = \int_{\mathcal{N}^{d+2}} \mathrm{d}\mathcal{L}(g, f_d, A^{G_b}). \tag{37}$$

The fermion model is exactly soluble when

$$-\mathrm{d}\mathcal{L}(g, f_d, A^{G_b}) \overset{1}{=} \frac{1}{2}\Big(\mathrm{Sq}^2 f_d + f_d e_2(A^{G_b})\Big). \tag{38}$$

In this case, the action amplitude is always 1. But such an equation has no solutions if we view g, f_d, and e_2 as independent cochains, since in general $\frac{1}{2}(\mathrm{Sq}^2 f_d + f_d e_2)$ is not a coboundary.

So to obtain an exactly soluble model, we further assume f_d to be functions of g, A:

$$f_d = n_d(g, A^{G_b}). \tag{39}$$

This process is called decorating symmetry point-defects (described by (g, A^{G_b})) with fermion particles (described by f_d)[17], or simply fermion decoration. This is also called trivializing the cocycle $\mathrm{Sq}^2 f_d + f_d e_2$ (see Section VIII C).

We require $n_d(g, A^{G_b})$ to be G_b-gauge invariant

$$n_d(g_i, A^{G_b}_{ij}) = n_d(g_i h_i, h_i^{-1} A^{G_b}_{ij} h_j). \tag{40}$$

We can use such a gauge transformation to set $g_i = 1$: $n_d(g_i, A^{G_b}_{ij}) = n_d(1, a^{G_b}_{ij})$ where

$$a^{G_b}_{ij} = g_i A^{G_b}_{ij} g_j^{-1}. \tag{41}$$

Thus n_d is a cocycle $n_d \in Z^d(\mathcal{B}G_b, \mathbb{Z}_2)$. Similarly, we can choose $\mathcal{L}(g, f_d, A^{G_b})$ to be

$$\mathcal{L}(g, f_d, A^{G_b}) \equiv \nu_{d+1}(a^{G_b}) + \frac{1}{2} f_d \xi_1(a^{G_b}, A^{G_b}), \tag{42}$$

where $\xi_1(a^{G_b}, A^{G_b})$ is given by

$$\mathrm{d}\xi_1(a^{G_b}, A^{G_b}) \overset{2}{=} [e_2(a^{G_b}) - e_2(A^{G_b})]. \tag{43}$$

Here we have assumed that $e_2(a^{G_b}) - e_2(A^{G_b})$ is always a coboundary.

The exact solubility condition becomes

$$-\mathrm{d}\nu_{d+1}(a^{G_b}) \overset{1}{=} \frac{1}{2}\Big(\mathrm{Sq}^2[n_d(a^{G_b})] + n_d(a^{G_b}) e_2(a^{G_b})\Big). \tag{44}$$

where $\nu_{d+1} \in C^{d+1}(\mathcal{B}G_b; \mathbb{R}/\mathbb{Z})$ is a cochain on $\mathcal{B}G_b$. With proper choices of $n_d(a^{G_b})$ and $e_2(a^{G_b})$, $\mathrm{Sq}^2[n_d(a^{G_b})] + e_2(a^{G_b}) n_d(a^{G_b})$ can be a coboundary and the above equation has solutions. The above is nothing but the twisted cocycle condition for group super-cohomology first derived by Gu and Wen in Ref. 17 (for the case $e_2 = 0$ and $d = 2, 3$).

This way we obtain an exactly soluble local fermionic model

$$\begin{aligned} &Z(\mathcal{M}^{d+1}, A^{G_b}, A^{\mathbb{Z}_2^f}) \\ &= \sum_{g\in C^0(\mathcal{M}^{d+1};G_b); f_d = n_d(a^{G_b})} \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(a^{G_b}) + \frac{1}{2} f_d \xi_1(a^{G_b}, A^{G_b}) + \frac{1}{2} f_d A^{\mathbb{Z}_2^f}} \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \\ &\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}), \quad f_d \overset{2}{=} \tilde f_d \text{ on } \mathcal{M}^{d+1}. \end{aligned} \tag{45}$$

The path integral sums over the G^b-valued 0-cochains g and $\mathbb{Z}_2$-valued d-cocycles f_d on $\mathcal{M}^{d+1}$, that satisfy $f_d \overset{2}{=}$

$n_d(a^{G_b})$. On $\mathcal{N}^{d+2}$, $\tilde f_d$ is a $\mathbb{Z}_2$-valued d-cocycles that satisfy $\tilde f_d \overset{2}{=} f_d \overset{2}{=} n_d(a^{G_b})$ on the boundary. In other words, $\tilde f_d$ may not be equal to $n_d(a^{G_b})$ on $\mathcal{N}^{d+2}$, where $n_d(a^{G_b})$ is not even defined. The $f_d \overset{2}{=} n_d(a^{G_b})$ condition on the boundary can be imposed by an energy penalty term in the Lagrangian. Thus g and f_d on $\mathcal{M}^{d+1}$ are the dynamical fields of the above local fermionic lattice model (in the bosonized form).

The above model is well defined only if $\mathcal{M}^{d+1}$ and A^{G_b} satisfy

$$\mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}) \overset{2,\mathrm{d}}{=} 0, \tag{46}$$

so that the twisted spin structure $A^{\mathbb{Z}_2^f}$ can be defined. This implies that the fermion in our model is described by a representation of $G_f = Z_2 \leftthreetimes_{e_2} G_b$, where G_f is a Z_2^f extension of G_b as determined by the 2-cocycle $e_2 \in H^2(\mathcal{B}G_b;\mathbb{Z}_2)$ (see Ref. 41).

Equation (45) is the first main result of this paper:

> Equation (45) describes a local fermionic system (in a bosonized form) where the full fermion symmetry is $G_f = Z_2^f \leftthreetimes_{e_2} G_b$. Such a fermionic model realizes a fermionic SPT state obtained by fermion decoration construction.

The above generalizes the previous results of Ref. 17 and 34 from $G_f = Z_2^f \times G_b$ to $G_f = Z_2^f \leftthreetimes_{e_2} G_b$.

The partition function (45) can be evaluated exactly

$$\begin{aligned} &Z(\mathcal{M}^{d+1}, A^{G_b}, A^{\mathbb{Z}_2^f}) \\ &= |G_b|^{N_v} \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(A^{G_b}) + \frac12 f_d A^{\mathbb{Z}_2^f}} \\ &\qquad \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \\ &\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}),\quad f_d \overset{2}{=} \tilde f_d \quad \text{on } \mathcal{M}^{d+1}. \end{aligned} \tag{47}$$

The volume independent[46,47] part of the above partition function is the SPT invariant

$$\begin{aligned} &Z^{\mathrm{top}}(\mathcal{M}^{d+1}, A^{G_b}, A^{\mathbb{Z}_2^f}) \\ &= \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(A^{G_b}) + \frac12 n_d(A^{G_b}) A^{\mathbb{Z}_2^f}} \\ &\quad \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \\ &\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}),\quad \tilde f_d \overset{2}{=} n_d(A^{G_b}) \quad \text{on } \mathcal{M}^{d+1}. \end{aligned} \tag{48}$$

Eqn. (48) is another key result of this paper. In this paper, we will assume that the SPT invariant eqn. (48) can resolve all different fermionic SPT phases. In other words, the two fermion SPT states belong to the same phase iff they have the same SPT invariant.

To show the above result let us consider the action amplitude

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\, S(\mathcal{M}^{d+1}, a^{G_b}, A^{G_b}, A^{\mathbb{Z}_2^f})} \\ &= \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(a^{G_b}) + \frac12 n_d(a^{G_b}) \xi_1(a^{G_b}, A^{G_b}) + \frac12 n_d(a^{G_b}) A^{\mathbb{Z}_2^f}} \\ &\quad \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \\ &\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}),\quad \tilde f_d \overset{2}{=} n_d(a^{G_b}) \quad \text{on } \mathcal{M}^{d+1}. \end{aligned} \tag{49}$$

where a^{G_b} is given in eqn. (5). We like to show that the action ampltude is indepent of g_i.

We note that

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\, S(\mathcal{M}^{d+1}, a^{G_b}, A^{G_b}, A^{\mathbb{Z}_2^f})} \mathrm{e}^{-\mathrm{i}\, S(\mathcal{M}^{d+1}, A^{G_b}, A^{G_b}, A^{\mathbb{Z}_2^f})} \\ &= \mathrm{e}^{\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(a^{G_b}) + \frac12 n_d(a^{G_b}) \xi_1(a^{G_b}, A^{G_b}) + \frac12 n_d(a^{G_b}) A^{\mathbb{Z}_2^f}} \\ &\quad \mathrm{e}^{-\mathrm{i}\, 2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(A^{G_b}) + \frac12 n_d(A^{G_b}) A^{\mathbb{Z}_2^f}} \\ &\quad \mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \end{aligned} \tag{50}$$

where $\mathcal{N}^{d+2} = I \times \mathcal{M}^{d+1}$. Since a_G is given by eqn. (5), we can have G_b-valued 1-cochain $\tilde a^{G_b}$ on $\mathcal{N}^{d+2}$ such that $\tilde a^{G_b}$ become a^{G_b} and A^{G_b} on the two boundaries of $\mathcal{N}^{d+2}$. Now we can choose $\tilde f_d = n_d(\tilde a^{G_b})$ on $\mathcal{N}^{d+2}$ so that it becomes $n_d(A^{G_b})$ and $n_d(a^{G_b})$ on the two boundaries. Using the relation

$$\begin{aligned} &\mathrm{d}\Big[\nu_{d+1}(\tilde a^{G_b}) + \frac12 n_d(\tilde a^{G_b}) \xi_1(\tilde a^{G_b}, A^{G_b}) + \frac12 n_d(\tilde a^{G_b}) A^{\mathbb{Z}_2^f}\Big] \\ &= -\frac12 [\mathrm{Sq}^2 n_d(\tilde a^{G_b}) + n_d(\tilde a^{G_b}) e_2(\tilde a^{G_b})] \\ &\quad + \frac12 n_d(\tilde a^{G_b}) [e_2(\tilde a^{G_b}) - e_2(A^{G_b})] \\ &\quad + \frac12 n_d(\tilde a^{G_b}) [\mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b})]. \\ &\overset{1}{=} -\frac12 [\mathrm{Sq}^2 \tilde f_d + \tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)], \end{aligned} \tag{51}$$

from eqn. (50), we see that

$$\mathrm{e}^{\mathrm{i}\, S(\mathcal{M}^{d+1}, a^{G_b}, A^{G_b}, A^{\mathbb{Z}_2^f})} = \mathrm{e}^{\mathrm{i}\, S(\mathcal{M}^{d+1}, A^{G_b}, A^{G_b}, A^{\mathbb{Z}_2^f})} \tag{52}$$

The exactly soluble model (45) systematically realizes a large class of fermionic SPT phases with any onsite symmetry G_f and in any dimensions. This class of fermionic SPT phases is described by the data (now written more precisely in terms of cochains on $\mathcal{B}G_b$)

$$\begin{aligned} \bar e_2(\bar a_{01}^{G_b}, \bar a_{12}^{G_b}) &\in Z^2(\mathcal{B}G_b;\mathbb{Z}_2), \\ \bar n_d(\bar a_{01}^{G_b}, \cdots, \bar a_{d-1,d}^{G_b}) &\in Z^d(\mathcal{B}G_b;\mathbb{Z}_2), \\ \bar\nu_{d+1}(\bar a_{01}^{G_b}, \cdots, \bar a_{d,d+1}^{G_b}) &\in C^{d+1}(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z}), \\ -\mathrm{d}\bar\nu_{d+1}(\bar a^{G_b}) &\overset{1}{=} \frac12 [\mathrm{Sq}^2 \bar n_d(\bar a^{G_b}) + \bar n_d(\bar a^{G_b}) \bar e_2(\bar a^{G_b})]. \end{aligned} \tag{53}$$

Here $\bar a^{G_b}$ is the canonical 1-conchain on $\mathcal{B}G_b$ (for details, see Ref. 41 and Appendix L). Also $e_2(a^{G_b})$, $n_d(a^{G_b})$ and $\nu_{d+1}(a^{G_b})$ in (45) are pullbacks of $\bar e_2$, $\bar n_d$ and $\bar\nu_{d+1}$ in (53) by the homomorphism $\phi: \mathcal{M}^{d+1} \to \mathcal{B}G_b$:

$$e_2 \overset{2}{=} \phi^* \bar e_2,\quad n_d \overset{2}{=} \phi^* \bar n_d,\quad \nu_{d+1} \overset{2}{=} \phi^* \bar\nu_{d+1}. \tag{54}$$

We see that, for a fixed fermion symmetry G_f, the constructed SPT states are labeled by $[\bar n_d(\bar a^{G_b}), \bar\nu_{d+1}(\bar a^{G_b})]$

(where $\bar e_2$ is fixed), that satisfying the conditions (53). However, different pairs $[\bar n_d(\bar a^{G_b}), \bar\nu_{d+1}(\bar a^{G_b})]$ can some times label the same fermionic SPT states. Those pairs that label the same SPT state are called equivalent. The equivalence relations are partially generated by the following two kinds of transformations:

1. a transformation generated by a d-cochain $\bar\eta_d \in C^d(\mathcal{B}G_b;\mathbb{R}/\mathbb{Z})$:

$$\bar n_d(\bar a^{G_b}) \to \bar n_d(\bar a^{G_b}),$$
$$\bar\nu_{d+1}(\bar a^{G_b}) \to \bar\nu_{d+1}(\bar a^{G_b}) + \mathrm{d}\bar\eta_d(\bar a^{G_b}). \tag{55}$$

2. a transformation generated by a $(d-1)$-cochain $\bar u_{d-1} \in C^{d-1}(\mathcal{B}G_b;\mathbb{Z}_2)$

$$\bar n_d(\bar a^{G_b}) \to \bar n_d(\bar a^{G_b}) + \mathrm{d}\bar u_{d-1}(\bar a^{G_b}),$$
$$\bar\nu_{d+1}(\bar a^{G_b}) \to \bar\nu_{d+1}(\bar a^{G_b}) + \frac{1}{2}\mathbb{S}\mathrm{q}^2\bar u_{d-1}(\bar a^{G_b}) \tag{56}$$
$$+ \frac{1}{2}\bar u_{d-1}(\bar a^{G_b}) \underset{d-2}{\smile} \bar n_d(\bar a^{G_b}) + \frac{1}{2}\bar u_{d-1}(\bar a^{G_b})\bar e_2(\bar a^{G_b}).$$

More detailed discussions are given in the next Section VII.

VII. A DESCRIPTION BASED ON HIGHER GROUPS

The data (53) that characterizes the exactly soluble model (45) has a higher group description. The higher group description is more compact, and allows us to see the equivalence relation between the data more clearly. For an introduction on higher groups, see Ref. 41 and Appendix L.

A. The higher group data for fermionic SPT models

The exactly soluble model (45) and the related fermionic SPT state is characterized by the following data:

1. A particular higher group $\mathcal{B}_f(Z_2,2;Z_2,d)$, determined by its $\mathbb{Z}_2$-valued canonical cochains $\bar e_2$ and $\bar f_d$, satisfying $\mathrm{d}\bar e_2 \overset{2}{=} 0$ and $\mathrm{d}\bar f_d \overset{2}{=} 0$ (see Ref. 41 and Appendix L).

2. A particular $\mathbb{R}/\mathbb{Z}$-valued $(d+2)$-cocycle

$$\bar\omega_{d+2}(\bar f_d,\bar e_2) \overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar f_d + \frac{1}{2}\bar f_d\bar e_2 \tag{57}$$

on the higher group $\mathcal{B}_f(Z_2,2;Z_2,d)$.

3. Different trivialization homomorphisms $\varphi : \mathcal{B}G_b \to \mathcal{B}_f(Z_2,2;Z_2,d)$, so that $\varphi^*\bar\omega_{d+2}$ is a $\mathbb{R}/\mathbb{Z}$-valued $(d+2)$-coboundary on $\mathcal{B}G_b$. (We like to remark that different choices of φ lead to different choices $n_d \in Z^d(\mathcal{B}G_b;\mathbb{Z}_2)$ in (53). If for some choices of φ, the last equation in (53) has no solution for the corresponding n_d, then φ is not a trivialization homomorphisms.)

4. Different choices of the trivialization $\bar\nu_{d+1}(\bar a^{G_b})$ that satisfy $-\mathrm{d}\bar\nu_{d+1}(\bar a^{G_b}) \overset{1}{=} \varphi^*\bar\omega_{d+2}(\bar f_d,\bar e_2)$.

To roughly understand the above result, we loosely rewrite eqn. (45) as

$$Z(\mathcal{M}^{d+1}, A^{G_b}, A^{\mathbb{Z}_2^f})$$
$$= \sum_{g\in C^0(\mathcal{M}^{d+1};G_b); f_d = n_d(a^{G_b})} \mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(a^{G_b})}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d + \tilde f_d e_2},$$
$$\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}),\quad f_d \overset{2}{=} \tilde f_d \ \text{ on } \mathcal{M}^{d+1}. \tag{58}$$

The first two pieces of data determine the term on $\mathcal{N}^{d+2}$: $\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d + \tilde f_d e_2}$, which is always fixed. In fact, $\frac12\mathrm{Sq}^2\tilde f_d + \frac12\tilde f_d e_2$ can be viewed as the pullback of $\bar\omega_{d+2}$ by a simplicial homomorphism $\phi_N : \mathcal{N}^{d+2} \to \mathcal{B}_f(Z_2,2;Z_2,d)$:

$$\frac{1}{2}\mathrm{Sq}^2 f_d + \frac{1}{2} f_d e_2 = \phi_N^*\bar\omega_{d+2}(\bar f_d,\bar e_2). \tag{59}$$

The next two pieces of data determine the term on $\mathcal{M}^{d+1}$: $\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(a^{G_b})}$. First, $n_d(a^{G_b})$ and $e_2(a^{G_b})$ in (53) are the pullback of $\bar f_d$ and $\bar e_2$ on $\mathcal{B}_f(Z_2,2;Z_2,d)$ by the trivialization homomorphism $\varphi : \mathcal{B}G_b \to \mathcal{B}_f(Z_2,2;Z_2,d)$ and by $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_b$. Thus if we only consider the pull back φ:

$$\frac{1}{2}\mathrm{Sq}^2\bar n_d(\bar a^{G_b}) + \frac{1}{2}\bar n_d(\bar a^{G_b})\bar e_2(\bar a^{G_b})$$
$$= \varphi^*(\frac{1}{2}\mathrm{Sq}^2\bar f_d + \frac{1}{2}\bar f_d\bar e_2), \tag{60}$$

we also require $\varphi^*(\frac12\mathrm{Sq}^2\bar f_d + \frac12\bar f_d\bar e_2)$ to be a coboundary on $\mathcal{B}G_b$, *i.e.* there is a cochain $\bar\nu_{d+1}(\bar a^{G_b})$ on $\mathcal{B}G_b$ such that

$$-\mathrm{d}\bar\nu_{d+1}(\bar a^{G_b}) \overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar n_d(\bar a^{G_b}) + \frac{1}{2}\bar n_d(\bar a^{G_b})\bar e_2(\bar a^{G_b}). \tag{61}$$

We see that the above higher group description recovers eqn. (53). Therefore, the exactly soluble models (45) and the corresponding fermionic SPT states are characterized by a pair $(\varphi,\bar\nu_{d+1})$, a trivialization homomorphism and a trivialization. The different trivialization homomorphisms φ correspond to different choices of $\bar n_d(\bar a^{G_b})$ and $\bar e_2(\bar a^{G_b})$. The different trivializations $\bar\nu_{d+1}$ differ by $d+1$-cocycles on $\mathcal{B}G_b$.

B. SPT invariant from the higher group description

The higher group description $(\varphi,\bar\nu_{d+1})$ also determines the SPT invariant of the corresponding fermionic SPT state with a bosonic symmetry G_b. In the following, we will discuss this SPT invariant in more detail. Given $(\varphi,\bar\nu_{d+1})$, we have a $\mathbb{Z}_2$-valued 2-cocycle $\bar e_2(\bar A^{G_b})$ and a $\mathbb{Z}_2$-valued d-cocycle $\bar n_d(\bar A^{G_b})$ on $\mathcal{B}G_b$, as the pull back by φ from the canonical cocycles $\bar e_2$ and $\bar f_d$ on $\mathcal{B}_f(Z_2,2;Z_2,d)$. The fermionic SPT invariant is obtained for a closed complex $\mathcal{M}^{d+1}$ which has a property that, for any $\mathbb{Z}_2$-valued d-cocycle f_d on $\mathcal{M}^{d+1}$, there exists a complex $\mathcal{N}^{d+1}$ with a $\mathbb{Z}_2$-valued d-cocycle $\tilde f_d$ on it such that $\partial\mathcal{N}^{d+1}=\mathcal{M}^d$ and $\tilde f_d \overset{2}{=} f_d$ on the boundary $\mathcal{M}^d$. To obtain the SPT invariant, we also need to choose a homomorphism $\phi:\mathcal{M}^{d+1}\to\mathcal{B}G_b$ and a $\mathbb{Z}_2$-valued 2-cochain $A^{\mathbb{Z}_2^f}$ on $\mathcal{M}^d$, such that the following condition is satisfied

$$
\begin{aligned}
\mathrm{d}A^{\mathbb{Z}_2^f} &\overset{2}{=} \mathrm{w}_2+\mathrm{w}_1^2+\phi^*\bar e_2(\bar A^{G_b}).\\
&\overset{2}{=} \mathrm{w}_2+\mathrm{w}_1^2+\phi^*\varphi^*\bar e_2. \qquad (62)
\end{aligned}
$$

For such choices of $\mathcal{M}^{d+1},\phi,A^{\mathbb{Z}_2^f}$, the fermionic SPT invariant is given by

$$
\begin{aligned}
&Z^{\text{top}}_{\varphi,\bar\nu_{d+1}}(\mathcal{M}^{d+1},\phi,A^{Z_2})\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar\nu_{d+1}(\bar A^{G_b})+\frac12[\phi^*\bar n_d(\bar A^{G_b})]A^{\mathbb{Z}_2^f}}\\
&\quad \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d+\tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)},\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar\nu_{d+1}(\bar A^{G_b})+\frac12(\phi^*\varphi^*\bar f_d)A^{\mathbb{Z}_2^f}}\\
&\quad \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d+\tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)},\\
&\tilde f_d \overset{2}{=} \phi^*\bar n_d(\bar A^{G_b}) \ \text{ on } \mathcal{M}^{d+1}=\partial\mathcal{N}^{d+2}. \qquad (63)
\end{aligned}
$$

We like to remark that the above discussions are valid regardless if we have time reversal symmetry or not. Without time reversal symmetry, $\mathcal{M}^{d+1}$ must be orientable, and $\mathrm{w}_1 \overset{2}{=} 0$. With time reversal symmetry, $\mathcal{M}^{d+1}$ can be non-orientable. Also w_1 is contained in A^{G_b}, since G_b contain the group Z_2^T of time reversal symmetry.

C. Equivalent relations between the labels $(\varphi,\bar\nu_{d+1})$

It is possible that the SPT states labeled by different pairs $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$, *i.e.* by different triples $[\bar e_2(\bar a^{G_b}),\bar f_d(\bar a^{G_b}),\bar\nu_{d+1}(a^{G_b})]$ and $[\bar e_2'(\bar a^{G_b}),\bar f_d'(\bar a^{G_b}),\bar\nu'_{d+1}(a^{G_b})]$, are the same SPT states since the two triples may give rise to the same SPT invariant (see eqn. (48) and eqn. (63)). In this case, we say that the two pairs are equivalent. [Note that the homomorphisim $\varphi:\mathcal{B}G_b\to\mathcal{B}(Z_2,2;Z_2,d)$ determines $\bar e_2(\bar a^{G_b})$ and $\bar f_d(\bar a^{G_b})$ on $\mathcal{B}G_b$.]

What are the equivalent relations for the pairs $(\varphi,\bar\nu_{d+1})$? In the following, we are going to propose that $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ are equivalent if they can be homotopically connected. However, it is possible that such an equivalent relation is not large enough. It may be possible that $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ can described the same fermionic SPT state even if they are not homotopically connected.

Mathematically, $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ are homotopically connected if

1. There exists a homomorphism $\Phi: I\times\mathcal{B}G_b\to\mathcal{B}_f(Z_2,2;Z_2,d)$ such that on the two boundaries, Φ reduces to φ and φ'.

2. There exists a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain $\bar\mu_{d+1}$ on $I\times\mathcal{B}G_b$ such that $-\mathrm{d}\bar\mu_{d+1}\overset{1}{=}\Phi^*\bar\omega_{d+2}$ and $\bar\mu_{d+1}$ reduces to $\bar\nu_{d+1}$ and $\bar\nu'_{d+1}$ on the two boundaries.

In the following, we like to show that homotopically connected $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ give rise to the same SPT invariant (63), and hence the same fermionic SPT order.

We note that

$$
\begin{aligned}
&\frac{Z^{\text{top}}_{\varphi,\bar\nu_{d+1}}(\mathcal{M}^{d+1},\phi,A^{Z_2})}{Z^{\text{top}}_{\varphi',\bar\nu'_{d+1}}(\mathcal{M}^{d+1},\phi,A^{Z_2})}\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar\nu_{d+1}(\bar A^{G_b})+\frac12(\phi^*\varphi^*\bar f_d)A^{\mathbb{Z}_2^f}}\\
&\quad \mathrm{e}^{-\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar\nu'_{d+1}(\bar A^{G_b})+\frac12(\phi^*\varphi'^*\bar f_d)A^{\mathbb{Z}_2^f}}\\
&\quad \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d+\tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}. \qquad (64)
\end{aligned}
$$

where $\mathcal{N}^{d+2}=I\times\mathcal{M}^{d+1}$. The homomorphism $\phi:\mathcal{M}^{d+1}\to\mathcal{B}G_b$ induce a natural homomorphism $\hat\phi:\mathcal{N}^{d+2}\to I\times\mathcal{B}G_b$. We choose $\tilde f_d$ on $\mathcal{N}^{d+2}$ to be $\tilde f_d=\hat\phi^*\Phi^*\bar f_d$. We note that $\tilde f_d$ becomes $\phi^*\varphi^*\bar f_d$ and $\phi^*(\varphi')^*\bar f_d$ on the two boundaries of $\mathcal{N}^{d+2}$. We also have a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{N}^{d+2}$: $\mu_{d+1}=\hat\phi^*\bar\mu_{d+1}$. We note that μ_{d+1} becomes $\phi^*\bar\nu_{d+1}$ and $\phi^*\bar\nu'_{d+1}$ on the two boundaries of $\mathcal{N}^{d+2}$. Therefore

$$
\begin{aligned}
&\frac{Z^{\text{top}}_{\varphi,\bar\nu_{d+1}}(\mathcal{M}^{d+1},\phi,A^{Z_2})}{Z^{\text{top}}_{\varphi',\bar\nu'_{d+1}}(\mathcal{M}^{d+1},\phi,A^{Z_2})}\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{N}^{d+2}}\mathrm{d}\left[\hat\phi^*\bar\mu_{d+1}+\frac12\tilde f_dA^{\mathbb{Z}_2^f}\right]}\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2\tilde f_d+\tilde f_d(\mathrm{w}_2+\mathrm{w}_1^2)}\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{N}^{d+2}}\mathrm{d}\hat\phi^*\bar\mu_{d+1}+\frac12(\mathrm{Sq}^2\tilde f_d+\tilde f_d\hat\phi^*\Phi^*\bar e_2)}=1. \qquad (65)
\end{aligned}
$$

Before ending this section, we like to introduce a larger equivalence relation. $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ are *cobordant* if

1. There exist a complex $\mathcal{C}_b$ such that $\partial\mathcal{C}_b=\mathcal{B}_f(Z_2,2;Z_2,d)\sqcup-\mathcal{B}_f(Z_2,2;Z_2,d)$ and a homomorphism $\Phi:\mathcal{C}_b\to\mathcal{B}_f(Z_2,2;Z_2,d)$ such that on the two boundaries, Φ reduces to φ and φ'.

2. There exists a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain $\bar\mu_{d+1}$ on $\mathcal{C}_b$ such that $-\mathrm{d}\bar\mu_{d+1}\overset{1}{=}\Phi^*\bar\omega_{d+2}$ and $\bar\mu_{d+1}$ reduces to $\bar\nu_{d+1}$ and $\bar\nu'_{d+1}$ on the two boundaries.

It is very tempting to regard two cobordant labels as equivalent. However, so far we cannot show that two cobordant labels give rise to the same SPT invariant.

D. Stacking and Abelian group structure of fermionic SPT phases

We have seen that the higher dimensional bosonization is closely related to a higher group $\mathcal{B}_f(Z_2,2;Z_2,d)$, and a particular choice of a $\mathbb{R}/\mathbb{Z}$-valued cocycle $\frac{1}{2}\mathrm{Sq}^2\bar f_d+\frac{1}{2}\bar f_d\bar e_2$ on the higher group. Such a choice of cocycle has an important additive property

$$\frac{1}{2}\mathrm{Sq}^2(\bar f_d+\bar f_d')+\frac{1}{2}(\bar f_d+\bar f_d')\bar e_2 \overset{1,\,d}{=}\Big(\frac{1}{2}\mathrm{Sq}^2\bar f_d+\frac{1}{2}\bar f_d\bar e_2\Big)+\Big(\frac{1}{2}\mathrm{Sq}^2\bar f_d'+\frac{1}{2}\bar f_d'\bar e_2\Big). \tag{66}$$

This additive property insures that the fermionic SPT phases can also be added so that the collection of fermionic SPT phases actually form an Abelian group. The addition of two SPT states physically corresponds to stacking two SPT states one on top another, which implies that the SPT phases should always have an Abelian group structure.

For two SPT phases described by $(\bar n_d(a^{G_b}),\bar\nu_{d+1}(a^{G_b}))$ and $(\bar n_d'(a^{G_b}),\bar\nu_{d+1}'(a^{G_b}))$ (with the same $\bar e_2(a^{G_b})$), they add following a twisted addition rule

$$(\bar n_d,\bar\nu_{d+1})+(\bar n_d',\bar\nu_{d+1}') =(\bar n_d+\bar n_d',\bar\nu_{d+1}+\bar\nu_{d+1}'+\frac{1}{2}\bar n_d \underset{d-1}{\smile} \bar n_d'). \tag{67}$$

VIII. A MORE GENERAL CONSTRUCTION FOR FERMIONIC SPT STATES

A. With time reversal symmetry

The action amplitude

$$\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\mathcal{L}(g,f_d,A^{G_b})+\frac{1}{2}f_dA^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)} \tag{68}$$

in (32) contains bosonic field g and fermionic field f_d. The bosonic field couples to a G_b-connection A^{G_b} and the fermionic field couples to a spin connection $A^{\mathbb{Z}_2^f}$ (for the twisted spin structure). More generally, the action amplitude has the following gauge invariance

$$\mathrm{w}_2+\mathrm{w}_1^2\to \mathrm{w}_2+\mathrm{w}_1^2+\mathrm{d}u_1,\quad A^{\mathbb{Z}_2^f}\to A^{\mathbb{Z}_2^f}+u_1, \tag{69}$$

where u_1 is a $\mathbb{Z}_2$ valued 1-cochain. Such a gauge invariance ensure the proper coupling between the fermion current and the spin connection.

To construct more general fermionic SPT states and to include time reversal symmetry, like what we did in Section IV D, we generalize g and f_d and allow them to couple to A^{G_b} and $A^{\mathbb{Z}_2^f}$ as well as the space-time connection $A^O\in O_\infty$. We can package the three connections $A^{G_b}_{ij}$, $A^{\mathbb{Z}_2^f}_{ij}$, and A^O_{ij} into a single connection $A^{G_{fO}}_{ij}\in G_{fO}$ where

$$G_{fO}=G_f^0\leftthreetimes O_\infty. \tag{70}$$

Here full fermion symmetry group is given by

$$G_f=G_f^0\leftthreetimes Z_2^T. \tag{71}$$

i.e. G_f^0 is the fermion symmetry with time reversal removed.

So a more general action amplitude can have the following form

$$\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\mathcal{L}(g,f_d,A^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)}, \tag{72}$$

where we may choose the field g to have its value in G_{fO}. Now g_i transforms non-trivially under space-time transformation. More precisely, under space time transformation O_{d+1}, g_i transforms as

$$g_i\to g_ih,\quad h\in G_{fO}. \tag{73}$$

where h has a property that under the natural projection

$$G_{fO}\xrightarrow{\pi}G_{fO}/G_f^0=O_\infty, \tag{74}$$

h becomes $h^O=\pi(h)\in O_\infty$, such that h^O is in the O_{d+1} subgroup of O_∞. The symmetry twist is now described by $A^{G_{fO}}_{ij}\in G_{fO}$ on each link (ij), such that, under the projection π, $A^{G_{fO}}_{ij}$ become $A^O_{ij}=\pi(A^{G_{fO}}_{ij})\in O_{d+1}\subset O_\infty$ and A^O_{ij} is the connection that describe the tangent bundle of the space-time M^{d+1}.

The above action amplitude should be invariant under the following gauge transformation

$$\mathrm{w}_2+\mathrm{w}_1^2\to \mathrm{w}_2+\mathrm{w}_1^2+\mathrm{d}u_1,\quad A^{G_{fO}}_{ij}\to A^{G_{fO}}_{ij}(-)^{(u_1)_{ij}} \tag{75}$$

It also has the following gauge invariance

$$g_i\to g_ih_i,\quad A^{G_{fO}}_{ij}\to h_i^{-1}A^{G_{fO}}_{ij}h_j,\quad g_i,h_i\in G_{fO}. \tag{76}$$

This leads to a more general bosonized fermion theory

$$Z(\mathcal{M}^{d+1},A^{G_{fO}}) \tag{77}$$
$$=\sum_{g\in C^0(\mathcal{M}^{d+1};G_{fO});f_d\in Z^d(\mathcal{M}^{d+1};\mathbb{Z}_2)}\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\mathcal{L}(g,f_d,A^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)}.$$

This allows us to introduce effective dynamical variables on the links given by

$$a^{G_{fO}}_{ij}=g_iA^{G_{fO}}_{ij}g_j^{-1},\quad g_i\in G_{fO}. \tag{78}$$

Using the effective dynamical variables we can construct a local fermionic exactly soluble models (in the bosonized form) that describes the fermion decoration

$$Z(\mathcal{M}^{d+1},A^{G_{fO}}) \tag{79}$$
$$=\sum_{g\in C^0(\mathcal{M}^{d+1};G_{fO});f_d=n_d(a^{G_{fO}})}\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(a^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)}.$$

where $\mathcal{M}^{d+1}$ is the boundary of $\mathcal{N}^{d+2}$, f_d on $\mathcal{N}^{d+2}$ satisfy $\mathrm{d}f_d \overset{2}{=} 0$, and f_d on $\mathcal{N}^{d+2}$ is an extension of f_d on $\mathcal{M}^{d+1}$. The above model can realize more general fermionic SPT states, which are constructed using the following data:

$$\begin{aligned}
\bar n_d(\bar a^{G_{fO}}) &\in Z^d(\mathcal{B}G_{fO};\mathbb{Z}_2);\\
\bar \nu_{d+1}(\bar a^{G_{fO}}) &\in C^{d+1}(\mathcal{B}G_{fO};\mathbb{R}/\mathbb{Z});\\
-\mathrm{d}\bar \nu_{d+1}(\bar a^{G_{fO}}) &\overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar n_d(\bar a^{G_{fO}})\\
&\quad + \frac{1}{2}\bar n_d(\bar a^{G_{fO}})[\bar{\mathrm{w}}_2(\bar a^{O}) + \bar{\mathrm{w}}_1^2(\bar a^{O})]
\end{aligned} \tag{80}$$

where $\bar a^O_{ij} = \pi(\bar a^{G_{fO}}_{ij}) \in O_\infty$ (see eqn. (74)), and n_d, ν_{d+1} in (79) are the pullbacks of $\bar n_d$, $\bar \nu_{d+1}$. Here we like to stress that the time reversal transformation has a non-trivial action on the value of $\bar\nu_{d+1}$: $\bar \nu_{d+1} \overset{T}{\to} -\bar\nu_{d+1}$. Thus d is defined with this action and should be more precisely written as $\mathrm{d}_{\mathrm{w}_1}$ (see Appendix A).

Eqn. (79) is exactly soluble since its action amplitude is independent of dynamic field g_i:

$$\begin{aligned}
&\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(a^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2+\mathrm{w}_1^2)}\\
&= \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(A^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2+\mathrm{w}_1^2)}
\end{aligned} \tag{81}$$

(Note that $a^{G_{fO}}$ and $A^{G_{fO}}$ only differ by a gauge transformation.) Thus, the partition function can be calculated exactly (see Sec. VIII C 2)

$$\begin{aligned}
&Z(\mathcal{M}^{d+1}, A^{G_{fO}})\\
&= |G_{fO}|^{N_v}\,\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(A^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2+\mathrm{w}_1^2)}\\
&f_d = n_d(A^{G_{fO}}) \ \text{ on } \mathcal{M}^{d+1} = \mathcal{N}^{d+2}.
\end{aligned} \tag{82}$$

This leads to the SPT invariant

$$\begin{aligned}
&Z^{\mathrm{top}}(\mathcal{M}^{d+1}, A^{G_{fO}})\\
&= \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(A^{G_{fO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d(\mathrm{w}_2+\mathrm{w}_1^2)}\\
&f_d = n_d(A^{G_{fO}}) \ \text{ on } \mathcal{M}^{d+1} = \mathcal{N}^{d+2}.
\end{aligned} \tag{83}$$

Eqns. (79) and (83) are the second main result of this paper:

> For fermion systems with full fermion symmetry $G_f = Z_2^f \leftthreetimes G_b$ where G_b contains time reversal symmetry, the fermions transform as representations of $G_{fO_{d+1}} = G_f^0 \leftthreetimes O_{d+1}$ (for imaginary time), under combined space-time rotation and internal symmetry transformation. Using the data $[\bar n_d(\bar a^{G_{fO}}), \bar\nu_{d+1}(\bar a^{G_{fO}})]$ in eqn. (80), we can construct an exactly soluble fermionic model (79) (in a bosonized form), which realizes a fermionic SPT state characterize by the SPT invariant (83).

Here we would like to stress that

> to describe the symmetry of a fermion system, it is not only important to specify the full fermion symmetry group G_f, it is also important to specify the group $G_{fO_{d+1}} = G_f \leftthreetimes O_{d+1}$ for the combined space-time rotation and internal symmetry transformation.

Our model for fermionic SPT state uses the information on how fermions transform under the combined space-time rotation and internal symmetry transformation.

B. Without time reversal symmetry

If there is no time reversal symmetry, we can choose the dynamical variable on each vertex to be $g_i \in G_{fSO} \equiv G_f \leftthreetimes SO_\infty$. Under space time transformation SO_{d+1}, g_i transforms as

$$g_i \to g_i h, \quad h \in G_{fSO}. \tag{84}$$

h also satisfy that under the natural projection $\pi : G_{fSO} \to G_{fSO}/G_f = SO_\infty$, h become $h^{SO} = \pi(h) \in SO_{d+1} \subset SO_\infty$.

The symmetry twist now is described by $A^{G_{fSO}}_{ij} \in G_{fSO}$ on each link (ij), such that $A^{SO}_{ij} = \pi(A^{G_{fSO}}_{ij}) \in SO_{d+1} \subset SO_\infty$ and A^{SO}_{ij} is the connection that describe the tangent bundle of the space-time M^{d+1}. The effective dynamical variables on the links are given by

$$a^{G_{fSO}}_{ij} = g_i A^{G_{fSO}}_{ij} g_j^{-1}. \tag{85}$$

Using the effective dynamical variables we can construct a local fermionic exactly soluble models (in the bosonized form)

$$\begin{aligned}
&Z(\mathcal{M}^{d+1}, A^{G_{fSO}})\\
&= \sum_{g\in C^0(\mathcal{M}^{d+1};G_{fSO}); f_d = n_d(a^{G_{fSO}})} \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(a^{G_{fSO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d \mathrm{w}_2}.
\end{aligned} \tag{86}$$

where $\mathcal{M}^{d+1}$ is orientable and is the regular boundary of orientable $\mathcal{N}^{d+2}$ and w_2 is the second Stiefel-Whitney class on $\mathcal{N}^{d+2}$. The above models are constructed using the following data:

$$\begin{aligned}
\bar n_d(\bar a^{G_{fSO}}) &\in Z^d(\mathcal{B}G_{fSO};\mathbb{Z}_2),\\
\bar \nu_{d+1}(\bar a^{G_{fO}}) &\in C^{d+1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z});\\
-\mathrm{d}\bar \nu_{d+1}(\bar a^{G_{fSO}}) &\overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar n_d(\bar a^{G_{fSO}})\\
&\quad + \frac{1}{2}\bar n_d(\bar a^{G_{fSO}})\bar{\mathrm{w}}_2(\bar a^{SO}),
\end{aligned} \tag{87}$$

where $\bar a^{SO}_{ij} \in SO_\infty$ is obtained from $\bar a^{G_{fSO}}_{ij}$ by the natural projection $G_{fSO} \to SO_\infty$. Again n_d and ν_{d+1} in (86) are the pullbacks of $\bar n_d$ and $\bar\nu_{d+1}$. The SPT invariant for the constructed model (86) is

$$\begin{aligned}
&Z^{\mathrm{top}}(\mathcal{M}^{d+1}, A^{G_{fSO}})\\
&= \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(A^{G_{fSO}})}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d + f_d \mathrm{w}_2}\\
&f_d = n_d(A^{G_{fSO}}) \ \text{ on } \mathcal{M}^{d+1} = \mathcal{N}^{d+2}.
\end{aligned} \tag{88}$$

Eqns. (86) and (88) are the third main result of this paper:

For fermion systems with full fermion symmetry $G_f = Z_2^f \leftthreetimes G_b$ where G_b contains no time reversal symmetry, the fermions transform as representations of $G_{fSO_{d+1}} = G_f \leftthreetimes SO_{d+1}$ (for imaginary time), under combined space-time rotation and internal symmetry transformation. Using the data $[\bar n_d(\bar a^{G_{fSO}}), \bar\nu_{d+1}(\bar a^{G_{fSO}})]$ in (87)), we can construct an exactly soluble fermionic model (86) (in a bosonized form). Such a fermionic model realizes a fermionic SPT state characerized by the SPT invariant (88).

C. A description based on higher groups

Again, the data (87) that characterizes the exactly soluble model (86) has a higher group description. The basic idea is the same as that introduced in Section VII, however, in this section we will elaborate in more detail.

1. Without time reversal symmetry

The exactly soluble model (86) and the related fermionic SPT state is characterized by the following data:

1. A particular higher group $\mathcal{B}_f(SO_\infty, 1; Z_2, d)$, determined by its $\mathbb{Z}_2$-valued canonical cochain $\mathrm{d}\bar f_d \overset{2}{=} 0$ (see Ref. 41 and Appendix L).
2. A particular $\mathbb{R}/\mathbb{Z}$-valued $(d+2)$-cocycle
$$\bar\omega_{d+2} \overset{1}{=} \frac{1}{2}\mathrm{Sq}^2 \bar f_d + \frac{1}{2}\bar f_d \bar{\mathrm{w}}_2(\bar a^{SO}) \tag{89}$$
on the higher group $\mathcal{B}_f(SO_\infty, 1; Z_2, d)$.
3. Different trivialization homomorphisms $\varphi : \mathcal{B}G_{fSO} \to \mathcal{B}_f(SO_\infty, 1; Z_2, d)$, where $G_{fSO} = G_f \leftthreetimes SO_\infty$, so that $\varphi^*\bar\omega_{d+2}$ is a $\mathbb{R}/\mathbb{Z}$-valued $(d+2)$-coboundary on $\mathcal{B}G_{fSO}$. (We like to remark that different choices of φ lead to different choices $n_d \in Z^d(\mathcal{B}G_{fSO}; \mathbb{Z}_2)$ in (87). If for some choices of φ, the last equation in (87) has no solution for the corresponding n_d, then φ is not a trivialization homomorphisms.)
4. Different choices of the trivialization $\bar\nu_{d+1}(\bar a^{G_{fSO}})$ that satisfy $-\mathrm{d}\bar\nu_{d+1}(\bar a^{G_{fSO}}) \overset{1}{=} \varphi^*\bar\omega_{d+2}(\bar f_d, \bar a^{SO})$.

To understand the above result, we note that the first two pieces of data determine the term on $\mathcal{N}^{d+2}$: $\mathrm{e}^{\mathrm{i}\pi\int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 f_d + f_d \mathrm{w}_2}$, which is always fixed. In fact, $\frac{1}{2}\mathrm{Sq}^2 f_d + \frac{1}{2} f_d \mathrm{w}_2$ can be viewed as the pullback of $\bar\omega_{d+2}$ by a simplicial homomorphism $\phi_N : \mathcal{N}^{d+2} \to \mathcal{B}_f(SO_\infty, 1; Z_2, d)$:
$$\frac{1}{2}\mathrm{Sq}^2 f_d + \frac{1}{2} f_d \mathrm{w}_2(a^{SO}) = \phi_N^* \bar\omega_{d+2}(\bar f_d, \bar a^{SO}). \tag{90}$$
Here a^{SO} is the pullback of $\bar a^{SO}$: $a^{SO} = \phi_N^* \bar a^{SO}$. We require ϕ_N to be a homomorphism such that $a^{SO} = \phi_N^* \bar a^{SO}$ is the connection that describes the tangent bundle of $\mathcal{N}^{d+2}$. In this case, $\mathrm{Sq}^2 f_d + f_d \mathrm{w}_2(a^{SO})$ is a $\mathbb{Z}_2$-valued coboundary on $\mathcal{N}^{d+2}$.

The next two pieces of data determine the term on $\mathcal{M}^{d+1}$: $\mathrm{e}^{\mathrm{i}2\pi\int_{\mathcal{M}^{d+1}} \nu_{d+1}(a^{G_{fSO}})}$. First, $n_d(a^{G_{fSO}})$ and $\mathrm{w}_2(a^{SO})$ in (87) are the pullback of $\bar f_d$ and $\bar{\mathrm{w}}_2(\bar a^{SO})$ on $\mathcal{B}_f(SO_\infty, 1; Z_2, d)$ by the trivialization homomorphism $\varphi : \mathcal{B}G_{fSO} \to \mathcal{B}_f(SO_\infty, 1; Z_2, d)$. (To be more precise, $n_d(a^{G_{fSO}})$ and $\mathrm{w}_2(a^{SO})$ in (87) are pullbacks of $\bar n_d(\bar a^{G_{fSO}})$ and $\bar{\mathrm{w}}_2(\bar a^{SO})$ by $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_{fSO}$.) Thus
$$\begin{aligned}&\frac{1}{2}\mathrm{Sq}^2 \bar n_d(\bar a^{G_{fSO}}) + \frac{1}{2}\bar n_d(\bar a^{G_{fSO}})\bar{\mathrm{w}}_2(\bar a^{SO}) \\ &= \varphi^*[\frac{1}{2}\mathrm{Sq}^2 \bar f_d + \frac{1}{2}\bar f_d \bar{\mathrm{w}}_2(\bar a^{SO})].\end{aligned} \tag{91}$$
We also require $\varphi^*[\frac{1}{2}\mathrm{Sq}^2 \bar f_d + \frac{1}{2}\bar f_d \bar{\mathrm{w}}_2(\bar a^{G_{SO}})]$ to be a coboundary on $\mathcal{B}G_{fSO}$. *i.e.* there is a cochain $\bar\nu_{d+1}(\bar a^{G_{fSO}})$ on $\mathcal{B}G_{fSO}$ such that
$$\begin{aligned}-\mathrm{d}\bar\nu_{d+1}(\bar a^{G_{fSO}}) \overset{1}{=}& \frac{1}{2}\mathrm{Sq}^2 \bar n_d(\bar a^{G_{fSO}}) \\ &+ \frac{1}{2}\bar n_d(\bar a^{G_{fSO}})\bar{\mathrm{w}}_2(\bar a^{SO})\end{aligned} \tag{92}$$

Therefore, the exactly soluble models (86) and the corresponding fermionic SPT states are characterized by a pair $(\varphi, \bar\nu_{d+1})$, a trivialization homomorphism and a trivialization. The different trivialization homomorphisms φ correspond to different choices of $n_d(a^{G_{fSO}})$. The different trivializations $\bar\nu_{d+1}$ differ by $d+1$-cocycles on $\mathcal{B}G_{fSO}$.

In fact, the exactly soluble models (86) can be written explicitly using the higher group data:
$$\begin{aligned}Z(\mathcal{M}^{d+1}, A^{G_{fSO}}) &= \sum_{\phi_M} \mathrm{e}^{\mathrm{i}2\pi\left(\int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}\right)}, \\ &- \mathrm{d}\bar\nu_{d+1} \overset{1}{=} \varphi^*\bar\omega_{d+2},\end{aligned} \tag{93}$$
Here ϕ_M is a simplicial homomorphism $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_{fSO}$, such that $\phi_M^* \bar a^{G_{fSO}}$ is gauge equivalent to $A^{G_{fSO}}$:
$$\bar a_{ij}^{G_{fSO}} = g_i A_{ij}^{G_{fSO}} g_j^{-1}. \tag{94}$$
Also, ϕ_N is a simplicial homomorphism $\phi_N : \mathcal{N}^{d+2} \to \mathcal{B}(G_{fSO}, 1; Z_2, d)$, such that, when restrict to the boundary of $\mathcal{N}^{d+2}$, $\phi_N = \varphi\phi_M$.

2. An exact evaluation of the partition function

Let us examine the action amplitude $\mathrm{e}^{\mathrm{i}2\pi\int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \mathrm{i}2\pi\int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}}$ for different ϕ_M but a fixed φ. ϕ_N is chosen such that it is given by $\varphi\phi_M$ at the boundary $\partial\mathcal{N}^{d+2}$. We know that $\frac{1}{2}\mathrm{Sq}^2 f_d + \frac{1}{2} f_d \mathrm{w}_2(a^{SO}) = \phi_N^*\bar\omega_{d+2}$ is a $\mathbb{R}/\mathbb{Z}$-valued

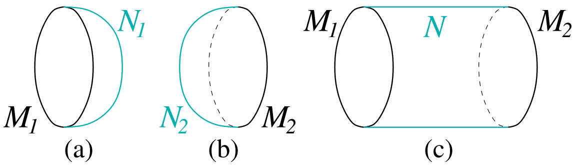

FIG. 2. (Color online) Three space-time $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_1 \sqcup -\mathcal{M}_2$, plus their extensions. $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}$. The ratio of the action amplitudes (which are pure U_1 phases) for space-time (a) and (b): $Z(a)/Z(b) = Z(a)Z^*(b)$ is given by the action amplitude for space-time (c): $Z(a)Z^*(b) = Z(c)$.

coboundary on $\mathcal{N}^{d+2}$, thus the value of $\mathrm{e}^{\mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}}$ only depends on ϕ_N on the boundary $\partial\mathcal{N}^{d+2}$, *i.e.* only depends on $\varphi\phi_M$. Therefore, the action amplitude $\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}}$ is a function of ϕ_M.

We call $\phi_{M_1} : \mathcal{M}_1^{d+1} \to \mathcal{B}G_{fSO}$ and $\phi_{M_2} : \mathcal{M}_2^{d+1} \to \mathcal{B}G_{fSO}$ cobordant, if there exists a homomorphism $\varphi_N : \mathcal{N}^{d+2} \to \mathcal{B}G_{fSO}$ such that $\partial\mathcal{N}^{d+2} = \mathcal{M}_1^{d+1} \sqcup \mathcal{M}_2^{d+1}$, $\varphi_N = \phi_{M_1}$ on the boundary $\mathcal{M}_1^{d+1}$, and $\varphi_N = \phi_{M_2}$ on the boundary $\mathcal{M}_2^{d+1}$ (see Fig. 2(c)). We can show that *two cobordant ϕ_{M_1} and ϕ_{M_2} have the same action amplitude.* This is because the difference (the ratio) of the action amplitudes is given by (see Fig. 2)

$$\mathrm{e}^{\mathrm{i}\,2\pi [\int_{\mathcal{M}_1^{d+1}} \phi_{M_1}^* \bar\nu_{d+1} - \int_{\mathcal{M}_2^{d+1}} \phi_{M_2}^* \bar\nu_{d+1}] + \mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}} \tag{95}$$

where $\phi_N = \varphi\varphi_N$. Because $\varphi_N = \phi_{M_1}$ or $\varphi_N = \phi_{M_2}$ on the two boundaries of $\mathcal{N}^{d+2}$, we have

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\,2\pi [\int_{\mathcal{M}_1^{d+1}} \phi_{M_1}^* \bar\nu_{d+1} - \int_{\mathcal{M}_2^{d+1}} \phi_{M_2}^* \bar\nu_{d+1}]} \\ =&\mathrm{e}^{\mathrm{i}\,2\pi [\int_{\mathcal{M}_1^{d+1}} \varphi_N^* \bar\nu_{d+1} - \int_{\mathcal{M}_2^{d+1}} \varphi_N^* \bar\nu_{d+1}]} \\ =&\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \mathrm{d}\varphi_N^* \bar\nu_{d+1}} \end{aligned} \tag{96}$$

Now the difference (the ratio) of the action amplitudes is given by

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \mathrm{d}\varphi_N^* \bar\nu_{d+1} + \phi_N^* \bar\omega_{d+2}} \\ =&\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \mathrm{d}\varphi_N^* \bar\nu_{d+1} + \varphi_N^* \varphi^* \bar\omega_{d+2}} \\ =&\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \varphi_N^* (\mathrm{d}\bar\nu_{d+1} + \varphi^* \bar\omega_{d+2})} = 1. \end{aligned} \tag{97}$$

We like to remark that in the action amplitude $\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}}$, the homomorphism $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_{fSO}$ usually cannot be extended to a homomorphism $\varphi_N : \mathcal{N}^{d+2} \to \mathcal{B}G_{fSO}$ (*i.e.* ϕ_M is not cobordant to a trivial homomorphism). As a result the action amplitude is not equal to 1. If the homomorphism $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_{fSO}$ can be extend to $\mathcal{N}^{d+2}$, the action amplitude will be equal to one. If ϕ_{M_1} and ϕ_{M_2} are cobordant, then ϕ_{M_1} and ϕ_{M_2} on $\mathcal{M}_1$ and $\mathcal{M}_2$ in Fig. 2(c) can be extended to $\mathcal{N}$. In this case, the action amplitude for space-time Fig. 2(c) will be equal to one.

In our exactly soluble model (93), the homomorphism ϕ_M is determined by $a^{G_{fSO}}$ which in turn is given by the dynamical fields g_i on vertices and the background field $A^{G_{fSO}}$ in the links (see (85)). For a fixed $A^{G_{fSO}}$, the different homomorphisms ϕ_M are all cobordant to each other, and the corresponding action amplitudes are all equal to each other. Therefore, the partition function for our model (93) can be calculated exactly

$$\begin{aligned} Z(\mathcal{M}^{d+1}, A^{G_{fSO}}) &= \sum_{\phi_M} \mathrm{e}^{\mathrm{i}\,2\pi \left(\int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2} \right)} \\ &= V^{N_v} \mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(A^{G_{fSO}}) + \mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 f_d + f_d \mathrm{w}_2} \end{aligned} \tag{98}$$

where V is the volume of G_{fSO} and N_v the number of vertices in $\mathcal{M}^{d+1}$. We see that the fermionic SPT state realized by (93) [which is labeled by a pair $(\varphi, \bar\nu_{d+1})$] is characterized by the SPT invariant

$$\begin{aligned} &Z^{\mathrm{top}}(\mathcal{M}^{d+1}, A^{G_{fSO}}) \\ =&\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \nu_{d+1}(A^{G_{fSO}}) + \mathrm{i}\,\pi \int_{\mathcal{N}^{d+2}} \mathrm{Sq}^2 f_d + f_d \mathrm{w}_2}, \end{aligned} \tag{99}$$

where $f_d \overset{2}{=} n_d(A^{G_{fSO}})$ on $\partial\mathcal{N}^{d+2}$ and

$$\begin{aligned} -\mathrm{d}\nu_{d+1}(A^{G_{fSO}}) \overset{1}{=} &\frac{1}{2} \mathrm{Sq}^2 n_d(A^{G_{fSO}}) \\ &+ \frac{1}{2} n_d(A^{G_{fSO}}) \mathrm{w}_2(A^{SO}). \end{aligned} \tag{100}$$

Here $A_{ij}^{SO} = \pi(A_{ij}^{G_{fSO}}) \in SO_\infty$ and A^{SO} must be the connection for the tangent bundle on $\mathcal{M}^{d+1}$.

Equation (99) can also be rewritten as

$$\begin{aligned} &Z^{\mathrm{top}}(\mathcal{M}^{d+1}, A^{G_{fSO}}) \\ =&\mathrm{e}^{\mathrm{i}\,2\pi \int_{\mathcal{M}^{d+1}} \phi_M^* \bar\nu_{d+1} + \mathrm{i}\,2\pi \int_{\mathcal{N}^{d+2}} \phi_N^* \bar\omega_{d+2}}, \\ &\phi_N|_{\partial\mathcal{N}^{d+2}} = \varphi\phi_M, \end{aligned} \tag{101}$$

where the homomorphism $\phi_M : \mathcal{M}^{d+1} \to \mathcal{B}G_{fSO}$ is determined by the background field $A^{G_{fSO}}$ via

$$A^{G_{fSO}} = \phi_M^* \bar a^{G_{fSO}}. \tag{102}$$

Also ϕ_N is a homomorphism $\phi_N : \mathcal{N}^{d+2} \to \mathcal{B}_f(SO_\infty, 1; \mathbb{Z}_2, d)$, such that, on the boundary $\mathcal{M}^{d+1} = \partial\mathcal{N}^{d+2}$, ϕ_N is given by $\varphi\phi_M$, where φ is a homomorphism $\varphi : \mathcal{B}G_{fSO} \to \mathcal{B}_f(SO_\infty, 1; \mathbb{Z}_2, d)$.

3. Equivalent relations between the labels $(\varphi, \bar\nu_{d+1})$

It is possible that the SPT states labeled by different pairs $(\varphi, \bar\nu_{d+1})$ and $(\varphi', \bar\nu'_{d+1})$, *i.e.* by different pairs $[n_d(A^{G_{fSO}}), \nu_{d+1}(A^{G_{fSO}})]$ and $[n'_d(A^{G_{fSO}}), \nu'_{d+1}(A^{G_{fSO}})]$, are the same SPT states since the two pair may give rise to the same SPT invariant. In this case, we say that the two pairs are equivalent.

What are the equivalent relations for the pairs $(\varphi, \bar\nu_{d+1})$ or $[n_d(A^{G_{fSO}}), \nu_{d+1}(A^{G_{fSO}})]$? Here is our proposal: $(\varphi, \bar\nu_{d+1})$ and $(\varphi', \bar\nu'_{d+1})$ are equivalent if

1. There exists a $\mathbb{R}/\mathbb{Z}$-valued $d+2$-cocycle $\bar\Omega_{d+2}$ on $I\times\mathcal{B}_f(SO_\infty,1;Z_2,d)$ such that when restricted on the two boundaries of $I\times\mathcal{B}_f(SO_\infty,1;Z_2,d)$, $\bar\Omega_{d+2}$ becomes $\bar\omega_{d+2}$ in (117).

2. There exists a homomorphism $\Phi: I\times\mathcal{B}G_{fSO}\to I\times\mathcal{B}_f(SO_\infty,1;Z_2,d)$ such that on the two boundaries, Φ reduces to φ and φ'.

3. There exists a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain $\bar\mu_{d+1}$ on $I\times\mathcal{B}G_{fSO}$ such that $-\mathrm{d}\bar\mu_{d+1}\overset{1}{=}\Phi^*\bar\Omega_{d+2}$ and $\bar\mu_{d+1}$ reduces to $\bar\nu_{d+1}$ and $\bar\nu'_{d+1}$ on the two boundaries.

In the following, we like to show that equivalent $(\varphi,\bar\nu_{d+1})$ and $(\varphi',\bar\nu'_{d+1})$ give rise to the same SPT invariant (99), and hence the same SPT order. We note that

$$
\begin{aligned}
&\frac{Z^{\text{top}}_{\varphi,\bar\nu_{d+1}}(\mathcal{M}^{d+1},\phi,A^{G_{fSO}})}{Z^{\text{top}}_{\varphi',\bar\nu'_{d+1}}(\mathcal{M}^{d+1},\phi,A^{G_{fSO}})}\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi^*\bar\nu_{d+1}(\bar A^{G_{fSO}})-\phi^*\bar\nu'_{d+1}(\bar A^{G_{fSO}})}\\
&\quad\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)}.
\end{aligned}
\tag{103}
$$

where $\mathcal{N}^{d+2}=I\times\mathcal{M}^{d+1}$. The homomorphism $\phi:\mathcal{M}^{d+1}\to\mathcal{B}G_b$ induce a natural homomorphism $\hat\phi:\mathcal{N}^{d+2}\to I\times\mathcal{B}G_b$. We choose f_d on $\mathcal{N}^{d+2}$ to be $f_d=\hat\phi^*\Phi^*\bar f_d$. We note that f_d becomes $n_d=\phi^*\varphi^*\bar f_d$ and $n'_d=\phi^*(\varphi')^*\bar f_d$ on the two boundaries of $\mathcal{N}^{d+2}$. We also have a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{N}^{d+2}$: $\mu_{d+1}=\hat\phi^*\bar\mu_{d+1}$. We note that μ_{d+1} becomes $\phi^*\bar\nu_{d+1}$ and $\phi^*\bar\nu'_{d+1}$ on the two boundaries of $\mathcal{N}^{d+2}$. Therefore (using eqn. (87))

$$
\begin{aligned}
&\frac{Z^{\text{top}}_{\varphi,\bar\nu_{d+1}}(\mathcal{M}^{d+1},\phi,A^{G_{fSO}})}{Z^{\text{top}}_{\varphi',\bar\nu'_{d+1}}(\mathcal{M}^{d+1},\phi,A^{G_{fSO}})}\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{N}^{d+2}}\mathrm{d}(\hat\phi^*\bar\mu_{d+1})}\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2 f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)}=1,
\end{aligned}
\tag{104}
$$

which complete our proof (see (101)).

The fermionic SPT states can also be labeled by $[\bar n_d(\bar a^{G_{fSO}}),\bar\nu_{d+1}(\bar a^{G_{fSO}})]$ where $\bar n_d(\bar a^{G_{fSO}})$ is a $\mathbb{Z}_2$-valued d-cocycle and $\bar\nu_{d+1}(\bar a^{G_{fSO}})$ is a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{B}G_{fSO}$. They are functions of canonical cocycle $\bar a^{G_{fSO}}$ on $\mathcal{B}G_{fSO}$. In terms of $[\bar n_d(\bar a^{G_{fSO}}),\bar\nu_{d+1}(\bar a^{G_{fSO}})]$, the equivalence relations are partially generated by the following two transformations (see (A29)):

1. Transformation generated by d-cochain $\bar\eta_d(\bar a^{G_{fSO}})\in C^{d-1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$

$$
\begin{aligned}
\bar n_d(\bar a^{G_{fSO}})&\to\bar n_d(\bar a^{G_{fSO}})\\
\bar\nu_{d+1}(\bar a^{G_{fSO}})&\to\bar\nu_{d+1}(\bar a^{G_{fSO}})+\mathrm{d}\bar\eta_d(\bar a^{G_{fSO}}).
\end{aligned}
\tag{105}
$$

2. Transformation generated by $d-1$-cochain $\bar u_d\in C^{d-1}(\mathcal{B}G_{fSO};\mathbb{Z}_2)$

$$
\bar n_d(\bar a^{G_{fSO}})\to\bar n_d(\bar a^{G_{fSO}})+\mathrm{d}\bar u_{d-1}(\bar a^{G_{fSO}})
\tag{106}
$$

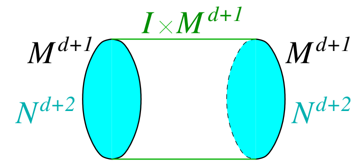

FIG. 3. (Color online) The boundary of $I\times\mathcal{N}^{d+2}$ is given by $\mathcal{N}^{d+2}\sqcup I\times\mathcal{M}^{d+1}\sqcup\mathcal{N}^{d+2}$. The boundary of $I\times\mathcal{M}^{d+1}$ is given by $\mathcal{M}^{d+1}\sqcup\mathcal{M}^{d+1}$.

$$
\begin{aligned}
&\bar\nu_{d+1}(\bar a^{G_{fSO}})\to\bar\nu_{d+1}(\bar a^{G_{fSO}})+\frac{1}{2}\bar u_{d-1}(\bar a^{G_{fSO}})\bar{\mathrm{w}}_2(\bar a^{SO})\\
&+\frac{1}{2}\mathbb{Sq}^2\bar u_{d-1}(\bar a^{G_{fSO}})+\frac{1}{2}\mathrm{d}\bar u_{d-1}(\bar a^{G_{fSO}})\underset{d-1}{\smile}\bar n_d(\bar a^{G_{fSO}}).
\end{aligned}
$$

We can show the above two transformation generate equivalent relations since they do not change the SPT invariant (99).

In fact, we have a more general equivalent relation in terms of $[n_d(a^{G_{fSO}}),\nu_{d+1}(a^{G_{fSO}})]$ in (87). Here $n_d(a^{G_{fSO}})=\phi_M^*\bar n_d(\bar a^{G_{fSO}})$ and $\nu_{d+1}(a^{G_{fSO}})=\phi_M^*\bar\nu_{d+1}(\bar a^{G_{fSO}})$ are a d-cocycle and a $d+1$-cochain on space-time $\mathcal{M}^{d+1}$. $[n_d(a^{G_{fSO}}),\nu_{d+1}(a^{G_{fSO}})]$ and $[n'_d(a^{G_{fSO}}),\nu'_{d+1}(a^{G_{fSO}})]$ produce the same SPT invariant if they satisfy (see (A29)):

1. Equivalence relation generated by d-cochain $\eta_d\in C^{d-1}(\mathcal{M}^{d+1};\mathbb{R}/\mathbb{Z})$

$$
\begin{aligned}
n'_d(a^{G_{fSO}})&\overset{2}{=}n_d(a^{G_{fSO}})\\
\nu'_{d+1}(a^{G_{fSO}})&\overset{1}{=}\nu_{d+1}(a^{G_{fSO}})+\mathrm{d}\eta_d.
\end{aligned}
\tag{107}
$$

2. Equivalence relation generated by $d-1$-cochain $u_d\in C^{d-1}(\mathcal{M}^{d+1};\mathbb{Z}_2)$

$$
\begin{aligned}
&n'_d(a^{G_{fSO}})\overset{2}{=}n_d(a^{G_{fSO}})+\mathrm{d}u_{d-1}
\end{aligned}
\tag{108}
$$

$$
\begin{aligned}
&\nu'_{d+1}(a^{G_{fSO}})\overset{1}{=}\nu_{d+1}(a^{G_{fSO}})+\frac{1}{2}u_{d-1}\mathrm{w}_2(a^{SO})\\
&+\frac{1}{2}\mathbb{Sq}^2u_{d-1}+\frac{1}{2}\mathrm{d}u_{d-1}\underset{d-1}{\smile}n_d(a^{G_{fSO}}).
\end{aligned}
$$

Because u_{d-1} and η_d are cochains on $\mathcal{M}^{d+1}$, which do not have to be the pullbacks of cochains $\bar u_{d-1}(\bar a^{G_{fSO}})$ and $\bar\eta_{d-1}(\bar a^{G_{fSO}})$ on $\mathcal{B}G_{fSO}$. So the above equivalent relation is more general.

We can show that above $[n_d(a^{G_{fSO}}),\nu_{d+1}(a^{G_{fSO}})]$ and $[n'_d(a^{G_{fSO}}),\nu'_{d+1}(a^{G_{fSO}})]$ produce the same action amplitude (thus the same SPT invariant). So the above relations are indeed equivalent relations.

We first introduce a $\mathbb{Z}_2$-valued cochain Ω_{d+2} on $I\times\mathcal{N}^{d+2}$

$$
\Omega_{d+2}\overset{1}{=}\frac{1}{2}\mathbb{Sq}^2F_d+\frac{1}{2}F_d\mathrm{w}_2
\tag{109}
$$

where F_d is a $\mathbb{Z}_2$-valued cocycle on $I\times\mathcal{N}^{d+2}$ given by

$$
F_d=f_d+\mathrm{d}U_{d-1}
\tag{110}
$$

and U_{d-1} is a $\mathbb{Z}_2$-valued cochain on $I\times\mathcal{N}^{d+2}$. We know that the boundary of $I\times\mathcal{N}^{d+2}$ has three pieces $\mathcal{N}^{d+2}$, $I\times\mathcal{M}^{d+1}$ and $\mathcal{N}^{d+2}$. U_{d-1} is chosen such that it becomes 0 on one of the $\mathcal{N}^{d+2}$, and becomes u_{d-1} on the boundary of the second $\mathcal{N}^{d+2}$: $\mathcal{M}^{d+1}=\partial\mathcal{N}^{d+2}$. Using (A26), we see that Ω_{d+2} is actually a cocycle on $I\times\mathcal{N}^{d+2}$. Therefore, we have (see Fig. 3)

$$1=\mathrm{e}^{\mathrm{i}\,2\pi\int_{I\times\mathcal{N}^{d+2}}\mathrm{d}\Omega_{d+2}}=\mathrm{e}^{\mathrm{i}\,\pi\int_{I\times\mathcal{M}^{d+2}}\mathbb{S}\mathrm{q}^2F_d+F_d\mathrm{w}_2}\times \mathrm{e}^{-\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d\mathrm{w}_2}\mathrm{e}^{+\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f'_d+f'_d\mathrm{w}_2}, \quad (111)$$

where $f'_d\overset{2}{=}f_d+\mathrm{d}U_{d-1}$ on $\mathcal{N}^{d+2}$. On the boundary $\mathcal{M}^{d+1}=\partial\mathcal{N}^{d+2}$, $f'_d\overset{2}{=}f_d+\mathrm{d}U_{d-1}$ becomes $n'_d\overset{2}{=}n_d+\mathrm{d}u_{d-1}$ (see (106)).

Next we calculate $\mathrm{e}^{\mathrm{i}\,\pi\int_{I\times\mathcal{M}^{d+2}}\mathbb{S}\mathrm{q}^2F_d+F_d\mathrm{w}_2}$ (using (A29)):

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\,\pi\int_{I\times\mathcal{M}^{d+2}}\mathbb{S}\mathrm{q}^2(f_d+\mathrm{d}U_{d-1})+(f_d+\mathrm{d}U_{d-1})\mathrm{w}_2}\\ &=\mathrm{e}^{\mathrm{i}\,\pi\int_{I\times\mathcal{M}^{d+2}}\mathrm{Sq}^2f_d+f_d\mathrm{w}_2+\mathrm{d}\left(\mathbb{S}\mathrm{q}^2U_{d-1}+\mathrm{d}U_{d-1}\underset{d-1}{\smile}n_d+U_{d-1}\mathrm{w}_2\right)}\\ &=\mathrm{e}^{\mathrm{i}\,2\pi\int_{I\times\mathcal{M}^{d+2}}\mathrm{d}\left(\nu_{d+1}+\frac12\mathbb{S}\mathrm{q}^2U_{d-1}+\frac12\mathrm{d}U_{d-1}\underset{d-1}{\smile}n_d+\frac12U_{d-1}\mathrm{w}_2\right)}\\ &=\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+2}}\nu_{d+1}}\mathrm{e}^{-\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+2}}\nu'_{d+1}}\end{aligned}\quad (112)$$

Combined with (111), we find

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+2}}\nu_{d+1}+\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d\mathrm{w}_2}\\ &=\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+2}}\nu'_{d+1}+\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f'_d+f'_d\mathrm{w}_2}\end{aligned}\quad (113)$$

The action amplitudes are indeed the same.

We like to remark that $[n_d(a^{G_{fSO}}),\nu_{d+1}(a^{G_{fSO}})]$ and $[n'_d(a^{G_{fSO}}),\nu'_{d+1}(a^{G_{fSO}})]$ not related by the above two types of transformations may still describe the same SPT phase. To really show $[n_d(a^{G_{fSO}}),\nu_{d+1}(a^{G_{fSO}})]$ and $[n'_d(a^{G_{fSO}}),\nu'_{d+1}(a^{G_{fSO}})]$ describe different SPT phases, we need to show they produce different SPT invariants.

4. Stacking and Abelian group structure of fermionic SPT phases

We have seen that the higher dimensional bosonization is closely related to a higher group $\mathcal{B}_f(SO_\infty,1;Z_2,d)$, and a particular choice of a $\mathbb{R}/\mathbb{Z}$-valued cocycle $\frac12\mathrm{Sq}^2\bar f_d+\frac12\bar f_d\bar{\mathrm{w}}_2$ on the higher group. Such a choice of cocycle has an important additive property

$$\begin{aligned}&\frac12\mathrm{Sq}^2(\bar f_d+\bar f'_d)+\frac12(\bar f_d+\bar f'_d)\bar{\mathrm{w}}_2\\ &\overset{1,d}{=}\Big(\frac12\mathrm{Sq}^2\bar f_d+\frac12\bar f_d\bar{\mathrm{w}}_2\Big)+\Big(\frac12\mathrm{Sq}^2\bar f'_d+\frac12\bar f'_d\bar{\mathrm{w}}_2\Big),\end{aligned}\quad (114)$$

which insures that the fermionic SPT phases form an Abelian group, as required physically by the stacking operation of SPT states.

For two SPT phases described by $(\bar n_d,\bar\nu_{d+1})$ and $(\bar n'_d,\bar\nu'_{d+1})$, they add following a twisted addition rule

$$\begin{aligned}&(\bar n_d,\bar\nu_{d+1})+(\bar n'_d,\bar\nu'_{d+1})\\ &=(\bar n_d+\bar n'_d,\bar\nu_{d+1}+\bar\nu'_{d+1}+\frac12\bar n_d\underset{d-1}{\smile}\bar n'_d).\end{aligned}\quad (115)$$

This allows us to extract the group $\mathrm{fSPT}_{d+1}(G_{fSO})$ given by the stacking of fermionic SPTs with symmetry G^f. Clearly there is homomorphism

$$\mathrm{fSPT}_{d+1}(G_{fSO})\to H^d(\mathcal{B}G_{fSO};\mathbb{Z}_2)$$
$$(\bar n_d,\nu_{d+1})\mapsto\bar n_d.$$

Not every $\bar n_d$ allows a solution of $\bar\nu_{d+1}$; the image of the above homomorphism is a subgroup of $H^d(\mathcal{B}G_{fSO};\mathbb{Z}_2)$, which will be called the obstruction-free subgroup, denoted by $BH^d(\mathcal{B}G_{fSO};\mathbb{Z}_2)$. The kernel of the above homomorphism is a quotient group of $H^{d+1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$. This is because of the extra gauge transformation $\frac12(\mathbb{S}\mathrm{q}^2\bar u_{d-1}+\bar u_{d-1}\mathrm{w}_2)$ when $\bar n_d=0$. Let $\Gamma\subset H^{d+1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$ be the subgroup generated by $\frac12(\mathbb{S}\mathrm{q}^2\bar u_{d-1}+\bar u_{d-1}\bar{\mathrm{w}}_2),\forall\bar u_{d-1}\in\mathcal{C}^{d-1}(\mathcal{B}G_{fSO};\mathbb{Z}_2)$, we have the following exact sequence for group extension:

$$\begin{aligned}&0\to H^{d+1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})/\Gamma\to\mathrm{fSPT}_{d+1}(G_{fSO})\\ &\to BH^{d+1}(\mathcal{B}G_{fSO};\mathbb{Z}_2)\to 0\end{aligned}\quad (116)$$

whose corresponding group 2-cocycle in $H^2[BH^d(\mathcal{B}G_{fSO};\mathbb{Z}_2),H^{d+1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})/\Gamma]$ is given by $\frac12\bar n_d\underset{d-1}{\smile}\bar n'_d$.

If all the symmetry in the fermionic SPT states is gauged, we obtain a bosonic topological order with emergent fermions. We know that bosonic topological orders form a commutative monoid under the stacking operation $\boxtimes$[46]. However, For bosonic topological order with emergent fermions, they cannot have inverse for the stacking operation $\boxtimes$, and thus they are not invertible topological orders[19,46,49,50]. However, we may modify the stacking operation by allowing the pairs of fermions from the two stacked phases to condense (equivalently, identifying fermions from the two stacked phases). Such a modified stacking operation is discussed in detail in Ref. 37 and 51. We denote the modified stacking operation by $\boxtimes_f$. The gauged fermionic SPT states also form an Abelian group under the modified stacking operation $\boxtimes_f$, as in (115).

5. With time reversal symmetry

In the above, we discussed the situation without time-reversal symmetry. In the presence of time reversal symmetry, we have the following result. The exactly soluble model (79) and the related fermionic SPT state is characterized by the following data:

1. A particular higher group $\mathcal{B}_f(O_\infty,1;Z_2,d)$, determined by its $\mathbb{Z}_2$-valued canonical cochain $\mathrm{d}\bar f_d\overset{2}{=}0$ (see Ref. 41 and Appendix L).

2. A particular $\mathbb{R}/\mathbb{Z}$-valued $d+2$-cocycle on

$$\bar\omega_{d+2}\overset{1}{=}\frac12\mathrm{Sq}^2\bar f_d+\frac12\bar f_d[\bar{\mathrm{w}}_2(\bar a^O)+\bar{\mathrm{w}}_1^2(\bar a^O)]\quad (117)$$

on $\mathcal{B}_f(O_\infty,1;Z_2,d)$.

3. Different trivialization homomorphisms φ : $\mathcal{B}G_{fO}\to\mathcal{B}_f(O_\infty,1;Z_2,d)$, where $G_{fO}=G_f\leftthreetimes O_\infty$.

4. Different choices of the trivialization $\nu_{d+1}(\bar a^{G_{fO}})$ that satisfy $-\mathrm{d}\nu_{d+1}(\bar a^{G_{fO}})\overset{1}{=}\varphi^*\bar\omega_{d+2}(\bar f_d,\bar a^O)$.

We like remark that in the above, the time reversal symmetry in O_∞ acts non-trivially on the value of ν_{d+1} (see Appendix A).

We can also use $[\bar n_d(\bar a^{G_{fO}}),\bar\nu_{d+1}(\bar a^{G_{fO}})]$ in (80) to label the fermion SPT states with time reversal symmetry. The equivalence relations are partially generated by the following two relations (see (A29)):

1. Equivalence relation generated by d-cochain $\bar\eta_d\in C^{d-1}(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$

$$\begin{aligned}&\bar n'_d(\bar a^{G_{fO}})\overset{2}{=}\bar n_d(\bar a^{G_{fO}})\\&\bar\nu_{d+1}(\bar a^{G_{fO}})\overset{1}{=}\bar\nu'_{d+1}(\bar a^{G_{fO}})+\mathrm{d}\bar\eta_d(\bar a^{G_{fO}})\end{aligned}\tag{118}$$

2. Equivalence relation generated by $d-1$-cochain $\bar u_{d-1}\in C^{d-1}(\mathcal{B}G_{fSO};\mathbb{Z}_2)$

$$\bar n'_d(\bar a^{G_{fO}})\overset{2}{=}\bar n_d(\bar a^{G_{fO}})+\mathrm{d}\bar u_{d-1}(\bar a^{G_{fO}})\tag{119}$$

$$\begin{aligned}&\bar\nu'_{d+1}(\bar a^{G_{fO}})\overset{1}{=}\bar\nu_{d+1}(\bar a^{G_{fO}})+\frac12\mathrm{d}\bar u_{d-1}(\bar a^{G_{fO}})\underset{d-1}{\smile}\bar n_d(\bar a^{G_{fO}})\\&+\frac12\mathbb{Sq}^2\bar u_{d-1}(\bar a^{G_{fO}})+\frac12\bar u_{d-1}(\bar a^{G_{fO}})[\mathrm{w}_2(a^O)+\mathrm{w}_1^2(a^O)]\end{aligned}$$

The corresponding SPT invariant is given by

$$\begin{aligned}&Z^{\text{top}}(\mathcal{M}^{d+1},A^{G_{fO}})\\&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\nu_{d+1}(A^{G_{fO}})+\mathrm{i}\,\pi\int_{\mathcal{N}^{d+2}}\mathrm{Sq}^2f_d+f_d(\mathrm{w}_2+\mathrm{w}_1^2)},\\&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^{d+1}}\phi_M^*\bar\nu_{d+1}+\mathrm{i}\,2\pi\int_{\mathcal{N}^{d+2}}\phi_N^*\bar\omega_{d+2}},\\&f_d|_{\partial\mathcal{N}^{d+2}}=n_d,\quad \phi_N|_{\partial\mathcal{N}^{d+2}}=\varphi\phi_M\end{aligned}\tag{120}$$

IX. 1+1D FERMIONIC SPT STATES

In this and next a few sections, we are going to apply our theory to study some simple fermionic SPT phases. In 1+1D, the SPT invariant dose not depend on n_1 (see (99)). Thus the different 1+1D fermionic SPT states are labeled by cocycles ν_2. After quotient out the equivalence relations, we find that 1+1D fermionic SPT states from fermion decoration are classified by $H^2(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$ without time reversal symmetry and by $H^2[\mathcal{B}G_{fO};(\mathbb{R}/\mathbb{Z})_T]$ with time reversal symmetry, where $(\mathbb{R}/\mathbb{Z})_T$ remind us that the time reversal symmetry in G_{fO} has a non-trivial action on $\mathbb{R}/\mathbb{Z}\overset{T}{\to}-\mathbb{R}/\mathbb{Z}$.

Since $G_{fSO}=G_f\leftthreetimes SO_\infty$, $H^2[\mathcal{B}G_{fO};(\mathbb{R}/\mathbb{Z})_T]$ is given by a quotient of a subset of

$$\begin{aligned}&H^1[\mathcal{B}SO_\infty;H^1(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})]\oplus H^2(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})\oplus\\&H^2(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})\end{aligned}\tag{121}$$

(see Appendix F). Using the universal coefficient theorem (E4) and (G6), we find that $H^1[\mathcal{B}SO_\infty;H^1(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})]=0$. Also $H^2(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})$ does not involve symmetry G_f can only correspond to fermionic invertible topological order. In 1+1D, we believe that fermion decoration construction produces all fermionic SPT states. Thus 1+1D fermionic SPT states are classified by a subset of $H^2(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})$ without time reversal symmetry. This is consistent with the result obtained by 1+1D bosonization:

> 1+1D fermionic SPT states with on-site symmetry G_f are classified by $H^2(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})$ without time reversal symmetry.

With time reversal symmetry, $H^2[\mathcal{B}G_{fO};(\mathbb{R}/\mathbb{Z})_T]$ is given by a quotient of a subset of

$$\begin{aligned}&H^1[\mathcal{B}O_\infty;H^1(\mathcal{B}G_f^0;\mathbb{R}/\mathbb{Z})_T]\oplus H^2(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z})\oplus\\&H^2(\mathcal{B}G_f^0;\mathbb{R}/\mathbb{Z})\end{aligned}\tag{122}$$

where the time reversal symmetry in O_∞ may have a non-trivial action on $H^1(\mathcal{B}G_f;\mathbb{R}/\mathbb{Z})$ (see Appendix F). In the above, we have used the fact that $G_{fO}=G_f^0\leftthreetimes O_\infty$, where G_f^0 is the fermionic symmetry group after removing the time reversal symmetry . The above result is consistent with the result from 1d bosonization:

> 1+1D fermionic SPT states with on-site symmetry G_f are classified by $H^2[\mathcal{B}G_f;(\mathbb{R}/\mathbb{Z})_T]$ with time reversal symmetry.

We like to remark that the 1d topological p-wave superconductor[39] is a fermionic invertible topological order. It is not a fermionic SPT state.

X. FERMIONIC $Z_2\times Z_2^f$-SPT STATE

In this section, we are going to study the simplest fermionic SPT phases, where the fermion symmetry is given by $G_f=Z_2\times Z_2^f$ symmetry. In this case, $G_b=Z_2$ and $e_2=0$. We will consider 2+1D and in 3+1D systems. A double-layer superconductor with layer exchange symmetry can realize $Z_2\times Z_2^f$ symmetry.

Our calculation contains three steps: (1) we first calculate the $\mathbb{Z}_2$-valued cocycle $\bar n_d$; (2) we then compute the $\mathbb{R}/\mathbb{Z}$-valued cochain $\bar\nu_{d+1}$; (3) last we construct the corresponding SPT invariant trying to identify distinct SPT phases labeled by $(\bar n_d,\bar\nu_{d+1})$. Our calculation also come with two flavors: (1) without extension of SO_∞, and (2) with extension of SO_∞. Since our approaches are constructive, both of the above two flavors produce exactly soluble fermionic models that realize various SPT states. However, the approach with extension of SO_∞ is more complete, *i.e.* it produces all the SPT phases produced by the approach without extension of SO_∞.

A. 2+1D

1. Without extension of SO_∞

Calculate $\bar n_2$: We note that cohomology ring $H^*(\mathcal{B}Z_2;\mathbb{Z}_2)$ is generated by 1-cocycle $\bar a^{\mathbb{Z}_2}$. Thus $\bar n_2 \in H^2(\mathcal{B}Z_2;\mathbb{Z}_2) = \mathbb{Z}_2$ has two choices: $\bar n_2 = \alpha_n(\bar a^{\mathbb{Z}_2})^2$, $\alpha_n = 0,1$.

Calculate $\bar\nu_3$: Next, we consider $\mathrm{d}\bar\nu_3$ in (53). Since $\bar e_2 = 0$, only the term $\frac12\mathrm{Sq}^2\bar n_2 = \frac12\bar n_2^2$ is non-zero. The term $\frac12\bar n_2^2 = \frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2})^4$ is a cocycle in $Z^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z})$. Since $H^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z}) = 0$, thus $\frac12\bar n_2^2$ is always a coboundary in $B^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z})$. Therefore (53) has solution for all choices of $\bar n_2$. Noticing that (see (A32))

$$\begin{aligned}
&\frac12\mathrm{d}(\bar a^{\mathbb{Z}_2})^3\\
&=\frac12(\mathrm{d}\bar a^{\mathbb{Z}_2})(\bar a^{\mathbb{Z}_2})^2-\frac12\bar a^{\mathbb{Z}_2}(\mathrm{d}\bar a^{\mathbb{Z}_2})\bar a^{\mathbb{Z}_2}+\frac12(\bar a^{\mathbb{Z}_2})^2\mathrm{d}\bar a^{\mathbb{Z}_2}\\
&=(\beta_2\bar a^{\mathbb{Z}_2})(\bar a^{\mathbb{Z}_2})^2-\bar a^{\mathbb{Z}_2}(\beta_2\bar a^{\mathbb{Z}_2})\bar a^{\mathbb{Z}_2}+(\bar a^{\mathbb{Z}_2})^2\beta_2\bar a^{\mathbb{Z}_2}\\
&\overset{2}{=}(\mathrm{Sq}^1\bar a^{\mathbb{Z}_2})(\bar a^{\mathbb{Z}_2})^2+\bar a^{\mathbb{Z}_2}(\mathrm{Sq}^1\bar a^{\mathbb{Z}_2})\bar a^{\mathbb{Z}_2}+(\bar a^{\mathbb{Z}_2})^2\mathrm{Sq}^1\bar a^{\mathbb{Z}_2}\\
&\overset{2}{=}(\bar a^{\mathbb{Z}_2})^4,
\end{aligned}\tag{123}$$

we find that the solution has a form

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_n}{4}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3,\quad \alpha_\nu=0,1. \tag{124}$$

where $\frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3 \in H^3(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z})$.

We like to remark that in order for $\frac14(\bar a^{\mathbb{Z}_2})^3$ to be well defined mod 1, we need to view the $\mathbb{Z}_2$-valued $\bar a^{\mathbb{Z}_2}$ as $\mathbb{Z}$-valued with values 0 and 1. Let us use $\bar a^{\mathbb{Z}_2}$ denote such a map from $\mathbb{Z}_2$-valued to $\mathbb{Z}$-valued. Thus more precisely, we have

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_n}{4}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3,\quad \alpha_\nu=0,1. \tag{125}$$

As a result, $Z_2\times Z_2^f$ SPT states are labeled by (α_n,α_ν). Thus, there are 4 different $Z_2\times Z_2^f$ fermionic SPT states from fermion decoration.

SPT invariant: The four obtained $Z_2\times Z_2^f$ fermionic SPT states labeled by $\alpha_n,\alpha_\nu = 0,1$ are realized by the following local fermionic model (in the bosonized form as in (45))

$$\begin{aligned}
&Z(\mathcal{M}^3,A^{G_b}) \\
&=\sum_{g\in C^0(\mathcal{M}^4;\mathbb{Z}_2);f_2\overset{2}{=}\alpha_n(a^{\mathbb{Z}_2})^2} \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^3}\frac{\alpha_n+2\alpha_\nu}{2}(a^{\mathbb{Z}_2})^3+\alpha_n(a^{\mathbb{Z}_2})^2A^{\mathbb{Z}_2^f}+\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2}.
\end{aligned}\tag{126}$$

Its SPT invariant is

$$\begin{aligned}
&Z^{\mathrm{top}}(\mathcal{M}^3,A^{G_b})\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_n+2\alpha_\nu}{4}(A^{\mathbb{Z}_2})^3+\frac{\alpha_n}{2}(A^{\mathbb{Z}_2})^2A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2},\\
&f_2\big|_{\partial\mathcal{N}^4}\overset{2}{=}n_2,\qquad \mathrm{d}A^{\mathbb{Z}_2}\overset{2}{=}\mathrm{w}_2.
\end{aligned}\tag{127}$$

2. With extension of SO_∞

Calculate $\bar n_2$: First $G_{fSO} = Z_2\times Spin_\infty$, where Z_2^f is contained in $Spin_\infty$. Since $\bar n_2 \in H^2[\mathcal{B}(Z_2\times Spin_\infty);\mathbb{Z}_2]$, $\bar n_2$ can be written as a combination of $\bar a^{\mathbb{Z}_2}$, $\bar{\mathrm{w}}_1$ and $\bar{\mathrm{w}}_2$. However, for $Spin_\infty$ $\bar{\mathrm{w}}_1 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} 0$. Thus $\bar n_2$ is given by

$$\bar n_2 \overset{2}{=} \alpha_n(\bar a^{\mathbb{Z}_2})^2. \tag{128}$$

Thus $\bar n_2$ has two choices: $\alpha_n = 0,1$.

Calculate $\bar\nu_3$: Next, we consider $\bar\nu_3(\bar a^{G_{fSO}})$ in (87) which becomes

$$\begin{aligned}
-\mathrm{d}\bar\nu_3 &\overset{1}{=} \frac12\mathrm{Sq}^2\bar n_2+\frac12\bar n_2\,\mathrm{d}\bar A^{\mathbb{Z}_2^f}\\
&\overset{1}{=} \frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2})^4+\frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2})^2\,\mathrm{d}\bar A^{\mathbb{Z}_2^f}
\end{aligned}\tag{129}$$

where we have labeled $\bar a_{ij}^{G_{fSO}} \in G_{fSO} = Z_2\times(Z_2^f\leftthreetimes SO_\infty)$ by a triple $(\bar a_{ij}^{Z_2},\bar A_{ij}^{\mathbb{Z}_2^f},\bar a_{ij}^{SO})$. We have used the fact that $\bar{\mathrm{w}}_2(a^{SO})$ is a coboundary: $\bar{\mathrm{w}}_2 \overset{2}{=} \mathrm{d}\bar A^{\mathbb{Z}_2^f}$. The solution of the above equation has a form

$$\bar\nu_3(\bar a^{\mathbb{Z}_2},\bar A^{\mathbb{Z}_2^f}) \overset{1}{=} \frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_n}{4}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2})^2\bar A^{\mathbb{Z}_2^f} \tag{130}$$

where $\alpha_\nu = 0,1$.

SPT invariant: The SPT invariant is given by

$$\begin{aligned}
&Z^{\mathrm{top}}(\mathcal{M}^3,A^{G_{fSO}})\\
&=\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_\nu}{2}(A^{\mathbb{Z}_2})^3+\frac{\alpha_n}{4}{A^{\mathbb{Z}_2}}^3+\frac{\alpha_n}{2}(A^{\mathbb{Z}_2})^2A^{\mathbb{Z}_2^f}}\\
&\quad\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2},\\
&f_2\big|_{\partial\mathcal{N}^4}\overset{2}{=}\alpha_n(A^{Z_2})^2,\qquad \mathrm{d}A^{\mathbb{Z}_2}\overset{2}{=}\mathrm{w}_2.
\end{aligned}\tag{131}$$

where the background connection $A^{G_{fSO}}$ is labeled by a triple $(A^{\mathbb{Z}_2},A^{\mathbb{Z}_2^f},A^{SO})$. As a result, $Z_2\times Z_2^f$ SPT states from fermion decoration are labeled by (α_n,α_ν). It turns out that all those labels (α_n,α_ν) are inequivalent according to (105) and (106). Thus there are 4 different $Z_2\times Z_2^f$ fermionic SPT states from fermion decoration.

However, for non-interacting fermions, the 2+1D $Z_2\times Z_2^f$-SPT phases are labeled by $\mathbb{Z}$. After include interaction, $\mathbb{Z}$ reduces to $\mathbb{Z}_8$, and there are 8 different fermionic $Z_2\times Z_2^f$-SPT phases[52–55]. The extra fermionic SPT phases must come from the decoration of the topological p-wave superconducting chains.[35,36] In this paper, we only develop a generic theory for fermion decoration, which misses some of the $Z_2\times Z_2^f$ fermionic SPT phases. We hope to develop a generic theory for the decoration of the topological p-wave superconducting chains in future.

B. 3+1D

1. Without extension of SO_∞

Calculate $\bar n_3$: $\bar n_3 \in H^3(\mathcal{B}Z_2;\mathbb{Z}_2) = \mathbb{Z}_2$ has two choices: $\bar n_3 = \alpha_n(\bar a^{\mathbb{Z}_2})^3$, $\alpha_n = 0,1$. These two $\bar n_3$'s are not equivalent.

Calculate $\bar\nu_4$: $\bar\nu_4$ is obtained from (53), which has a form

$$-\mathrm{d}\bar\nu_4 \overset{1}{=} \frac{\alpha_n}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^3. \tag{132}$$

It turns out that $\frac{1}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^3 \overset{1}{=} \frac{1}{2}(\bar a^{\mathbb{Z}_2})^5$ (see (M5)), which is a non-trivial element in $H^5(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$. Thus $\bar\nu_4$ has no solution when $\alpha_n = 1$ and α_n must be zero. When $\bar n_3 = 0$, $\bar\nu_4$ has only one inequivalent solution since $H^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z}) = 0$.

2. *With extension of SO_∞*

Calculate $\bar n_3$: From $G_{fSO} = Z_2 \times Spin_\infty$, we see that $\bar n_3 \in H^3(G_{fSO};\mathbb{Z}_2)$ is generated by $\bar a^{\mathbb{Z}_2}$ and Stiefel-Whitney class $\bar{\mathrm{w}}_n$. For $Spin_\infty$, $\bar{\mathrm{w}}_1 = \bar{\mathrm{w}}_2 = 0$. Also, $\mathrm{Sq}^1\bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_1\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_3$. Since $\bar{\mathrm{w}}_2$ is a coboundary for $Spin_\infty$, $\bar{\mathrm{w}}_3$ is also a coboundary. Thus, $\bar n_3$ is given by

$$\bar n_3 \overset{2}{=} \alpha_n(\bar a^{\mathbb{Z}_2})^3, \qquad \alpha_n = 0,1. \tag{133}$$

Calculate $\bar\nu_4$: $\bar\nu_4$ is obtained from

$$-\mathrm{d}\bar\nu_4 \overset{1,\mathrm{d}}{=} \frac{\alpha_n}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^3. \tag{134}$$

Since $\frac{1}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^3 \overset{1}{=} \frac{1}{2}(\bar a^{\mathbb{Z}_2})^5$ (see (M5)) is the non-trivial element in $H^5(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$. Thus $\bar\nu_4$ has solution only when $\alpha_n = 0$.

When $\bar n_3 = 0$, $\bar\nu_4$ has a form

$$\bar\nu_4(\bar a^{G_{fSO}}) \overset{1}{=} \frac{\alpha_{\nu,1}}{2}(\bar a^{\mathbb{Z}_2})^4 + \frac{\alpha_{\nu,2}}{2}\bar{\mathrm{w}}_4 + \alpha_\nu p_1,$$
$$\alpha_{\nu,1},\alpha_{\nu,2} = 0,1, \qquad \alpha_\nu \in [0,1), \tag{135}$$

where p_1 is the first Pontryagin class. However, $\frac{1}{2}(\bar a^{\mathbb{Z}_2})^4$ is a coboundary in $B^4(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$.

Also, in 3+1D space-time $\mathcal{M}^4$, $\mathrm{w}_4 \overset{2,\mathrm{d}}{=} \mathrm{w}_2^2 + \mathrm{w}_1^4$ (see Appendix I 4). Since $\mathcal{M}^4$ is orientable spin manifold, $\mathrm{w}_2^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^4 \overset{2,\mathrm{d}}{=} 0$, we also have $\mathrm{w}_4 \overset{2,\mathrm{d}}{=} 0$. Last α_ν is not quantized and different values of α_ν's are connected and belong to the same phase. Thus the above solutions are equivalent. We find that there is only one trivial fermionic $Z_2 \times Z_2^f$ SPT phases in 3+1D from fermion decoration. This agrees with the result in Ref. 17.

Ref. 18 and 19 showed that there is only one trivial fermionic $Z_2 \times Z_2^f$-SPT phases in 3+1D. Our result is also consistent with that.

XI. FERMIONIC Z_4^f-SPT STATE

In this section, we are going to study fermionic SPT phases with $G_f = Z_4^f$ symmetry in 2+1D and in 3+1D. Such a symmetry can be realized by a charge-$2e$ superconductor of electrons where the Z_4^f symmetry is generated by 180° S_z-spin rotation. Another way to realize the Z_4^f symmetry is via charge-$4e$ superconductors of electrons. This kind of fermionic SPT states is beyond the approach in Ref. 17, 35, and 36 which only deal with G_f of the form $G_f = \mathbb{Z}_2^f \times G_b$.

For fermion systems with bosonic symmetry $G_b = Z_2$, the full fermionic symmetry G_f is an extension of G_b by Z_2^f. The bosonic symmetry $G_b = Z_2$ has two extensions described by $\bar e_2 \in H^2(\mathcal{B}Z_2;\mathbb{Z}_2) = \mathbb{Z}_2$. For $\bar e_2 \overset{2}{=} 0$, the extension is $G_f = Z_2^f \times Z_2$. The corresponding fermion SPT phases are discussed in the last section. For $\bar e_2 \overset{2}{=} (\bar a^{\mathbb{Z}_2})^2$, the extension is $G_f = Z_2^f \leftthreetimes Z_2 = Z_4^f$. We will discuss the corresponding fermionic SPT phases in this section.

For $G_f = Z_4^f$, the group G_{fSO} is an extension of SO_∞ by Z_4^f:

$$G_{fSO} = Z_4^f \leftthreetimes_{e_2} SO_\infty = (Z_2^f \leftthreetimes Z_2) \leftthreetimes SO_\infty. \tag{136}$$

The possible extensions of SO_∞ by Z_4 are labeled by $\bar e_2 \in H^2(\mathcal{B}SO_\infty;\mathbb{Z}_4) = \mathbb{Z}_2$ which is generated by $\bar e_2 \overset{4}{=} 2\bar{\mathrm{w}}_2(\bar a^{SO})$.

The links in the simplicial complex $\mathcal{B}G_{fSO}$ are labeled by $\bar a_{ij}^{G_{fSO}} \in G_{fSO}$ (see Ref. 41 and Appendix L). We may label the elements $\bar a_{ij}^{G_{fSO}} \in G_{fSO}$ by a pair $\bar a_{ij}^{SO} \in SO_\infty$ and $\bar a_{ij}^{Z_4^f} \in \mathbb{Z}_4^f$:

$$a^{G_{fSO}} = (a^{Z_4^f}, a^{SO}). \tag{137}$$

This allows us to introduce two projections $\pi(\bar a_{ij}^{G_{fSO}}) = \bar a_{ij}^{SO}$ and $\sigma(\bar a_{ij}^{G_{fSO}}) = \bar a_{ij}^{Z_4^f}$ (see Appendix N). Thus we can also label the links using a pair $(\bar a_{ij}^{Z_4^f}, \bar a_{ij}^{SO})$. Although $\mathrm{w}_2(\bar a^{SO})$ is a cocycle in $C^2(\mathcal{B}SO_\infty;\mathbb{Z}_2)$, $2\mathrm{w}_2(\pi(\bar a^{G_{fSO}}))$, when viewed as a function of $\bar a^{G_{fSO}}$, is a coboundary in $B^2(\mathcal{B}G_{fSO};\mathbb{Z}_4)$. In other words, the two canonical 1-cochain on $\mathcal{B}G_{fSO}$, $\bar a_{ij}^{SO}$ and $\bar a_{ij}^{Z_4^f}$, are related by (see (N13))

$$\mathrm{d}\bar a^{Z_4^f} \overset{4}{=} 2\bar{\mathrm{w}}_2(\bar a^{SO}). \tag{138}$$

We can write $\bar a^{Z_4^f}$ as

$$\bar a^{Z_4^f} \overset{4}{=} \bar a^{Z_2} + 2\bar A^{\mathbb{Z}_2^f} \tag{139}$$

where $\bar a^{Z_2}$ and $\bar A^{\mathbb{Z}_2^f}$ are $\mathbb{Z}_2$-valued 1-cochains. We see

$$\bar a^{Z_4^f} \overset{2}{=} \bar a^{Z_2}, \qquad \mathrm{d}\bar a^{Z_2} \overset{2}{=} 0. \tag{140}$$

Equation (140) implies that

$$\beta_2\bar a^{Z_2} \overset{2}{=} \bar{\mathrm{w}}_2(\bar a^{SO}) + \mathrm{d}\bar A^{\mathbb{Z}_2^f}. \tag{141}$$

which can be rewritten as (see (A32))

$$\mathrm{Sq}^1\bar a^{Z_2} \overset{2}{=} (\bar a^{Z_2})^2 \overset{2}{=} \bar{\mathrm{w}}_2(\bar a^{SO}) + \mathrm{d}\bar A^{\mathbb{Z}_2^f}. \tag{142}$$

A. 2+1D

1. Without extension of SO_∞

Calculate $\bar n_2$: First, $\bar n_2 \in H^2(\mathcal{B}Z_2;\mathbb{Z}_2)=\mathbb{Z}_2$. It has two choices: $\bar n_2=\alpha_n(\bar a^{\mathbb{Z}_2})^2$, $\alpha_n=0,1$.
Calculate $\bar\nu_3$: Similar to the last section, $\bar\nu_3$ satisfies

$$-\mathrm{d}\bar\nu_3 \overset{1}{=} \frac{\alpha_n^2}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^2+\frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2})^2\bar e_2 \overset{1}{=} 0. \tag{143}$$

Thus, $\bar\nu_3$ has two solutions:

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3,\quad \alpha_\nu=0,1, \tag{144}$$

since $H^3(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2$.
SPT invariant: The four Z_4^f fermionic SPT states labeled by $\alpha_n,\alpha_\nu=0,1$ have the following SPT invariant

$$\begin{aligned}&Z^{\mathrm{top}}(\mathcal{M}^3,A^{Z_2},A^{\mathbb{Z}_2^f})\\&=\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_\nu}{2}(A^{\mathbb{Z}_2})^3+\frac{\alpha_n}{2}(A^{\mathbb{Z}_2})^2A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2},\\&\mathrm{d}A^{\mathbb{Z}_2^f}\overset{2}{=}\mathrm{w}_2+(A^{\mathbb{Z}_2})^2,\qquad f_2|_{\partial\mathcal{N}^4}\overset{2}{=}\alpha_n(A^{\mathbb{Z}_2})^2\end{aligned} \tag{145}$$

where the space-time $\mathcal{M}^3$ is orientable and $\mathrm{w}_1\overset{2}{=}0$. However, as we will see below, the four Z_4^f fermionic SPT states all belong to the same phase.

On 2+1D space-time manifold, $\mathrm{w}_2+\mathrm{w}_1^2\overset{2,\mathrm{d}}{=}0$ (see Appendix I3). The Z_4^f fermionic symmetry requires the space-time $\mathcal{M}^3$ to be a orientable manifold with $\mathrm{w}_2+(A^{\mathbb{Z}_2})^2\overset{2,\mathrm{d}}{=}0$ and $\mathrm{w}_1\overset{2,\mathrm{d}}{=}0$. Thus $(A^{\mathbb{Z}_2})^2$ is always a coboundary: $(A^{\mathbb{Z}_2})^2=\mathrm{d}u_1$.

Let us write $f_2=\tilde f_2+\alpha_n\mathrm{d}u_1$ where $\tilde f_2\overset{2}{=}0$ on $\partial\mathcal{N}^4$, which implies $\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathbb{Sq}^2\tilde f_2+\tilde f_2\mathrm{w}_2}=1$. The SPT invariant now becomes (see (A29))

$$\begin{aligned}&Z^{\mathrm{top}}(\mathcal{M}^3,A^{Z_2},A^{\mathbb{Z}_2^f})\\&=\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_\nu}{2}A^{\mathbb{Z}_2}\mathrm{d}u_1+\frac{\alpha_n}{2}\mathrm{d}u_1A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^3}\alpha_n(\mathbb{Sq}^2u_1+u_1\mathrm{w}_2)}\\&=\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^3}\alpha_n[u_1(\mathrm{w}_2+\mathrm{d}u_1)+(u_1\mathrm{d}u_1+u_1\mathrm{w}_2)]}=1.\end{aligned} \tag{146}$$

We see that the SPT invariant is independent of α_n,α_ν. Thus at the end, we get only one Z_4^f fermionic SPT phase, which is the trivial SPT phase.

2. With extension of SO_∞

Calculate $\bar n_2$: With extension of SO_∞, in general, $\bar n_2(\bar a^{G_{fSO}})\in H^2(\mathcal{B}G_{fSO};\mathbb{Z}_2)$ is given by [using the pair $(\bar a^{SO},\bar a^{Z_4^f})$ to label $\bar a^{G_{fSO}}$]

$$\bar n_2(\bar a^{G_{fSO}})\overset{2}{=}\alpha_{n,1}\bar{\mathrm{w}}_2(\bar a^{SO})+\alpha_{n,2}(\bar a^{Z_2})^2, \tag{147}$$

$\alpha_{n,1},\alpha_{n,2}=0,1$. The above can be reduced to

$$\bar n_2(\bar a^{G_{fSO}})\overset{2}{=}\alpha_n(\bar a^{Z_2})^2 \tag{148}$$

$\alpha_n=0,1$, due to the relation (142). So $\bar n_2(\bar a^{G_{fSO}})$ has two choices, and $H^2(\mathcal{B}G_{fSO};\mathbb{Z}_2)=\mathbb{Z}_2$.
Calculate $\bar\nu_3$: Next, we consider ν_3 in (87) which becomes

$$-\mathrm{d}\bar\nu_3\overset{1}{=}\frac{1}{2}\mathrm{Sq}^2\bar n_2+\frac{1}{2}\bar n_2\bar{\mathrm{w}}_2\overset{1}{=}\frac{\alpha_n}{2}(\bar a^{Z_2})^2\mathrm{d}\bar A^{\mathbb{Z}_2^f}, \tag{149}$$

where we have used (142) and (148). We find that $\bar\nu_3$ is given by

$$\bar\nu_3\overset{1}{=}\frac{\alpha_\nu}{2}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_n}{2}(\bar a^{Z_2})^2\bar A^{\mathbb{Z}_2^f},\qquad \alpha_\nu=0,1. \tag{150}$$

SPT invariant: This leads to the SPT invariant

$$\begin{aligned}&Z^{\mathrm{top}}(\mathcal{M}^3,A^{G_{fSO}})\\&=\mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_\nu}{2}(A^{\mathbb{Z}_2})^3+\frac{\alpha_n}{2}(A^{\mathbb{Z}_2})^2A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2},\\&f_2|_{\partial\mathcal{N}^4}\overset{2}{=}\alpha_n(A^{\mathbb{Z}_2})^2,\quad \mathrm{d}A^{\mathbb{Z}_2^f}\overset{2}{=}\mathrm{w}_2+(A^{\mathbb{Z}_2})^2,\end{aligned} \tag{151}$$

which as calculated above is always an identity. Thus, there is only one trivial Z_4^f SPT fermionic phase in 2+1D. This agrees with the result obtained in Ref. 37.

The Z_4^f symmetry was denoted by $G_-(C)$ symmetry in Ref. 26. There, it was found that for non-interacting fermion systems with $Z_4^f=G_-(C)$ symmetry, the SPT phases in 2+1D is classified by $\mathbb{Z}$. The result from Ref. 37 indicates that all of those non-interacting fermion Z_4^f-SPT states actually correspond to trivial SPT states in the presence of interactions.

B. 3+1D

1. Without extension of SO_∞

Calculate $\bar n_3$: $\bar n_3\in H^3(\mathcal{B}Z_2;\mathbb{Z}_2)$ has two choices:

$$\bar n_3=\alpha_n(\bar a^{\mathbb{Z}_2})^3, \tag{152}$$

$\alpha_n=0,1$, since $H^3(\mathcal{B}Z_2;\mathbb{Z}_2)=\mathbb{Z}_2$.
Calculate $\bar\nu_4$: Next we want to solve

$$-\mathrm{d}\bar\nu_4\overset{1}{=}\frac{1}{2}(\mathrm{Sq}^2\bar n_3+\bar n_3\bar e_2)\overset{1}{=}0. \tag{153}$$

where we have used (M5). Since $H^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z})=0$, the solution of (153) is unique $\bar\nu_4\overset{1}{=}0$.
SPT invariant: This leads to the SPT invariant

$$\begin{aligned}Z(\mathcal{M}^3,A^{\mathbb{Z}_4^f})&=\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n(A^{\mathbb{Z}_2})^3A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2f_2+f_2\mathrm{w}_2},\\ f_3|_{\partial\mathcal{N}^4}&\overset{2}{=}\alpha_n(A^{\mathbb{Z}_2})^3,\end{aligned} \tag{154}$$

where we have written $A^{\mathbb{Z}_4^f}$ as

$$A^{\mathbb{Z}_4^f}\overset{4}{=}A^{\mathbb{Z}_2}+2A^{\mathbb{Z}_2^f}, \tag{155}$$

which satisfies

$$\begin{aligned}&\mathrm{d}A^{\mathbb{Z}_4^f}\overset{4}{=}2\mathrm{w}_2,\\&\mathrm{d}A^{\mathbb{Z}_2}\overset{2}{=}0,\qquad \mathrm{d}A^{\mathbb{Z}_2^f}\overset{2}{=}\mathrm{w}_2+\beta_2A^{\mathbb{Z}_2}\overset{2}{=}\mathrm{w}_2+(A^{\mathbb{Z}_2})^2.\end{aligned} \tag{156}$$

2. With extension of SO_∞

Calculate $\bar n_3$: In general, $\bar n_3(\bar a^{G_{fSO}}) \in H^3(\mathcal{B}G_{fSO};\mathbb{Z}_2)$ can be written as

$$\bar n_3(\bar a^{G_{fSO}}) \overset{2}{=} \alpha_n \bar{\mathrm{w}}_3(\bar a^{SO}) + \alpha_n'(\bar a^{Z_2})^3 + \alpha_n''\bar{\mathrm{w}}_2(\bar a^{SO})\bar a^{Z_2}, \quad \alpha_n,\alpha_n',\alpha_n'' = 0,1. \tag{157}$$

However, from $\bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} (\bar a^{Z_2})^2$, we find that $\bar{\mathrm{w}}_2 \bar a^{Z_2} \overset{2,\mathrm{d}}{=} (\bar a^{Z_2})^3$ and $\mathrm{Sq}^1(\bar{\mathrm{w}}_2 + (\bar a^{Z_2})^2) \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_1\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} 0$ (see Appendix I and notice $\bar{\mathrm{w}}_1 \overset{2}{=} 0$). Thus the above expression for $n_3(\bar a^{G_{fSO}})$ is reduced to

$$n_3(\bar a^{G_{fSO}}) \overset{2}{=} \alpha_n (\bar a^{Z_2})^3, \quad \alpha_n = 0,1. \tag{158}$$

There are two choices of n_3.
Calculate $\bar\nu_4$: Next, we consider $\bar\nu_4$ in (87) which becomes (see (142) and (M5))

$$\begin{aligned} -\mathrm{d}\bar\nu_4(\bar a^{G_{fSO}}) &\overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar n_3(\bar a^{G_{fSO}}) + \frac{1}{2}\bar n_3(\bar a^{G_{fSO}})\bar{\mathrm{w}}_2(\bar a^{SO}) \\ &\overset{1}{=} \frac{\alpha_n}{2}(\bar a^{Z_2})^5 + \frac{\alpha_n}{2}(\bar a^{Z_2})^3\bar{\mathrm{w}}_2(\bar a^{SO}) \\ &\overset{1}{=} \frac{\alpha_n}{2}(\bar a^{Z_2})^3\mathrm{d}\bar A^{\mathbb{Z}_2^f}. \end{aligned} \tag{159}$$

From $\frac{1}{2}\mathrm{w}_2 \overset{1,\mathrm{d}}{=} \frac{1}{2}(\bar a^{Z_2})^2$, we obtain that $\frac{1}{2}\mathrm{w}_2(\bar a^{Z_2})^2 \overset{1,\mathrm{d}}{=} \frac{1}{2}(\bar a^{Z_2})^4$ and $\frac{1}{2}(\mathrm{w}_2)^2 \overset{1,\mathrm{d}}{=} \frac{1}{2}(\bar a^{Z_2})^2\mathrm{w}_2$. Thus $\frac{1}{2}(\bar{\mathrm{w}}_2)^2 \overset{1,\mathrm{d}}{=} \frac{1}{2}(\bar a^{Z_2})^4$. Since $H^4(\mathcal{B}Z_2;\mathbb{R}/\mathbb{Z}) = 0$, we find that $\frac{1}{2}(\bar a^{Z_2})^4 \overset{1,\mathrm{d}}{=} 0$. We also find (see Appendix I) $0 \overset{2,\mathrm{d}}{=} \mathrm{Sq}^1(\bar{\mathrm{w}}_2 + (\bar a^{Z_2})^2) \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_1\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_3$ and $0 \overset{2,\mathrm{d}}{=} \mathbb{S}\mathrm{q}^2(\bar{\mathrm{w}}_2 + (\bar a^{Z_2})^2) = \bar{\mathrm{w}}_2^2 + (\bar a^{Z_2})^4$. To summarize, we have

$$\frac{1}{2}(\bar a^{Z_2})^4 \overset{1,\mathrm{d}}{=} \frac{1}{2}\bar{\mathrm{w}}_2^2 \overset{1,\mathrm{d}}{=} \frac{1}{2}\bar a^{Z_2}\bar{\mathrm{w}}_3 \overset{1,\mathrm{d}}{=} \frac{1}{2}(\bar a^{Z_2})^2\mathrm{w}_2 \overset{1,\mathrm{d}}{=} 0. \tag{160}$$

This allows us to conclude that $\bar\nu_4$ must have a form

$$\bar\nu_4 \overset{1}{=} \frac{\alpha_n}{2}(\bar a^{Z_2})^3\bar A^{\mathbb{Z}_2^f} + \frac{\alpha_\nu}{2}\bar p_1. \tag{161}$$

But on $\mathcal{M}^4$, we have $p_1 \overset{2,\mathrm{d}}{=} \mathrm{w}_2^2 \overset{2,\mathrm{d}}{=} 0$. Thus the pullback of $\bar\nu_4$ to $\mathcal{M}^4$ reduces to

$$\nu_4 \overset{1}{=} \frac{\alpha_n}{2}(a^{\mathbb{Z}_2})^3 A^{\mathbb{Z}_2^f}. \tag{162}$$

SPT invariant: The corresponding fermionic model (in the bosonized form) is given by

$$\begin{aligned} &Z(\mathcal{M}^4, A^{G_{fSO}}) \\ &= \sum_{g_i\in G_{fO}} \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n(a^{\mathbb{Z}_2})^3A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2 f_2 + f_2(\mathrm{w}_2+\mathrm{w}_1^2)}, \\ &f_3\big|_{\partial\mathcal{N}^5} \overset{2}{=} \alpha_n(a^{\mathbb{Z}_2})^3, \end{aligned} \tag{163}$$

which lead to a SPT invariant given by

$$\begin{aligned} &Z^{\mathrm{top}}(\mathcal{M}^4, A^{G_{fSO}}) \\ &= \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n(A^{\mathbb{Z}_2})^3A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2 f_2 + f_2\mathrm{w}_2}, \\ &f_3\big|_{\partial\mathcal{N}^5} \overset{2}{=} \alpha_n(A^{\mathbb{Z}_2})^3, \quad \mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + (A^{\mathbb{Z}_2})^2, \end{aligned} \tag{164}$$

where $A^{G_{fSO}}$ is labeled by $(A^{\mathbb{Z}_2}, A^{\mathbb{Z}_2^f}, A^{SO})$ (see (137) and (139)). This agrees with (154)

Is the above SPT invariant trivial or not trivial? One way to show the non-trivialness is to change $A^{\mathbb{Z}_2^f}$ by a $\mathbb{Z}_2$-valued 1-cocycle $A_0^{\mathbb{Z}_2^f}$. In this case $Z^{\mathrm{top}}(\mathcal{M}^4, A^{G_{fSO}})$ changes by a factor

$$\tilde Z^{\mathrm{top}} = \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n(A^{\mathbb{Z}_2})^3A_0^{\mathbb{Z}_2^f}} = \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n\mathrm{w}_2A^{\mathbb{Z}_2}A_0^{\mathbb{Z}_2^f}} \tag{165}$$

where we have used $\mathrm{w}_2 \overset{2,\mathrm{d}}{=} (A^{\mathbb{Z}_2})^2$. Since

$$\begin{aligned} \mathrm{w}_2A^{\mathbb{Z}_2}A_0^{\mathbb{Z}_2^f} &\overset{2,\mathrm{d}}{=} \mathrm{Sq}^2(A^{\mathbb{Z}_2}A_0^{\mathbb{Z}_2^f}) \overset{2,\mathrm{d}}{=} (\mathrm{Sq}^1A^{\mathbb{Z}_2})(\mathrm{Sq}^1A_0^{\mathbb{Z}_2^f}) \\ &\overset{2,\mathrm{d}}{=} (\frac{1}{2}\mathrm{d}A^{\mathbb{Z}_2})(\frac{1}{2}\mathrm{d}A_0^{\mathbb{Z}_2^f}). \end{aligned} \tag{166}$$

We find

$$\tilde Z^{\mathrm{top}} = \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{M}^4}\alpha_n(\frac{1}{2}\mathrm{d}A^{\mathbb{Z}_2})(\frac{1}{2}\mathrm{d}A_0^{\mathbb{Z}_2^f})} = 1, \tag{167}$$

which is independent of α_n. This suggests that $\alpha_n = 0,1$ describes the same SPT phases.

C. $Z_4^f \times Z_2^T$-SPT model

We have seen that the model (163) describes a trivial Z_4^f-SPT phase even when $\alpha_n = 1$. However, we note that the model (163) actually has a $Z_4^f \times Z_2^T$ symmetry. It turns out that the model (163) realizes a non-trivial $Z_4^f \times Z_2^T$-SPT phase when $\alpha_n = 1$.

To physically detect that non-trivial $Z_4^f \times Z_2^T$-SPT phase, we note that $n_3 = \alpha_n(A^{\mathbb{Z}_2})^3$ is the 3-cocycle fermion current. Let us put the Z_4^f-SPT state on space-time $S_t^1 \times M^3$, where $A^{G_{fSO}}$ on $S_t^1 \times M^3$ is the pull-back of a $A^{G_{fSO}}$ on the space M^3. In this case, using $(A^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_2$ and $\mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_2$ on M^3, we can rewrite $(A^{\mathbb{Z}_2})^3$ on M^3 as $A^{\mathbb{Z}_2}\mathrm{w}_2 \overset{2,\mathrm{d}}{=} A^{\mathbb{Z}_2}\mathrm{w}_1^2$. Now we can choose $M^3 = \mathbb{R}P^2 \times S^1$. Then we have

$$\int_{M^3} n_3 \overset{2}{=} \alpha_n \int_{\mathbb{R}P^2\times S^1} A^{\mathbb{Z}_2}\mathrm{w}_1^2 \overset{2}{=} \alpha_n \int_{S_z^1} A^{\mathbb{Z}_2}. \tag{168}$$

We find that, if we put the Z_4^f-SPT state on space $\mathbb{R}P^2 \times S^1$, the fermion number in the ground state will be given by $N_f \overset{2}{=} \alpha_n \int_{S^1} A^{\mathbb{Z}_2}$. Thus adding a Z_4^f symmetry twist around S_z^1 will change the fermion number in the ground state by α_n mod 2. When $\alpha_n = 1$, the non-trivial change in the fermion number indicates the non-trivialness of the SPT phase.

To realize such a fermionic $Z_4^f \times Z_2^T$-SPT phase in 3+1D, we start with a symmetry breaking state that break the Z_4^f to Z_2^f. The order-parameter has a Z_2-value. We then consider a random configuration of order-parameter in 3d space. A fermion decoration that gives rise to the $\alpha_n = 1$ phase is realized by binding a fermion worldline to $*(*W_3)^3$. Here W_3 is the 3-dimensional domain wall of the Z_2-order-parameter. $*$ is the Poincaré dual. Thus $*W_3$ is a 1-cocycle. $*(*W_3)^3$ is the Poincaré dual of a 3-cocycle which is a closed loop. The fermion worldline is attached to such a loop.

XII. FERMIONIC $Z_2^f \times Z_2^T$-SPT STATE

The fermion symmetry $Z_2^f \times Z_2^T$ is realized by electron superconductors with coplanar spin polarization. The time reversal Z_2^T is generated by the standard time reversal followed by a 180° spin rotation.

For $G_f = Z_2^T \times Z_2^f$, G_{fO} is a Z_2^f extension of O_∞: $G_{fO} = Z_2^f \leftthreetimes O_\infty$. Such kind of extensions are classified by $H^2(\mathcal{B}O_\infty;\mathbb{Z}_2) = \mathbb{Z}_2^2$ which is generated by w_1^2 and w_2. For fermion symmetry $Z_2^f \times Z_2^T$, we should choose the extension $G_{fO} = Z_2^f \leftthreetimes_{\mathrm{w}_2+\mathrm{w}_1^2} O_\infty = Pin_\infty^-$. According to Appendix N, this implies that, on $\mathcal{B}Pin_\infty^-$, the canonical 1-cochain $\bar a^{G_{fO}}$ satisfy a relation

$$\bar{\mathrm{w}}_2(\bar a^{G_{fO}}) + \bar{\mathrm{w}}_1^2(\bar a^{G_{fO}}) \overset{2}{=} \mathrm{d}\bar A^{\mathbb{Z}_2^f}. \tag{169}$$

A. 2+1D

1. Without extension of O_∞

Calculate $\bar n_2$: From $\bar n_2 \in H^2(\mathcal{B}Z_2^T;\mathbb{Z}_2) = \mathbb{Z}_2$, we obtain

$$\bar n_2 \overset{2,\mathrm{d}}{=} \alpha_n(\bar a^{Z_2^T})^2 \overset{2,\mathrm{d}}{=} \alpha_n \bar{\mathrm{w}}_1^2, \quad \alpha_n = 0,1. \tag{170}$$

Calculate $\bar\nu_3$: Next, we consider $\bar\nu_3$ in (53). Since $\bar e_2 = 0$, only the term $\frac12\mathrm{Sq}^2\bar n_2 = \frac12\bar n_2^2$ is non-zero. The term $\frac12\bar n_2^2 = \frac{\alpha_n}{2}(\bar a^{\mathbb{Z}_2^T})^4$ is a cocycle in $\mathcal{Z}^4(Z_2^T;(\mathbb{R}/\mathbb{Z})_T)$. Note that now Z_2^T has a non-trivial action on the value $\mathbb{R}/\mathbb{Z}$ and the differential operator d is modified by this non-trivial action [which corresponds to the cases of non-trivial α_i in Ref. 41 and Appendix L, see (L18)]. This modifies the group cohomology. Since $H^4(\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2$, and $\frac12(a^{\mathbb{Z}_2^T})^4$ is the non-trivial cocycle in $H^4(\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T)$, (53) has solution only when $\alpha_n = 0$. In case $\alpha_n = 0$, there is only one solution $\nu_3 \overset{1}{=} 0$, since $H^3(\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T) = 0$.

From (126), we see that when $\alpha_n = 1$, the action amplitude is not real, and breaks the time reversal symmetry. This is why $\alpha_n = 1$ is not a solution for time-reversal symmetric cases.

2. With extension of O_∞

Calculate $\bar n_2$: Due to the relation (169) on $\mathcal{B}G_{fO} = \mathcal{B}Pin_\infty^-$, $H^2(\mathcal{B}Pin_\infty^-) = \mathbb{Z}_2$ which is generated by $\bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_1^2$. Therefore, $\bar n_2 \in H^2(\mathcal{B}Pin_\infty^-;\mathbb{Z}_2) = \mathbb{Z}_2$ has two choices

$$n_2 \overset{2}{=} \alpha_n \bar{\mathrm{w}}_1^2, \quad \alpha_n = 0,1. \tag{171}$$

Calculate $\bar\nu_3$: Next, we consider $\bar\nu_3$ that satisfy (80) which becomes, in the present case,

$$\begin{aligned} -\mathrm{d}\bar\nu_3 &\overset{1}{=} \frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^4 + \bar{\mathrm{w}}_1^2(\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_1^2)]. \\ &\overset{1}{=} \frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^4 + \bar{\mathrm{w}}_1^2\,\mathrm{d}\bar A^{\mathbb{Z}_2^f}]. \end{aligned} \tag{172}$$

where we have used

$$\mathrm{d}\bar A^{\mathbb{Z}_2^f} \overset{2}{=} \bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_1^2. \tag{173}$$

$\frac12\bar{\mathrm{w}}_1^4$ is a non-trivial cocycle in $H^4(\mathcal{B}Pin_\infty^-;(\mathbb{R}/\mathbb{Z})_T)$. Thus (80) has solution only when $\alpha_n = 0$. In this case, there are four solutions

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_\nu}{2}\bar{\mathrm{w}}_3 + \frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_1^3, \quad \alpha_\nu,\tilde\alpha_\nu = 0,1. \tag{174}$$

SPT invariant: The corresponding SPT invariant is given by

$$\begin{aligned} &Z^{\mathrm{top}}(\mathcal{M}^3, A^{G_{fO}}) \\ &= \mathrm{e}^{\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\frac{\alpha_\nu}{2}\mathrm{w}_3(A^O)+\frac{\tilde\alpha_\nu}{2}\mathrm{w}_1^3(A^O)}\,\mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2 f_2 + f_2\mathrm{w}_2}, \\ &f_2|_{\partial\mathcal{N}^4} \overset{2}{=} 0 \ \to\ \mathrm{e}^{\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2 f_2 + f_2\mathrm{w}_2} = 1, \end{aligned} \tag{175}$$

where the background connection $A^{G_{fO}}$ is labeled by $(A^{\mathbb{Z}_2^f}, A^O)$. On 2+1D space-time $\mathcal{M}^3$, we always have $\mathrm{w}_2 + \mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} 0$ and $\mathrm{w}_1^3 \overset{2,\mathrm{d}}{=} \mathrm{w}_2\mathrm{w}_1 \overset{2,\mathrm{d}}{=} \mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0$ (see Appendix I 3). The above four solutions give rise to the same SPT invariant and the same fermionic $Z_2^f \times Z_2^T$ SPT phase.

For non-interacting fermions, there is no non-trivial fermionic $Z_2^f \times Z_2^T$-SPT phase. The above result implies that, for interacting fermions, fermion decoration also fail to produce any non-trivial fermionic $Z_2^f \times Z_2^T$-SPT phase. The spin cobordism consideration[18,19] tells us that there is no non-trivial fermionic $Z_2^f \times Z_2^T$-SPT phase, even beyond the fermion decoration construction.

B. 3+1D

1. Without extension of O_∞

Calculate $\bar n_3$: From $H^3(\mathcal{B}Z_2^T;\mathbb{Z}_2) = \mathbb{Z}_2$, we find that:

$$\bar n_3 = \alpha_n(\bar a^{\mathbb{Z}_2^T})^3 = \alpha_n \bar{\mathrm{w}}_1^3, \tag{176}$$

$\alpha_n = 0,1$.

Calculate $\bar\nu_4$: Next, we consider $\bar\nu_4$ in (53), which has a form

$$-\mathrm{d}\bar\nu_4 \overset{1}{=} \frac{\alpha_n}{2}\mathrm{Sq}^2\bar{\mathrm{w}}_1^3 \overset{1}{=} \frac{\alpha_n}{2}\bar{\mathrm{w}}_1^5 \tag{177}$$

where (M5) is used. It turns out that $\frac12\bar{\mathrm{w}}_1^5$ is a trivial element in $H^5(\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T)=0$:

$$\frac12\bar{\mathrm{w}}_1^5 \overset{1}{=} \mathrm{d}_{\mathrm{w}_1}\bar\eta_4. \tag{178}$$

To calculate $\bar\eta_4$, we first note that (see (A9))

$$(\mathrm{d}_{\bar{\mathrm{w}}_1}\frac14)_{01} = \frac14(-)^{(\bar{\mathrm{w}}_1)_{01}} - \frac14 = -\frac12(\bar{\mathrm{w}}_1)_{01}, \tag{179}$$

or

$$\mathrm{d}_{\bar{\mathrm{w}}_1}\frac14 \overset{1}{=} \frac12\bar{\mathrm{w}}_1. \tag{180}$$

We also note that

$$\bar{\mathrm{w}}_1^4 \overset{2}{=} (\mathrm{Sq}^1\bar{\mathrm{w}}_1)^2 \overset{2}{=} (\beta_2\bar{\mathrm{w}}_1)^2, \tag{181}$$

where $\beta_2\bar{\mathrm{w}}_1$ is a $\mathbb{Z}$-valued cocycle. Thus

$$\frac12\bar{\mathrm{w}}_1^5 \overset{1}{=} \frac12\bar{\mathrm{w}}_1(\beta_2\bar{\mathrm{w}}_1)^2 \overset{1}{=} (\mathrm{d}_{\bar{\mathrm{w}}_1}\frac14)(\beta_2\bar{\mathrm{w}}_1)^2 \overset{1}{=} \mathrm{d}_{\bar{\mathrm{w}}_1}[\frac14(\beta_2\bar{\mathrm{w}}_1)^2]. \tag{182}$$

or

$$\bar\nu_4 = \frac14(\beta_2\bar{\mathrm{w}}_1)^2 \tag{183}$$

We see that $\bar\nu_4$ has a form

$$\bar\nu_4 \overset{1,\mathrm{d}}{=} \frac{\alpha_n}{4}(\beta_2\bar{\mathrm{w}}_1)^2 + \frac{\alpha_\nu}{2}\bar{\mathrm{w}}_1^4 \quad \alpha_\nu = 0,1. \tag{184}$$

2. With extension of O_∞

Calculate $\bar n_3$: $\bar n_3 \in H^3(\mathcal{B}Pin^-_\infty;\mathbb{Z}_2)$ may have a form

$$n_3 \overset{2}{=} \alpha_n\bar{\mathrm{w}}_1^3 + \alpha_n'\bar{\mathrm{w}}_1\bar{\mathrm{w}}_2 + \alpha_n''\bar{\mathrm{w}}_3, \qquad \alpha_n,\alpha_n,\alpha_n'' = 0,1. \tag{185}$$

From (169), we have $\bar{\mathrm{w}}_1\bar{\mathrm{w}}_2 \overset{2}{=} \bar{\mathrm{w}}_1^3$ and $0 \overset{2}{=} \mathrm{Sq}^1(\bar{\mathrm{w}}_1^2+\bar{\mathrm{w}}_2) \overset{2}{=} \bar{\mathrm{w}}_1\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_3$. Thus $\bar n_3$ has two choices

$$\bar n_3 \overset{2}{=} \alpha_n\bar{\mathrm{w}}_1^3, \qquad \alpha_n = 0,1. \tag{186}$$

Calculate $\bar\nu_4$: Next, we consider $\bar\nu_4$ that satisfy (80) which becomes, after using (M5)

$$\begin{aligned}-\mathrm{d}\bar\nu_4 &\overset{1}{=} \frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^5 + \bar{\mathrm{w}}_1^3(\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_1^2)].\\ &\overset{1}{=} \alpha_n(\mathrm{d}\bar\eta_4 + \frac12\bar{\mathrm{w}}_1^3\mathrm{d}\bar A^{\mathbb{Z}_2^f}).\end{aligned} \tag{187}$$

where we have used (169) and (178).

From $\frac12\bar{\mathrm{w}}_2 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_1^2$, we obtain that $\frac12\bar{\mathrm{w}}_2\bar{\mathrm{w}}_1^2 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_1^4$ and $\frac12\bar{\mathrm{w}}_2^2 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_1^2\bar{\mathrm{w}}_2$. We also find (see Appendix I) $0 \overset{2,\mathrm{d}}{=} \mathrm{Sq}^1(\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_1^2) \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_1\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_3$ and $0 \overset{2,\mathrm{d}}{=} \mathbb{S}\mathrm{q}^2(\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_1^2) = \bar{\mathrm{w}}_2^2+\bar{\mathrm{w}}_1^4$. To summarize, we have

$$\frac12\bar{\mathrm{w}}_1^4 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_2^2 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_1^2\bar{\mathrm{w}}_2 \overset{1,\mathrm{d}}{=} \frac12\bar{\mathrm{w}}_1\bar{\mathrm{w}}_3 \overset{1,\mathrm{d}}{=} \frac12\bar p_1. \tag{188}$$

where $\bar p_1(a^O)$ is the first Pontryagin class (see (J2)).

Thus (80) has a solution of form

$$\begin{aligned}\bar\nu_4 &\overset{1}{=} \alpha_n(\frac14(\beta_2\bar{\mathrm{w}}_1)^2 + \frac12\bar{\mathrm{w}}_1^3\bar A^{\mathbb{Z}_2^f}) + \frac{\alpha_\nu}{2}\bar{\mathrm{w}}_1^4 + \frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_4,\\ &\alpha_\nu,\tilde\alpha_\nu = 0,1,\end{aligned} \tag{189}$$

SPT invariant: The corresponding SPT invariant is given by

$$\begin{aligned}&Z^{\text{top}}(\mathcal{M}^4, A^{G_{fO}}) = \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2 f_3 + f_3(\mathrm{w}_2+\mathrm{w}_1^2)}\\ &\qquad \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4}\alpha_n(\frac14(\beta_2\mathrm{w}_1)^2+\frac12\mathrm{w}_1^3A^{\mathbb{Z}_2^f})+\frac{\alpha_\nu}{2}\mathrm{w}_1^4+\frac{\tilde\alpha_\nu}{2}\mathrm{w}_4},\\ &f_3|_{\partial\mathcal{N}^5} \overset{2}{=} \alpha_n\mathrm{w}_1^3, \quad \mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2+\mathrm{w}_1^2,\end{aligned} \tag{190}$$

where the background connection $A^{G_{fO}}$ is labeled by $(A^{\mathbb{Z}_2^f}, A^O)$. In 3+1D space-time, we have some additional relations (I16). When combined with (188), we find

$$\begin{aligned}&\frac12\mathrm{w}_1^4 \overset{1,\mathrm{d}}{=} \frac12\mathrm{w}_2^2 \overset{1,\mathrm{d}}{=} \frac12\mathrm{w}_1^2\mathrm{w}_2 \overset{1,\mathrm{d}}{=} \frac12\mathrm{w}_1\mathrm{w}_3 \overset{1,\mathrm{d}}{=} \frac12 p_1 \overset{1,\mathrm{d}}{=} 0,\\ &\frac12\mathrm{w}_4 \overset{1,\mathrm{d}}{=} 0.\end{aligned} \tag{191}$$

Therefore, the SPT invariant is independent of α_ν and $\tilde\alpha_\nu$, and they fail to label different $Z_2^f\times Z_2^T$-SPT phases. Also from $\mathrm{w}_1^2+\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$ and $\mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$ on the 3+1D space-time, we see that $\mathrm{w}_1^3 \overset{2,\mathrm{d}}{=} 0$. Thus n_3 is always a $\mathbb{Z}_2$-valued coboundary when pulled back on $\mathcal{M}^4$. As a result, the SPT invariant is independent of α_n and it fails to label different $Z_2^f\times Z_2^T$-SPT phases (see (108)). Therefore, there is only one trivial $Z_2^f\times Z_2^T$ SPT phase.

The fermionic $Z_2^f\times Z_2^T$ symmetry is denoted by G_+^T in Ref. 26. It was found that for non-interacting fermions, there is no non-trivial fermionic $Z_2^f\times Z_2^T$-SPT phase in 3+1D.[24–26] The spin cobordism consideration[18,19] also tells us that there is no non-trivial fermionic $Z_2^f\times Z_2^T$-SPT phase, even beyond the fermion decoration construction.

XIII. FERMIONIC $Z_4^{T,f}$-SPT STATE

In this section, we consider the fermionic SPT states with $G_b = Z_2^T$ and a non-trivial $\bar e_2 = (\bar a^{\mathbb{Z}_2^T})^2 = \bar{\mathrm{w}}_1^2$ in $H^2(\mathcal{B}Z_2^T;\mathbb{Z}_2)=\mathbb{Z}_2$. In this case, the full fermionic symmetry is $G_f = Z_4^{T,f}$. For $G_f = Z_4^{T,f}$, G_{fO} is a Z_2^f extension of O_∞: $G_{fO} = Z_2^f \leftthreetimes_{\mathrm{w}_2} O_\infty = Pin^+_\infty$. This implies that on $\mathcal{B}G_{fO} = \mathcal{B}Pin^+_\infty$, we have a relation (see Appendix N)

$$\bar{w}_2(\bar{a}^{G_{fO}}) \overset{2}{=} d\bar{A}^{\mathbb{Z}_2^f}. \tag{192}$$

The fermion symmetry $Z_4^{T,f}$ is realized by charge $2e$ electron superconductors with spin-orbital couplings.

A. 2+1D

1. Without extension of O_∞

Calculate $\bar{n}_2$: First, the possible $\bar{n}_2$'s have a form

$$\bar{n}_2 = \alpha_n(\bar{a}^{\mathbb{Z}_2^T})^2 \overset{2}{=} \alpha_n \bar{w}_1^2, \tag{193}$$

where $\alpha_n = 0, 1$.

However, after $\bar{n}_2$ is pulled back on $\mathcal{M}^3$, it becomes $n_2 \overset{2}{=} \alpha_n w_1^2$. On 2+1D manifold, we have $w_2 + w_1^2 \overset{2,d}{=} 0$, and on a Pin$^+$ manifold we have $w_2 \overset{2,d}{=} 0$. Thus, on a 2+1D Pin$^+$ manifold, w_1^2 is always a coboundary. The two solutions of n_2 are equivalent (see (108)) and we can choose $\alpha_n = 0$.

Calculate $\bar{\nu}_3$: Now $\bar{\nu}_3$ is obtained from $d\bar{\nu}_3 \overset{1}{=} 0$. $\bar{\nu}_3$ has only one solution $\bar{\nu}_3 \overset{1}{=} 0$, since $H^3(\mathcal{B}Z_2^T; (\mathbb{R}/\mathbb{Z})_T) = 0$.

2. With extension of O_∞

Calculate $\bar{n}_2$: Due to the relation (192), $H^2(\mathcal{B}Pin^+_\infty) = \mathbb{Z}_2$ which is generated by $\bar{w}_1^2$. Thus $\bar{n}_2 \in H^2(\mathcal{B}Pin^+_\infty; \mathbb{Z}_2)$ has two choices $\bar{n}_2 \overset{2}{=} \alpha_n \bar{w}_1^2$, $\alpha_n = 0, 1$.

On a 2+1D space-time $\mathcal{M}^3$ with $G_{fO} = Z_2^f \leftthreetimes_{\bar{w}_2} O_\infty = Pin^+_\infty$ connection $A^{G_{fO}}$, the connection can be viewed as a pullback from the canonical 1-cochain $\bar{a}^{G_{fO}}$ on $\mathcal{B}G_{fO}$. Such a 2+1D space-time satisfies $w_2 \overset{2,d}{=} 0$ and is a Pin$^+$ manifold. Since any 3-manifold $\mathcal{M}^3$ satisfies $w_2 + w_1^2 \overset{2,d}{=} 0$ (see Appendix I), therefore, $\mathcal{M}^3$ with $Z_2^f \leftthreetimes_{\bar{w}_2} O_\infty = Pin^+_\infty$ connection satisfies $w_2 \overset{2,d}{=} w_1^2 \overset{2,d}{=} 0$. The pullback of $\bar{n}_2$ on $\mathcal{M}^3$ is given by $n_2 \overset{2}{=} \alpha_n w_1^2$. We see that, after pulled back to $\mathcal{M}^3$, $\alpha_n = 1$ and $\alpha_n = 1$ are equivalent. We may choose $\alpha_n = 0$.

Calculate $\bar{\nu}_3$: Next, we consider $\bar{\nu}_3$ that satisfy $-d\bar{\nu}_3 \overset{1}{=} 0$. There are four solutions

$$\bar{\nu}_3 \overset{1}{=} \frac{\alpha_\nu}{2}\bar{w}_3 + \frac{\tilde{\alpha}_\nu}{2}\bar{w}_1^3, \quad \alpha_\nu, \tilde{\alpha}_\nu = 0, 1. \tag{194}$$

After pulled back to $\mathcal{M}^3$, $\bar{\nu}_3$ becomes $\nu_3 \overset{1}{=} \frac{\alpha_\nu}{2} w_3 + \frac{\tilde{\alpha}_\nu}{2} w_1^3$. But on 2+1D space-time $\mathcal{M}^3$, we have a relation $w_3 \overset{2,d}{=} 0$ (see (I12)). Combined the $w_2 \overset{2,d}{=} w_1^2 \overset{2,d}{=} 0$ obtained above, we find that the above four solutions differ only by coboundaries, which give rise to the same fermionic $Z_4^{T,f}$ SPT phase. Therefore the fermion decoration construction fails to produce a non-trivial fermionic $Z_4^{T,f}$ SPT phase. Thus, the fermion decoration construction fail to produces any non-trivial fermionic $Z_4^{T,f}$-SPT state.

In fact, 2+1D $Z_4^{T,f}$ fermionic SPT phases is classified by $\mathbb{Z}_2$[18,19]. The non-trivial $Z_4^{T,f}$ SPT phase can be realized as a $p + ip$ superconductor for spin-up fermions stacking with a $p - ip$ superconductor for spin-down fermions[56,57]. It has the following special property: After we gauge the Z_2^f symmetry, we obtain a Z_2^f gauge theory with $G_b = Z_2^T$ symmetry. In 2+1D, Z_2^f charge e, Z_2^f-flux, and their bound state em are all point-like topological excitations. The time-reversal symmetry in this case exchanges the bosonic e and m, and em is a Kramers doublet[29,33,58].

B. 3+1D

1. Without extension of O_∞

Calculate $\bar{n}_3$: $\bar{n}_3 \in H^3(\mathcal{B}Z_2^T; \mathbb{Z}_2) = \mathbb{Z}_2$ is given

$$\bar{n}_3 \overset{2,d}{=} \alpha_n(\bar{a}^{\mathbb{Z}_2^T})^3 \overset{2,d}{=} \alpha_n \bar{w}_1^3. \tag{195}$$

For the 3+1D space-time with a Pin$^+$ structure (*i.e.* $w_2 \overset{2,d}{=} 0$), it turns out that, even when $w_2 \overset{2,d}{=} 0$, w_1^3 is still non-trivial. So $n_3 \overset{1}{=} \alpha_n w_1^3$ indeed has two choices.

Calculate $\bar{\nu}_4$: Next we want to solve

$$\begin{aligned} -d\bar{\nu}_4 &\overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar{n}_3 + \frac{1}{2}\bar{n}_3\bar{e}_2 \\ &\overset{1}{=} \frac{\alpha_n}{2}\mathbb{S}\mathrm{q}^2(\bar{a}^{\mathbb{Z}_2^T})^3 + \frac{\alpha_n}{2}(\bar{a}^{\mathbb{Z}_2^T})^5 \overset{1}{=} 0, \end{aligned} \tag{196}$$

where we have used (M5)). Since $H^4(\mathcal{B}Z_2^T; (\mathbb{R}/\mathbb{Z})_T) = \mathbb{Z}_2$, ν_4 has two solutions

$$\bar{\nu}_4 \overset{1}{=} \frac{\alpha_\nu}{2}(\bar{a}^{\mathbb{Z}_2^T})^4 \overset{1}{=} \frac{\alpha_\nu}{2}\bar{w}_1^4.$$

On 3+1D space-time with $w_2 \overset{2,d}{=} 0$, $\frac{1}{2}w_1^4$ is still a non-trivial $\mathbb{R}/\mathbb{Z}$-valued cocycle. The pullback on $\mathcal{M}^4$, $\nu_4 \overset{1}{=} \frac{\alpha_\nu}{2} w_1^4$, is still non-trivial.

2. With extension of O_∞

Calculate $\bar{n}_3$: Due to the relation (192), and $0 \overset{2,d}{=} \mathrm{Sq}^1\bar{w}_2 \overset{2,d}{=} \bar{w}_1\bar{w}_2 + \bar{w}_3 \overset{2,d}{=} \bar{w}_3$, we find that $H^2(\mathcal{B}Pin^+_\infty; \mathbb{Z}_2) = \mathbb{Z}_2$ is generated by $\bar{w}_1^3$. Thus $\bar{n}_3 \in H^2(\mathcal{B}Pin^+_\infty; \mathbb{Z}_2)$ has two choices $\bar{n}_3 \overset{2}{=} \alpha_n \bar{w}_1^3$, $\alpha_n = 0, 1$.

Calculate $\bar{\nu}_4$: Next, we consider $\bar{\nu}_4$ that satisfy

$$\begin{aligned} -d\bar{\nu}_4 &\overset{1}{=} \frac{\alpha_n}{2}[\bar{w}_1^5 + \bar{w}_1^3(\bar{w}_2 + \bar{w}_1^2)] \\ &\overset{1}{=} \frac{\alpha_n}{2}\bar{w}_1^3 d\bar{A}^{\mathbb{Z}_2^f}. \end{aligned} \tag{197}$$

Since $\bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} 0$, there are eight solutions

$$\bar{\nu}_4 \overset{1}{=} \frac{\alpha_n}{2}\bar{\mathrm{w}}_1^3 \bar{A}^{\mathbb{Z}_2^f} + \frac{\alpha_{\nu,1}}{2}\bar{\mathrm{w}}_1^4 + \frac{\alpha_{\nu,2}}{2}\bar{\mathrm{w}}_4 + \frac{\alpha_{\nu,3}}{2}\bar{p}_1, \quad \alpha_{\nu,i} = 0,1. \tag{198}$$

But on 3+1D space-time $\mathcal{M}^3$ with $\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$, we have relations $\mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0$, $\mathrm{w}_4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^4$, and $p_1 \overset{2}{=} \mathrm{w}_2^2 \overset{2}{=} 0$ (see (I16)). So the pullback of $\bar{\nu}_4$ on $\mathcal{M}^4$ becomes

$$\nu_4 \overset{1}{=} \frac{\alpha_n}{2}\mathrm{w}_1^3 \bar{A}^{\mathbb{Z}_2^f} + \frac{\alpha_\nu}{2}\mathrm{w}_1^4, \quad \alpha_\nu = 0,1. \tag{199}$$

SPT invariant: The corresponding SPT invariant is given by

$$Z^{\text{top}}(\mathcal{M}^4, A^{G_{fO}}) = \mathrm{e}^{\,\mathrm{i}\,\pi \int_{\mathcal{N}^5} \mathrm{Sq}^2 f_3 + f_3(\mathrm{w}_2+\mathrm{w}_1^2)} \mathrm{e}^{\,\mathrm{i}\,2\pi \int_{\mathcal{M}^4} \frac{\alpha_n}{2}\mathrm{w}_1^3 \bar{A}^{\mathbb{Z}_2^f} + \frac{\alpha_\nu}{2}\mathrm{w}_1^4},$$
$$f_3\big|_{\partial\mathcal{N}^5} \overset{2}{=} \alpha_n \mathrm{w}_1^3, \quad \mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2, \tag{200}$$

where the background connection $A^{G_{fO}}$ is labeled by $(A^{\mathbb{Z}_2^f}, A^O)$. We see that fermion decoration construction produces four different fermionic $Z_4^{T,f}$-SPT phases.

For non-interacting fermion systems, the $Z_4^{T,f}$-SPT phases are classified by $\mathbb{Z}$.[24–26] However, after include interaction, Ref. 18 and 19 found that the $Z_4^{T,f}$-SPT phases are classified by $\mathbb{Z}_{16}$. Thus fermion decoration construction does not produce all the $Z_4^{T,f}$-SPT phases.

XIV. FERMIONIC $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$-SPT STATE – INTERACTING TOPOLOGICAL INSULATORS

The symmetry group

$$G_f = (U_1^f \rtimes_\phi Z_4^{T,f})/Z_2 \tag{201}$$

is the symmetry group for topological insulator, *i.e.* for electron systems with time reversal $Z_4^{T,f}$ and charge conservation U_1^f symmetries. Such a symmetry can be realized by electron systems with spin-orbital coupling.

In the above expression, ϕ is a homomorphism $\phi: Z_4^{T,f} \to \mathrm{Aut}(U_1^f)$. Let T be the generator of $Z_4^{T,f}$, then $\phi(T)$ changes an element in U_1^f to its inverse. The semi-direct product $U_1^f \rtimes_\phi Z_4^{T,f}$ is defined using such an automorphism $\phi(T)$. Z_2 in $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$ is generated by the product of the π-rotation in U_1^f and T^2. It is in the center of $U_1^f \rtimes_\phi Z_4^{T,f}$.

The symmetry group can be written as

$$G_f = U_1^f \leftthreetimes_{\bar{\varepsilon}_2,\phi} Z_2^T. \tag{202}$$

In other words, the elements in G_f can be labeled by $(\bar{a}^{(\mathbb{R}/\mathbb{Z})^f}, \bar{a}^{\mathbb{Z}_2^T})$, $\bar{a}^{(\mathbb{R}/\mathbb{Z})^f} \in (\mathbb{R}/\mathbb{Z})^f$ and $\bar{a}^{\mathbb{Z}_2^T} \in \mathbb{Z}_2^T = \{0,1\}$, such that

$$(\bar{a}_1^{(\mathbb{R}/\mathbb{Z})^f}, \bar{a}_1^{\mathbb{Z}_2^T})(\bar{a}_2^{(\mathbb{R}/\mathbb{Z})^f}, \bar{a}_2^{\mathbb{Z}_2^T}) = \Big(\bar{a}_1^{(\mathbb{R}/\mathbb{Z})^f} + \phi(\bar{a}_1^{\mathbb{Z}_2^T}) \circ \bar{a}_2^{(\mathbb{R}/\mathbb{Z})^f} + \bar{\varepsilon}_2(\bar{a}_1^{\mathbb{Z}_2^T}, \bar{a}_2^{\mathbb{Z}_2^T}), \bar{a}_1^{\mathbb{Z}_2^T} + \bar{a}_2^{\mathbb{Z}_2^T}\Big) \tag{203}$$

where

$$\bar{\varepsilon}_2(\bar{a}_1^{\mathbb{Z}_2^T}, \bar{a}_2^{\mathbb{Z}_2^T}) = \frac{1}{2}\bar{a}_1^{\mathbb{Z}_2^T}\bar{a}_2^{\mathbb{Z}_2^T}, \quad \phi(0) = 1,\ \phi(1) = -1. \tag{204}$$

In terms of cochains, the above can be rewritten as

$$\bar{\varepsilon}_2 = \frac{1}{2}(\bar{a}^{\mathbb{Z}_2^T})^2. \tag{205}$$

We may write U_1^f as $\mathbb{Z}_2^f \leftthreetimes_{\bar{c}_1} U_1$, *i.e.* write

$$\bar{a}^{(\mathbb{R}/\mathbb{Z})^f} = \frac{1}{2}\bar{a}^{\mathbb{R}/\mathbb{Z}} + \frac{1}{2}\bar{A}^{\mathbb{Z}_2^f} \tag{206}$$

where $\bar{a}^{\mathbb{R}/\mathbb{Z}} \in \mathbb{R}/\mathbb{Z} = (-\frac{1}{2}, \frac{1}{2}]$ and $\bar{A}^{\mathbb{Z}_2^f} \in \mathbb{Z}_2^f = \{0,1\}$. From

$$\begin{aligned}
\bar{a}_1^{(\mathbb{R}/\mathbb{Z})^f} + \bar{a}_2^{(\mathbb{R}/\mathbb{Z})^f} &= \frac{1}{2}\bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \frac{1}{2}\bar{a}_2^{\mathbb{R}/\mathbb{Z}} + \frac{1}{2}\bar{A}_1^{\mathbb{Z}_2^f} + \frac{1}{2}\bar{A}_2^{\mathbb{Z}_2^f} \\
&= \frac{1}{2}(\bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} - \lfloor \bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} \rfloor) \\
&\quad + \frac{1}{2}(\bar{A}_1^{\mathbb{Z}_2^f} + \frac{1}{2}\bar{A}_2^{\mathbb{Z}_2^f} + \lfloor \bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} \rfloor) \\
&= \frac{1}{2}(\bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} - \lfloor \bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} \rfloor) \\
&\quad + \frac{1}{2}(\bar{A}_1^{\mathbb{Z}_2^f} + \frac{1}{2}\bar{A}_2^{\mathbb{Z}_2^f} + \bar{c}_1(\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}})),
\end{aligned} \tag{207}$$

we see that

$$\bar{c}_1(\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}}) = \lfloor \bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \bar{a}_2^{\mathbb{R}/\mathbb{Z}} \rfloor \tag{208}$$

which is the first Chern class of U_1. Note that $\bar{c}_1(\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}})$ is a smooth function of $\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}}$ near $\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}} = 0$. But it has discontinuities away from $\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}} = 0$.

Now we can label elements in G_f by triples $(\bar{A}^{\mathbb{Z}_2^f}, \bar{a}^{\mathbb{R}/\mathbb{Z}}, \bar{a}^{\mathbb{Z}_2^T})$. The group multiplication is

$$\begin{aligned}
&(\bar{A}_1^{\mathbb{Z}_2^f}, \bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_1^{\mathbb{Z}_2^T})(\bar{A}_2^{\mathbb{Z}_2^f}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{Z}_2^T}) \\
&= \Big(\bar{A}_1^{\mathbb{Z}_2^f} + \bar{A}_2^{\mathbb{Z}_2^f} + 2\bar{\varepsilon}_2(\bar{a}_1^{\mathbb{Z}_2^T}, \bar{a}_2^{\mathbb{Z}_2^T}) + \bar{c}_1(\bar{a}_1^{\mathbb{R}/\mathbb{Z}}, \bar{a}_2^{\mathbb{R}/\mathbb{Z}}), \\
&\qquad \bar{a}_1^{\mathbb{R}/\mathbb{Z}} + \phi(\bar{a}_1^{\mathbb{Z}_2^T}) \circ \bar{a}_2^{\mathbb{R}/\mathbb{Z}},\ \bar{a}_1^{\mathbb{Z}_2^T} + \bar{a}_2^{\mathbb{Z}_2^T}\Big)
\end{aligned} \tag{209}$$

We see that G_f can also be written as

$$G_f = Z_2^f \leftthreetimes_{\bar{e}_2} (U_1 \rtimes_\phi Z_2^T), \tag{210}$$

where

$$\bar{e}_2 \overset{2,\mathrm{d}}{=} \bar{c}_1 + (\bar{a}^{\mathbb{Z}_2^T})^2 \overset{2,\mathrm{d}}{=} \bar{c}_1 + \bar{\mathrm{w}}_1^2. \tag{211}$$

For fermion symmetry $G_f = (U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$, the corresponding extended group G_{fO} can be written as

$$G_{fO} = G_f^0 \leftthreetimes_{\bar{e}_2'} O_\infty = U_1^f \leftthreetimes_{\bar{e}_2'} O_\infty = Z_2^f \leftthreetimes_{\bar{e}_2'} (U_1 \rtimes_\phi O_\infty)$$

$$\bar\varepsilon_2' \overset{1}{=} \frac{1}{2}\bar{\rm w}_2,\quad \bar e_2' \overset{2}{=} \bar{\rm w}_2+\bar c_1. \tag{212}$$

where G_f^0 is the fermion symmetry with time reversal removed: $G_f^0 = U_1^f$. We like to mention that $\bar\varepsilon_2' \in H^2(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z})$ which describes how we extend O_∞ by U_1^f and $\bar e_2' \in H^2(\mathcal{B}(U_1 \rtimes_\phi O_\infty);\mathbb{Z}_2^f)$ which describes how we extend $U_1 \rtimes_\phi O_\infty$ by Z_2^f.

We may view $\bar\varepsilon_2'(\bar a^O)$ as $\bar\varepsilon_2'(\pi^{U_1}(\bar a^{G_{fO}}))$ an element in $H^2(\mathcal{B}G_{fO};(\mathbb{R}/\mathbb{Z})_T)$, where π^{U_1} is the projection $G_{fO} \xrightarrow{\pi^{U_1}} O_\infty$. Then $\bar\varepsilon_2'(\pi^{U_1}(\bar a^{G_{fO}}))$ is a trivial element in $H^2(\mathcal{B}G_{fO};(\mathbb{R}/\mathbb{Z})_T)$ (*i.e.* is a $(\mathbb{R}/\mathbb{Z})_T$-valued coboundary, see Appendix N):

$$\frac{1}{2}\bar{\rm w}_2(\pi^{U_1}(\bar a^{G_{fO}})) \overset{1}{=} {\rm d}\bar\eta_1(\bar a^{G_{fO}}),\\ \bar\eta_1 \in C^1(\mathcal{B}G_{fO},(\mathbb{R}/\mathbb{Z})_T). \tag{213}$$

Similarly, we may view $\bar e_2'$ as a trivial element in $H^2(\mathcal{B}G_{fO};\mathbb{Z}_2)$:

$$\bar{\rm w}_2(\pi^{Z_2^f}(\bar a^{G_{fO}})) + \bar c_1(\pi^{Z_2^f}(\bar a^{G_{fO}})) \overset{2}{=} {\rm d}\bar u_1(\bar a^{G_{fO}}),\\ \bar u_1 \in C^1(\mathcal{B}G_{fO},\mathbb{Z}_2), \tag{214}$$

where $\pi^{Z_2^f}$ is the projection $G_{fO} \xrightarrow{\pi^{Z_2^f}} U_1 \rtimes O_\infty$. Equations, (213) and (214) imply that although $\bar c_1$ mod 2 can be a non-trivial $\mathbb{Z}_2$-valued cocycle, $\frac{1}{2}\bar c_1$ is always a $(\mathbb{R}/\mathbb{Z})_T$-valued coboundary on $\mathcal{B}G_{fO}$.

In other words, on space-time $\mathcal{M}^{d+1}$, we have a G_{fO} connection $a^{G_{fO}}$. The G_{fO} connection can be labeled in two ways

$$a^{G_{fO}} = (a^{(\mathbb{R}/\mathbb{Z})^f}, a^O) = (A^{\mathbb{Z}_2^f}, a^{\mathbb{R}/\mathbb{Z}}, a^O) \tag{215}$$

using the two expressions of G_{fO} (212). In the above a^O is the connection of the tangent bundle of the space-time. The above results implies that, if the a^O can be lifted to a $a^{G_{fO}}$ connection, then $\frac{1}{2}{\rm w}_2(a^O)$ on $\mathcal{M}^{d+1}$ is a $(\mathbb{R}/\mathbb{Z})_T$-valued coboundary and ${\rm w}_2+c_1$ is a $\mathbb{Z}_2$-valued coboundary. Here O has a non-trivial action on the $(\mathbb{R}/\mathbb{Z})_T$ coefficient: $\mathbb{R}/\mathbb{Z} \xrightarrow{T} -\mathbb{R}/\mathbb{Z}$, as indicated by the subscript T.

A. 2+1D

1. *Without extension of O_∞*

Calculate $\bar n_2$: To construct fermionic $(U_1^f \rtimes Z_4^{T,f})/Z_2$ SPT states using fermion decoration, we need to find $\bar n_2 \in H^2(\mathcal{B}G_b;\mathbb{Z}_2) = H^2(\mathcal{B}(U_1 \rtimes Z_2^T);\mathbb{Z}_2)$. Notice that

$$H^2[\mathcal{B}(U_1 \rtimes Z_2^T);\mathbb{Z}_2] \rightarrowtail H^2(\mathcal{B}Z_2^T;\mathbb{Z}_2)\oplus\\ H^1[\mathcal{B}Z_2^T;H^1(\mathcal{B}U_1,\mathbb{Z}_2)] \oplus H^2(\mathcal{B}U_1;\mathbb{Z}_2) \tag{216}$$

We can construct $\bar n_2(\bar a^{G_b})$ using flat Z_2^T-connection $a^{Z_2^T}$ and nearly flat U_1 connection $a^{\mathbb{R}/\mathbb{Z}}$:

$$n_2(a^{G_b}) \overset{2}{=} \alpha_n(\bar a^{\mathbb{Z}_2^T})^2 + \tilde\alpha_n \bar c_1 \overset{2}{=} \alpha_n \bar{\rm w}_1^2 + \tilde\alpha_n \bar c_1,\\ \alpha_n,\tilde\alpha_n = 0,1. \tag{217}$$

When $\alpha_n = 1$, we decorate the intersection of Z_2^T symmetry-breaking domain walls by fermion. When $\tilde\alpha_n = 1$, we decorate the 2π-flux of bosonic U_1 (which is the π-flux of bosonic U_1^f) by fermion.

We note that the space-time $\mathcal{M}^{d+1}$ has a twisted spin structure

$${\rm d}A^{\mathbb{Z}_2^f} \overset{2}{=} {\rm w}_2 + {\rm w}_1^2 + e_2 \overset{2}{=} {\rm w}_2 + c_1. \tag{218}$$

So an electron system with time reversal and charge conservation symmetry $G_f = (U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$ can only lives on space-time with trivial ${\rm w}_2 + c_1$. A 2+1D space-time $\mathcal{M}^3$ also satisfies ${\rm w}_2 + {\rm w}_1^1 \overset{2,{\rm d}}{=} 0$. Thus decorating the intersection of Z_2^T symmetry-breaking domain walls and decorating the 2π-flux of bosonic U_1 give rise to the same SPT phase. The inequivalent pullback of $\bar n_2$ to $\mathcal{M}^3$ has a form

$$n_2 \overset{2}{=} \alpha_n {\rm w}_1^2,\qquad \alpha_n = 0,1. \tag{219}$$

So n_2 has two choices.

Calculate $\bar\nu_3$: Next, we calculate $\bar\nu_3(\bar a^{G_b})$ from (53), which becomes

$$-{\rm d}\bar\nu_3 \overset{1}{=} \frac{\alpha_n}{2}[\bar{\rm w}_1^4 + \bar{\rm w}_1^2(\bar{\rm w}_1^2 + \bar c_1)].\\ \overset{1}{=} \frac{\alpha_n}{2}\bar c_1 \bar{\rm w}_1^2 \tag{220}$$

$\frac{1}{2}\bar c_1 \bar{\rm w}_1^2$ is a coboundary in $H^4(\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T)$:

$$\frac{1}{2}\bar c_1 \bar{\rm w}_1^2 \overset{1}{=} {\rm d}\frac{1}{4}\bar c_1 \bar{\rm w}_1. \tag{221}$$

We first note that $\frac{1}{4}\bar c_1 \bar{\rm w}_1$ is a product of three values: a $\mathbb{R}/\mathbb{Z}$-value $\frac{1}{4}$, a $\mathbb{Z}$-value in $\bar c_1$, and a $\mathbb{Z}_2$-value in $\bar{\rm w}_1$. The time reversal Z_2^T has a non-trivial action on the $\mathbb{R}/\mathbb{Z}$-value $\frac{1}{4} \to -\frac{1}{4}$ and a non-trivial action on the $\mathbb{Z}$-value $\mathbb{Z} \to -\mathbb{Z}$. Thus time reversal Z_2^T acts on the Chern class $\bar c_1 \to -\bar c_1$. Therefore, Z_2^T acts trivially on the combination $\frac{1}{4}\bar{\rm w}_1\bar c_1$, and d in ${\rm d}\frac{1}{4}\bar{\rm w}_1\bar c_1$ is the ordinary differentiation operator, not the ${\rm d}_{{\rm w}_1}$ in (A9). Now using (A32), we find

$${\rm d}\frac{1}{4}\bar{\rm w}_1\bar c_1 = \frac{1}{2}(\beta_2\bar{\rm w}_1)c_1 \overset{1}{=} \frac{1}{2}\bar{\rm w}_1^2\bar c_1. \tag{222}$$

This means that $\bar\nu_3$ has solution for all two cases $\alpha_n = 0,1$:

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_n}{4}\bar{\rm w}_1\bar c_1. \tag{223}$$

We note that

$$H^3[\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T] \rightarrowtail H^3[\mathcal{B}Z_2^T;H^0(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus$$

$$
\begin{aligned}
&H^2[\mathcal{B}Z_2^T;H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus H^1[\mathcal{B}Z_2^T;H^2(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&H^0[\mathcal{B}Z_2^T;H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\\
&=H^3[\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T]\oplus H^2[\mathcal{B}Z_2^T;\mathbb{Z}]\oplus H^0[\mathcal{B}Z_2^T;(\mathbb{Z})_T]\\
&=0 \qquad (224)
\end{aligned}
$$

Thus, for each $\bar n_2$, there is only one solution of $\bar\nu_3$ since $H^3[\mathcal{B}(U_1\rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T]=0$.

2. *With extension of O_∞*

Calculate $\bar n_2$: To construct fermionic $G_f=(U_1^f\rtimes_\phi Z_4^{T,f})/Z_2$ SPT states using fermion decoration and extension of O_∞ , we first calculate $\bar n_2\in H^2(\mathcal{B}G_{fO};\mathbb{Z}_2)=H^2(\mathcal{B}(U_1^f\leftthreetimes_{\bar\varepsilon_2'}O_\infty);\mathbb{Z}_2)$. $\bar n_2$ has a form

$$\bar n_2\overset{2,\mathrm{d}}{=}\alpha_n\bar{\mathrm{w}}_1^2+\tilde\alpha_n\bar{\mathrm{w}}_2,\qquad \alpha_n,\tilde\alpha_n=0,1. \tag{225}$$

We do not have the $\bar c_1$ term due to the relation $\bar c_1\overset{2,\mathrm{d}}{=}\bar{\mathrm{w}}_2$.

However, on 2+1D manifold, $\mathrm{w}_2\overset{2,\mathrm{d}}{=}\mathrm{w}_1^2$. Thus the pullback of $\bar n_2$ on space-time $\mathcal{M}^3$ has a simpler form

$$n_2\overset{2,\mathrm{d}}{=}\alpha_n\mathrm{w}_1^2,\qquad \alpha_n=0,1. \tag{226}$$

Calculate $\bar\nu_3$: Next, we calculate $\bar\nu_3(\bar a^{G_{fO}})$ from (80), which becomes

$$
\begin{aligned}
-\mathrm{d}\bar\nu_3&\overset{1}{=}\frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^4+\bar{\mathrm{w}}_1^2(\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_1^2)].\\
&\overset{1}{=}\frac{\alpha_n}{2}\bar{\mathrm{w}}_1^2\bar c_1
\end{aligned}
$$

Similarly, we find $\bar\nu_3\in H^3(\mathcal{B}(U_1^f\leftthreetimes_{\bar\varepsilon_2'}O_\infty);(\mathbb{R}/\mathbb{Z})_T)$ to have a form

$$\bar\nu_3\overset{1}{=}\frac{\alpha_n}{4}\bar{\mathrm{w}}_1\bar c_1+\frac{\alpha_\nu}{2}\bar{\mathrm{w}}_1^3+\frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_3. \tag{227}$$

The term $\frac12\bar{\mathrm{w}}_1\bar c_1\overset{1,\mathrm{d}}{=}\frac12\bar{\mathrm{w}}_1\bar{\mathrm{w}}_2$ is not included since $\frac12\bar{\mathrm{w}}_2$ is a coboundary (see (213)).

In 2+1D space-time, we have $\mathrm{w}_1^2+\mathrm{w}_2\overset{2,\mathrm{d}}{=}\mathrm{w}_3\overset{2,\mathrm{d}}{=}0$ (see (I12)). This also implies that w_1^3 is a coboundary. So the pullback of $\bar\nu_3$ on $\mathcal{M}^3$ is reduced to

$$\nu_3\overset{1}{=}\frac{\alpha_n}{4}\mathrm{w}_1c_1. \tag{228}$$

SPT invariant: The above ν_3 gives rise to two fermionic $(U_1^f\rtimes_\phi Z_4^{T,f})/Z_2$-SPT states, whose SPT invariant is given by

$$
\begin{aligned}
&Z^{\mathrm{top}}(\mathcal{M}^4,A^{G_{fO}})\\
&=\mathrm{e}^{\mathrm{i}2\pi\int_{\mathcal{M}^4}\frac{\alpha_n}{4}\mathrm{w}_1c_1}\,\mathrm{e}^{\mathrm{i}\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2f_2+f_2(\mathrm{w}_2+\mathrm{w}_1^2)},\\
&\mathrm{d}A^{\mathbb{Z}_2^f}\overset{2}{=}\mathrm{w}_2+c_1(A^{U_1}),\quad f_2|_{\partial\mathcal{N}^4}\overset{2}{=}\alpha_n\mathrm{w}_1^2,\\
&\alpha_n=0,1. \qquad (229)
\end{aligned}
$$

Here we label $A^{G_{fO}}\in U_1^f\leftthreetimes_{\bar\varepsilon_2'}O_\infty$ by $(A^{U_1^f},A^O)$, and label $A^{U_1^f}$ by $A_{ij}^{U_1^f}=\frac12A_{ij}^{U_1}+\frac12A_{ij}^{\mathbb{Z}_2^f}$, where $A_{ij}^{U_1^f},A_{ij}^{U_1}\in[0,1)$ and $A_{ij}^{\mathbb{Z}_2^f}=0,1$. The 2+1D space-time and its G_{fO} connection satisfies (213) and (214).

For non-interacting fermion systems, the $(U_1^f\rtimes_\phi Z_4^{T,f})/Z_2$-SPT phases (the topological insulators) are classified by $\mathbb{Z}_2$.[24–26] In the above, we show that, after including interaction and via fermion decoration, the resulting interacting topological insulators are still described by $\mathbb{Z}_2$.

B. 3+1D

1. *Without extension of O_∞*

Calculate $\bar n_3$: $\bar n_3(\bar a^{G_b})\in H^2(\mathcal{B}G_b;\mathbb{Z}_2)=H^2(\mathcal{B}(U_1\rtimes Z_2^T);\mathbb{Z}_2)$ has a form

$$\bar n_3\overset{2}{=}\alpha_n\bar{\mathrm{w}}_1^3+\tilde\alpha_n\bar{\mathrm{w}}_1\bar c_1 \tag{230}$$

On a 3+1D space-time $\mathcal{M}^4$ with $\mathrm{w}_2\overset{2,\mathrm{d}}{=}c_1$, we have $\mathrm{w}_1c_1\overset{2,\mathrm{d}}{=}\mathrm{w}_1\mathrm{w}_2\overset{2,\mathrm{d}}{=}0$ (see (I16)). Thus after pulled back on $\mathcal{M}^4$, $\bar n_3$ reduces to

$$n_3\overset{2}{=}\alpha_n\mathrm{w}_1^3,\qquad \alpha_n=0,1. \tag{231}$$

Calculate $\bar\nu_4$: $\bar\nu_4(\bar a^{G_b})$ is calculated from (53), which becomes

$$
\begin{aligned}
-\mathrm{d}\bar\nu_4&\overset{1}{=}\frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^5+\bar{\mathrm{w}}_1^3(\bar{\mathrm{w}}_1^2+\bar c_1)].\\
&\overset{1}{=}\frac{\alpha_n}{2}\bar c_1\bar{\mathrm{w}}_1^3
\end{aligned}
$$

To see if $\frac12\bar c_1\bar{\mathrm{w}}_1^3$ is a non-trivial cocycle, we note that

$$
\begin{aligned}
&H^5[\mathcal{B}(U_1\rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T]\rightarrowtail H^5[\mathcal{B}Z_2^T;H^0(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&H^4[\mathcal{B}Z_2^T;H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus H^3[\mathcal{B}Z_2^T;H^2(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&H^2[\mathcal{B}Z_2^T;H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus H^1[\mathcal{B}Z_2^T;H^4(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&\quad H^0[\mathcal{B}Z_2^T;H^5(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\\
&=H^5[\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T]\oplus H^4[\mathcal{B}Z_2^T;\mathbb{Z}]\oplus H^2[\mathcal{B}Z_2^T;(\mathbb{Z})_T]\oplus\\
&\quad H^5(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2\oplus\mathbb{Z} \qquad (232)
\end{aligned}
$$

The $\mathbb{Z}$ in $H^4[\mathcal{B}Z_2^T;\mathbb{Z}]$ comes from $H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})=\mathbb{Z}$. The unit in $\mathbb{Z}$ correspond to the generator of $H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})$: the $\mathbb{R}/\mathbb{Z}$-valued nearly-flat 1-cochain $\bar a^{\mathbb{R}/\mathbb{Z}}$. The time-reversal Z_2^T has a non-trivial action on the $\mathbb{R}/\mathbb{Z}$-value: $\mathbb{R}/\mathbb{Z}\to-\mathbb{R}/\mathbb{Z}$. It also has a non-trivial action on U_1: $\bar a^{\mathbb{R}/\mathbb{Z}}\to-\bar a^{\mathbb{R}/\mathbb{Z}}$. So the total action of Z_2^T is given by $\bar a^{\mathbb{R}/\mathbb{Z}}\to\bar a^{\mathbb{R}/\mathbb{Z}}$. Thus the action of Z_2^T on the coefficient $\mathbb{Z}$ is trivial. In this case $H^4[\mathcal{B}Z_2^T;\mathbb{Z}]=\mathbb{Z}_2$. It is generated by $\bar\eta_5$ that satisfy

$$\mathrm{d}\bar\eta_5\overset{1}{=}\bar c_1\beta_2(\bar{\mathrm{w}}_1^3)\overset{1}{=}\frac12\bar c_1\,\mathrm{d}_{\mathrm{w}_1}(\bar{\mathrm{w}}_1^3). \tag{233}$$

Thus

$$\bar\eta_5=\frac12\bar c_1\bar{\mathrm{w}}_1^3. \tag{234}$$

The $\mathbb{Z}$ in $H^2[\mathcal{B}Z_2^T;\mathbb{Z}_T]$ comes from $H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$. The unit in $\mathbb{Z}$ correspond to the generator of $H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})$: the $\mathbb{R}/\mathbb{Z}$-valued nearly-flat 3-cochain $\bar a^{\mathbb{R}/\mathbb{Z}}c_1$, where c_1 is the first Chern class of $\bar a^{\mathbb{R}/\mathbb{Z}}$. The total action of Z_2^T is given by $\bar a^{\mathbb{R}/\mathbb{Z}}c_1 \to -\bar a^{\mathbb{R}/\mathbb{Z}}c_1$. Thus the action of Z_2^T on the coefficient $\mathbb{Z}$ is non-trivial $\mathbb{Z}\to -\mathbb{Z}$. This is why we denote $\mathbb{Z}$ as $\mathbb{Z}_T$. In this case $H^2[\mathcal{B}Z_2^T;\mathbb{Z}_T] = 0$.

So, $H^5[\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T]$ is generated by $\frac{1}{2}\bar{\mathrm{w}}_1^3\bar c_1$ and $\bar a^{\mathbb{R}/\mathbb{Z}}\bar c_1^2$. We see that $\frac{1}{2}\bar{\mathrm{w}}_1^3\bar c_1$ is a non-trivial cocycle in $H^5[\mathcal{B}(Z_2^T \rtimes U_1);(\mathbb{R}/\mathbb{Z})_T]$ This means that $\bar\nu_4$ has solution only when $\alpha_n = 0$. In this case, $\bar\nu_4$ is given by the cocycles in $H^4[\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T]$. We note that

$$\begin{aligned}
&H^4[\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T] \rightarrowtail H^4[\mathcal{B}Z_2^T;H^0(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&H^3[\mathcal{B}Z_2^T;H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus H^2[\mathcal{B}Z_2^T;H^2(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus\\
&H^1[\mathcal{B}Z_2^T;H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\oplus H^0[\mathcal{B}Z_2^T;H^4(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})]\\
&= H^4[\mathcal{B}Z_2^T;(\mathbb{R}/\mathbb{Z})_T]\oplus H^3[\mathcal{B}Z_2^T;\mathbb{Z}]\oplus H^1[\mathcal{B}Z_2^T;\mathbb{Z}_T]\\
&= \mathbb{Z}_2\oplus\mathbb{Z}_2 \qquad (235)
\end{aligned}$$

The $\mathbb{Z}$ in $H^3[\mathcal{B}Z_2^T;\mathbb{Z}]$ comes from $H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$. The unit in $\mathbb{Z}$ correspond to the generator of $H^1(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})$: the $\mathbb{R}/\mathbb{Z}$-valued nearly-flat 1-cochain $\bar a^{\mathbb{R}/\mathbb{Z}}$. The time-reversal Z_2^T has a non-trivial action on the $\mathbb{R}/\mathbb{Z}$-value: $\mathbb{R}/\mathbb{Z} \to -\mathbb{R}/\mathbb{Z}$. It also has a non-trivial action on U_1: $\bar a^{\mathbb{R}/\mathbb{Z}} \to -\bar a^{\mathbb{R}/\mathbb{Z}}$. So the total action of Z_2^T is given by $\bar a^{\mathbb{R}/\mathbb{Z}} \to \bar a^{\mathbb{R}/\mathbb{Z}}$. Thus the action of Z_2^T on the coefficient $\mathbb{Z}$ is trivial. In this case $H^3[\mathcal{B}Z_2^T;\mathbb{Z}] = 0$.

The $\mathbb{Z}$ in $H^1[\mathcal{B}Z_2^T;\mathbb{Z}_T]$ comes from $H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$. The unit in $\mathbb{Z}$ correspond to the generator of $H^3(\mathcal{B}U_1;\mathbb{R}/\mathbb{Z})$: the $\mathbb{R}/\mathbb{Z}$-valued nearly-flat 3-cochain $\bar a^{\mathbb{R}/\mathbb{Z}}c_1$, where c_1 is the first Chern class of $\bar a^{\mathbb{R}/\mathbb{Z}}$. The total action of Z_2^T is given by $\bar a^{\mathbb{R}/\mathbb{Z}}c_1 \to -\bar a^{\mathbb{R}/\mathbb{Z}}c_1$. Thus the action of Z_2^T on the coefficient $\mathbb{Z}$ is non-trivial $\mathbb{Z}\to-\mathbb{Z}$. This is why we denote $\mathbb{Z}$ as $\mathbb{Z}_T$. In this case $H^1[\mathcal{B}Z_2^T;\mathbb{Z}_T] = \mathbb{Z}_2$.

So, $H^4[\mathcal{B}(U_1 \rtimes Z_2^T);(\mathbb{R}/\mathbb{Z})_T]$ is generated by $\frac{1}{2}\bar c_1^2$ and $\frac{1}{2}(\bar a^{Z_2^T})^4$. $\bar\nu_4$ may have a form

$$\bar\nu_4 \overset{1}{=} \frac{\alpha_\nu}{2}\bar{\mathrm{w}}_1^4 + \frac{\tilde\alpha_\nu}{2}\bar c_1^2, \qquad \alpha_\nu,\tilde\alpha_\nu = 0,1. \qquad (236)$$

2. *With extension of O_∞*

Calculate $\bar n_3$: Let us first calculate $\bar n_3 \in H^2(\mathcal{B}G_{fO};\mathbb{Z}_2) = H^3(\mathcal{B}(U_1^f \leftthreetimes_{\bar\varepsilon_2'} O_\infty);\mathbb{Z}_2)$. $\bar n_3$ has a form

$$\bar n_3 \overset{2,\mathrm{d}}{=} \alpha_n\bar{\mathrm{w}}_1^3 + \tilde\alpha_n\bar{\mathrm{w}}_1\bar{\mathrm{w}}_2, \qquad \alpha_n,\tilde\alpha_n = 0,1. \qquad (237)$$

We do not have the $\bar{\mathrm{w}}_1\bar c_1$ term due to the relation $\bar c_1 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_2$.

However, on 3+1D manifold, $\mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$. Thus the pullback of $\bar n_3$ on space-time $\mathcal{M}^3$ has a simpler form

$$n_3 \overset{2,\mathrm{d}}{=} \alpha_n\mathrm{w}_1^3, \qquad \alpha_n = 0,1. \qquad (238)$$

Calculate $\bar\nu_4$: Next, we calculate $\bar\nu_4(\bar a^{G_{fO}})$ from (80), which becomes

$$\begin{aligned}
-\mathrm{d}\bar\nu_4 &\overset{1}{=} \frac{\alpha_n}{2}[\bar{\mathrm{w}}_1^5 + \bar{\mathrm{w}}_1^3(\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_1^2)]\\
&\overset{1}{=} \frac{\alpha_n}{2}\bar{\mathrm{w}}_1^3\bar{\mathrm{w}}_2 \overset{1}{=} \frac{\alpha_n}{2}\bar{\mathrm{w}}_1^3\bar c_1 + \frac{\alpha_n}{2}\bar{\mathrm{w}}_1^3\mathrm{d}\bar u_1, \qquad (239)
\end{aligned}$$

where we have used (214). We note that $\frac{1}{2}\bar{\mathrm{w}}_1^3$ is a $(\mathbb{R}/\mathbb{Z})_T$-valued coboundary:

$$\frac{1}{2}\bar{\mathrm{w}}_1^3 \overset{1}{=} \mathrm{d}\eta_2, \qquad (240)$$

where Z_2^T has a non-trivial action on the value $(\mathbb{R}/\mathbb{Z})_T$. Thus $\frac{1}{2}\bar{\mathrm{w}}_1^3\bar{\mathrm{w}}_2$ is also a coboundary

$$\frac{1}{2}\bar{\mathrm{w}}_1^3\bar{\mathrm{w}}_2 \overset{1}{=} \mathrm{d}(\bar\eta_2\bar c_1 + \frac{1}{2}\bar{\mathrm{w}}_1^3\bar u_1). \qquad (241)$$

We find $\bar\nu_4$ to have a form

$$\begin{aligned}
\bar\nu_4 &\overset{1}{=} \alpha_n(\bar\eta_2\bar c_1 + \frac{1}{2}\bar{\mathrm{w}}_1^3\bar u_1) + \frac{\alpha_{\nu,1}}{2}\bar{\mathrm{w}}_1^4 + \frac{\alpha_{\nu,2}}{2}\bar{\mathrm{w}}_2^2 + \frac{\alpha_{\nu,3}}{2}\bar{\mathrm{w}}_4\\
&\quad + \frac{\alpha_{\nu,4}}{2}\bar{\mathrm{w}}_1^2\bar{\mathrm{w}}_2 + \frac{\alpha_{\nu,5}}{2}\bar{\mathrm{w}}_1\bar{\mathrm{w}}_3, \qquad \alpha_{\nu,i} = 0,1. \qquad (242)
\end{aligned}$$

The terms $\bar p_1$, $\bar{\mathrm{w}}_2\bar c_1$, and $\bar{\mathrm{w}}_1^2\bar c_1$ are not included since $\bar{\mathrm{w}}_2^2 \overset{2}{=} \bar p_1$ and $\bar{\mathrm{w}}_2 \overset{2}{=} \bar c_1$.

In 3+1D space-time, we have $\mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_3 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_4 \overset{2,\mathrm{d}}{=} 0$ (see (I16)). Thus after pulled back to space-time $\mathcal{M}^4$, $\bar\nu_4$ reduces to

$$\begin{aligned}
&\nu_4 \overset{1}{=} \alpha_n(\eta_2 c_1 + \frac{1}{2}\mathrm{w}_1^3 u_1) + \frac{\alpha_\nu}{2}\mathrm{w}_1^4 + \frac{\tilde\alpha_\nu}{2}\mathrm{w}_2 c_1\\
&\quad \alpha_\nu,\tilde\alpha_\nu = 0,1. \qquad (243)
\end{aligned}$$

We note that $\mathrm{w}_2 c_1 \overset{2,\mathrm{d}}{=} \mathrm{w}_2^2 \overset{2,\mathrm{d}}{=} p_1$.

From (213), we see that $\frac{1}{2}\mathrm{w}_2$ is a coboundary. So one might expect the term $\frac{1}{2}\mathrm{w}_2 c_1$ to be a coboundary, and can be dropped. $\frac{1}{2}\mathrm{w}_2 c_1$ is indeed a coboundary if we view $\frac{1}{2}\mathrm{w}_2$ as a $(\mathbb{R}/\mathbb{Z})_T$-valued cocycle, where subscript T indicate the non-trivial action by time-reversal. (Note that $\frac{1}{2}\mathrm{w}_2$ can also be viewed as a $\mathbb{R}/\mathbb{Z}$-valued cocycle, and in this case it may not be a coboundary.) Since c_1 is a $\mathbb{Z}_T$-valued cocycle, this implies that $\frac{1}{2}\mathrm{w}_2 c_1$ is a coboundary if we view it as $\mathbb{R}/\mathbb{Z}$-valued cocycle, where the time-reversal acts trivially. But $\frac{\tilde\alpha_\nu}{2}\mathrm{w}_2 c_1$ in ν_4 is viewed as a $(\mathbb{R}/\mathbb{Z})_T$ valued cocycle, which may be non-trivial.

SPT invariant: The above ν_4 gives rise to two fermionic $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$-SPT states, whose SPT invariant is given by

$$\begin{aligned}
Z^{\mathrm{top}}(\mathcal{M}^4, A^{G_{fO}}) &= \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^4}\alpha_n(\eta_2 c_1+\frac{1}{2}\mathrm{w}_1^3 u_1)+\frac{\alpha_\nu}{2}\mathrm{w}_1^4+\frac{\tilde\alpha_\nu}{2}p_1}\\
&\quad \mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^5}\mathrm{Sq}^2 f_3 + f_3(\mathrm{w}_2+\mathrm{w}_1^2)},\\
&\mathrm{d}A^{Z_2^f} \overset{2}{=} \mathrm{w}_2 + c_1(A^{U_1}), \quad f_3\big|_{\partial\mathcal{N}^4} \overset{2}{=} \alpha_n\mathrm{w}_1^3,\\
&\alpha_n,\alpha_\nu = 0,1. \qquad (244)
\end{aligned}$$

Here we label $A^{G_{fO}} \in U_1^f \leftthreetimes_{\bar{e}_2'} O_\infty$ by $(A^{U_1^f}, A^O)$, and label $A^{U_1^f}$ by $A_{ij}^{U_1^f} = \frac{1}{2}A_{ij}^{U_1} + \frac{1}{2}A_{ij}^{\mathbb{Z}_2^f}$, where $A_{ij}^{U_1^f}, A_{ij}^{U_1} \in [0,1)$ and $A_{ij}^{\mathbb{Z}_2^f} = 0,1$. The 3+1D space-time and its G_{fO} connection satisfies (213) and (214).

In the above, we show that, after including interaction and via fermion decoration, we obtain 8 types of interacting topological insulators (including the trivial type). For non-interacting fermion systems, the $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$-SPT phases (the topological insulators) in 3+1D are classified by $\mathbb{Z}_2$.[24–26] We also know that bosonic Z_2^T-SPT phases are classified by $\mathbb{Z}_2^2$,[28–31] which are generated by bosonic Z_2^T-SPT states with SPT invariants $\frac{1}{2}\mathrm{w}_1^4$, $\frac{1}{2}p_1$.[30,31] Our above result indicates that the 4 bosonic Z_2^T-SPT phases correspond to 4 different fermionic SPT phases when the bosons are formed by electron-hole pairs. $\mathbb{Z}_2$ from free fermions and $\mathbb{Z}_2^2$ from bosonic electron-hole pairs leads to the total of 8 fermionic $(U_1^f \rtimes_\phi Z_4^{T,f})/Z_2$-SPT phases in 3+1D, obtained via fermion decoration construction. This is consistent with the previous physical argument[59] and the cobordism calculation.[18,19]

XV. FERMIONIC SU_2^f-SPT STATE

In this section, we are going to study fermionic SPT phases with $G_f = SU_2^f$ symmetry in 2+1D and in 3+1D. Such a symmetry can be realized by a charge-$2e$ spin-singlet superconductor of electrons. For non-interacting electron, there is no non-trivial SU_2^f-SPT phase in 3+1D.

For fermion systems with bosonic symmetry $G_b = SO_3$, the full fermionic symmetry G_f is an extension of G_b by Z_2^f. The extension $G_f = SU_2^f = Z_2^f \leftthreetimes_{\mathrm{w}_2^{SO_3}} SO_3$ is characterized by

$$\bar{e}_2 \overset{2}{=} \bar{\mathrm{w}}_2^{SO_3} \in H^2(\mathcal{B}SO_3;\mathbb{Z}_2). \tag{245}$$

For $G_f = SU_2^f$, the group G_{fSO} is an extension of SO_∞ by SU_2^f:

$$G_{fSO} = SU_2^f \leftthreetimes_{e_2} SO_\infty = Z_2^f \leftthreetimes_{\bar{e}_2'} (SO_3 \times SO_\infty). \tag{246}$$

where $\bar{e}_2' \in H^2(\mathcal{B}(SO_3 \times SO_\infty);\mathbb{Z}_2)$ is given by

$$\bar{e}_2' \overset{2}{=} \bar{\mathrm{w}}_2(\bar{a}^{SO}) + \bar{\mathrm{w}}_2^{SO_3}(\bar{a}^{SO_3}). \tag{247}$$

On $\mathcal{B}G_{fSO}$, $\bar{e}_2'$ is trivialized

$$\bar{\mathrm{w}}_2(\bar{a}^{SO}) + \bar{\mathrm{w}}_2^{SO_3}(\bar{a}^{SO_3}) \overset{2}{=} \mathrm{d}\bar{A}^{\mathbb{Z}_2^f}, \tag{248}$$

where a G_{fSO} connection is labeled by

$$a^{G_{fSO}} = (A^{\mathbb{Z}_2^f}, a^{SO_3}, a^{SO}). \tag{249}$$

A. 2+1D

1. *Without extension of SO_∞*

Calculate $\bar{n}_2$: First, $\bar{n}_2 \in H^2(\mathcal{B}SO_3;\mathbb{Z}_2) = \mathbb{Z}_2$. It has two choices:

$$\bar{n}_2 = \alpha_n \bar{\mathrm{w}}_2^{SO_3}, \qquad \alpha_n = 0,1. \tag{250}$$

Calculate $\bar{\nu}_3$: Similar to the last section, $\bar{\nu}_3$ satisfies

$$-\mathrm{d}\bar{\nu}_3 \overset{1}{=} \frac{\alpha_n^2}{2}\mathrm{Sq}^2\bar{\mathrm{w}}_2^{SO_3} + \frac{\alpha_n}{2}\bar{\mathrm{w}}_2^{SO_3}\bar{e}_2 \overset{1}{=} 0. \tag{251}$$

Thus, $\bar{\nu}_3$ has solutions classified by $\mathbb{Z}$:

$$\bar{\nu}_3 \overset{1}{=} \alpha_\nu \omega_3^{SO_3}, \quad \alpha_\nu \in \mathbb{Z}, \tag{252}$$

since $H^3(\mathcal{B}SO_3;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$ (see (G7)).
SPT invariant: The SU_2^f fermionic SPT states labeled by α_n, α_ν have the following SPT invariant

$$\begin{aligned} &Z^{\text{top}}(\mathcal{M}^3, A^{Z_2}, A^{\mathbb{Z}_2^f}) \\ &= \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{\mathcal{M}^3}\alpha_\nu\omega_3^{SO_3} + \frac{\alpha_n}{2}\mathrm{w}_2^{SO_3}A^{\mathbb{Z}_2^f}}\,\mathrm{e}^{\,\mathrm{i}\,\pi\int_{\mathcal{N}^4}\mathrm{Sq}^2 f_2 + f_2\mathrm{w}_2}, \\ &\mathrm{d}A^{\mathbb{Z}_2^f} \overset{2}{=} \mathrm{w}_2 + \mathrm{w}_2^{SO_3}, \qquad f_2\big|_{\partial\mathcal{N}^4} \overset{2}{=} \alpha_n\mathrm{w}_2^{SO_3} \end{aligned} \tag{253}$$

where the space-time $\mathcal{M}^3$ is orientable and $\mathrm{w}_1 \overset{2}{=} 0$. However, as we will see below, the SU_2^f fermionic SPT phase are only labeled by $\alpha_\nu \in \mathbb{Z}$.

On 2+1D space-time manifold, $\mathrm{w}_2 + \mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} 0$ (see Appendix I3). The SU_2^f fermionic symmetry requires the space-time $\mathcal{M}^3$ to be a orientable manifold with $\mathrm{w}_2 + \mathrm{w}_2^{SO_3} \overset{2,\mathrm{d}}{=} 0$ and $\mathrm{w}_1 \overset{2,\mathrm{d}}{=} 0$. Thus $n_2 \overset{2,\mathrm{d}}{=} \mathrm{w}_2^{SO_3}$ is always a coboundary, and $\alpha_n = 0,1$ describe the same SPT phase.

2. *With extension of SO_∞*

Calculate $\bar{n}_2$: With extension of SO_∞, in general, $\bar{n}_2(\bar{a}^{G_{fSO}}) \in H^2(\mathcal{B}G_{fSO};\mathbb{Z}_2)$ is given by [using the triple $(\bar{A}^{\mathbb{Z}_2^f}, \bar{a}^{SO_3}, \bar{a}^{SO})$ to label $\bar{a}^{G_{fSO}}$]

$$\bar{n}_2(\bar{a}^{G_{fSO}}) \overset{2}{=} \alpha_{n,1}\bar{\mathrm{w}}_2(\bar{a}^{SO}) + \alpha_{n,2}\bar{\mathrm{w}}_2^{SO_3}(\bar{a}^{SO_3}), \tag{254}$$

$\alpha_{n,1}, \alpha_{n,2} = 0,1$. The above can be reduced to

$$\bar{n}_2(\bar{a}^{G_{fSO}}) \overset{2}{=} \alpha_n\bar{\mathrm{w}}_2 \tag{255}$$

$\alpha_n = 0,1$, due to the relation (248). So $\bar{n}_2(\bar{a}^{G_{fSO}})$ has two choices. But on 2+1D orientable space-time $\mathcal{M}^3$, $\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$. Thus after pulled back to $\mathcal{M}^3$, w_2 is a coboundary. Thus n_2 is always trivial.
Calculate $\bar{\nu}_3$: Next, we consider ν_3 in (87) which becomes $\mathrm{d}\bar{\nu}_3 \overset{1}{=} 0$. We find that $\bar{\nu}_3$ is given by

$$\bar{\nu}_3 \overset{1}{=} \alpha_\nu\bar{\omega}_3^{SO_3} + \frac{\tilde{\alpha}_\nu}{2}\bar{\mathrm{w}}_3 \qquad \alpha_\nu \in \mathbb{Z}, \quad \tilde{\alpha}_\nu = 0,1. \tag{256}$$

However, $\bar{\mathrm{w}}_3 \overset{2,\mathrm{d}}{=} \beta_2\bar{\mathrm{w}}_2$ (see (A32) and (I7)) So the term $\frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_3$ can be rewritten as

$$\frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_3 \overset{1,\mathrm{d}}{=} \frac{\tilde\alpha_\nu}{2}\beta_2\bar{\mathrm{w}}_2. \tag{257}$$

In this form, since $\beta_2\bar{\mathrm{w}}_2$ is a $\mathbb{Z}$-valued cocycle, $\tilde\alpha_\nu$ do not have to be quantized as $\tilde\alpha_\nu = 0, 1$. $\tilde\alpha_\nu$ can take any real values and $\frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_3$ is still a $\mathbb{R}/\mathbb{Z}$-valued cocycle. Thus $\tilde\alpha_\nu$ is not quantized and can be tuned to zero. Thus we drop the $\frac{\tilde\alpha_\nu}{2}\bar{\mathrm{w}}_3$ term. Thus, The fermionic SU_2^f-SPT phase obtained via fermion decoration is classified by $\alpha_n \in \mathbb{Z}$.

B. 3+1D

1. Without extension of SO_∞

Calculate $\bar n_3$: $\bar n_3 \in H^3(\mathcal{B}SO_3;\mathbb{Z}_2)$ has two choices:

$$\bar n_3 = \alpha_n \bar{\mathrm{w}}_3^{SO_3} \tag{258}$$

$\alpha_n = 0, 1$, since $H^3(\mathcal{B}SO_3;\mathbb{Z}_2) = \mathbb{Z}_2$.
Calculate $\bar\nu_4$: Next we want to solve

$$\begin{aligned} -\mathrm{d}\bar\nu_4 &\overset{1}{=} \frac{1}{2}(\mathrm{Sq}^2\bar n_3 + \bar n_3\bar e_2). \\ &\overset{1}{=} \frac{\alpha_n}{2}(\mathrm{Sq}^2\bar{\mathrm{w}}_3^{SO_3} + \bar{\mathrm{w}}_3^{SO_3}\bar{\mathrm{w}}_2^{SO_3}) \overset{1}{=} \mathrm{d}\bar s_4^{SO_3}, \end{aligned} \tag{259}$$

where we have used (see (I7))

$$\mathrm{Sq}^2\bar{\mathrm{w}}_3^{SO_3} \overset{2}{=} \bar{\mathrm{w}}_2^{SO_3}\bar{\mathrm{w}}_3^{SO_3} + \mathrm{d}\bar s_4^{SO_3}(\bar a^{SO_3}) \tag{260}$$

Since $H^4(\mathcal{B}SO_3;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$, the solution of $\bar\nu_4$ has a form

$$\bar\nu_4 \overset{1}{=} \frac{\alpha_\nu}{2}\bar{\mathrm{w}}_4^{SO_3} + \frac{\alpha_n}{2}\bar s_4^{SO_3}, \quad \alpha_\nu = 0, 1. \tag{261}$$

2. With extension of SO_∞

Calculate $\bar n_3$: In general, $\bar n_3(\bar a^{G_{fSO}}) \in H^3(\mathcal{B}G_{fSO};\mathbb{Z}_2)$ can be written as

$$\begin{aligned} \bar n_3(\bar a^{G_{fSO}}) &\overset{2}{=} \alpha_n\bar{\mathrm{w}}_3 + \alpha_n'\bar{\mathrm{w}}_3^{SO_3}, \\ &\alpha_n, \alpha_n' = 0, 1. \end{aligned} \tag{262}$$

However, from $\bar{\mathrm{w}}_2 \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_2^{SO_3}$, we find that $\mathrm{Sq}^1(\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_2^{SO_3}) \overset{2,\mathrm{d}}{=} \bar{\mathrm{w}}_3 + \bar{\mathrm{w}}_2^{SO_3} \overset{2,\mathrm{d}}{=} 0$ (see (I7) and notice $\bar{\mathrm{w}}_1 \overset{2}{=} \bar{\mathrm{w}}_1^{SO_3} \overset{2}{=} 0$). Furthermore, on a orientable 4-manifold $\mathcal{M}^4$, $\mathrm{w}_3^{SO_3} \overset{2,\mathrm{d}}{=} \mathrm{w}_3$ is always a coboundary. From Appendix I 4, we have

$$\begin{aligned} a^{Z_2}\mathrm{w}_3 &\overset{2,\mathrm{d}}{=} (a^{Z_2})^2\mathrm{w}_2 \overset{2,\mathrm{d}}{=} \mathrm{Sq}^2(a^{Z_2})^2 \\ &\overset{2,\mathrm{d}}{=} (a^{Z_2})^4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1(a^{Z_2})^3 \overset{2,\mathrm{d}}{=} 0. \end{aligned} \tag{263}$$

Since $a^{\mathbb{Z}_2}$ can be an arbitrary $\mathbb{Z}_2$-valued 1-cocycle, we find $\mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0$. Thus the above expression for $n_3(\bar a^{G_{fSO}})$ is reduced to

$$n_3(\bar a^{G_{fSO}}) \overset{2}{=} 0. \tag{264}$$

Calculate $\bar\nu_4$: Next, we consider $\bar\nu_4$ in (87) which becomes $\mathrm{d}\bar\nu_4(\bar a^{G_{fSO}}) \overset{1}{=} 0$. We find that

$$\bar\nu_4 \overset{1}{=} 0. \tag{265}$$

Note that we do not have the $\bar p_1$ term since $\bar p_1$ is $\mathbb{Z}$-valued and its coefficient is not quantized. Also on $\mathcal{M}^4$, we have $\mathrm{w}_4 \overset{2,\mathrm{d}}{=} \mathrm{w}_2^2 \overset{2,\mathrm{d}}{=} p_1$. So we do not have the w_4 and w_2^2 terms. Due to (248), we also do not have the $\mathrm{w}_2\mathrm{w}_2^{SO_3}$ and $(\mathrm{w}_2^{SO_3})^2$ terms. Thus there is no non-trivial fermionic SU_2^f-SPT phases in 3+1D from fermion decoration.

XVI. FERMIONIC $Z_2 \times Z_4 \times Z_2^f$-SPT STATE

In this section, we are going to study fermionic SPT phases, where the fermion symmetry is given by $G_f = Z_2 \times Z_4 \times Z_2^f$ symmetry. For non-interaction fermions, there is no non-trivial $Z_2\times Z_4\times Z_2^f$-SPT phases in 3+1D. For $G_f = Z_2 \times Z_4 \times Z_2^f$, the corresponding $G_{fSO} = G_f \leftthreetimes SO_\infty$ is given by $G_{fSO} = Z_2 \times Z_4 \times Spin_\infty$. $\bar{\mathrm{w}}_2$ is trivialized on $\mathcal{B}G_{fSO}$:

$$\bar{\mathrm{w}}_2(a^{SO}) \overset{2}{=} \mathrm{d}\bar A^{\mathbb{Z}_2^f}, \tag{266}$$

where we have labeled the G_{fSO} connection as

$$a^{G_{fSO}} = (A^{\mathbb{Z}_2^f}, a^{\mathbb{Z}_2}, a^{\mathbb{Z}_4}, a^{SO}). \tag{267}$$

where $a^{\mathbb{Z}_4}$ is a $\mathbb{Z}_4$-valued 1-cocycle.

A. 2+1D

Calculate $\bar n_2$: First $\bar n_2 \in H^2(\mathcal{B}G_{fSO};\mathbb{Z}_2)$ is given by

$$\begin{aligned} \bar n_2 &\overset{2}{=} \alpha_{n,1}(\bar a^{\mathbb{Z}_2})^2 + \alpha_{n,2}(\bar a^{\mathbb{Z}_4})^2 + \alpha_{n,3}\bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}, \\ &\alpha_{n,i} = 0, 1. \end{aligned} \tag{268}$$

Calculate $\bar\nu_3$: Next, we consider $\bar\nu_3(\bar a^{G_{fSO}})$ in (87) which becomes

$$\begin{aligned} -\mathrm{d}\bar\nu_3 &\overset{1}{=} \frac{1}{2}\mathrm{Sq}^2\bar n_2 + \frac{1}{2}\bar n_2\,\mathrm{d}\bar A^{\mathbb{Z}_2^f} \\ &\overset{1}{=} \frac{\alpha_{n,1}}{2}(\bar a^{\mathbb{Z}_2})^4 + \frac{\alpha_{n,2}}{2}(\bar a^{\mathbb{Z}_4})^4 + \frac{\alpha_{n,3}}{2}(\bar a^{\mathbb{Z}_2})^2(\bar a^{\mathbb{Z}_4})^2 \\ &\quad + \frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}[\mathrm{d}(\bar a^{\mathbb{Z}_2} \underset{1}{\smile} \bar a^{\mathbb{Z}_4})]\bar a^{\mathbb{Z}_4} \\ &\quad + \frac{\alpha_{n,1}\alpha_{n,2}}{2}\mathrm{d}[(\bar a^{\mathbb{Z}_2})^2 \underset{1}{\smile} (\bar a^{\mathbb{Z}_4})^2] \\ &\quad + \frac{\alpha_{n,1}\alpha_{n,3}}{2}\mathrm{d}[(\bar a^{\mathbb{Z}_2})^2 \underset{1}{\smile} \bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}] \\ &\quad + \frac{\alpha_{n,2}\alpha_{n,3}}{2}\mathrm{d}[(\bar a^{\mathbb{Z}_4})^2 \underset{1}{\smile} \bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}] \\ &\quad + [\frac{\alpha_{n,1}}{2}(\bar a^{\mathbb{Z}_2})^2 + \frac{\alpha_{n,2}}{2}(\bar a^{\mathbb{Z}_4})^2 + \frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}]\mathrm{d}\bar A^{\mathbb{Z}_2^f}. \end{aligned} \tag{269}$$

The solution of the above equation has a form

$$\bar\nu_3 \overset{1}{=} \frac{\alpha_{n,1}}{4}\bar a^{\mathbb{Z}_2}\beta_2\bar a^{\mathbb{Z}_2} + \frac{\alpha_{n,2}}{4}\bar a^{\mathbb{Z}_4}\beta_2\bar a^{\mathbb{Z}_2} + \frac{\alpha_{n,3}}{4}\bar a^{\mathbb{Z}_4}\beta_2\bar a^{\mathbb{Z}_2}$$

$$+\frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_2}\underset{1}{\smile}\bar a^{\mathbb{Z}_4})\bar a^{\mathbb{Z}_4}$$
$$+\frac{\alpha_{n,1}\alpha_{n,2}}{2}(\bar a^{\mathbb{Z}_2})^2\underset{1}{\smile}(\bar a^{\mathbb{Z}_4})^2$$
$$+\frac{\alpha_{n,1}\alpha_{n,3}}{2}(\bar a^{\mathbb{Z}_2})^2\underset{1}{\smile}\bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4} \quad (270)$$
$$+\frac{\alpha_{n,2}\alpha_{n,3}}{2}(\bar a^{\mathbb{Z}_4})^2\underset{1}{\smile}\bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}$$
$$+[\frac{\alpha_{n,1}}{2}(\bar a^{\mathbb{Z}_2})^2+\frac{\alpha_{n,2}}{2}(\bar a^{\mathbb{Z}_4})^2+\frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}\bar a^{\mathbb{Z}_4}]\bar A^{\mathbb{Z}_2^f}$$
$$+\frac{\alpha_{\nu,1}}{2}(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_{\nu,2}}{2}\bar a^{\mathbb{Z}_4}\beta_2\bar a^{\mathbb{Z}_2}+\frac{\alpha_{\nu,3}}{4}(\bar a^{\mathbb{Z}_4})^3.$$

where $\alpha_{\nu,i}|_{i=1,2}=0,1$ and $\alpha_{\nu,3}=0,1,2,3$. $\alpha_{n,i},\alpha_{\nu,i}$ label 128 different SPT phases, which can be divided into 8 classes with 16 SPT phases in each class. The 8 classes are labeled by $\alpha_{n,i}$. The 16 SPT phases in each class only differ by stacking the bosonic $Z_2\times Z_4$-SPT phases realized by fermion pairs.

B. 3+1D

Calculate $\bar n_3$: Since $G_{fSO}=\mathbb{Z}_2\times Z_4\times Spin_\infty$, $\bar n_3\in H^3(G_{fSO};\mathbb{Z}_2)$ is generated by $\bar a^{\mathbb{Z}_2}$, $\bar a^{\mathbb{Z}_4}$, and Stiefel-Whitney class w_n. For $Spin_\infty$, $\bar{\mathrm{w}}_1=\bar{\mathrm{w}}_2=0$. Also, $\mathrm{Sq}^1\bar{\mathrm{w}}_2\overset{2,\mathrm{d}}{=}\bar{\mathrm{w}}_1\bar{\mathrm{w}}_2+\bar{\mathrm{w}}_3\overset{2,\mathrm{d}}{=}\bar{\mathrm{w}}_3$. Since $\bar{\mathrm{w}}_2$ is a coboundary for $Spin_\infty$, $\bar{\mathrm{w}}_3$ is also a coboundary. Thus, $\bar n_3$ is given by

$$\bar n_3\overset{2}{=}\alpha_{n,1}(\bar a^{\mathbb{Z}_2})^3+\alpha_{n,2}(\bar a^{\mathbb{Z}_4})^3+\alpha_{n,3}\bar a^{\mathbb{Z}_4}(\bar a^{\mathbb{Z}_2})^2+\alpha_{n,4}\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_4})^2,\qquad \alpha_{n,i}=0,1. \quad (271)$$

Calculate $\bar\nu_4$: $\bar\nu_4$ is calculated from (87)

$$\begin{aligned}\mathrm{d}\bar\nu_4 &\overset{1,\mathrm{d}}{=}\frac{\alpha_{n,1}}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_2})^3+\frac{\alpha_{n,2}}{2}\mathrm{Sq}^2(\bar a^{\mathbb{Z}_4})^3\\&\quad+\frac{\alpha_{n,3}}{2}\mathrm{Sq}^2\bar a^{\mathbb{Z}_4}(\bar a^{\mathbb{Z}_2})^2+\frac{\alpha_{n,4}}{2}\mathrm{Sq}^2\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_4})^2\\&\overset{1,\mathrm{d}}{=}\frac{\alpha_{n,1}}{2}(\bar a^{\mathbb{Z}_2})^5+\frac{\alpha_{n,2}}{2}(\bar a^{\mathbb{Z}_4})^5\\&\quad+\frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_4}(\bar a^{\mathbb{Z}_2})^4+\frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_4})^4,\end{aligned}$$

where we have used (M6). $\frac{1}{2}(\bar a^{\mathbb{Z}_2})^5$, $\frac12(\bar a^{\mathbb{Z}_4})^5$, and $\frac{\alpha_{n,3}}{2}\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_4})^4$ are non-trivial cocycles in $H^5(\mathcal{B}G_{fSO};\mathbb{R}/\mathbb{Z})$. But $\frac12\bar a^{\mathbb{Z}_4}(\bar a^{\mathbb{Z}_2})^4$ is a coboundary

$$\frac12\bar a^{\mathbb{Z}_4}(\beta_2\bar a^{\mathbb{Z}_2})(\beta_2\bar a^{\mathbb{Z}_2})\overset{1}{=}\frac14\bar a^{\mathbb{Z}_4}(\mathrm{d}\bar a^{\mathbb{Z}_2})\beta_2\bar a^{\mathbb{Z}_2}\overset{1}{=}\mathrm{d}(\frac14\bar a^{\mathbb{Z}_4}\bar a^{\mathbb{Z}_2}\beta_2\bar a^{\mathbb{Z}_2}) \quad (272)$$

Thus ν_4 has a form

$$\bar\nu_4(\bar a^{G_{fSO}})\overset{1}{=}\frac{\alpha_n}{4}\bar a^{\mathbb{Z}_4}\bar a^{\mathbb{Z}_2}\beta_2\bar a^{\mathbb{Z}_2}+\frac{\alpha_{\nu,1}}{2}\bar a^{\mathbb{Z}_2}(\bar a^{\mathbb{Z}_4})^3+\frac{\alpha_{\nu,2}}{2}\bar a^{\mathbb{Z}_4}(\bar a^{\mathbb{Z}_2})^3,\qquad \alpha_{\nu,i}=0,1. \quad (273)$$

We find that there are eight fermionic $Z_2\times Z_4\times Z_2^f$-SPT phases in 3+1D from fermion decoration. Those phases can be divided into two groups of four, and the four phases in each group differ by bosonic $Z_2\times Z_4$-SPT phases from fermion pairs.

Ref. 21 has calculated $Z_2\times Z_4\times Z_2^f$-SPT phases using cobordism approach and found $\mathbb{Z}_2\times\mathbb{Z}_4$ phases. On the other hand, non-interacting fermions can only realize the trivial $Z_2\times Z_4\times Z_2^f$-SPT phase.[24–26] So the intrinsic fermionic $Z_2\times Z_4\times Z_2^f$-SPT phase (the phases that cannot be realized by bosonic fermion pairs) cannot come from non-interacting fermions.[60] In the above, we see that the intrinsic fermionic $Z_2\times Z_4\times Z_2^f$-SPT phases can all come from fermion decoration.

After posting this paper, a paper Ref. 61 with related results appeared. As stressed in this paper, the symmetry in fermion systems is not described by $G_b\leftthreetimes Z_2^f$, but more precisely by a structure $Z_2^f\leftthreetimes G_b\leftthreetimes SO_\infty$:

$$1\to Z_2^f\to G_b\leftthreetimes SO_\infty\to SO_\infty\to 1. \quad (274)$$

Previously, the fermionic SPT orders with symmetry $G_f=G_b\times Z_2^f$ were studied[17,35,36]. This paper generalizes the symmetry to the more general case $G_f=G_b\leftthreetimes Z_2^f$, or more generally $Z_2^f\leftthreetimes G_b\leftthreetimes SO_\infty$. The paper Ref. 61 also include the cases with more general symmetry $G_f=G_b\leftthreetimes Z_2^f$, but not as general as $Z_2^f\leftthreetimes G_b\leftthreetimes SO_\infty$.

XVII. SUMMARY

In this paper, we construct exactly soluble models to systematical realize a large class of fermionic SPT states. The constructed path integrals and the corresponding fermionic SPT phases are labeled by some data. Those data can be described in a compact form using terminology of higher group $\mathcal{B}(\Pi_1,1;\Pi_2,2;\cdots)$ (see Appendix L for details). We note that, for a d-group $\mathcal{B}(G,1;Z_2,d)$ (*i.e.* a complex with only one vertex), its links are labeled by group elements $g\in G$. This gives rise to the so called canonical G-valued 1-cochain $\bar a$ on the complex $\mathcal{B}(G,1;Z_2,d)$. On each d-simplex in $\mathcal{B}(G,1;Z_2,d)$ we also have a Z_2 label. This gives us the canonical $\mathbb{Z}_2$-valued d-cochain $\bar f_d$ on the complex $\mathcal{B}(G,1;Z_2,d)$. The condition $\mathrm{d}\bar f_d\overset{2}{=}0$ gives us a a particular higher group $\mathcal{B}_f(O_\infty,1;Z_2,d)$ (see Ref. 41 and Appendix L). Now, we are ready to state our results:

1. **Characterization data without time reversal**: For unitary symmetry G_f, the fermionic SPT phases obtained via fermion decoration are described by a pair (φ,ν_{d+1}), where $\varphi:\mathcal{B}(G_f\leftthreetimes SO_\infty,1)\to\mathcal{B}_f(SO_\infty,1;Z_2,d)$ is a homomorphism between two higher groups and ν_{d+1} is a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{B}(G_f\leftthreetimes SO_\infty,1)$ that trivializes the pullback of a $\mathbb{R}/\mathbb{Z}$-valued $d+2$-cocycle $\bar\omega_{d+2}=\frac12\mathrm{Sq}^2\bar f_d+\frac12\bar f_d\bar{\mathrm{w}}_2$ on $\mathcal{B}_f(SO_\infty,1;Z_2,d)$, *i.e.* $-\mathrm{d}\nu_{d+1}=\mathrm{d}\varphi^*\bar\omega_{d+2}$. Here $\bar f_d$ is the canonical d-cochain on

$\mathcal{B}_f(SO_\infty, 1; Z_2, d)$, and $\bar{\mathrm{w}}_n$ is the n^{th} Stiefel-Whitney class constructed from canonical d-cochain $\bar{a}$ on $\mathcal{B}_f(SO_\infty, 1; Z_2, d)$. Note that $\bar{a}$ can be viewed as the SO_∞ connection of a SO_∞ bundle over $\mathcal{B}_f(SO_\infty, 1; Z_2, d)$.

2. **Characterization data with time reversal**: In the presence of time reversal symmetry $G_f = G_f^0 \leftthreetimes Z_2^T$, we find that the fermionic SPT phases obtained via fermion decoration are described by a pair (φ, ν_{d+1}), where $\varphi : \mathcal{B}(G_f^0 \leftthreetimes O_\infty, 1) \to \mathcal{B}_f(O_\infty, 1; Z_2, d)$ is a homomorphism between two higher groups and ν_{d+1} is a $\mathbb{R}/\mathbb{Z}$-valued $d+1$-cochain on $\mathcal{B}(G_f^0 \leftthreetimes O_\infty, 1)$ that trivializes the pull-back of a $\mathbb{R}/\mathbb{Z}$-valued $d+2$-cocycle $\bar{\omega}_{d+2} = \frac{1}{2}\mathrm{Sq}^2 \bar{f}_d + \frac{1}{2}\bar{f}_d(\bar{\mathrm{w}}_2 + \bar{\mathrm{w}}_1^2)$ on $\mathcal{B}_f(O_\infty, 1; Z_2, d)$, *i.e.* $-\mathrm{d}\nu_{d+1} = \mathrm{d}\varphi^* \bar{\omega}_{d+2}$.

3. **Model construction and SPT invariant**: Using the data (φ, ν_{d+1}), we can write down the explicit re-triangulation invariant path integral that describes a local fermion model (in bosonized form) that realizes the corresponding SPT phase (see (93), (86), and (79)). We can also write down the SPT invariant[18,30,31,43,44] that characterize the resulting fermionic SPT phase (see (101), (99), and (120)). Those bosonized fermion path integrals, (93), (86), and (79), and the corresponding SPT invariants, (101), (99), and (120), are the main results of this paper.

4. **Equivalence relation**: Only the pairs (φ, ν_{d+1}) that give rise to distinct SPT invariants correspond to distinct SPT phases. The pairs (φ, ν_{d+1}) that give rise to the same SPT invariant are regarded as equivalent. In particular, two homotopically connected φ's are equivalent and two ν_{d+1}'s differ by a coboundary are equivalent.

The data (φ, ν_{d+1}) cover all the fermion SPT states obtained via fermion decoration. But they do not include the fermion SPT states obtained via decoration of chains of 1+1D topological p-wave superconducting states, but may include some fermion SPT states obtained via decoration of sheets of 2+1D topological p-wave superconducting state.

After we post this paper, a related and more general treatment of fermionic SPT phases Ref. 61 was posted. We thank Zheng-Cheng Gu and Juven Wang for many very helpful discussions. XGW is supported by NSF Grant No. DMR-1506475 and DMS-1664412.

Appendix A: Space-time complex, cochains, and cocycles

In this paper, we consider models defined on a space-time lattice. A space-time lattice is a triangulation of the D-dimensional space-time M^D, which is denoted by $\mathcal{M}^D$. We will also call the triangulation $\mathcal{M}^D$ as a space-time complex, which is formed by simplices – the vertices, links, triangles, *etc* . We will use $i, j, \cdots$ to label vertices of the space-time complex. The links of the complex (the 1-simplices) will be labeled by $(i, j), (j, k), \cdots$. Similarly, the triangles of the complex (the 2-simplices) will be labeled by $(i, j, k), (j, k, l), \cdots$.

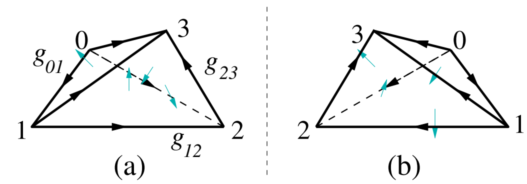

FIG. 4. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

In order to define a generic lattice theory on the space-time complex $\mathcal{M}^D$ using local Lagrangian term on each simplex, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.[28,62,63] A branching structure is a choice of orientation of each link in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 4).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming links, and the second vertex is the vertex with only one incoming link, *etc* . So the simplex in Fig. 4a has the following vertex ordering: $0, 1, 2, 3$.

The branching structure also gives the simplex (and its sub-simplices) a canonical orientation. Fig. 4 illustrates two 3-simplices with opposite canonical orientations compared with the 3-dimension space in which they are embedded. The blue arrows indicate the canonical orientations of the 2-simplices. The black arrows indicate the canonical orientations of the 1-simplices.

Given an Abelian group $(\mathbb{M}, +)$, an n-cochain f_n is an assignment of values in $\mathbb{M}$ to each n-simplex, for example a value $f_{n;i,j,\cdots,k} \in \mathbb{M}$ is assigned to n-simplex $(i, j, \cdots, k)$. So *a cochain f_n can be viewed as a bosonic field on the space-time lattice.*

$\mathbb{M}$ can also be viewed a $\mathbb{Z}$-module (*i.e.* a vector space with integer coefficient) that also allows scaling by an integer:

$$x + y = z, \quad x * y = z, \quad mx = y, \\ x, y, z \in \mathbb{M}, \quad m \in \mathbb{Z}. \tag{A1}$$

The direct sum of two modules $\mathbb{M}_1 \oplus \mathbb{M}_2$ (as vector spaces) is equal to the direct product of the two modules (as sets):

$$\mathbb{M}_1 \oplus \mathbb{M}_2 \overset{\text{as set}}{=} \mathbb{M}_1 \times \mathbb{M}_2 \tag{A2}$$

We like to remark that a simplex $(i, j, \cdots, k)$ can have two different orientations. We can use $(i, j, \cdots, k)$ and

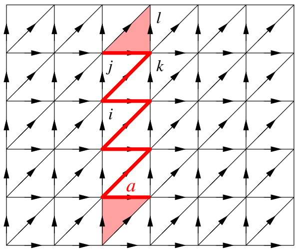

FIG. 5. (Color online) A 1-cochain a has a value 1 on the red links: $a_{ik} = a_{jk} = 1$ and a value 0 on other links: $a_{ij} = a_{kl} = 0$. $\mathrm{d}a$ is non-zero on the shaded triangles: $(\mathrm{d}a)_{jkl} = a_{jk} + a_{kl} - a_{jl}$. For such 1-cochain, we also have $a \smile a = 0$. So when viewed as a $\mathbb{Z}_2$-valued cochain, $\beta_2 a \neq a \smile a$ mod 2.

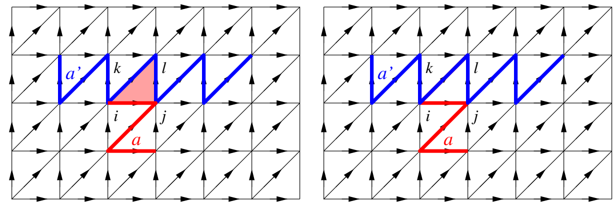

FIG. 6. (Color online) A 1-cochain a has a value 1 on the red links, Another 1-cochain a' has a value 1 on the blue links. On the left, $a \smile a'$ is non-zero on the shade triangles: $(a \smile a')_{ijl} = a_{ij} a'_{jl} = 1$. On the right, $a' \smile a$ is zero on every triangle. Thus $a \smile a' + a' \smile a$ is not a coboundary.

$(j, i, \cdots, k) = -(i, j, \cdots, k)$ to denote the same simplex with opposite orientations. The value $f_{n;i,j,\cdots,k}$ assigned to the simplex with opposite orientations should differ by a sign: $f_{n;i,j,\cdots,k} = -f_{n;j,i,\cdots,k}$. So to be more precise f_n is a linear map f_n : n-simplex $\to \mathbb{M}$. We can denote the linear map as $\langle f_n, n\text{-simplex}\rangle$, or

$$\langle f_n, (i, j, \cdots, k)\rangle = f_{n;i,j,\cdots,k} \in \mathbb{M}. \tag{A3}$$

More generally, a *cochain* f_n is a linear map of n-chains:

$$f_n : n\text{-chains} \to \mathbb{M}, \tag{A4}$$

or (see Fig. 5)

$$\langle f_n, n\text{-chain}\rangle \in \mathbb{M}, \tag{A5}$$

where a *chain* is a composition of simplices. For example, a 2-chain can be a 2-simplex: (i, j, k), a sum of two 2-simplices: $(i, j, k) + (j, k, l)$, a more general composition of 2-simplices: $(i, j, k) - 2(j, k, l)$, *etc* . The map f_n is linear respect to such a composition. For example, if a chain is m copies of a simplex, then its assigned value will be m times that of the simplex. $m = -1$ correspond to an opposite orientation.

We will use $C^n(\mathcal{M}^D; \mathbb{M})$ to denote the set of all n-cochains on $\mathcal{M}^D$. $C^n(\mathcal{M}^D; \mathbb{M})$ can also be viewed as a set all $\mathbb{M}$-valued fields (or paths) on $\mathcal{M}^D$. Note that $C^n(\mathcal{M}^D; \mathbb{M})$ is an Abelian group under the $+$-operation.

The total space-time lattice $\mathcal{M}^D$ correspond to a D-chain. We will use the same $\mathcal{M}^D$ to denote it. Viewing f_D as a linear map of D-chains, we can define an "integral" over $\mathcal{M}^D$:

$$\int_{\mathcal{M}^D} f_D \equiv \langle f_D, \mathcal{M}^D\rangle \tag{A6}$$
$$= \sum_{(i_0,i_1,\cdots,i_D)} s_{i_0 i_1 \cdots i_D} (f_D)_{i_0,i_1,\cdots,i_D}.$$

Here $s_{i_0 i_1 \cdots i_D} = \pm 1$, such that a D-simplex in the D-chain $\mathcal{M}^D$ is given by $s_{i_0 i_1 \cdots i_D}(i_0, i_1, \cdots, i_D)$.

We can define a derivative operator d acting on an n-cochain f_n, which give us an $n+1$-cochain (see Fig. 5):

$$\langle \mathrm{d} f_n, (i_0 i_1 i_2 \cdots i_{n+1})\rangle$$
$$= \sum_{m=0}^{n+1} (-)^m \langle f_n, (i_0 i_1 i_2 \cdots \hat{i}_m \cdots i_{n+1})\rangle \tag{A7}$$

where $i_0 i_1 i_2 \cdots \hat{i}_m \cdots i_{n+1}$ is the sequence $i_0 i_1 i_2 \cdots i_{n+1}$ with i_m removed, and $i_0, i_1, i_2 \cdots i_{n+1}$ are the ordered vertices of the $(n+1)$-simplex $(i_0 i_1 i_2 \cdots i_{n+1})$.

A cochain $f_n \in C^n(\mathcal{M}^D; \mathbb{M})$ is called a *cocycle* if $\mathrm{d} f_n = 0$. The set of cocycles is denoted by $Z^n(\mathcal{M}^D; \mathbb{M})$. A cochain f_n is called a *coboundary* if there exist a cochain f_{n-1} such that $\mathrm{d} f_{n-1} = f_n$. The set of coboundaries is denoted by $B^n(\mathcal{M}^D; \mathbb{M})$. Both $Z^n(\mathcal{M}^D; \mathbb{M})$ and $B^n(\mathcal{M}^D; \mathbb{M})$ are Abelian groups as well. Since $\mathrm{d}^2 = 0$, a coboundary is always a cocycle: $B^n(\mathcal{M}^D; \mathbb{M}) \subset Z^n(\mathcal{M}^D; \mathbb{M})$. We may view two cocycles differ by a coboundary as equivalent. The equivalence classes of cocycles, $[f_n]$, form the so called cohomology group denoted by

$$H^n(\mathcal{M}^D; \mathbb{M}) = Z^n(\mathcal{M}^D; \mathbb{M})/B^n(\mathcal{M}^D; \mathbb{M}), \tag{A8}$$

$H^n(\mathcal{M}^D; \mathbb{M})$, as a group quotient of $Z^n(\mathcal{M}^D; \mathbb{M})$ by $B^n(\mathcal{M}^D; \mathbb{M})$, is also an Abelian group.

When we study systems with time reversal symmetry, we need to consider cochains with *local value*. To define cochains with local value, we first note that a manifold $\mathcal{M}^D$ has a Stiefel-Whitney class w_1 to describe its orientation structure. On each link (ij) of $\mathcal{M}^D$, w_1 has two values $(\mathrm{w}_1)_{ij} = 0, 1$. For a cochain f with local values, we cannot directive compare their values on different simplexes, say $f_{ijk\cdots}$ and $f_{lmn\cdots}$. To compare $f_{ijk\cdots}$ and $f_{lmn\cdots}$, we need to use w_1 to parallel transport the value $f_{lmn\cdots}$ with base point l to the value with base point i: $f_{lmn\cdots} \to (\mathrm{w}_1)_{il} f_{lmn\cdots}$ then we can compare $f_{ijk\cdots}$ and $(\mathrm{w}_1)_{il} f_{lmn\cdots}$.

Such a interpretation of the value of a cochain will modify our definition of derivative operator

$$\langle \mathrm{d}_{\mathrm{w}_1} f_n, (i_0 i_1 i_2 \cdots i_{n+1})\rangle$$
$$= (-)^{(\mathrm{w}_1)_{01}} \langle f_n, (i_1 i_2 \cdots i_{n+1})\rangle$$

$$+\sum_{m=1}^{n+1}(-)^m\langle f_n,(i_0i_1i_2\cdots\hat{i}_m\cdots i_{n+1})\rangle \quad \text{(A9)}$$

we note that on the right-hand-side of the above equation, all the terms have the base point 0, except the first term which has a base point 1.

In this paper, on orientable manifold M^D, we will choose $\mathrm{w}_1 \overset{2}{=} 0$, and the cochains will all be the cochains with global values and the derivative operator is defined by (A7). On the other hand, on unorientable manifold M^D, the cochains will all be the cochains with local values and the derivative operator is defined by (A9). One can show that d and $\mathrm{d}_{\mathrm{w}_1}$ have the same local properties. We will drop the subscript w_1 in $\mathrm{d}_{\mathrm{w}_1}$ and write $\mathrm{d}_{\mathrm{w}_1}$ as d.

For the $\mathbb{Z}_N$-valued cocycle x_n, $\mathrm{d}x_n \overset{N}{=} 0$. Thus

$$\beta_N x_n \equiv \frac{1}{N}\mathrm{d}x_n \quad \text{(A10)}$$

is a $\mathbb{Z}$-valued cocycle. Here β_N is Bockstein homomorphism.

We notice the above definition for cochains still makes sense if we have a non-Abelian group $(G,\cdot)$ instead of an Abelian group $(\mathbb{M},+)$, however the differential d defined by (A7) will not satisfy $\mathrm{d}\circ\mathrm{d}=1$, except for the first two d's. That is to say, one may still make sense of 0-cocycle and 1-cocycle, but no more further naively by formula (A7). For us, we only use non-Abelian 1-cocycle in this article. Thus it is OK. Non-Abelian cohomology is thoroughly studied in mathematics motivating concepts such as gerbe.

From two cochains f_m and h_n, we can construct a third cochain p_{m+n} via the cup product (see Fig. 6):

$$\begin{aligned} p_{m+n} &= f_m \smile h_n, \\ \langle p_{m+n},(0\to m+n)\rangle &= \langle f_m,(0\to m)\rangle\times \\ &\quad \langle h_n,(m\to m+n)\rangle, \end{aligned} \quad \text{(A11)}$$

where $i\to j$ is the consecutive sequence from i to j:

$$i\to j \equiv i, i+1,\cdots,j-1,j. \quad \text{(A12)}$$

Note that the above definition applies to cochains with global. If h_n has a local value, we then have

$$\begin{aligned} p_{m+n} &= f_m \smile h_n, \\ \langle p_{m+n},(0\to m+n)\rangle &= (-)^{(\mathrm{w}_1)_{0m}}\langle f_m,(0\to m)\rangle\times \\ &\quad \langle h_n,(m\to m+n)\rangle, \end{aligned} \quad \text{(A13)}$$

The cup product has the following property

$$\mathrm{d}(h_n\smile f_m) = (\mathrm{d}h_n)\smile f_m + (-)^n h_n\smile(\mathrm{d}f_m) \quad \text{(A14)}$$

for cochains with global or local values. We note that the above is a local relation. Locally, we can always choose a gauge to make $\mathrm{w}_1 \overset{2}{=} 0$. Thus, the local relations are valid for cochains with both global and local values.

We see that $h_n\smile f_m$ is a cocycle if both f_m and h_n are cocycles. If both f_m and h_n are cocycles, then $f_m\smile h_n$ is a coboundary if one of f_m and h_n is a coboundary. So the cup product is also an operation on cohomology groups $\smile: H^m(M^D;\mathbb{M})\times H^n(M^D;\mathbb{M})\to H^{m+n}(M^D;\mathbb{M})$. The cup product of two *cocycles* has the following property (see Fig. 6)

$$f_m\smile h_n = (-)^{mn}h_n\smile f_m + \text{coboundary} \quad \text{(A15)}$$

We can also define higher cup product $f_m \underset{k}{\smile} h_n$ which gives rise to a $(m+n-k)$-cochain[64]:

$$\begin{aligned} &\langle f_m \underset{k}{\smile} h_n,(0,1,\cdots,m+n-k)\rangle \\ &= \sum_{0\le i_0<\cdots<i_k\le n+m-k}(-)^p\langle f_m,(0\to i_0, i_1\to i_2,\cdots)\rangle\times \\ &\qquad \langle h_n,(i_0\to i_1, i_2\to i_3,\cdots)\rangle, \end{aligned} \quad \text{(A16)}$$

and $f_m \underset{k}{\smile} h_n = 0$ for $k<0$ or for $k>m$ or n. Here $i\to j$ is the sequence $i,i+1,\cdots,j-1,j$, and p is the number of permutations to bring the sequence

$$0\to i_0, i_1\to i_2,\cdots; i_0+1\to i_1-1, i_2+1\to i_3-1,\cdots \quad \text{(A17)}$$

to the sequence

$$0\to m+n-k. \quad \text{(A18)}$$

For example

$$\begin{aligned} \langle f_m \underset{1}{\smile} h_n,(0\to m+n-1)\rangle &= \sum_{i=0}^{m-1}(-)^{(m-i)(n+1)}\times \\ \langle f_m,(0\to i, i+n\to m+n-1)\rangle &\langle h_n,(i\to i+n)\rangle. \end{aligned} \quad \text{(A19)}$$

We can see that $\underset{0}{\smile}=\smile$. Unlike cup product at $k=0$, the higher cup product of two cocycles may not be a cocycle. For cochains f_m, h_n, we have

$$\begin{aligned} \mathrm{d}(f_m \underset{k}{\smile} h_n) &= \mathrm{d}f_m \underset{k}{\smile} h_n + (-)^m f_m \underset{k}{\smile} \mathrm{d}h_n + \\ &(-)^{m+n-k} f_m \underset{k-1}{\smile} h_n + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m \end{aligned} \quad \text{(A20)}$$

If h_n has a local value, we then have

$$\begin{aligned} &\langle f_m \underset{k}{\smile} h_n,(0,1,\cdots,m+n-k)\rangle \\ &= \sum_{0\le i_0<\cdots<i_k\le n+m-k}(-)^p(-)^{(\mathrm{w}_1)_{0i_0}}\langle f_m,(0\to i_0, i_1\to i_2,\cdots)\rangle\times \\ &\qquad \langle h_n,(i_0\to i_1, i_2\to i_3,\cdots)\rangle, \end{aligned} \quad \text{(A21)}$$

Let f_m and h_n be cocycles and c_l be a chain, from (A20) we can obtain

$$\begin{aligned} \mathrm{d}(f_m \underset{k}{\smile} h_n) &= (-)^{m+n-k} f_m \underset{k-1}{\smile} h_n \\ &\quad + (-)^{mn+m+n} h_n \underset{k-1}{\smile} f_m, \end{aligned}$$

$$
\begin{aligned}
&\mathrm{d}(f_m \underset{k}{\smile} f_m) = [(-)^k + (-)^m] f_m \underset{k-1}{\smile} f_m, \\
&\mathrm{d}(c_l \underset{k-1}{\smile} c_l + c_l \underset{k}{\smile} \mathrm{d}c_l) = \mathrm{d}c_l \underset{k}{\smile} \mathrm{d}c_l \\
&\quad - [(-)^k - (-)^l](c_l \underset{k-2}{\smile} c_l + c_l \underset{k-1}{\smile} \mathrm{d}c_l). \qquad \text{(A22)}
\end{aligned}
$$

From (A22), we see that, for $\mathbb{Z}_2$-valued cocycles z_n,

$$
\mathrm{Sq}^{n-k}(z_n) \equiv z_n \underset{k}{\smile} z_n \qquad \text{(A23)}
$$

is always a cocycle. Here Sq is called the Steenrod square. More generally $h_n \underset{k}{\smile} h_n$ is a cocycle if $n+k = \text{odd}$ and h_n is a cocycle. Usually, the Steenrod square is defined only for $\mathbb{Z}_2$-valued cocycles or cohomology classes. Here, we like to define a generalized Steenrod square for $\mathbb{M}$-valued cochains c_n:

$$
\mathbb{Sq}^{n-k} c_n \equiv c_n \underset{k}{\smile} c_n + c_n \underset{k+1}{\smile} \mathrm{d}c_n. \qquad \text{(A24)}
$$

From (A22), we see that

$$
\begin{aligned}
\mathrm{d}\mathbb{Sq}^k c_n &= \mathrm{d}(c_n \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} \mathrm{d}c_n) \qquad \text{(A25)} \\
&= \mathbb{Sq}^k \mathrm{d}c_n + (-)^n \begin{cases} 0, & k = \text{odd} \\ 2\mathbb{Sq}^{k+1} c_n & k = \text{even} \end{cases}.
\end{aligned}
$$

In particular, when c_n is a $\mathbb{Z}_2$-valued cochain, we have

$$
\mathrm{d}\mathbb{Sq}^k c_n \overset{2}{=} \mathbb{Sq}^k \mathrm{d}c_n. \qquad \text{(A26)}
$$

Next, let us consider the action of $\mathbb{Sq}^k$ on the sum of two $\mathbb{M}$-valued cochains c_n and c_n':

$$
\begin{aligned}
&\mathbb{Sq}^k(c_n + c_n') = \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + \\
&\quad c_n \underset{n-k}{\smile} c_n' + c_n' \underset{n-k}{\smile} c_n + c_n \underset{n-k+1}{\smile} \mathrm{d}c_n' + c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + [1 + (-)^k] c_n \underset{n-k}{\smile} c_n' \\
&\quad - (-)^{n-k}[-(-)^{n-k} c_n' \underset{n-k}{\smile} c_n + (-)^n c_n \underset{n-k}{\smile} c_n'] \\
&\quad + c_n \underset{n-k+1}{\smile} \mathrm{d}c_n' + c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + [1 + (-)^k] c_n \underset{n-k}{\smile} c_n' \\
&+ (-)^{n-k}[\mathrm{d}c_n' \underset{n-k+1}{\smile} c_n + (-)^n c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n] \\
&- (-)^{n-k} \mathrm{d}(c_n' \underset{n-k+1}{\smile} c_n) + c_n \underset{n-k+1}{\smile} \mathrm{d}c_n' + c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + [1 + (-)^k] c_n \underset{n-k}{\smile} c_n' \\
&\quad + [1 + (-)^k] c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n - (-)^{n-k} \mathrm{d}(c_n' \underset{n-k+1}{\smile} c_n) \\
&\quad - [(-)^{n-k+1} \mathrm{d}c_n' \underset{n-k+1}{\smile} c_n - c_n \underset{n-k+1}{\smile} \mathrm{d}c_n'] \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + [1 + (-)^k] c_n \underset{n-k}{\smile} c_n' \\
&\quad + [1 + (-)^k] c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n - (-)^{n-k} \mathrm{d}(c_n' \underset{n-k+1}{\smile} c_n) \\
&\quad - \mathrm{d}(\mathrm{d}c_n' \underset{n-k+2}{\smile} c_n) + \mathrm{d}c_n' \underset{n-k+2}{\smile} \mathrm{d}c_n \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n' + \mathrm{d}c_n' \underset{n-k+2}{\smile} \mathrm{d}c_n \\
&\quad + [1 + (-)^k][c_n \underset{n-k}{\smile} c_n' + c_n' \underset{n-k+1}{\smile} \mathrm{d}c_n] \\
&\quad - (-)^{n-k} \mathrm{d}(c_n' \underset{n-k+1}{\smile} c_n) - \mathrm{d}(\mathrm{d}c_n' \underset{n-k+2}{\smile} c_n). \qquad \text{(A27)}
\end{aligned}
$$

We see that, if one of the c_n and c_n' is a cocycle,

$$
\mathbb{Sq}^k(c_n + c_n') \overset{2,\mathrm{d}}{=} \mathbb{Sq}^k c_n + \mathbb{Sq}^k c_n'. \qquad \text{(A28)}
$$

We also see that

$$
\begin{aligned}
&\mathbb{Sq}^k(c_n + \mathrm{d}f_{n-1}) \qquad \text{(A29)} \\
&= \mathbb{Sq}^k c_n + \mathbb{Sq}^k \mathrm{d}f_{n-1} + [1 + (-)^k] \mathrm{d}f_{n-1} \underset{n-k}{\smile} c_n \\
&\quad - (-)^{n-k} \mathrm{d}(c_n \underset{n-k+1}{\smile} \mathrm{d}f_{n-1}) - \mathrm{d}(\mathrm{d}c_n \underset{n-k+2}{\smile} \mathrm{d}f_{n-1}) \\
&= \mathbb{Sq}^k c_n + [1 + (-)^k][\mathrm{d}f_{n-1} \underset{n-k}{\smile} c_n + (-)^n \mathbb{Sq}^{k+1} f_{n-1}] \\
&+ \mathrm{d}[\mathbb{Sq}^k f_{n-1} - (-)^{n-k} c_n \underset{n-k+1}{\smile} \mathrm{d}f_{n-1} - \mathrm{d}c_n \underset{n-k+2}{\smile} \mathrm{d}f_{n-1}] \\
&= \mathbb{Sq}^k c_n + [1 + (-)^k][c_n \underset{n-k}{\smile} \mathrm{d}f_{n-1} + (-)^n \mathbb{Sq}^{k+1} f_{n-1}] \\
&+ \mathrm{d}[\mathbb{Sq}^k f_{n-1} - (-)^{n-k} \mathrm{d}f_{n-1} \underset{n-k+1}{\smile} c_n].
\end{aligned}
$$

Using (A30), we can also obtain the following result if $\mathrm{d}c_n = \text{even}$

$$
\begin{aligned}
&\mathbb{Sq}^k(c_n + 2c_n') \\
&\overset{4}{=} \mathbb{Sq}^k c_n + 2\mathrm{d}(c_n \underset{n-k+1}{\smile} c_n') + 2\mathrm{d}c_n \underset{n-k+1}{\smile} c_n' \\
&\overset{4}{=} \mathbb{Sq}^k c_n + 2\mathrm{d}(c_n \underset{n-k+1}{\smile} c_n') \qquad \text{(A30)}
\end{aligned}
$$

As another application, we note that, for a $\mathbb{M}$-valued cochain m_d and using (A20),

$$
\begin{aligned}
&\mathbb{Sq}^1(m_d) = m_d \underset{d-1}{\smile} m_d + m_d \underset{d}{\smile} \mathrm{d}m_d \\
&= \frac{1}{2}(-)^d[\mathrm{d}(m_d \underset{d}{\smile} m_d) - \mathrm{d}m_d \underset{d}{\smile} m_d] + \frac{1}{2} m_d \underset{d}{\smile} \mathrm{d}m_d \\
&= (-)^d \beta_2(m_d \underset{d}{\smile} m_d) - (-)^d \beta_2 m_d \underset{d}{\smile} m_d + m_d \underset{d}{\smile} \beta_2 m_d \\
&= (-)^d \beta_2 \mathbb{Sq}^0 m_d - 2(-)^d \beta_2 m_d \underset{d+1}{\smile} \beta_2 m_d \\
&= (-)^d \beta_2 \mathbb{Sq}^0 m_d - 2(-)^d \mathbb{Sq}^0 \beta_2 m_d \qquad \text{(A31)}
\end{aligned}
$$

This way, we obtain a relation between Steenrod square and Bockstein homomorphism, when m_d is a $\mathbb{Z}_2$-valued cochain

$$
\mathbb{Sq}^1(m_d) \overset{2}{=} \beta_2 m_d, \qquad \text{(A32)}
$$

where we have used $\mathbb{Sq}^0 m_d = m_d$ for $\mathbb{Z}_2$-valued cochain.

Appendix B: Almost cocycle and almost coboundary

A $\mathbb{R}/\mathbb{Z}$-valued cocycle c satisfies

$$
\mathrm{d}c \overset{1}{=} 0. \qquad \text{(B1)}
$$

A $\mathbb{R}/\mathbb{Z}$-valued almost-cocycle $\tilde{c}$ satisfies

$$\mathrm{d}\tilde{c} \overset{1}{\approx} 0. \tag{B2}$$

Physically, it is impossible to constrain an $\mathbb{R}/\mathbb{Z}$-value to exactly zero. So almost-cocycle is more relevant for our model construction. In this paper, when we say $\mathbb{R}/\mathbb{Z}$-valued cocycle, we really mean $\mathbb{R}/\mathbb{Z}$-valued almost-cocycle.

Similarly, a $\mathbb{R}/\mathbb{Z}$-valued almost-coboundary is given by

$$\mathrm{d}c + \epsilon, \tag{B3}$$

where ϵ is an almost-zero cochain.

The cohomology class $H^*_{\mathrm{a}}(M;\mathbb{R}/\mathbb{Z})$ for $\mathbb{R}/\mathbb{Z}$-valued almost-cocycles is different from the cohomology class $H^*(M;\mathbb{R}/\mathbb{Z})$ for $\mathbb{R}/\mathbb{Z}$-valued cocycles. For example, let us consider $H^1_{\mathrm{a}}(S^2;\mathbb{R}/\mathbb{Z})$. We parametrize S^2 by the polar angle θ, and the azimuthal angle ϕ. The 1-cochain

$$\tilde{a}^{\mathbb{R}/\mathbb{Z}} = \frac{1-\cos(\theta)}{4\pi}\mathrm{d}\phi \tag{B4}$$

is not a cocycle since $\mathrm{d}\tilde{a}^{\mathbb{R}/\mathbb{Z}} \neq 0$ mod 1. But it is a 1-almost-cocycle since

$$\mathrm{d}\tilde{a}^{\mathbb{R}/\mathbb{Z}} \approx c_1, \tag{B5}$$

and c_1 is a $\mathbb{Z}$-valued. $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ is not an 1-almost-coboundary, since c_1 generates $H^2(S^2;\mathbb{Z})$. In fact, $\tilde{a}^{\mathbb{R}/\mathbb{Z}}$ is the generator of $H^1_{\mathrm{a}}(S^2;\mathbb{R}/\mathbb{Z})$ and we have

$$H^1_{\mathrm{a}}(S^2;\mathbb{R}/\mathbb{Z}) \cong H^2(S^2;\mathbb{Z}) = \mathbb{Z}. \tag{B6}$$

In contrast $H^1(S^2;\mathbb{R}/\mathbb{Z}) = 0$.

In general, for an almost-cocycle $\tilde{c}$,

$$\lfloor \mathrm{d}\tilde{c} \rceil = C \tag{B7}$$

is a $\mathbb{Z}$-valued cocycle. The non-trivialness of $\tilde{c}$ is given by the non-trivialness of C. Thus we have

$$H^n_{\mathrm{a}}(M;\mathbb{R}/\mathbb{Z}) \cong H^{n+1}(M;\mathbb{Z}). \tag{B8}$$

In this paper, we will drop the subscript and write $H^n_{\mathrm{a}}(M;\mathbb{R}/\mathbb{Z})$ as $H^n(M;\mathbb{R}/\mathbb{Z})$.

Appendix C: Some additional discussion of fermion decoration

1. Another form of exactly soluble local fermionic models

We can also write the path integral (45) as one on M^{d+1}. To do so, we introduce a new $\mathbb{Z}_2$-valued $(d-1)$-cochain field b_{d-1} that satisfy

$$\mathrm{d}b_{d-1} \overset{2}{=} f_d \tag{C1}$$

to write the integrand of $\int_{N^{d+2}}$ as a total derivative (using (A26))

$$\begin{aligned}&\mathrm{d}[\mathbb{S}\mathrm{q}^2 b_{d-1} + b_{d-1}(\mathrm{w}_2+\mathrm{w}_1^2)]\\ &\overset{2}{=} \mathrm{Sq}^2 n_d + n_d(\mathrm{w}_2+\mathrm{w}_1^2).\end{aligned} \tag{C2}$$

This way, we can change the path integral (45) to one on M^{d+1} only:

$$Z(M^{d+1},A^{G_b}) = \sum_{g\in C^0(M^{d+1};G_b); f_d \overset{2}{=} n_d(a^{G_b})} \mathrm{e}^{\,\mathrm{i}\,2\pi\int_{M^{d+1}} \nu_{d+1}(a^{G_b}) + \frac12 \mathbb{S}\mathrm{q}^2 b_{d-1} + \frac12 b_{d-1}e_2}, \tag{C3}$$

where b_{d-1} is a function of f_d as determined from (C1). The summation $\sum_{g\in C^0(M^{d+1};G_b); f_d \overset{2}{=} n_d(a^{G_b})}$ (*i.e.* the path integral) is over a G_b-valued 0-cochain field g and $\mathbb{Z}_2$-valued d-cochain field f_d. But f_d subject to a constrain $f_d \overset{2}{=} n_d(a^{G_b}) = n_d(g_i A^{G_b}_{ij} g_j^{-1})$, which can be imposed as an energy penalty. The term $\frac12 \mathbb{S}\mathrm{q}^2 b_{d-1}$ makes the current f_d a fermion current (*i.e.* makes the field f_d to describe a fermion).

In order for (C3) to be well defined, the action amplitude $\mathrm{e}^{\,\mathrm{i}\,2\pi\int_{M^{d+1}} \nu_{d+1}(a^{G_b}) + \frac12[\mathbb{S}\mathrm{q}^2 b_{d-1} + b_{d-1}e_2(a^{G_b})]}$ should be a function of f_d and does not depend on which solution b_{d-1} of $\mathrm{d}b_{d-1} \overset{2}{=} f_d$ that we choose. Different solutions can differ by a cocycle $\bar{b}_{d-1}$. Using (A27), we find that

$$\begin{aligned}&\mathbb{S}\mathrm{q}^2(b_{d-1}+\bar{b}_{d-1}) + (b_{d-1}+\bar{b}_{d-1})e_2 - \mathbb{S}\mathrm{q}^2 b_{d-1} - b_{d-1}e_2\\ &\overset{2,\mathrm{d}}{=} \mathbb{S}\mathrm{q}^2(\bar{b}_{d-1}) + \bar{b}_{d-1}e_2(a^{G_b})\\ &\overset{2,\mathrm{d}}{=} \bar{b}_{d-1}[\mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b})].\end{aligned} \tag{C4}$$

Thus the path integral is well defined only on M^{d+1} with a symmetry twist A such that $\mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}) \overset{2,\mathrm{d}}{=} 0$. This implies that f_d describes a fermion. This also implies that the fermion is described by a representation of $G_f = Z_2 \leftthreetimes_{e_2} G_b$ (see Ref. 41).

2. Exactly soluble local bosonic models with emergent fermions

If we treat the field b_{d-1} in (C3) as an independent dynamical field (instead of as a function of f_d), then we will get a very different theory:

$$Z(M^{d+1},A^{G_b}) = \sum_{g\in C^0(M^{d+1};G_b);\, \mathrm{d}b_{d-1} \overset{2}{=} n_d(a^{G_b})} \mathrm{e}^{\,\mathrm{i}\,\pi\int_{M^{d+1}} 2\nu_{d+1}(a^{G_b}) + \mathbb{S}\mathrm{q}^2 b_{d-1} + b_{d-1}e_2(a^{G_b})}, \tag{C5}$$

The new path integral (C5) sums over the 0-cochains g_i and $(d-1)$-cochains b_{d-1} satisfying $\mathrm{d}b_{d-1} = n_d(a^{G_b})$. Such a model is actually a local bosonic model. The local bosonic model has emergent fermions whose current is given by $f_d = \mathrm{d}b_{d-1} - n_d(a^{G_b})$.

The above local bosonic model has a G_b symmetry. Thus the model describes symmetry G_b enriched topological order with emergent fermions. If we break the

G_b symmetry, the model describes a Z_2-topological order with emergent fermion. Such a Z_2-topological is described by a Z_2 gauge theory where the Z_2 charge is fermionic. In the presence of G_b symmetry, the emergent fermions carry fractionalized symmetry quantum number. Since the partition function of the local bosonic model vanishes when $\mathrm{w}_2 + \mathrm{w}_1^2 + e_2(A^{G_b}) \neq 0$, we conclude that the emergent fermions carry representations of $G_f = Z_2 \leftthreetimes_{e_2} G_b$. The partition function of the local bosonic model vanishes when $\mathrm{w}_2+\mathrm{w}_1^2+e_2(A^{G_b}) \neq 0$. We may view (45) as a fermionic model with G_f symmetry, and the bosonic model (C5) as the Z_2^f gauged fermionic model. If we gauge all the symmetry G_f in the fermionic model (45), we will obtain a higher gauge theory as described in Ref. 41:

$$Z(M^{d+1}) = \sum_{a^{G_b}\in C^1(M^{d+1};G_b);\, \mathrm{d}b_{d-1}=n_d(a^{G_b})} \mathrm{e}^{\,\mathrm{i}\,\pi \int_{M^{d+1}} 2\nu_{d+1}(a^{G_b})+\mathbb{Sq}^2 b_{d-1}+b_{d-1}e_2(a^{G_b})}. \tag{C6}$$

Despite their similarity, the two local bosonic models (C5) and (C6) are very different. In (C5), the dynamical fields are g, b_{d-1} (with $a^{G_b}_{ij} = g_i A^{G_b}_{ij} g_j^{-1}$), where g is a G_b-valued 0-cochain living on vertices. In contrast, in (C5), the dynamical fields are a^{G_b}, b_{d-1} where a^{G_b} is a G_b-valued 1-cochain living on links.

3. Another connection to higher gauge theory

There is another connection to higher gauge theory. After gauging all the G_f symmetry in the fermionic model (45), we obtain a local bosonic model (C6). Such a local bosonic model is a higher gauge theory with emergent fermion. Such a higher gauge theory is characterized by a higher group and its cocycle. Thus the data that characterizes a fermionic SPT phase is closely related to a higher group and its cocycle.

In fact the local bosonic theory (C6) is characterized by a higher group $\mathcal{B}_{n_d}(G_b, 1; Z_2^f, d-1)$ and its higher-group cocycle ω_{d+1}. The field content a^{G_b}, b_{d-1} and their conditions

$$\begin{aligned}(\delta a^{G_b})_{ijk} &\equiv a^{G_b}_{ij} a^{G_b}_{jk} a^{G_b}_{ki} = 1\\ \mathrm{d}b_{d-1} &\overset{2}{=} n_d(a^{G_b}),\end{aligned} \tag{C7}$$

determine the higher group $\mathcal{B}_{n_d}(G_b, 1; Z_2^f, d-1)$, which can be viewed as a space with homotopy groups $\pi_1 = G_b$, $\pi_{d-1} = Z_2^f$, and other $\pi_n = 0$.

The local bosonic model (C6) and (C5) are exactly soluble if the Lagrangian is given by a higher-group cocycle satisfying $\mathrm{d}\omega_{d+1} \overset{1}{=} 0$ (*i.e.* $\omega_{d+1} \in Z^{d+1}[\mathcal{B}_{n_d}(G_b, 1; Z_2^f, d-1); \mathbb{R}/\mathbb{Z}]$):

$$\begin{aligned}&\omega_{d+1}(a^{G_b}, b_{d-1})\\ &= \nu_{d+1}(a^{G_b}) + \frac{1}{2}[\mathbb{Sq}^2 b_{d-1} + n_d(a^{G_b}) e_2(a^{G_b})],\end{aligned} \tag{C8}$$

(See Ref. 41 and Appendix L for an introduction on higher groups and higher-group cocycles.)

Appendix D: Operations on modules

The tensor-product operation $\otimes_R$ and the torsion-product operation Tor_1^R act on R-modules $\mathbb{M}, \mathbb{M}', \mathbb{M}''$. Here R is a ring and a R-module is like a vector space over R (*i.e.* we can "multiply" a "vector" in $\mathbb{M}$ by an element of R, and two "vectors" in $\mathbb{M}$ can add.) The tensor-product operation $\otimes_R$ has the following properties:

$$\begin{aligned}&\mathbb{M}\otimes_{\mathbb{Z}}\mathbb{M}' \simeq \mathbb{M}'\otimes_{\mathbb{Z}}\mathbb{M},\\ &(\mathbb{M}'\oplus\mathbb{M}'')\otimes_R\mathbb{M} = (\mathbb{M}'\otimes_R\mathbb{M})\oplus(\mathbb{M}''\otimes_R\mathbb{M}),\\ &\mathbb{M}\otimes_R(\mathbb{M}'\oplus\mathbb{M}'') = (\mathbb{M}\otimes_R\mathbb{M}')\oplus(\mathbb{M}\otimes_R\mathbb{M}'');\\ &\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{M}\simeq\mathbb{M}\otimes_{\mathbb{Z}}\mathbb{Z}=\mathbb{M},\\ &\mathbb{Z}_n\otimes_{\mathbb{Z}}\mathbb{M}\simeq\mathbb{M}\otimes_{\mathbb{Z}}\mathbb{Z}_n=\mathbb{M}/n\mathbb{M},\\ &\mathbb{Z}_n\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}\simeq\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}_n=0,\\ &\mathbb{Z}_m\otimes_{\mathbb{Z}}\mathbb{Z}_n=\mathbb{Z}_{\langle m,n\rangle},\\ &\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}=0,\\ &\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}=\mathbb{R}.\end{aligned} \tag{D1}$$

The torsion-product operation Tor_1^R has the following properties:

$$\begin{aligned}&\mathrm{Tor}^1_R(\mathbb{M},\mathbb{M}')\simeq\mathrm{Tor}^1_R(\mathbb{M}',\mathbb{M}),\\ &\mathrm{Tor}^1_R(\mathbb{M}'\oplus\mathbb{M}'',\mathbb{M})=\mathrm{Tor}^1_R(\mathbb{M}',\mathbb{M})\oplus\mathrm{Tor}^1_R(\mathbb{M}'',\mathbb{M}),\\ &\mathrm{Tor}^1_R(\mathbb{M},\mathbb{M}'\oplus\mathbb{M}'')=\mathrm{Tor}^1_R(\mathbb{M},\mathbb{M}')\oplus\mathrm{Tor}^1_R(\mathbb{M},\mathbb{M}'')\\ &\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{M})=\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{M},\mathbb{Z})=0,\\ &\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{M})=\{m\in\mathbb{M}|nm=0\},\\ &\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_n,\\ &\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n)=\mathbb{Z}_{\langle m,n\rangle},\\ &\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z},\mathbb{R}/\mathbb{Z})=0.\end{aligned} \tag{D2}$$

These expressions allow us to compute the tensor-product $\otimes_R$ and the torsion-product Tor^1_R. We will use abbreviated Tor to denote $\mathrm{Tor}^1_{\mathbb{Z}}$.

In addition to $\otimes_{\mathbb{Z}}$ and Tor, we also have Ext and Hom operations on modules. Ext operation is given by

$$\begin{aligned}&\mathrm{Ext}^1_R(\mathbb{M}'\oplus\mathbb{M}'',\mathbb{M})=\mathrm{Ext}^1_R(\mathbb{M}',\mathbb{M})\oplus\mathrm{Ext}^1_R(\mathbb{M}'',\mathbb{M}),\\ &\mathrm{Ext}^1_R(\mathbb{M},\mathbb{M}'\oplus\mathbb{M}'')=\mathrm{Ext}^1_R(\mathbb{M},\mathbb{M}')\oplus\mathrm{Ext}^1_R(\mathbb{M},\mathbb{M}'')\\ &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{M})=0,\\ &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{M})=\mathbb{M}/n\mathbb{M},\\ &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{Z})=\mathbb{Z}_n,\\ &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z})=0,\\ &\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n)=\mathbb{Z}_{\langle m,n\rangle},.\end{aligned} \tag{D3}$$

The Hom operation on modules is given by

$$\mathrm{Hom}_R(\mathbb{M}'\oplus\mathbb{M}'',\mathbb{M})=\mathrm{Hom}_R(\mathbb{M}',\mathbb{M})\oplus\mathrm{Hom}_R(\mathbb{M}'',\mathbb{M}),$$

$$\begin{aligned}
&\mathrm{Hom}_R(\mathbb{M}, \mathbb{M}' \oplus \mathbb{M}'') = \mathrm{Hom}_R(\mathbb{M}, \mathbb{M}') \oplus \mathrm{Hom}_R(\mathbb{M}, \mathbb{M}''),\\
&\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{M}) = \mathbb{M},\\
&\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{M}) = \{m \in \mathbb{M} | nm = 0\},\\
&\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) = 0,\\
&\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n,\\
&\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{\langle m,n\rangle}. \qquad \text{(D4)}
\end{aligned}$$

We will use abbreviated Ext and Hom to denote $\mathrm{Ext}^1_{\mathbb{Z}}$ and $\mathrm{Hom}_{\mathbb{Z}}$.

Appendix E: Künneth formula and universal coefficient theorem

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex X and chain complex X'. The Künneth formula is expressed in terms of the tensor-product operation $\otimes_R$ and the torsion-product operation Tor_1^R described in the last section (see Ref. 65 page 247):

$$\begin{aligned}
0 &\to \bigoplus_{k=0}^{d} H^k(X, \mathbb{M}) \otimes_R H^{d-k}(X', \mathbb{M}')\\
&\to H^d(X \times X', \mathbb{M} \otimes_R \mathbb{M}')\\
&\to \bigoplus_{k=0}^{d+1} \mathrm{Tor}_R^1(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{M}')) \to 0, \qquad \text{(E1)}
\end{aligned}$$

where the exact sequence is split. Here R is a principle ideal domain and $\mathbb{M}, \mathbb{M}'$ are R-modules such that $\mathrm{Tor}_R^1(\mathbb{M}, \mathbb{M}') = 0$. We also require either
(1) $H_d(X, \mathbb{Z})$ and $H_d(X', \mathbb{Z})$ are finitely generated, or
(2) $\mathbb{M}'$ and $H_d(X', \mathbb{Z})$ are finitely generated.
(For example, $\mathbb{M}' = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_n \oplus \cdots \oplus \mathbb{Z}_m$ is finitely generated, with $R = \mathbb{Z}$.)

For more details on principal ideal domain and R-module, see the corresponding Wiki articles. Note that ring $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains. Also, $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are not finitely generate R-modules if $R = \mathbb{Z}$.

The Künneth formula works for topological cohomology where X and X' are treated as topological spaces.

As the first application of Künneth formula, we like to use it to calculate $H^*(X', \mathbb{M})$ from $H^*(X', \mathbb{Z})$, by choosing $R = \mathbb{M}' = \mathbb{Z}$. In this case, the condition $\mathrm{Tor}_R^1(\mathbb{M}, \mathbb{M}') = \mathrm{Tor}_{\mathbb{Z}}^1(\mathbb{M}, \mathbb{Z}) = 0$ is always satisfied. $\mathbb{M}$ can be $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}_n$ *etc* . So we have

$$\begin{aligned}
0 &\to \bigoplus_{k=0}^{d} H^k(X, \mathbb{M}) \otimes_{\mathbb{Z}} H^{d-k}(X', \mathbb{Z})\\
&\to H^d(X \times X', \mathbb{M})\\
&\to \bigoplus_{k=0}^{d} \mathrm{Tor}(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{Z})) \to 0 \qquad \text{(E2)}
\end{aligned}$$

Again, the exact sequence is split.

We can further choose X to be the space of one point in (E2), and use

$$H^d(X, \mathbb{M})) = \begin{cases} \mathbb{M}, & \text{if } d = 0,\\ 0, & \text{if } d > 0, \end{cases} \qquad \text{(E3)}$$

to reduce (E2) to

$$H^d(X, \mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} H^d(X, \mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{M}, H^{d+1}(X, \mathbb{Z})). \qquad \text{(E4)}$$

where X' is renamed as X. The above is a form of the universal coefficient theorem which can be used to calculate $H^*(X, \mathbb{M})$ from $H^*(X, \mathbb{Z})$ and the module $\mathbb{M}$.

Using the universal coefficient theorem, we can rewrite (E2) as

$$H^d(X \times X', \mathbb{M}) \simeq \bigoplus_{k=0}^{d} H^k[X, H^{d-k}(X', \mathbb{M})]. \qquad \text{(E5)}$$

The above is valid for topological cohomology.

Appendix F: Lyndon-Hochschild-Serre spectral sequence

The Lyndon-Hochschild-Serre spectral sequence (see Ref. 66 page 280,291, and Ref. 67) allows us to understand the structure of of the cohomology of a fiber bundle $F \to X \to B$, $H^*(X; \mathbb{M})$, from $H^*(F; \mathbb{M})$ and $H^*(B; \mathbb{M})$. In general, $H^d(X; \mathbb{M})$, when viewed as an Abelian group, contains a chain of subgroups

$$\{0\} = H_{d+1} \subset H_d \subset \cdots \subset H_0 = H^d(X; \mathbb{M}) \qquad \text{(F1)}$$

such that H_l/H_{l+1} is a subgroup of a factor group of $H^l[B, H^{d-l}(F; \mathbb{M})_B]$, *i.e.* $H^l[B, H^{d-l}(F; \mathbb{M})_B]$ contains a subgroup Γ^k, such that

$$\begin{aligned}
H_l/H_{l+1} &\subset H^l[B, H^{d-l}(F; \mathbb{M})_B]/\Gamma^l,\\
l &= 0, \cdots, d. \qquad \text{(F2)}
\end{aligned}$$

Note that $\pi_1(B)$ may have a non-trivial action on $\mathbb{M}$ and $\pi_1(B)$ may have a non-trivial action on $H^{d-l}(F; \mathbb{M})$ as determined by the structure $F \to X \to B$. We add the subscript B to $H^{d-l}(F; \mathbb{M})$ to indicate this action. We also have

$$\begin{aligned}
H_0/H_1 &\subset H^0[B, H^d(F; \mathbb{M})_B],\\
H_d/H_{d+1} &= H_d = H^d(B; \mathbb{M})/\Gamma^d. \qquad \text{(F3)}
\end{aligned}$$

In other words, all the elements in $H^d(X; \mathbb{M})$ can be one-to-one labeled by $(x_0, x_1, \cdots, x_d)$ with

$$x_l \in H_l/H_{l+1} \subset H^l[B, H^{d-l}(F; \mathbb{M})_B]/\Gamma^l. \qquad \text{(F4)}$$

Let $x_{l,\alpha}$, $\alpha = 1, 2, \cdots$, be the generators of H^l/H^{l+1}. Then we say $x_{i,\alpha}$ for all l, α are the generators of

$H^d(X;\mathbb{M})$. We also call H_l/H_{l+1}, $l=0,\cdots,d$, the generating sub-factor groups of $H^d(X;\mathbb{M})$.

The above result implies that we can use $(k_0,k_1,\cdots,k_d)$ with $k_l \in H^l[B,H^{d-l}(F;\mathbb{M})_B]$ to label all the elements in $H^d(X;\mathbb{M})$. However, such a labeling scheme may not be one-to-one, and it may happen that only some of $(k_0,k_1,\cdots,k_d)$ correspond to the elements in $H^d(X;\mathbb{M})$. But, on the other hand, for every element in $H^d(X;\mathbb{M})$, we can find a $(k_0,k_1,\cdots,k_d)$ that corresponds to it. Such a relation can be described by an injective map

$$H^d(F \leftthreetimes B;\mathbb{M}) \rightarrowtail \bigoplus_{l=0}^{d} H^l[B,H^{d-l}(F;\mathbb{M})_B] \tag{F5}$$

For the special case $X=B\times F$, $(k_0,k_1,\cdots,k_d)$ will give us an one-to-one labeling of the elements in $H^d(B\times F;\mathbb{M})$. In fact

$$H^d(B\times F;\mathbb{M}) = \bigoplus_{l=0}^{d} H^l[B,H^{d-l}(F;\mathbb{M})]. \tag{F6}$$

Appendix G: The ring of $H^*(\mathcal{B}SO_\infty;\mathbb{Z})$

The ring $H^*(\mathcal{B}SO_n;\mathbb{Z}_2)$ has a simple structure:

$$H^*(\mathcal{B}SO_n;\mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{w}_2,\mathrm{w}_3,\cdots,\mathrm{w}_n]. \tag{G1}$$

According to Ref. 68, the ring $H^*(\mathcal{B}SO_n;\mathbb{Z})$ is given by

$$H^*(\mathcal{B}SO_\infty;\mathbb{Z}) = \mathbb{Z}[\{p_i\},\{\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)\},X_n]/\sim, \tag{G2}$$

where $\mathbb{Z}[\{p_i\},\{\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)\},X_n]$ is a polynomial ring generated by p_i and $\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)$, $0<i\le\lfloor\frac{n-1}{2}\rfloor$, $0<i_1<i_2<\cdots\le\lfloor\frac{n-1}{2}\rfloor$, with integer coefficients. Here $p_i\in H^{4i}(\mathcal{B}SO_\infty;\mathbb{Z})$ is the Pontryagin class of dimension $4i$ and $\mathrm{w}_i\in H^i(\mathcal{B}SO_\infty;\mathbb{Z}_2)$ is the Stiefel-Whitney class of dimension i. Since $\mathrm{Tor}H^d(\mathcal{B}G,\mathbb{R}/\mathbb{Z}) = \mathrm{Tor}H^{d+1}(\mathcal{B}G;\mathbb{Z})$ (see, for example, Ref. 69), the natural map $H^d(\mathcal{B}G;\mathbb{Z}_2)\to \mathrm{Tor}H^d(\mathcal{B}G,\mathbb{R}/\mathbb{Z})$ induces the Bockstein homomorphism $H^d(\mathcal{B}G;\mathbb{Z}_2)\to H^{d+1}(\mathcal{B}G;\mathbb{Z})$: $\beta_2 : H^i(\mathcal{B}SO_\infty;\mathbb{Z}_2)\to H^{i+1}(\mathcal{B}SO_\infty;\mathbb{Z})$. Note that $f\in H^i(\mathcal{B}SO_\infty;\mathbb{Z}_2)$ satisfies $\mathrm{d}f \overset{2}{=} 0$. Thus $\frac{1}{2}\mathrm{d}f \overset{1}{=} 0$, or $\frac{1}{2}\mathrm{d}f$ is an integral cocycle. This allows us to write the Bockstein homomorphism as

$$\beta_2 f = \frac{1}{2}\mathrm{d}f \in H^{i+1}(\mathcal{B}SO_\infty;\mathbb{Z}). \tag{G3}$$

To obtain the ring $H^*(\mathcal{B}SO_\infty;\mathbb{Z})$ from a polynomial ring generated by p_i and $\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)$, we need to quotient out certain equivalence relations $\sim$. The equivalence relations $\sim$ contain

$$\begin{aligned}
&2\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)=0, \quad p_n \overset{2}{=} \mathrm{w}_{2n}^2, \quad X_n=\beta_2\mathrm{w}_{2k} \text{ if } n=2k+1, \quad X_n^2=p_k \text{ if } n=2k,\\
&\beta_2 w(I)\beta_2 w(J) = \sum_{k\in I} \beta_2\mathrm{w}_{2k}\ \beta_2\mathrm{w}[(I-\{k\})\cup J-(I-\{k\})\cap J]\ p[(I-\{k\})\cap J],
\end{aligned} \tag{G4}$$

where $I=\{i_1,i_2,\cdots\}$, $\mathrm{w}(I)=\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots$, and $p(I)=p_{i_1}p_{i_2}\cdots$. Many last kind of the equivalence relations are trivial identities. The first non-trivial equivalence relations appears at dimension 14:

$$\begin{aligned}
&\beta_2(\mathrm{w}_4\mathrm{w}_2)\beta_2\mathrm{w}_6 = \beta_2\mathrm{w}_4\beta_2(\mathrm{w}_6\mathrm{w}_2)+\beta_2\mathrm{w}_2\beta_2(\mathrm{w}_6\mathrm{w}_4),\\
&\beta_2(\mathrm{w}_4\mathrm{w}_2)\beta_2(\mathrm{w}_4\mathrm{w}_2) = p_1\beta_2\mathrm{w}_4\beta_2\mathrm{w}_4 + p_2\beta_2\mathrm{w}_2\beta_2\mathrm{w}_2.
\end{aligned} \tag{G5}$$

We see that there are no effective equivalence relations of the last kind for dimensions less than 14. So for low dimensions,

$$\begin{aligned}
&H^0(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z},\\
&H^1(\mathcal{B}SO_\infty;\mathbb{Z})=0,\\
&H^2(\mathcal{B}SO_\infty;\mathbb{Z})=0,\\
&H^3(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}_2=\{\beta_2\mathrm{w}_2\},\\
&H^4(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}=\{p_1\},\\
&H^5(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}_2=\{\beta_2\mathrm{w}_4\},\\
&H^6(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}_2=\{\beta_2\mathrm{w}_2\beta_2\mathrm{w}_2\},\\
&H^7(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}_2^{\oplus 3}=\{\beta_2\mathrm{w}_6,\beta_2(\mathrm{w}_2\mathrm{w}_4),p_1\beta_2\mathrm{w}_2\},\\
&H^8(\mathcal{B}SO_\infty;\mathbb{Z})=\mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}_2=\{p_1^2,p_2,\beta_2\mathrm{w}_2\beta_2\mathrm{w}_4\}.
\end{aligned} \tag{G6}$$

In the above, we also list the basis (or generators) of cohomology classes. Using (B8), the above allows us to obtain

$$\begin{aligned}
&H^0_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=0,\\
&H^1_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=0,\\
&H^2_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2=\{\tfrac{1}{2}\mathrm{w}_2\},\\
&H^3_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=\mathbb{Z}=\{\omega_3\},\\
&H^4_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2=\{\tfrac{1}{2}\mathrm{w}_4\},\\
&H^5_{\mathrm{a}}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z})=\mathbb{Z}_2=\{\tfrac{1}{2}\mathrm{w}_2\mathrm{w}_3)\},
\end{aligned} \tag{G7}$$

$$H^6_{\rm a}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{\oplus 3} = \{\frac12 \mathrm{w}_6, \frac12 \mathrm{w}_2\mathrm{w}_4, \frac12 p_1\mathrm{w}_2\},$$
$$H^7_{\rm a}(\mathcal{B}SO_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}_2 = \{\omega_3 p_1, \omega_7, \frac12 \mathrm{w}_3\mathrm{w}_4\}.$$

where ω_{4n-1} is a $\mathbb{R}/\mathbb{Z}$-valued almost-cocycle on $\mathcal{B}SO_\infty$ (the gravitational Chern-Simons term)

$$\mathrm{d}\omega_{4n-1} = p_n. \tag{G8}$$

The above basis give rise to the basis in (G6) through the natural map β: $H^d(\mathcal{B}G,\mathbb{R}/\mathbb{Z}) \to H^{d+1}(\mathcal{B}G,\mathbb{Z})$.

Appendix H: The ring of $H^*(\mathcal{B}O_\infty;\mathbb{Z})$

The ring $H^*(\mathcal{B}O_n;\mathbb{Z}_2)$ is given by

$$H^*(\mathcal{B}O_n;\mathbb{Z}_2) = \mathbb{Z}_2[\mathrm{w}_1,\mathrm{w}_2,\cdots,\mathrm{w}_n]. \tag{H1}$$

and the ring $H^*(\mathcal{B}O_\infty;\mathbb{Z})$ is given by[68]

$$H^*(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}[\{p_i\},\{\beta_2(\mathrm{w}_1^\epsilon \mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)\}]/\sim, \tag{H2}$$

where $\mathbb{Z}[\{p_i\},\{\beta_2(\mathrm{w}_1^\epsilon \mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)\}]$ is a polynomial ring with integer coefficients, generated by p_i and $\beta_2(\mathrm{w}_1^\epsilon \mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)$, $\epsilon=0,1$, $0<i\le\lfloor\frac n2\rfloor$, and $0<i_1<i_2<\cdots\le\lfloor\frac n2\rfloor$. Here $p_i\in H^{4i}(\mathcal{B}O_\infty;\mathbb{Z})$ is the Pontryagin class of dimension $4i$ and $\mathrm{w}_i\in H^i(\mathcal{B}O_\infty;\mathbb{Z}_2)$ is the Stiefel-Whitney class of dimension i.

To obtain the ring $H^*(\mathcal{B}O_\infty;\mathbb{Z})$ from a polynomial ring generated by p_i and $\beta_2(\mathrm{w}_1^\epsilon \mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)$, we need to quotient out certain equivalence relations $\sim$. The equivalence relations $\sim$ contain

$$2\beta_2(\mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots)=0,\quad \beta_2(\mathrm{w}_1\mathrm{w}_n)=0,\quad p_n \overset{2}{=} \mathrm{w}_{2n}^2,\quad (\beta_2\mathrm{w}_n)^2\big|_{n=\text{even}} = p_{n/2}\beta_2\mathrm{w}_1,$$
$$\beta_2\mathrm{w}(I)\beta_2\mathrm{w}(J) = \sum_{k\in I}\beta_2\mathrm{w}_{2k}\ \beta_2\mathrm{w}[(I-\{k\})\cup J-(I-\{k\})\cap J]\ p[(I-\{k\})\cap J], \tag{H3}$$

where $I=\{\frac{\epsilon}{2},i_1,i_2,\cdots\}$, $\mathrm{w}(I)=\mathrm{w}_1^\epsilon \mathrm{w}_{2i_1}\mathrm{w}_{2i_2}\cdots$, and $p(I)=\mathrm{w}_1^\epsilon p_{i_1}p_{i_2}\cdots$. Many last kind of the equivalence relations are trivial identities. The non-trivial equivalence relations for dimension 9 and less are given by:

$$[\beta_2(\mathrm{w}_1\mathrm{w}_2)]^2 = (\beta_2\mathrm{w}_2)^2\beta_2\mathrm{w}_1 + (\beta_2\mathrm{w}_1)^2 p_1,$$
$$\beta_2(\mathrm{w}_1\mathrm{w}_2)\beta_2\mathrm{w}_4 = \beta_2\mathrm{w}_2\beta_2(\mathrm{w}_1\mathrm{w}_4) + \beta_2\mathrm{w}_1\beta_2(\mathrm{w}_2\mathrm{w}_4) \tag{H4}$$

So for low dimensions,

$$\begin{aligned}
&H^0(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z},\\
&H^1(\mathcal{B}O_\infty;\mathbb{Z}) = 0,\\
&H^2(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2 = \{\beta_2\mathrm{w}_1\},\\
&H^3(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2 = \{\beta_2\mathrm{w}_2\},\\
&H^4(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z} = \{(\beta_2\mathrm{w}_1)^2, p_1\},\\
&H^5(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2^{\oplus 2} = \{\beta_2\mathrm{w}_4, \beta_2\mathrm{w}_1\beta_2\mathrm{w}_2\},\\
&H^6(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2^{\oplus 3} = \{(\beta_2\mathrm{w}_2)^2, (\beta_2\mathrm{w}_1)^3, p_1\beta_2\mathrm{w}_1\},\\
&H^7(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2^{\oplus 5} = \{\beta_2\mathrm{w}_6, \beta_2(\mathrm{w}_2\mathrm{w}_4),\\
&\qquad p_1\beta_2\mathrm{w}_2, (\beta_2\mathrm{w}_1)^2\beta_2\mathrm{w}_2, \beta_2\mathrm{w}_1\beta_2\mathrm{w}_4\},\\
&H^8(\mathcal{B}O_\infty;\mathbb{Z}) = \mathbb{Z}_2^{\oplus 5}\oplus\mathbb{Z}^{\oplus 2} = \{\beta_2(\mathrm{w}_1\mathrm{w}_2\mathrm{w}_4), (\beta_2\mathrm{w}_1)^4,\\
&\qquad \beta_2\mathrm{w}_2\beta_2\mathrm{w}_4, \beta_2\mathrm{w}_1(\beta_2\mathrm{w}_2)^2, p_1(\beta_2\mathrm{w}_1)^2, p_1^2, p_2\}.
\end{aligned} \tag{H5}$$

In the above, we also list the basis (or generators) of cohomology classes. Due to the relation (B8), the above allows us to obtain

$$\begin{aligned}
&H^0_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z},\\
&H^1_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 = \{\frac12\mathrm{w}_1\},\\
&H^2_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 = \{\frac12\mathrm{w}_2\},\\
&H^3_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z} = \{\frac12\mathrm{w}_1\beta_2\mathrm{w}_1, \omega_3\},\\
&H^4_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{\oplus 2} = \{\frac12\mathrm{w}_4, \frac12\mathrm{w}_1\beta_2\mathrm{w}_2\},\\
&H^5_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{\oplus 3} = \{\frac12\mathrm{w}_2\beta_2\mathrm{w}_2, \frac12\mathrm{w}_1(\beta_2\mathrm{w}_1)^2, \frac12\mathrm{w}_1p_1\},\\
&H^6_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{\oplus 5} = \{\frac12\mathrm{w}_6, \frac12\mathrm{w}_2\mathrm{w}_4, \frac12\mathrm{w}_2p_1,\\
&\qquad \frac12\mathrm{w}_2(\beta_2\mathrm{w}_1)^2, \frac12\mathrm{w}_1\beta_2\mathrm{w}_4\},\\
&H^7_{\rm a}(\mathcal{B}O_\infty;\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^{\oplus 5}\oplus\mathbb{Z}^{\oplus 2} = \{\frac12\mathrm{w}_1\mathrm{w}_2\mathrm{w}_4, \frac12\mathrm{w}_1(\beta_2\mathrm{w}_1)^3,\\
&\qquad \frac12\mathrm{w}_2\beta_2\mathrm{w}_4, \frac12\mathrm{w}_1(\beta_2\mathrm{w}_2)^2, \frac12\mathrm{w}_1p_1\beta_2\mathrm{w}_1, \omega_3p_1, \omega_7\}.
\end{aligned} \tag{H6}$$

Appendix I: Relations between cocycles and Stiefel-Whitney classes on a closed manifold

The cocycles and the Stiefel-Whitney classes on a closed manifold satisfy many relations. In this section, we will show how to generate those relations.

1. Introduction to Stiefel-Whitney classes

The Stiefel-Whitney classes $\mathrm{w}_i \in H^i(M^D;\mathbb{Z}_2)$ is defined for an O_n vector bundle on a d-dimensional space with $n \to \infty$. If the O_∞ vector bundle on d-dimensional space, M^D, happen to be the tangent bundle of M^D direct summed with a trivial ∞-dimensional vector bundle, then the corresponding Stiefel-Whitney classes are referred as the Stiefel-Whitney classes of the manifold M^D.

The Stiefel-Whitney classes of manifold behave well under the connected sum of manifolds. Let

$$\mathrm{w}(M) = 1 + \mathrm{w}_1(M) + \mathrm{w}_2(M) + \cdots \tag{I1}$$

be the total Stiefel-Whitney class of a manifold M. If we know $\mathrm{w}(M)$ and $\mathrm{w}(N)$, then we can obtain $\mathrm{w}(M\#N)$:

$$\mathrm{w}(M\#N) \overset{2,\mathrm{d}}{=} \mathrm{w}(M) + \mathrm{w}(N) - 1. \tag{I2}$$

Under the product of manifolds, we have

$$\mathrm{w}(M \times N) \overset{2,\mathrm{d}}{=} \mathrm{w}(M)\mathrm{w}(N). \tag{I3}$$

The Stiefel-Whitney numbers are non-oriented cobordism invariant. All the Stiefel-Whitney numbers of a smooth compact manifold vanish iff the manifold is the boundary of some smooth compact manifold. Here the manifold can be non-orientable.

The Stiefel-Whitney numbers and Pontryagin numbers are oriented cobordism invariant. All the Stiefel-Whitney numbers and Pontryagin numbers of a smooth compact orientable manifold vanish iff the manifold is the boundary of some smooth compact orientable manifold.

2. Relations between Stiefel-Whitney classes of the tangent bundle

For generic O_∞ vector bundle, the Stiefel-Whitney classes are all independent. However, the Stiefel-Whitney classes for a manifold (*i.e.* for the tangent bundle) are not independent and satisfy many relations.

To obtain those relations, we note that, for any O_∞ vector bundle, the total Stiefel-Whitney class $\mathrm{w} = 1 + \mathrm{w}_1 + \mathrm{w}_2 + \cdots$ is related to the total Wu class $u = 1 + u_1 + u_2 + \cdots$ through the total Steenrod square[70]:

$$\mathrm{w} \overset{2,\mathrm{d}}{=} Sq(u), \quad Sq = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots. \tag{I4}$$

Therefore, $\mathrm{w}_n \overset{2,\mathrm{d}}{=} \sum_{i=0}^n \mathrm{Sq}^i(u_{n-i})$. The Steenrod squares have the following properties:

$$\mathrm{Sq}^i(x_j) \overset{2,\mathrm{d}}{=} 0,\ i > j, \quad \mathrm{Sq}^j(x_j) \overset{2,\mathrm{d}}{=} x_j x_j, \quad \mathrm{Sq}^0 = 1, \tag{I5}$$

for any $x_j \in H^j(M^D;\mathbb{Z}_2)$. Thus

$$u_n \overset{2,\mathrm{d}}{=} \mathrm{w}_n + \sum_{i=1,2i\le n} \mathrm{Sq}^i(u_{n-i}). \tag{I6}$$

This allows us to compute u_n iteratively, using Wu formula

$$\begin{aligned} &\mathrm{Sq}^i(\mathrm{w}_j) \overset{2,\mathrm{d}}{=} 0, \quad i > j, \qquad \mathrm{Sq}^i(\mathrm{w}_i) \overset{2,\mathrm{d}}{=} \mathrm{w}_i\mathrm{w}_i, \\ &\mathrm{Sq}^i(\mathrm{w}_j) \overset{2,\mathrm{d}}{=} \mathrm{w}_i\mathrm{w}_j + \sum_{k=1}^{i} \frac{(j-i-1+k)!}{(j-i-1)!k!}\mathrm{w}_{i-k}\mathrm{w}_{j+k}, \quad i < j, \\ &\mathrm{Sq}^1(\mathrm{w}_j) \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_j + (j-1)\mathrm{w}_{j+1}, \end{aligned} \tag{I7}$$

and the Steenrod relation

$$\mathrm{Sq}^n(xy) \overset{2,\mathrm{d}}{=} \sum_{i=0}^n \mathrm{Sq}^i(x)\mathrm{Sq}^{n-i}(y). \tag{I8}$$

We find

$$\begin{aligned} &u_0 \overset{2,\mathrm{d}}{=} 1, \quad u_1 \overset{2,\mathrm{d}}{=} \mathrm{w}_1, \quad u_2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2 + \mathrm{w}_2, \\ &u_3 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_2, \quad u_4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4, \\ &u_5 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^3\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_2^2 + \mathrm{w}_1^2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_4, \\ &u_6 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2\mathrm{w}_2^2 + \mathrm{w}_1^3\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_4 + \mathrm{w}_2\mathrm{w}_4, \\ &u_7 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_3 + \mathrm{w}_1\mathrm{w}_3^2 + \mathrm{w}_1\mathrm{w}_2\mathrm{w}_4, \\ &u_8 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^8 + \mathrm{w}_2^4 + \mathrm{w}_1^2\mathrm{w}_3^2 + \mathrm{w}_1^2\mathrm{w}_2\mathrm{w}_4 + \mathrm{w}_1\mathrm{w}_3\mathrm{w}_4 + \mathrm{w}_4^2 \\ &\qquad + \mathrm{w}_1^3\mathrm{w}_5 + \mathrm{w}_3\mathrm{w}_5 + \mathrm{w}_1^2\mathrm{w}_6 + \mathrm{w}_2\mathrm{w}_6 + \mathrm{w}_1\mathrm{w}_7 + \mathrm{w}_8. \end{aligned} \tag{I9}$$

We note that the Steenrod squares form an algebra:

$$\begin{aligned} \mathrm{Sq}^a\mathrm{Sq}^b &= \sum_{j=0}^{[a/2]} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j, \\ &= \sum_{j=0}^{[a/2]} \frac{(b-j-1)!}{(a-2j)!(b-a+j-1)!}\mathrm{Sq}^{a+b-j}\mathrm{Sq}^j, \\ &0 < a < 2b. \end{aligned} \tag{I10}$$

which leads to the relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$.

If the O_∞ vector bundle on d-dimensional space, M^D, happen to be the tangent bundle of M^D, then the corresponding Wu class and the Steenrod square satisfy

$$\mathrm{Sq}^{D-j}(x_j) \overset{2,\mathrm{d}}{=} u_{D-j}x_j, \text{ for any } x_j \in H^j(M^D;\mathbb{Z}_2). \tag{I11}$$

We can generate many relations for cocycles and Stiefel-Whitney classes on a manifold using the above result:

1. If we choose x_j to be a combination of Stiefel-Whitney classes, plus the Sq operations them, the above will generate many relations between Stiefel-Whitney classes.

2. If we choose x_j to be a combination of Stiefel-Whitney classes and cocycles, plus the Sq operations them, the above will generate many relations between Stiefel-Whitney classes and cocycles.

As an application, we note that $\mathrm{Sq}^i(x_j) \overset{2,\mathrm{d}}{=} 0$ if $i > j$. Therefore $u_i x_{D-i} \overset{2,\mathrm{d}}{=} 0$ for any $x_{D-i} \in H^{D-i}(M^D;\mathbb{Z}_2)$ if $i > D-i$. Since $\mathbb{Z}_2$ is a field and according to the Poincaré duality, this implies that $u_i \overset{2,\mathrm{d}}{=} 0$ for $2i > D$. Also $\mathrm{Sq}^n \cdots \mathrm{Sq}^m(u_i) \overset{2,\mathrm{d}}{=} 0$ if $2i > D$. This also gives us relations among Stiefel-Whitney classes.

3. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 1-cocycle in 3-dimensions

On a 3-dimensional manifold, we can find many relations between Stiefel-Whitney classes:
(1) $u_2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2 + \mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$.
(2) $u_3 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$.
(3) $\mathrm{Sq}^1(u_2) \overset{2,\mathrm{d}}{=} 0$. Using $\mathrm{Sq}^1(\mathrm{w}_i) \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_i + (i+1)\mathrm{w}_{i+1}$, we find that $\mathrm{Sq}^1(\mathrm{w}_1^2 + \mathrm{w}_2) \overset{2,\mathrm{d}}{=} \mathrm{Sq}^1(\mathrm{w}_1)\mathrm{w}_1 + \mathrm{w}_1\mathrm{Sq}^1(\mathrm{w}_1) + \mathrm{Sq}^1(\mathrm{w}_2) \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0$.
This gives us three relations

$$\mathrm{w}_1^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_2, \quad \mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} \mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0. \tag{I12}$$

Let $a^{\mathbb{Z}_2}$ be a $\mathbb{Z}_2$-valued 1-cocycle. We can also find a relation between the Stiefel-Whitney classes and $a^{\mathbb{Z}_2}$:

$$\mathrm{w}_1(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{Sq}^1((a^{\mathbb{Z}_2})^2) \overset{2,\mathrm{d}}{=} 2(a^{\mathbb{Z}_2})^3 \overset{2,\mathrm{d}}{=} 0. \tag{I13}$$

There are six possible 3-cocycles that can be constructed from the Stiefel-Whitney classes and the 1-cocycle $a^{\mathbb{Z}_2}$:

$$\begin{array}{lll} (\mathrm{w}_1)^3, & \mathrm{w}_1\mathrm{w}_2, & \mathrm{w}_3, \\ (a^{\mathbb{Z}_2})^3, & \mathrm{w}_1(a^{\mathbb{Z}_2})^2, & \mathrm{w}_1^2 a^{\mathbb{Z}_2}. \end{array} \tag{I14}$$

From the above relations, we see that only two of them are non-zero:

$$(a^{\mathbb{Z}_2})^3, \qquad \mathrm{w}_1^2 a^{\mathbb{Z}_2}. \tag{I15}$$

4. Relations between Stiefel-Whitney classes and a $\mathbb{Z}_2$-valued 1-cocycle in 4-dimensions

The relations between the Stiefel-Whitney classes for 4-dimensional manifold can be listed:
(1) $u_3 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0$.
(2) $u_4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_1\mathrm{w}_3 + \mathrm{w}_4 \overset{2,\mathrm{d}}{=} 0$.
(3) $\mathrm{Sq}^1(u_3) \overset{2,\mathrm{d}}{=} 0$, which implies $\mathrm{Sq}^1(\mathrm{w}_1\mathrm{w}_2) \overset{2,\mathrm{d}}{=} \mathrm{Sq}^1(\mathrm{w}_1)\mathrm{w}_2 + \mathrm{w}_1\mathrm{Sq}^1(\mathrm{w}_2) \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_1^2\mathrm{w}_2 + \mathrm{w}_1\mathrm{w}_3 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0$,
which can be summarized as

$$\mathrm{w}_1\mathrm{w}_2 \overset{2,\mathrm{d}}{=} 0, \quad \mathrm{w}_1\mathrm{w}_3 \overset{2,\mathrm{d}}{=} 0, \quad \mathrm{w}_1^4 + \mathrm{w}_2^2 + \mathrm{w}_4 \overset{2,\mathrm{d}}{=} 0. \tag{I16}$$

We also have many relations between the Stiefel-Whitney classes and $a^{\mathbb{Z}_2}$:
(1) $\mathrm{Sq}^1((a^{\mathbb{Z}_2})^3) \overset{2,\mathrm{d}}{=} (a^{\mathbb{Z}_2})^4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1(a^{\mathbb{Z}_2})^3$.
(2) $\mathrm{Sq}^1(\mathrm{w}_1(a^{\mathbb{Z}_2})^2) \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1[\mathrm{w}_1(a^{\mathbb{Z}_2})^2]$.
(3) $\mathrm{Sq}^1(\mathrm{w}_1^2 a^{\mathbb{Z}_2}) \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^3 a^{\mathbb{Z}_2}$.
(4) $\mathrm{Sq}^1(\mathrm{w}_2 a^{\mathbb{Z}_2}) \overset{2,\mathrm{d}}{=} (\mathrm{w}_1\mathrm{w}_2 + \mathrm{w}_3)a^{\mathbb{Z}_2} + \mathrm{w}_2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1\mathrm{w}_2 a^{\mathbb{Z}_2}$, which implies that $\mathrm{w}_3 a^{\mathbb{Z}_2} \overset{2,\mathrm{d}}{=} \mathrm{w}_2(a^{\mathbb{Z}_2})^2$.
(5) $\mathrm{Sq}^2((a^{\mathbb{Z}_2})^2) \overset{2,\mathrm{d}}{=} (a^{\mathbb{Z}_2})^4 \overset{2,\mathrm{d}}{=} (\mathrm{w}_1^2 + \mathrm{w}_2)(a^{\mathbb{Z}_2})^2$.
(6) $\mathrm{Sq}^2(\mathrm{w}_1 a^{\mathbb{Z}_2}) \overset{2,\mathrm{d}}{=} \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} (\mathrm{w}_1^2 + \mathrm{w}_2)\mathrm{w}_1 a^{\mathbb{Z}_2} \overset{2,\mathrm{d}}{=} \mathrm{w}_1^3 a^{\mathbb{Z}_2}$, which is the same as (2).
To summarize

$$\begin{array}{ll} \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_1^3 a^{\mathbb{Z}_2}, & (a^{\mathbb{Z}_2})^4 \overset{2,\mathrm{d}}{=} \mathrm{w}_1(a^{\mathbb{Z}_2})^3, \\ \mathrm{w}_2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} \mathrm{w}_3 a^{\mathbb{Z}_2}, & (a^{\mathbb{Z}_2})^4 + \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2 + \mathrm{w}_2(a^{\mathbb{Z}_2})^2 \overset{2,\mathrm{d}}{=} 0. \end{array} \tag{I17}$$

There are nine 4-cocycles that can be constructed from Stiefel-Whitney classes and a 1-cocycle $a^{\mathbb{Z}_2}$:

$$\begin{array}{lll} (a^{\mathbb{Z}_2})^4, & \mathrm{w}_1(a^{\mathbb{Z}_2})^3, & \mathrm{w}_1^2(a^{\mathbb{Z}_2})^2, \\ \mathrm{w}_2(a^{\mathbb{Z}_2})^2, & \mathrm{w}_1^3 a^{\mathbb{Z}_2}, & \mathrm{w}_3 a^{\mathbb{Z}_2}, \\ \mathrm{w}_1^4, & \mathrm{w}_2^2, & \mathrm{w}_4. \end{array} \tag{I18}$$

Only four of them are independent

$$\mathrm{w}_1^4, \quad \mathrm{w}_2^2, \quad \mathrm{w}_3 a^{\mathbb{Z}_2}, \quad \mathrm{w}_1^3 a^{\mathbb{Z}_2}. \tag{I19}$$

Appendix J: Relation between Pontryagin classes and Stiefel-Whitney classes

There is result due to Wu that relate Pontryagin classes and Stiefel-Whitney classes (see Ref. 71 Theorem C): Let B be a vector bundle over a manifold X, w_i be its Stiefel-Whitney classes and p_i its Pontryagin classes. Let ρ_4 be the reduction modulo 4 and θ_2 be the embedding of $\mathbb{Z}_2$ into $\mathbb{Z}_4$ (as well as their induced actions on cohomology groups). Then

$$\mathcal{P}_2(\mathrm{w}_{2i}) \overset{4,\mathrm{d}}{=} p_i + 2\Big(\mathrm{w}_1 Sq^{2i-1}\mathrm{w}_{2i} + \sum_{j=0}^{i-1} \mathrm{w}_{2j}\mathrm{w}_{4i-2j}\Big), \tag{J1}$$

where $\mathcal{P}_2$ is the Pontryagin square, which maps $x \in H^{2n}(X,\mathbb{Z}_2)$ to $\mathcal{P}_2(x) \in H^{4n}(X,\mathbb{Z}_4)$. The Pontryagin square has a property that $\mathcal{P}_2(x) \overset{2}{=} x^2$. Therefore

$$\mathcal{P}_2(\mathrm{w}_{2i}) \overset{2,\mathrm{d}}{=} \mathrm{w}_{2i}^2 \overset{2,\mathrm{d}}{=} p_i. \tag{J2}$$

Appendix K: Spin and Pin structures

Stiefel-Whitney classes can determine when a manifold can have a spin structure. The spin structure is defined only for orientable manifolds. The tangent bundle for an orientable manifold M^d is a SO_d bundle. The group SO_d has a central extension to the group $Spin(d)$. Note that $\pi_1(SO_d) = \mathbb{Z}_2$. The group $Spin(d)$ is the double covering of the group SO_d. A spin structure on M^D is a $Spin(d)$ bundle, such that under the group reduction $Spin(d) \to SO_d$, the $Spin(d)$ bundle reduces to the SO_d

bundle. Some manifolds cannot have such a lifting from SO_d tangent bundle to the $Spin(d)$ spinor bundle. The manifolds that have such a lifting is called spin manifold. A manifold is a spin manifold iff its first and second Stiefel-Whitney class vanishes $\mathrm{w}_1 = \mathrm{w}_2 = 0$.

For a non-orientable manifold N^d, the tangent bundle is a O_d bundle. The non-connected group O_d has two nontrivial central extensions (double covers) by Z_2 with different group structures, denoted by $Pin^+(d)$ and $Pin^-(d)$. So the O_d tangent bundle has two types of lifting to a Pin^+ bundle and a Pin^- bundle, which are called Pin^+ structure and Pin^- structure respectively. The manifolds with such lifting are called Pin^+ manifolds or Pin^- manifolds. We see that the concept of $Pin^\pm$ structure applies to both orientable and non-orientable manifolds. A manifold is a Pin^+ manifold iff $\mathrm{w}_2 = 0$. A manifold is a Pin^- manifold iff $\mathrm{w}_2 + \mathrm{w}_1^2 = 0$. If a manifold N^d does admit Pin^+ or Pin^- structures, then the set of isomorphism classes of Pin^+-structures (or Pin^--structures) can be labeled by elements in $H^1(N^d; \mathbb{Z}_2)$. For example $\mathbb{R}P^4$ admits two Pin^+-structures and no Pin^--structures since $\mathrm{w}_2(\mathbb{R}P^4) = 0$ and $\mathrm{w}_2(\mathbb{R}P^4) + \mathrm{w}_1^2(\mathbb{R}P^4) \neq 0$.

From (I2), we see that $M\#N$ is Pin$^+$ iff both M and N are Pin$^+$. Similarly, $M\#N$ is Pin$^-$ iff both M and N are Pin$^-$.

Appendix L: Higher group as simplicial complex

1. Higher group and its classifying space

Given a *topological space* K, we can triangulate it and use the resulting *complex* $\mathcal{K}$ to model it. If K is connected, we can choose the complex $\mathcal{K}$ to have only one vertex. We can even choose the one-vertex complex to be a *simplicial set*. Such a simplicial set is called a *higher group* if various Kan conditions are satisfied and the corresponding space K is called the *classifying space* of the higher group. More precisely, $\mathcal{K}$ is an n-group ($n \in \{1, 2, \dots\} \sqcup \{\infty\}$), if $\mathcal{K}$ satisfies Kan conditions Kan(m, j), i.e. the natural horn projection $\mathcal{K}_m \xrightarrow{p_j^m} \Lambda_j^m(\mathcal{K})$ is surjective, for all $0 \le j \le m$; and strict Kan conditions Kan$(m, j)!$, i.e. $\mathcal{K}_m \xrightarrow{p_j^m} \Lambda_j^m(\mathcal{K})$ is isomorphic, for all $0 \le j \le m$ and $m \ge n + 1$. Here, $\Lambda_j^m(\mathcal{K})$ denotes the set of (m, j)-horns in $\mathcal{K}$. We will use $\mathcal{B}$ to denote a higher group (*i.e.* a simplicial set).

Let us describe an explicit construction for such a higher group $\mathcal{B}$. As a simplicial set, $\mathcal{B}$ is described by a set of vertices $[\mathcal{B}]_0$, a set of links $[\mathcal{B}]_1$, a set of triangles $[\mathcal{B}]_2$, etc. The complex $\mathcal{B}$ is formally described by

$$[\mathcal{B}]_0 \overset{d_0,d_1}{\leftleftarrows} [\mathcal{B}]_1 \overset{d_0,d_1,d_2}{\leftleftarrows} [\mathcal{B}]_2 \overset{d_0,\dots,d_3}{\leftleftarrows} [\mathcal{B}]_3 \overset{d_0,\dots,d_4}{\leftleftarrows} [\mathcal{B}]_4, \quad \text{(L1)}$$

where d_i are the face maps, describing how the n-simplices are attached to a $(n-1)$-simplex.

As the set of vertices, $[\mathcal{B}]_0 = \{pt\}$, *i.e.* there is only one vertex. An link in $[\mathcal{B}]_1$ is labeled simply by a label a_{01} whose end points are both this point pt. Such labels from a group G. An triangle in $[\mathcal{B}]_2$ is labeled by its three links a_{01}, a_{12}, a_{02}, and possibly an additional label b_{012}. Such additional labels form an Abelian group Π_2. Thus an triangle is labeled by $(a_{01}, a_{12}, a_{02}; b_{012})$. We introduce a compact notation

$$s[012] = (a_{01}, a_{12}, a_{02}; b_{012}) \quad \text{(L2)}$$

to denote such a triangle. Similarly, an generic d-simplex is labeled by a label-set $s[0\cdots d] = (a_{ij}; b_{ijk}; \cdots)$.

We see that a higher group $\mathcal{B}$ in this model consists the data of a collection of groups $G, \Pi_2, \Pi_3, \cdots$, where G can be non-Abelian and Π_i's are Abelian. Both G and Π_i can be discrete or continuous. We denote such a higher group as $\mathcal{B}(G, 1; \Pi_2, 2; \Pi_3, 3; \cdots)$. With such a labeling of the simplices, such as $s[012] = (a_{01}, a_{12}, a_{02}; b_{012})$, the face map d_i can be expressed simply

$$\begin{aligned} d_0(a_{01}, a_{12}, a_{02}; b_{012}) &= a_{12}, \\ d_1(a_{01}, a_{12}, a_{02}; b_{012}) &= a_{02}, \\ d_2(a_{01}, a_{12}, a_{02}; b_{012}) &= a_{01}, \end{aligned} \quad \text{(L3)}$$

or

$$d_m s[0\cdots d] = s[0\cdots \hat{m} \cdots d] \quad \text{(L4)}$$

where $\hat{m}$ means that the m index is removed.

However, in order for the label-set $s[0\cdots d]$ to label a d-simplex in complex $\mathcal{B}(G, 1, \Pi_2, 2; \Pi_3, 3; \cdots)$, the labels a_{ij}, etc., in the set $s[0\cdots d]$ must satisfy certain conditions. Those conditions determine the structure of a higher group $\mathcal{B}(G, 1; \Pi_2, 2; \Pi_3, 3; \cdots)$. Such constructed higher group is a triangulation of a topological space K. Different higher groups give us a classification of homotopy types of topological spaces.

From our labeling of the simplices $s[0\cdots d] = (a_{ij}; b_{ijk}; \cdots)$, we can also introduce the canonical cochains on the higher group $\mathcal{B}(G, 1, \Pi_2, 2; \Pi_3, 3; \cdots)$. The canonical G-valued 1-cochain a is given by its evaluation on 1-simplices $s[01] = a_{01}$:

$$\langle a, a_{01} \rangle = a_{01}, \quad a_{01} \in G. \quad \text{(L5)}$$

The canonical Π_2-valued 2-cochain b is given by its evaluation on 2-simplices $s[012] = (a_{01}, a_{12}, a_{02}; b_{012})$:

$$\langle b, (a_{01}, a_{12}, a_{02}; b_{012}) \rangle = b_{012}, \quad b_{012} \in \Pi_2. \quad \text{(L6)}$$

The canonical Π_n-valued n-cochain x^n can be defined in a similar fashion.

The conditions satisfied by the labels a_{ij}, b_{ijk}, etc., in the set $s[0\cdots d]$ can be expressed as the conditions on those canonical cochains. In other words, we start with a chain complex of groups

$$G \xleftarrow{q_2} \Pi_2 \xleftarrow{q_3} \Pi_3 \xleftarrow{q_4} \dots \xleftarrow{q_k} \Pi_k,$$

and group actions $G \xrightarrow{\alpha_j} Aut(\Pi_j)$, where q_i are G-equivariant with G acting on G trivially, and

$$\alpha_j(a_{pq}a_{qr}a_{pr}^{-1}) = id. \tag{L7}$$

Then the structure and the definition of a higher group $\mathcal{B}_{n_3,\ldots,n_{k+1}}(G,1;\Pi_2,2;\ldots;\Pi_k,k)$ can be formulated via the conditions on the canonical cocycles inductively: given $k-1$ cocycles

$$\begin{aligned} &n_3 \in Z^3(G,(\Pi_2^0)^{\alpha_2}),\\ &n_4 \in Z^4(\mathcal{B}_{n_3}(G,1;\Pi_2,2),(\Pi_3^0)^{\alpha_3}),\\ &\ldots \\ &n_{k+1} \in Z^{k+1}(\mathcal{B}_{n_3,\ldots,n_k}(G,1;\Pi_2,2;\ldots;\Pi_{k-1},k-1),(\Pi_k^0)^{\alpha_k}) \end{aligned} \tag{L8}$$

where $\Pi_j^0 := \ker q_j \subset \Pi_j$ for $j=2,\ldots,k$,

$$\begin{aligned} X_d := \{&s[0\ldots d] = (x^1_{01},x^1_{02},\ldots,x^1_{d-1d};\\ &x^2_{012},\ldots,x^2_{(d-2)(d-1)d};\ldots;x^d_{0\ldots d})|\\ &x^1_{..} \in G, x^j_{..} \in \Pi_j,\\ &\mathrm{d}_{\alpha_j}x^j = q_{j+1}(x^{j+1}) + n_{j+1}(x^1;x^2;\ldots;x^{j-1}),\\ &\forall j=2,3,\ldots,d, \text{ and } \mathrm{d}x^1 = q_2(x^2).\} \end{aligned} \tag{L9}$$

is a k-group. Here we take all $\Pi_{\geq k+1}=0$ (thus $q_{\geq k+1}=0$) and all $n_{\geq k+2}=0$ in the general definition of X_d. Here

$$\begin{aligned} \mathrm{d}_{\alpha_j}x^j(s[0\ldots j+1] :&= \alpha_j(a_{01})\cdot x^j(s[1\ldots j+1])\\ &- x^j(s[02\ldots j+1]) + \ldots \end{aligned} \tag{L10}$$

Equation (L7) guarantees that $d_{\alpha_j}\circ d_{\alpha_j}=0$.

We notice that $\mathcal{B}_{n_3}(G,1;\Pi_2,2)$ is the 2-group constructed via cocycles n_3 with equations $dx^1 = q_2(x^2)$ and $\mathrm{d}_{\alpha_2}x^2 = n_3(x^1)$. But it is not in contradiction with the equation $\mathrm{d}_{\alpha_2}x^2 = q_3(x^3)+n_3(x^1)$ in the definition of X_d. First of all, n_3 is a cocycle, therefore it is possible that $\mathrm{d}_{\alpha_2}x^2 = n_3(x^1)$ has solutions. Secondly, $\mathrm{d}_{\alpha_2}x^2 = q_3(x^3)+n_3(x^1)$ is describing another set of solutions in X_d, which is also possible to be there. Why? If we apply d_{α_2} to it, we have $\mathrm{d}_{\alpha_2}q_3(x^3)=0$, but this holds naturally, since $\mathrm{d}_{\alpha_2}q_3(x^3) = q_3(\mathrm{d}_{\alpha_3}x^3) = q_3(n_4(x^1,x^2))=0$, no matter we have $\mathrm{d}_{\alpha_3}x^3 = q_4(x^4)+n_4(x^1,x^2)$ or $\mathrm{d}_{\alpha_3}x^3 = n_4(x^1,x^2)$. Thirdly, from both equations, we have $q_2(n_3(x^1)) = q_2(\mathrm{d}_{\alpha_2}x^2) = \mathrm{d}q_2(x^2) = \mathrm{d}\mathrm{d}x^1 = 0$, which is also fine since n_3 takes value in $\Pi_2^0=\ker q_2$. We thus can further understand these equations, which are not contradict to each other, inductively for higher k's.

Now let us prove that what we construct satisfies desired Kan conditions, therefore is a higher group. Notice that the horn space $\Lambda_j^m(X)$ has the same $(m-2)$-skeleton as X_m, thus to verify the Kan condition Kan(m,j), we only need to take care of $(m-1)$-faces. Since there is no non-trivial $\geq k+1$ faces, it is clear that Kan$(\geq k+2,j)$! are satisfied. Then Kan$(k+1,j)$! are satisfied for $0\leq j\leq k+1$ because the following equation,

$$\mathrm{d}_{\alpha_k}x^k = n_{k+1}(x^1;x^2;\ldots;x^{k-1}), \tag{L11}$$

implies that as long as we know any $k+1$ out of $k+2$ k-faces in the $(k+1)$-simplex $s[01\ldots k+1]$, then the other one is determined uniquely. Similarly, Kan$(m+1,j)$ are satisfied for $0\leq j\leq m<k$ because the following equation,

$$\mathrm{d}_{\alpha_m}x^m = q_{m+1}(x^{m+1}) + n_{m+1}(x^1;x^2;\ldots;x^{m-1}),$$

implies that any $m+1$ out the $m+2$ m-faces in the $(m+1)$-simplex $s[01\ldots m+1]$ determines the other one up to a choice of $q_{m+1}(x^{m+1})$. Thus we can always fill the $(m+1,j)$-horn and we have unique filling if and only if $q_{m+1}=0$.

Moreover, if two sets of canonical two chains n_j^X's and n_j^Y's differ by coboundaries valued in $\ker q_j$, then they define weak equivalent k-groups. More precisely, this is an inductive process: if $n_3^Y - n_3^X = \mathrm{d}n_2'$, and $n_2' \in \ker q_2$, then we let $f_2: x^2 \mapsto x^2 + n_2(x_1)$ and $f_1 = f_0 = id$; further using this truncated simplicial homomorphism, if $f^*n_4^Y - n_4^X = \mathrm{d}n_3'$, and $n_3' \in \ker q_3$, then we let $f_3: x^3 \mapsto x^3 + n_3'(x_1,x_2)$; further using $f_0,\ldots,f_3$, if $f^*n_5^Y - n_5^X = \mathrm{d}n_4'$, and $n_4' \in \ker q_4$, then we let $f_5: x^5 \mapsto x^5 + n_4'(x_1,x_2,x_3)$; $\ldots$ in the end, we obtain a simplicial homomorphism f made up by automorphisms f_j of $G^j\times\Pi_2^{\binom{j}{2}}\times\Pi_3^{\binom{j}{3}}\times\cdots\times\Pi_j$. This simplicial morphism is a weak equivalence $f: X\to Y$ of higher groups. Here X is the higher group defined by canonical chains n_j^X and Y the one defined by n_j^Y. This is because f introduces isomorphisms on homotopy groups of X and Y. Notice that X and Y both have the same homotopy groups: $\pi_0=0$, $\pi_1 = G/\mathrm{Im}q_2$, $\pi_2 = \Pi_2/\mathrm{Im}q_3$, $\ldots$. The construction above makes sure that f is a simplicial homomorphism, and it gives rise to isomorphisms when passing to homotopy groups.

Therefore, if we fix other canonical cocycles, up to weak equivalence, we may take $n_{k+1} \in H^{k+1}(\mathcal{B}_{n_3,\ldots,n_k}(G,1;\Pi_2,2;\ldots;\Pi_{k-1},k-1),(\Pi_k^0)^{\alpha_k})$, and if $q_k=0$, we may further assume that $n_{k+1} \in H^{k+1}(\mathcal{B}_{n_3,\ldots,n_k}(G,1;\Pi_2,2;\ldots;\Pi_{k-1},k-1),{\Pi_k}^{\alpha_k})$.

2. 3-group

In the following, we discuss a 3-group $\mathcal{B}(G,1,\Pi_2,2;\Pi_3,3)$ in more details. The missing labels $\Pi_n, n|_{n>3}$ mean that $\Pi_n = |_{n>3}0$. In order for $(a_{01},a_{12},a_{02};b_{012})$ to label a triangle in the complex, a_{01},a_{12},a_{02} must satisfy

$$(\delta a)_{012} \equiv a_{01}a_{12}a_{02}^{-1} = q_2(b_{012}). \tag{L12}$$

In terms of canonical cochains, the above condition can be rewritten as

$$\delta a = q_2(b). \tag{L13}$$

Here q_2 is a group homomorphism $q_2:\Pi_2\to G$. So only a_{01} and a_{12}, b_{012} are independent. The triangles in the complex are described by independent labels

$$s[012] = [a_{01},a_{12};b_{012}]. \tag{L14}$$

Therefore the set of triangles is given by $G^{\times 2}\times\Pi_2$. If a_{01}, a_{12}, a_{02} do not satisfy the above condition, then the three links a_{01}, a_{12}, a_{02} simply do not bound a triangle (*i.e.* there is a hole).

Similarly, for a tetrahedron $s[0123]$, the labels a_{ij} in the label-set $s[0123]$ all satisfy (L12) if we replace 012 by $i<j<k$. There an additional condition

$$(\mathrm{d}_{\alpha_2}b)_{0123} \equiv \alpha_2(a_{01})\cdot b_{123}-b_{023}+b_{013}-b_{012} \\ = q_3(c_{0123})+n_3(a_{01},a_{12},a_{23}), \qquad \text{(L15)}$$

where q_3 is a group homomorphism $q_3:\Pi_3\to\Pi_2$, α_2 is a group homomorphism $\alpha_2: G\to \mathrm{Aut}(\Pi_2)$, and $n_3\in Z^3(\mathcal{B}(G,1),(\Pi_2^0)^{\alpha_2})$. In terms of canonical cochains, the above can be rewritten as

$$\mathrm{d}_{\alpha_2}b=q_3(c)+n_3(a) \qquad \text{(L16)}$$

We see that a tetrahedron are described by independent labels

$$s[0123]=[a_{01},a_{12},a_{23};b_{012},b_{013},b_{023};c_{0123}]. \qquad \text{(L17)}$$

Therefore the set of tetrahedrons is given by $G^{\times 3}\times\Pi_2^{\times 3}\times\Pi_3$.

For a 4-simplex $s[01234]$, the labels a_{ij} and b_{ijkl} in the label-set $s[01234]$ all satisfy (L12) and (L15). There an additional condition

$$(\mathrm{d}_{\alpha_3}c)_{01234} \\ \equiv \alpha_3(a_{01})\cdot c_{1234}-c_{0234}+c_{0134}-c_{0124}+c_{0123} \\ = n_4(a_{01},a_{12},a_{23},a_{34},b_{012},b_{013},b_{023}) \qquad \text{(L18)}$$

where α_3 is a group homomorphism $\alpha_3: G\to\mathrm{Aut}(\Pi_3)$ and $n_4\in Z^4(\mathcal{B}(G,1;\Pi_2,2),(\Pi_3^0)^{\alpha_3})$ is a closed cochain. In terms of canonical cochains, the above can be rewritten as

$$\mathrm{d}_{\alpha_3}c=n_4(a,b). \qquad \text{(L19)}$$

In general, the 3-group $\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3)$ has the following sets of simplices:

$$pt \underset{}{\overset{d_0,d_1}{\Leftarrow}} G \overset{d_0,\dots,d_2}{\Lleftarrow} G^{\times 2}\times\Pi_2 \overset{d_0,\dots,d_3}{\longleftarrow} G^{\times 3}\times\Pi_2^{\times 3} \overset{d_0,\dots,d_4}{\longleftarrow} G^{\times 4}\times\Pi_2^{\times 6}\times\Pi_3 \overset{d_0,\dots,d_5}{\longleftarrow} G^{\times 5}\times\Pi_2^{\times 10}\times\Pi_3^{\times 10}\dots \qquad \text{(L20)}$$

The d-simplices form a set $G^{\times d}\times\Pi_2^{\times\binom{d}{2}}\times\Pi_3^{\times\binom{d}{3}}$. The d-simplices in $G^{\times d}\times\Pi_2^{\binom{d}{2}}\times\Pi_3^{\times\binom{d}{3}}$ are labeled by $\{a_{ij},b_{ijk},c_{ijkl}\}$, $i,j,k,l=0,1,\cdots,d$, that satisfy the conditions (L12) (after replacing 012 by $i<j<k$), (L15) (after replacing 0123 by $i<j<k<l$) and (L18) (after replacing 01234 by $i<j<k<l<m$).

We find that 3-groups $\mathcal{B}(G;\Pi_2;\Pi_3)$ are classified by the following data

$$G;\ \Pi_2,q_2,\alpha_2,n_3;\ \Pi_3,q_3,\alpha_3,n_4 \qquad \text{(L21)}$$

where G is a group, Π_2,Π_3 are Abelian groups, α_2,α_3 are group actions $\alpha_2: G\to\mathrm{Aut}(\Pi_2)$ and $\alpha_3: G\to\mathrm{Aut}(\Pi_3)$, $n_3\in Z^3(\mathcal{B}(G,1),(\Pi_2^0)^{\alpha_2})$, and $n_4\in Z^4(\mathcal{B}(G;\Pi_2);(\Pi_3^0)^{\alpha_3})$. When n_3,n_4 differ by a coboundary valued in $\ker q_2$, $\ker q_3$ respectively, the 3-groups are weak equivalent.

3. 3-group cocycle

In the following, we give an explicit description of 3-group cocycles, which are the cocycles on the complex $\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3)$. First, a d-dimensional 3-group cochain ν_d with value $\mathbb{M}$ is a function $\omega_d: G^{\times d}\times\Pi_2^{\binom{d}{2}}\times\Pi_3^{\binom{d}{3}}\to\mathbb{M}$. Then, using the face map (L4), we can define the differential operator d acting on the 3-group cochains as the following:

$$(\mathrm{d}\omega_d)(s[0\cdots d+1])=\sum_{m=0}^{d+1}(-)^m\omega_d(s[1\cdots\hat{m}\cdots d+1]), \qquad \text{(L22)}$$

With the above definition of d operator, we can define the 3-group cocycles as the 3-group cochains that satisfy $\mathrm{d}\omega_d=0$. Two different 3-group cocycles $\mathrm{d}\omega_d$ and $\mathrm{d}\omega_d'$ are equivalent if they are different by a 3-group coboundary $\mathrm{d}\nu_{d-1}$. The set of equivalent classes of d-dimensional 3-group cocycles is denoted by $H^d(\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3),\mathbb{M})$, which in fact forms an Abelian group.

In the above, we gave a quite general definition of k-group. In more standard definition, q_i is chosen to be $q_i=0$. Such $q_i=0$ k-group will be denoted by $\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3;\cdots)$. Its homotopy group are given by

$$\pi_n\big(\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3;\cdots)\big)=\Pi_n,\quad \Pi_0=G. \qquad \text{(L23)}$$

Usually, we can use the canonical cochains a,b,c *etc* to construct the cocycles on $\mathcal{B}(G,1;\Pi_2,2;\Pi_3,3)$. For example, on a 3-group $\mathcal{B}_0(G,1;\Pi_2,2;\Pi_3,3)$ defined via its canonical cochains: $\delta a=1$, $\mathrm{d}b=0$, $\mathrm{d}c=0$, a 3-group cocycle ω_d will be a cocycle on $\mathcal{B}_0(G,1;\Pi_2,2;\Pi_3,3)$. The expressions $b, b^2, \mathrm{Sq}^3 c$, *etc* are also cocycles on $\mathcal{B}_0(G.1;\Pi_2,2;\Pi_3,3)$.

4. Continuous group

The above discussions apply to both discrete and continuous groups. However, in order to construct principle bundle or higher principle bundle on space-time M, it is not enough to consider only strict simplicial homomorphisms from space-time complex $\mathcal{M}$ to $\mathcal{G}(G,1;\Pi_2,2;\dots)$ when G is a continuous group. The reason is that, for example, in the case of $\mathcal{G}(SU_2,1)=\mathcal{B}SU_2$, strict simplicial homomorphisms $\phi_{\text{strict}}:\mathcal{M}\to\mathcal{B}SU_2$ can only produce trivial principal SU_2-bundles on $M=|\mathcal{M}|$, which is the geometric realization of $\mathcal{M}$. We thus need to allow generalised morphisms ϕ_{gen} from $\mathcal{M}$ to $\mathcal{G}(G,1;\Pi_2,2;\dots)$, so that their pullback can produce non-trivial higher principal bundles on $\mathcal{M}$.

Let us explain this via an example: a generalized morphism $\mathcal{M}\xrightarrow{\phi}\mathcal{B}SU_2$ consists of a zigzag, $\mathcal{M}\xleftarrow{\chi}\tilde{\mathcal{M}}\xrightarrow{\tilde\phi}\mathcal{B}SU_2$, where both χ and $\tilde\phi$ are strict simplicial homomorphisms and χ is a weak equivalence. Here, we define $X\to Y$ between simplicial topological spaces being a weak equivalence if and only if their geometric realization $|X|$ and $|Y|$ are weakly homotopy equivalent (namely all homotopy groups are the same). Homotopy equivalence clearly implies weak homotopy equivalence. This coincides with usual weak equivalence between simplicial sets when both X and Y are simplicial sets (taking discrete topology). Then to present an SU_2-principal bundle P on $\mathcal{M}$, we take a good cover $\{U_\alpha\}$ (that is, all sorts of finite intersections $\cap U_\alpha$ are contractible), where P is trivial on each U_α. Then we take $\tilde{\mathcal{M}}$ to be the Cech groupoid $\sqcup_\alpha U_\alpha \Leftarrow \sqcup_{\alpha\beta}U_\alpha\cap U_\beta\dots$. The set of vertices of $\tilde{\mathcal{M}}$ is the set of points in all U_α's, *i.e.* the disjoined union $\sqcup_\alpha U_\alpha$. The set of links in $\tilde{\mathcal{M}}$ is given by the set of the points $x_{\alpha\beta}$ in $U_\alpha\cap U_\beta$, imagined as links linking $x_\alpha\in U_\alpha$ and $x_\beta\in U_\alpha$ which are actually the same point x in M but in different covers. The set of triangles in $\tilde{\mathcal{M}}$ is given by the set of the points $x_{\alpha\beta\gamma}\in U_\alpha\cap U_\beta\cap U_\gamma$. A point in $U_\alpha\cap U_\beta\cap U_\gamma$ can be viewed as a triple $x_\alpha\in U_\alpha$, $x_\beta\in U_\beta$, and $x_\gamma\in U_\gamma$, that correspond to the same point in M. The homomorphism $\tilde\phi$ is determined by $\tilde\phi_1$ with $\tilde\phi_1(x_{\alpha\beta})=a_{\alpha\beta}(x_{\alpha\beta})$, where $x_{\alpha\beta}\in U_\alpha\cap U_\beta$ and $a_{\alpha\beta}(x_{\alpha\beta})\in G=SU_2$ are the transition functions to glue P.

We take the so called abstract nerve $N(\tilde{\mathcal{M}})$ of the covering simplicial space $\tilde{\mathcal{M}}$, which is constructed as following: $N(\tilde{\mathcal{M}})_0$ is the index set I of the cover $\{U_\alpha\}$. We denote a vertex by v_α with $\alpha\in I$. A d-simplex $s[0,\dots,d]$ is a set $\{v_{\alpha_0},\dots,v_{\alpha_d}\}$ with $d\ge 0$ and $\alpha_0,\dots,\alpha_d\in I$, such that $U_{\alpha_0}\cap\dots\cap U_{\alpha_d}\neq\emptyset$. It is clear that there is a map $\tilde{\mathcal{M}}\xrightarrow{\chi'}N(\tilde{\mathcal{M}})$ by mapping all points in $U_{\alpha_0}\cap\dots\cap U_{\alpha_d}$ exactly back to the simplex $s[0,\dots,d]$. As long as M is paracompact and $\{U_i\}$ is a good cover, as we assumed, Borsuk Nerve Theorem ensures that M and $|N(\tilde{\mathcal{M}})|$ are homotopy equivalent. Segal[72] proved that in $|\tilde{\mathcal{M}}|$ and M are homotopy equivalent. Thus $|N(\tilde{\mathcal{M}})|$ and $|\tilde{\mathcal{M}}|$ are homotopy equivalent. Thus χ' is a weak equivalence.

On the other hand, from the simplicial set $\mathcal{M}$ which corresponds to a simplicial decomposition of M, we create a cover $U_{\mathcal{M}}$ by the star construction: we denote S the set of simplices in the simplicial decomposition given by $\mathcal{M}$. For $\sigma\in S$, the star of σ defined by $U_\sigma:=\cup_{\sigma\subset s,s\in S}s^0$ the union of interior of simplices having σ as a subface, is an open set in M. Clearly, for two vertices v_i and v_j, $U_{v_i}\cap U_{v_j}\neq\emptyset$ if and only if there is an edge e_{ij} linking v_i and v_j, and in this case, $U_{v_i}\cap U_{v_j}=U_{e_{ij}}$. In general, $U_{v_0}\cap\dots U_{v_d}\neq\emptyset$ if and only if there is a simplex $s[0,\dots,d]$ with vertices $v_0,\dots,v_d$, and in this case, $U_{v_0}\cap\dots\cap U_{v_d}=U_{[0,\dots,d]}$. It is then clear if we take the cover in $\tilde{\mathcal{M}}$ to be this particular open cover (which is a good cover), namely stars of all vertices in $\mathcal{M}$, then $N(\tilde{\mathcal{M}})$ is exactly $\mathcal{M}$. Thus we obtain a weak equivalence $\mathcal{M}\xleftarrow{\chi'}\tilde{\mathcal{M}}$.

Now the only problem left is that this cover $U_{\mathcal{M}}$ might not be fine enough to create non-trivial principal bundles. But this can be easily solved by taking a refinement of $U_{\mathcal{M}}$ on which transition functions glue to the desired principal bundle P. We denote the Cech groupoid of the refinement by $\hat{\mathcal{M}}$. Then there is a weak equivalence between Cech groupoids $\tilde{\mathcal{M}}\xleftarrow{\cong}\hat{\mathcal{M}}$. Then we have weak equivalences $\mathcal{M}\xleftarrow{\cong}\tilde{\mathcal{M}}\xleftarrow{\cong}\hat{\mathcal{M}}$. Thus we realize P as a generalised morphism, $\mathcal{M}\xleftarrow{\cong}\hat{\mathcal{M}}\to\mathcal{B}SU_2$.

Thus in our article, when we talk about homomorphisms between simplicial objects, we understand them as this correct version of morphisms, namely generalised morphisms when G is continuous or strict simplicial homomorphisms when G is discrete.

Appendix M: Calculate $a^3 \underset{1}{\smile} a^3$

Let a be a 1-cochain. We have

$$\langle a^3\underset{1}{\smile}a^n,(012\cdots n+2)\rangle \overset{2}{=} \sum\langle a^3,A_1\cup A_3\rangle\langle x_n,A_2\rangle$$
$$\overset{2}{=}(a^3)_{0,n,n+1,n+2}(a^n)_{01\cdots n}+(a^3)_{0,1,n+1,n+2}(a^n)_{12\cdots n+1}$$
$$+(a^3)_{0,1,2,n+2}(a^n)_{23\cdots n+2}$$
$$\overset{2}{=}a_{01}a_{12}\cdots a_{n+1,n+2}(a_{0n}+a_{1,n+1}+a_{2,n+2}). \tag{M1}$$

Thus

$$a^3\underset{1}{\smile}a^n \overset{2}{=}(a\underset{1}{\smile}a^n)a^2+a(a\underset{1}{\smile}a^n)a+a^2(a\underset{1}{\smile}a^n) \tag{M2}$$

When a is a $\mathbb{Z}_2$-valued 1-cocycle, the above allows us to obtain

$$a^3\underset{1}{\smile}a^3 \overset{2}{=}(a\underset{1}{\smile}a^3)a^2+a(a\underset{1}{\smile}a^3)a+a^2(a\underset{1}{\smile}a^3). \tag{M3}$$

Using (A20), we find $a\underset{1}{\smile}a^3\overset{2}{=}a^3\underset{1}{\smile}a$, and

$$a^3\underset{1}{\smile}a\overset{2}{=}(a\underset{1}{\smile}a)a^2+a(a\underset{1}{\smile}a)a+a^2(a\underset{1}{\smile}a)\overset{2}{=}a^3 \tag{M4}$$

This allows us to show

$$a^3 \underset{1}{\smile} a^3 = \mathrm{Sq}^2 a^3 \overset{2}{=} a^5. \tag{M5}$$

More generally, we can show that for $\mathbb{Z}_2$-valued 1-cocycles a_1 and a_2,

$$\mathrm{Sq}^2(a_1 a_2^2) \overset{2}{=} a_1 a_2^4. \tag{M6}$$

Appendix N: Group extension and trivialization

Consider an extension of a group H

$$A \to G \to H \tag{N1}$$

where A is an Abelian group with group multiplication given by $x + y \in A$ for $x, y \in A$. Such a group extension is denoted by $G = A \leftthreetimes H$. It is convenient to label the elements in G as (h, x), where $h \in H$ and $x \in A$. The group multiplication of G is given by

$$(h_1, x_1)(h_2, x_2) = (h_1 h_2, x_1 + \alpha(h_1) \circ x_2 + e_2(h_1, h_2)). \tag{N2}$$

where e_2 is a function

$$e_2 : H \times H \to A, \tag{N3}$$

and α is a function

$$\alpha : H \to \mathrm{Aut}(A). \tag{N4}$$

We see that group extension is defined via e_2 and α. The associativity

$$[(h_1, x_1)(h_2, x_2)](h_3, x_3) = (h_1, x_1)[(h_2, x_2)](h_3, x_3)] \tag{N5}$$

requires that

$$\begin{aligned} &x_1 + \alpha(h_1) \circ x_2 + e_2(h_1, h_2) + \alpha(h_1 h_2) \circ x_3 + e_2(h_1 h_2, h_3) \\ &= \alpha(h_1) \circ x_2 + \alpha(h_1)\alpha(h_2) \circ x_3 + \alpha(h_1) \circ e_2(h_2, h_3)] \\ &\quad + x_1 + e_2(h_1, h_2 h_3) \end{aligned} \tag{N6}$$

or

$$\alpha(h_1)\alpha(h_2) = \alpha(h_1 h_2) \tag{N7}$$

and

$$\begin{aligned} &e_2(h_1, h_2) - e_2(h_1, h_2 h_3) + e_2(h_1 h_2, h_3) \\ &- \alpha(h_1) \circ e_2(h_2, h_3) = 0. \end{aligned} \tag{N8}$$

Such a e_2 is a group 2-cocycle $e_2 \in H^2(\mathcal{B}H; A_\alpha)$, where H has a non-trivial action on the coefficient A as described by α. Also, α is a group homomorphism $\alpha : H \to \mathrm{Aut}(A)$. We see that the A extension from H to G is described by a group 2-cocycle e_2 and a homomorphism α. Thus we can more precisely denote the group extension by $G = A \leftthreetimes_{e_2, \alpha} H$.

Note that the homomorphism $\alpha : H \to \mathrm{Aut}(A)$ is in fact the action by conjugation in G,

$$\begin{aligned} &(h, 0)(\mathbf{1}, x) = (h, \alpha(h) \circ x) = (\mathbf{1}, \alpha(h) \circ x)(h, 0), \\ &\quad \Rightarrow (h, 0)(\mathbf{1}, x)(h, 0)^{-1} = (\mathbf{1}, \alpha(h) \circ x). \end{aligned} \tag{N9}$$

Thus, α is trivial if and only if A lies in the center of G. This case is called a central extension, where the action α will be omitted.

Our way to label group elements in G:

$$g = (h, x) \in G \tag{N10}$$

defines two projections of G:

$$\begin{aligned} &\pi : G \to H, \quad \pi(g) = h, \\ &\sigma : G \to A, \quad \sigma(g) = x. \end{aligned} \tag{N11}$$

π is a group homomorphism while σ is a generic function. Using the two projections, $g_1 g_2 = g_3$ can be written as

$$\begin{aligned} &(\pi(g_1), \sigma(g_1))(\pi(g_2), \sigma(g_2)) \\ &= [\pi(g_1)\pi(g_2), \sigma(g_1) + \alpha(\pi(g_1)) \circ \sigma(g_2) + e_2(\pi(g_1), \pi(g_2))] \\ &= (\pi(g_3), \sigma(g_3)) = (\pi(g_1 g_2), \sigma(g_1 g_2)) \end{aligned} \tag{N12}$$

We see that the group cocycle $e_2(h_1, h_2)$ in $H^2(\mathcal{B}H; A)$ can be pullback to give a group cocycle $e_2(\pi(g_1), \pi(g_2))$ in $H^2(\mathcal{B}G; A)$, and such a pullback is a coboundary

$$e_2(\pi(g_1), \pi(g_2)) = -\sigma(g_1) + \sigma(g_1 g_2) - \alpha(\pi(g_1)) \circ \sigma(g_2), \tag{N13}$$

i.e. an element in $B^2(\mathcal{B}G; A_\alpha)$, where G has a non-trivial action on the coefficient A as described by α.

The above result can be put in another form. Consider the homomorphism

$$\varphi : \mathcal{B}G \to \mathcal{B}H \tag{N14}$$

where $G = A \leftthreetimes_{e_2} H$, and e_2 is a A-valued 2-cocycle on $\mathcal{B}H$. The homomorphism φ sends an link of $\mathcal{B}G$ labeled by $a^G_{ij} \in G$ to an link of $\mathcal{B}H$ labeled by $a^H_{ij} = \pi(a^G_{ij}) \in H$. The pullback of e_2 by φ, $\varphi^* e_2$, is always a coboundary on $\mathcal{B}G$

The above discussion also works for continuous group, if we only consider a neighborhood near the group identity $\mathbf{1}$. In this case, $e_2(h_1, h_2)$ and $\alpha(h)$ are continuous functions on such a neighborhood. But globally, $e_2(h_1, h_2)$ and $\alpha(h)$ may not be continuous functions.

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