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Luttinger liquid theory of one-dimensional quantum systems ignores exponentially weak backscattering of particles. This endows Luttinger liquids with superfluid properties. The corresponding two-fluid hydrodynamic description available at present applies only to Galilean-invariant systems, whereas most experimental realizations of one-dimensional quantum liquids lack Galilean invariance. Here we develop the two-fluid theory of such quantum liquids. In the low-frequency limit the theory reduces to single-fluid hydrodynamics. However, the absence of Galilean invariance brings about three new transport coefficients. We obtain expressions for these coefficients in terms of the backscattering rate.

Introduction.— One-dimensional quantum liquids are commonly described in the framework of the so-called Luttinger liquid theory [1, 2]. In the simplest form of this theory the elementary excitations are non-interacting bosons. In this approximation the system does not equilibrate. Interactions between the excitations arise from corrections to the Luttinger liquid Hamiltonian, which are irrelevant perturbations in the renormalization group sense. The resulting scattering processes give rise to relaxation of the system to thermal equilibrium. The irrelevant perturbations scale as powers of energy, and thus the corresponding relaxation rate τ_{ex}^{-1} scales as a power of temperature T. Importantly, the collisions generated by these irrelevant perturbations conserve not only the number of particles, momentum, and energy of the system, but also an additional quantity J, which was first introduced in Ref. [1]. It was shown recently [3–5] that as a result Luttinger liquids behave as superfluids and, similarly to superfluid ^{4}He [6, 7], can be described by two-fluid hydrodynamics.

The reason the system behaves as a superfluid can be understood as follows. The physical meaning of the conserved quantity J is most transparent for quantum liquids composed of fermions, in which case J is the difference of the numbers of right- and left-moving particles. At $J \neq 0$ even in the absence of elementary excitations the quantum liquid has a finite momentum $p_F J$ and velocity $u_0 = p_F J/mN$. Here p_F is the Fermi momentum, and m and N are the mass and number of particles. At non-zero temperature the elementary excitations form a gas that moves with its own velocity u_{ex}. Thus, similar to superfluid ^{4}He, Luttinger liquid exhibits two types of macroscopic motion, and its flow should be described by two-fluid hydrodynamics.

Luttinger liquid theory accounts only for the low-energy excitations of the system. On the other hand, the scattering processes involving excitations with energies of order of the bandwidth D can backscatter fermions and thus violate conservation of J. The most efficient pro-cesses of this type involve holes near the bottom of the band [8], and thus have a rate that is exponentially small at low temperatures, $\tau^{-1} \propto e^{-D/T}$ [9–11]. Because of that the superfluid behavior of one-dimensional quantum liquids is limited to the frequency range $\tau^{-1} \ll \omega \ll \tau_{\mathrm{ex}}^{-1}$. At the lowest frequencies $\omega \ll \tau^{-1}$, backscattering processes lead to equilibration of the velocities, $u_{\mathrm{ex}} - u_0 \to 0$, and the quantum liquid behaves as a normal fluid.

Dynamics of one-dimensional quantum liquids at frequencies $\omega \ll \tau_{\mathrm{ex}}^{-1}$ is described by the two-fluid hydrodynamic theory [3–5], which was obtained by adapting the Landau's theory of superfluidity of liquid ^{4}He [6, 7] to one dimension. An important limitation of this approach is the assumption that the system is Galilean-invariant. On the other hand, for quantum liquids realized in solid state systems the underlying crystalline lattice plays an important role and usually leads to violation of Galilean invariance. Development of two-fluid hydrodynamic theory of one-dimensional quantum liquids in this regime is our main goal.

An important property of Galilean-invariant systems is that the particle number current j_n is given by the momentum density p divided by the particle mass m. Therefore, the collisions between the particles, which conserve momentum, do not affect j_n. This is in contrast with the energy current j_ε, which in addition to a contribution proportional to p contains a dissipative contribution, typically expressed in terms of the thermal conductivity and temperature gradient. In the absence of Galilean invariance j_n is not uniquely defined by momentum density and, similar to j_ε, has a dissipative component. The particle and energy currents appear in response to gradients of chemical potential μ and temperature T. In a linear approximation the dissipative parts of j_n and j_ε are related to the gradients $\partial_x \mu$ and $\partial_x T$ by a matrix of four transport coefficients, similar to thermoelectric coefficients in conductors [12]. Our two-fluid theory enables us to express these transport coefficients in terms of the relaxation time τ.

Luttinger liquid.—For simplicity, we consider a quantum liquid of spinless particles. In the Luttinger liquid approximation it is described in terms of the bosonic fields $\varphi(x)$ and $\theta(x)$, satisfying the commutation relation

$$[\varphi(x), \partial_{x'}\theta(x')] = i\pi\delta(x - x'). \tag{1}$$

The fields are subject to the boundary conditions

$$\varphi(L) = \varphi(0), \quad \theta(L) = \theta(0) - \pi J, \tag{2}$$

where L is the size of the system. The second condition formally defines the conserved quantity J, which for systems of fermions is the difference of the numbers of right- and left-moving particles. It will be convenient to also define a field $\vartheta(x)$, which satisfies periodic boundary conditions $\vartheta(L) = \vartheta(0)$ and is related to $\theta(x)$ by

$$\partial_x\theta = -\frac{\chi}{\hbar} + \partial_x\vartheta, \quad \chi = \pi\hbar\frac{J}{L}. \tag{3}$$

Here $\hbar$ is the Planck's constant, while χ has the dimension of momentum and is proportional to the density of the conserved quantity J.

The momentum of the Luttinger liquid can be expressed in terms of the bosonic fields as [1, 13]

$$P = -\hbar\int dx\left(n + \frac{\partial_x\varphi}{\pi}\right)\partial_x\theta = N\chi - \frac{\hbar}{\pi}\int dx\,\partial_x\varphi\,\partial_x\vartheta. \tag{4}$$

Here we denote the total number of particles by N, while $n = N/L$ is the average density. The first form of Eq. (4) is clear from the physical meaning of the bosonic fields: $\partial_x\varphi(x)/\pi$ is the density of particles at point x, measured from n, whereas $-\hbar\,\partial_x\theta(x)$ is the momentum of the liquid per particle. In the second form of Eq. (4) for a liquid of spinless fermions $N\chi$ can be expressed in terms of the Fermi momentum $p_F = \pi\hbar n$ as $p_F J$ and represents the momentum of the ground state with unequal numbers of the right- and left-moving particles. The remaining integral of $-(\hbar/\pi)\partial_x\varphi\,\partial_x\vartheta$ accounts for the momentum of the elementary excitations.

The Hamiltonian of the Luttinger liquid can be presented in the form [1, 2]

$$H = \int dx\left\{\frac{\hbar v}{2\pi}\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\varphi)^2\right]\right.$$
$$\left. + \alpha_\theta\partial_x\varphi(\partial_x\theta)^2 + \alpha_\varphi(\partial_x\varphi)^3 + \dots\right\}, \tag{5}$$

where v is the velocity of the elementary excitations and K is the Luttinger liquid constant. The coefficients α_θ and α_φ can be expressed in terms of the dependences of v and K on density as [11]

$$\alpha_\theta = \frac{\hbar}{2\pi^2}\,\partial_n\,(vK), \quad \alpha_\varphi = \frac{\hbar}{6\pi^2}\,\partial_n\left(\frac{v}{K}\right). \tag{6}$$

We omitted from the Hamiltonian (5) terms with scaling dimensions exceeding three.

In most applications the Hamiltonian of the Luttinger liquid is approximated by the first line of Eq. (5). In this case the elementary excitations are noninteracting bosons. The energy of the boson with momentum q is $\epsilon_q = v|q|$ [1, 2]. We will need to account for a correction to this expression arising at small but finite χ. To this end we substitute Eq. (3) into the cubic term in Eq. (5) proportional to α_θ. To linear order in χ this generates a correction to the quadratic Hamiltonian in the first line of Eq. (5) proportional to the momentum of the elementary excitations. As a result, the energy spectrum takes the form

$$\epsilon_q = v|q| + \frac{\chi}{m^*}q. \tag{7}$$

Here we have introduced the effective mass

$$m^* = \frac{\pi\hbar}{\partial_n(vK)}. \tag{8}$$

In the Galilean-invariant case the parameters v and K are not independent. Specifically, $vK = v_F$ where the Fermi velocity $v_F = \pi\hbar n/m$. In this case the effective mass (8) coincides with the mass m of the particles comprising the Luttinger liquid. Equation (7) can then be interpreted as the Galilean transformation of the original spectrum $\epsilon_q = v|q|$ to the frame moving with velocity χ/m.

Thermal equilibrium.—We now consider an equilibrium state of the Luttinger liquid. It is described by the number of particles N, the value of J, and the boson occupation numbers N_q, which take the usual Bose form

$$N_q = \left[\exp\left(\frac{\epsilon_q - uq}{T}\right) - 1\right]^{-1}. \tag{9}$$

The parameter u appears as a consequence of the conservation of momentum and can be thought of as the velocity of the gas of excitations. It is worth noting that the occupation numbers (9) implicitly depend on N and J, because the velocity of the excitations v in Eq. (7) is a function of density, while $\chi \propto J$.

Using the distribution function (9) it is straightforward to obtain the thermodynamic properties of the system to leading order in T/D. Below we will need expressions for the energy and momentum densities of the Luttinger liquid. We will limit our consideration to low-velocity flows and neglect contributions beyond linear order in u and χ. The energy density ε is obtained by adding contributions $\epsilon_q N_q$ of all excitations, resulting in

$$\varepsilon = \frac{\pi T^2}{6\hbar v}. \tag{10}$$

We obtain the momentum density p using the second form of Eq. (4), in which the second term is given by the sum of the contributions qN_q of all bosonic states. This yields

$$p = n\chi + \rho_{\rm ex}\left(u - \frac{\chi}{m^*}\right), \quad \rho_{\rm ex} = \frac{\pi T^2}{3\hbar v^3}. \tag{11}$$

We earlier interpreted u as the velocity of the gas of bosonic excitations of the Luttinger liquid. Given that the corresponding contribution to the momentum density is $\rho_{\rm ex}u$, one can interpret $\rho_{\rm ex}$ as the mass density of the gas of excitations.

Hydrodynamics of Luttinger liquids.—Let us now consider the dynamics of the Luttinger liquid at time scales much longer than $\tau_{\rm ex}$. In this regime the liquid is in local thermal equilibrium at every point. As a result, its state is fully described by the densities of the four conserved quantities: number of particles, energy, momentum, and J. Time evolution of these densities is described by four continuity equations expressing the conservation laws:

$$\partial_t n + \partial_x j_n = 0, \tag{12}$$
$$\partial_t \varepsilon + \partial_x j_\varepsilon = 0, \tag{13}$$
$$\partial_t p + \partial_x j_p = 0, \tag{14}$$
$$\partial_t \chi + \partial_x j_\chi = 0, \tag{15}$$

where we used χ defined in (3) instead of the density J/L. In order to complete the hydrodynamic description of the system we need to evaluate the currents j_n, j_ε, j_p, and j_χ. We will restrict ourselves to terms up to linear order in small parameters $\chi(x)$, $u(x)$, and $\delta n(x) = n(x) - N/L$.

To evaluate the particle current j_n we write the Heisenberg equation of motion for the operator of particle density $n(x) = N/L + \partial_x \varphi(x)/\pi$. This yields the evolution equation (12) in the operator form with

$$j_n(x) = \frac{i}{\pi\hbar}[\varphi(x), H] = -\frac{vK}{\pi}\partial_x \theta - \frac{2\alpha_\theta}{\hbar}\partial_x \varphi\, \partial_x \theta. \tag{16}$$

Next we substitute $\partial_x \theta$ in the form (3) and evaluate the expectation value of the operator (16) in the equilibrium state of the Luttinger liquid. Introducing a new effective mass

$$m_0 = \frac{\pi\hbar n}{vK} \tag{17}$$

and noting that the operator $(-\hbar/\pi)\partial_x \varphi \partial_x \vartheta$ is the momentum density of the bosonic excitations (see above), we obtain

$$j_n = \frac{n\chi}{m_0} + \frac{\rho_{\rm ex}}{m^*}\left(u - \frac{\chi}{m^*}\right). \tag{18}$$

For Galilean-invariant systems $vK = v_F$, and thus $m_0 = m$. In this case j_n must coincide with the ratio of momentum density p and the particle mass m. This is easily verified by substituting $m_0 = m^* = m$ into Eqs. (11) and (18).

The above procedure can be extended to the evaluation of the remaining three currents. To obtain j_ε one can write the equation of motion for the operator of Hamiltonian density, keeping both the quadratic and cubic terms in Eq. (5). After thermal averaging one obtains

$$j_\varepsilon = 2\varepsilon u + \varepsilon\,\frac{n\partial_n v}{v}\,\frac{\chi}{m_0}. \tag{19}$$

The evaluation of the momentum current j_p starts with the equation of motion for the operator of momentum density $-\hbar(n + \partial_x \varphi/\pi)\partial_x \theta$, cf. Eq. (4). The result is

$$j_p = m_0 v^2 \delta n + \varepsilon\,\frac{\partial_n(nv)}{v}. \tag{20}$$

Finally, the current j_χ is obtained by averaging the equation of motion for the operator $\partial_x \theta$ and using Eq. (3). This yields

$$j_\chi = \frac{m_0 v^2}{n}\delta n + \varepsilon\,\frac{\partial_n v}{v}. \tag{21}$$

It is easy to show that the above expressions for j_p and j_χ coincide with the pressure Π and chemical potential μ of the Luttinger liquid, respectively.

Compared to conventional hydrodynamics, the above theory includes an additional equation (15) originating from the conservation of J. In the Galilean-invariant case the resulting theory was interpreted [3–5] as two-fluid hydrodynamics, fully analogous to that of superfluid ^{4}He in three dimensions [6, 7]. The normal component of the fluid in this analogy is the gas of excitations characterized by velocity u, while the superfluid component describes the Fermi surface and moves with velocity χ/m. In the absence of Galilean invariance the full set of two-fluid hydrodynamic equations of the Luttinger liquid is given by the four continuity equations (12)–(15) along with the constitutive relations (18)–(21). The latter were derived to linear order in the deviations from the static equilibrium state, in which δn, χ and u vanish. To this approximation the system is described by two effective masses, m^* and m_0. The Galilean invariant limit is recovered at $m^* = m_0 = m$.

Hydrodynamics of one-dimensional quantum liquids.— As discussed above, Luttinger liquid approximation neglects the exponentially weak backscattering processes, which relax J to its equilibrium value. The latter is determined by the velocity u and can be found as follows. To leading order at $T \to 0$ the value of J is determined by the Gibbs distribution $w_J \propto e^{-(E - uP)/T}$, where $E = \pi\hbar vKJ^2/2L$ is the ground state energy of the Luttinger liquid, obtained by substituting Eq. (3) into the first line of Eq. (5), and the momentum $P = \pi\hbar NJ/L$ is obtained from Eq. (11). The maximum of this distribution gives the equilibrium value $J = uN/vK$ or, equivalently, $\chi = m_0 u$.

At time scales much longer than $\tau_{\rm ex}$ the deviation of the system from equilibrium is described by $\chi - m_0 u \neq 0$. For small deviations, relaxation to equilibrium proceeds with the exponentially small rate τ^{-1} [9–11], following the usual relaxation law

$$\frac{d}{dt}(\chi - m_0 u) = -\frac{1}{\tau}(\chi - m_0 u). \tag{22}$$

The relaxation processes obey the remaining three conservation laws. In particular, the momentum density is

conserved, $dp/dt = 0$. Using the expression (11), we obtain a linear relation between the $d\chi/dt$ and du/dt. This enables us to find $d\chi/dt$. We then find the time evolution equation for χ by substituting $d\chi/dt$ into the right-hand side of Eq. (15),

$$\partial_t \chi + \partial_x j_\chi = -\frac{1}{\tau} \frac{\rho_{\mathrm{ex}}}{m_0 n} (\chi - m_0 u). \qquad (23)$$

Here we accounted only for the leading at small T contribution in the right-hand side.

Equation (23) completes our generalizaiton of two-fluid hydrodynamics of one-dimensional quantum liquids to the non-Galilean-invariant case. The dynamics of the system at frequencies $\omega \ll \tau_{\mathrm{ex}}^{-1}$ is fully described by Eqs. (12)–(14) and (23) along with the constitutive relations (18)–(21). At $\tau^{-1} \ll \omega \ll \tau_{\mathrm{ex}}^{-1}$ the right-hand side of Eq. (23) can be neglected and the system is described by Eqs. (12)–(15). In this regime the system supports two sound modes, similar to the Galilean-invariant case [3–5]. On the other hand, at low frequencies $\omega \ll \tau^{-1}$ qualitatively new physics arises.

Dissipation in non-Galilean-invariant fluids.—In the low-frequency limit the backscattering processes are very effective at bringing the system to equilibrium characterized by only three conserved quantities: number of particles, energy, and momentum. The quantity J is no longer conserved; in a uniform fluid it takes the equilibrium value uN/vK. In a non-uniform fluid, the system approaches a local equilibrium at every point in space, with $\chi(x) = m_0 u(x)$. However, full local equilibrium cannot be achieved in a non-uniform system, i.e., the deviation from equilibrium $\delta\chi = \chi - m_0 u$ acquires a finite value proportional to the gradients of physical parameters of the system, such as the temperature and chemical potential. As a result, the system behaves as a conventional fluid described the hydrodynamic equations (12)–(14), with the constitutive relations containing dissipative contributions proportional to $\partial_x T$ and $\partial_x \mu$.

We now apply our two-fluid description to evaluate dissipative components of the currents. We start by using Eq. (11) to bring $\delta\chi$ to the form $\delta\chi = (m_0/\rho_{\mathrm{ex}})\{[n + \rho_{\mathrm{ex}}(m_0^{-1} - m^{*-1})]\chi - p\}$. We then use Eqs. (14) and (23) to express $\partial_t \delta\chi$ as a linear combination of $\delta\chi$, $\partial_x j_p = \partial_x \Pi$, and $\partial_x j_\chi = \partial_x \mu$. At low frequencies the time derivative $\partial_t \delta\chi \sim \omega\delta\chi \ll \delta\chi/\tau$ and can be neglected. This enables us to express $\delta\chi$ as

$$\delta\chi = \tau \frac{m_0 s}{\rho_{\mathrm{ex}}} \partial_x T - \tau \left(1 - \frac{m_0}{m^*}\right) \partial_x \mu. \qquad (24)$$

Here we kept only the leading at $T \to 0$ terms and used the thermodynamic relation $\partial_x \Pi = n\partial_x \mu + s\partial_x T$ to express the gradient of pressure in terms of the gradients of temperature and chemical potential, with $s = \pi T/3\hbar v$ being the entropy density of the Luttinger liquid.

The particle and energy currents given by Eqs. (18) and (19) are linear in u and χ. One can get further insight into the physics associated with these currents by expressing them instead in terms of the momentum density (11) and $\delta\chi = \chi - m_0 u$. Due to conservation of momentum, the parts of j_n and j_ε proportional to p are not affected by the relaxation processes and represent the equilibrium contributions to the currents. The components of the currents proportional to $\delta\chi$ correspond to dissipative contributions. In the low-frequency limit, the latter can be expressed in terms of the gradients of temperature and chemical potential with the aid of Eq. (24). This yields

$$j_n = \frac{p}{m_0} - \gamma_{11}\partial_x \mu - \gamma_{12}\frac{\partial_x T}{T}, \qquad (25)$$

$$j_\varepsilon = \frac{p}{m_0}\frac{\partial_n(n^2 v)}{n^2 v}\varepsilon - \gamma_{21}\partial_x \mu - \gamma_{22}\frac{\partial_x T}{T}, \qquad (26)$$

where

$$\gamma_{11} = \tau\rho_{\mathrm{ex}}\left(\frac{1}{m^*} - \frac{1}{m_0}\right)^2, \quad \gamma_{22} = 2\tau\varepsilon v^2,$$

$$\gamma_{12} = \gamma_{21} = 2\tau\varepsilon\left(\frac{1}{m^*} - \frac{1}{m_0}\right).$$

Hydrodynamics of one-dimensional quantum liquids at $\omega \ll \tau^{-1}$ is fully described by the set of three continuity equations (12)–(14) along with the expressions (20), (25), and (26) for the currents.

In the Galilean-invariant case, $m^* = m_0 = m$, the dissipative contribution to the particle current vanishes, and Eq. (25) recovers the expected relation $j_n = p/m$. The energy current contains both the equilibrium and dissipative contributions, with the latter defining the thermal conductivity of the liquid $\kappa = \gamma_{22}/T$, cf. Ref. [14]. In the absence of Galilean invariance the dissipation is described by the matrix of four coefficients γ_{ij}. The latter relates dissipative components of j_n and j_ε to $\partial_x \mu$ and $\partial_x T/T$, and is analogous to the martix of thermoelectric coefficients in conductors [12]. The coefficient γ_{11} is analogous to electrical conductivity. The off-diagonal martix elements satisfy the Onsager relation, $\gamma_{12} = \gamma_{21}$.

Summary.—We have developed a hydrodynamic theory of non-Galilean-invariant one-dimensional quantum liquids, which applies at frequencies $\omega \ll \tau_{\mathrm{ex}}^{-1}$. The fluid is described by the three usual continuity equations (12)–(14), expressing conservation of particle number, energy, and momentum, and the additional evolution equation (23). At small deviations from equilibrium the dynamics of the system is characterized by two effective masses, m^* and m_0, both of which coincide with the particle mass in the Galilean-invariant case. At the lowest frequencies, $\omega \ll \tau^{-1}$, our theory reduces to ordinary hydrodynamics. However, the absence of Galilean invariance results in the emergence of new transport coefficients, analogous to conductivity and thermoelectric coefficients.

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