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Phys. Rev. Applied **6**, 014017 — Published 29 July 2016

DOI: 10.1103/PhysRevApplied.6.014017

Electric Power Generation from Earth's Rotation Through its Own Magnetic Field

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(Dated: June 27, 2016)

Abstract

We examine electric power generation from Earth's rotation through its own non-rotating magnetic field (that component of the field symmetric about Earth's rotation axis). There is a simple general proof that this is impossible. However, we identify a loophole in that proof and show that voltage could be continuously generated in a low-magnetic-Reynolds number conductor rotating with Earth, provided magnetically permeable material were used to ensure $\mathrm{curl}(\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$ within the conductor, where $\mathbf{B_0}$ derives from the axially symmetric component of Earth's magnetic flux density and $\mathbf{v}$ is Earth's rotation velocity at the conductor's location. We solve the relevant equations for one laboratory realization, and from this solution predict voltage magnitude and sign dependence on system dimensions and orientation relative to Earth's rotation. The effect, which would be available nearly globally with no intermittency, requires testing and further examination to see if it could be scaled to practical emission–free power generation.

I. INTRODUCTION

Barnett showed in 1912 that when an axially symmetric electromagnet rotates about its north-south axis, its magnetic field does not rotate with the magnet [1], resolving a controversy [2] that had its origins in Faraday's interpretation of his rotating disk experiment [3, 4]. Here, in Section II we first review Faraday's results and carefully address the definition of electromotive force (emf) and the historical meaning of saying a magnetic field "rotates with the magnet." We then describe the compelling experimental evidence for the non-rotation of axially symmetric magnetic fields. In Section III, we consider the particular case of the non-rotation of the axisymmetric component of Earth's magnetic field. The rotation of Earth's surface through that non-rotating component yields a steady $\mathbf{v} \times \mathbf{B}$ force that one might hope to use to generate electric power. However, in Section IV we present a simple, and seemingly general, proof that power generation in this way is impossible.

Nonetheless, in Section V we show that this proof has a loophole, suggesting that continuous power generation is possible if two unusual conditions are both met. These two conditions will not simultaneously hold in any typical natural or laboratory circuit, but it is possible to create them together. The first condition is that the current path must lie within a magnetically permeable conductor the topology of which is such that $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq \mathbf{0}$ in its interior, where $\mathbf{B}$ is the magnetic flux density and $\mathbf{v}$ is the velocity of the conductor. The second is that this conductor must have a magnetic Reynolds number $R_m \ll 1$, which on a laboratory scale excludes all common metal and mu-metal conductors.

In Sections VI to IX, we fully calculate one realization of such a system: a low-R_m magnetically permeable cylindrical shell. Section VI first considers the case when the shell is stationary ($\mathbf{v} = \mathbf{0}$) with respect to a constant background magnetic field (with zero background electric field). Then the current density $\mathbf{J} = \mathbf{0}$ and it is straightforward to derive the corresponding magnetic flux density $\mathbf{B_0}$ within the shell. We prove that in general, $\nabla \times (\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$, so that if the shell's composition, dimensions, and velocity can be chosen to yield $R_m \ll 1$, the system would fulfill the necessary conditions for emf generation.

In Sections VII through IX, we demonstrate that for this system these conditions are also sufficient. In Section VII, we show that if the shell is put into motion transverse to its long axis, $\mathbf{B_0}$ can no longer be a solution. We calculate the $\mathbf{B}$ that does satisfy the induction equation within the moving shell when $\mathbf{v} \neq \mathbf{0}$ and $R_m \ll 1$, and find that the time-dependent part of $\mathbf{B}$ goes to zero extremely rapidly. In Section VIII, we find that there remains a time-independent solution, given by $\mathbf{B_0}$ plus a series of perturbation terms scaled by successive powers of R_m. We use these results in Section IX to derive an expression for the emf generated in the shell. Section X provides an intuitive discussion of the results of the previous calculations.

In Section XI, we present a parallel analysis in a frame co-moving with the translating shell, and demonstrate that there is a net non-zero Poynting vector flux delivering power into the shell. We show in Section XII that in the frame in which the shell is translating at $\mathbf{v}$, a magnetic braking term arises in Poynting's theorem, and the generated electrical power equals the braking loss from Earth's rotational kinetic energy.

We make quantitative predictions for this system in Section XIII, including the striking prediction that the voltage generated should change sign when the cylindrical shell (together with its attached leads and voltmeter) is rotated by 180°. Section XIV begins the discussion of whether such systems might be scaled up to generate useful amounts of electric power.

II. HISTORICAL BACKGROUND AND DEFINITIONS

In December 1831, Faraday experimented with a conducting disk rotating near a magnet [3, 4]. The disk connected via brushes to a simple galvanometer, with leads running to the disk's axle and edge. The galvanometer circuit was stationary in the laboratory. Current flowed when the magnet was stationary and the disk rotated, or when the disk and magnet rotated coaxially along the magnet's north-south axis of symmetry; but not when the magnet rotated about this axis and the disk was stationary [3, 5].

Faraday subsequently experimented with a rotating conducting magnet connected to a galvanometer via brushes on the magnet's axle and rim [5, 6]. Current flowed when the magnet rotated around its north-south axis but the galvanometer circuit remained stationary, or when the magnet was stationary but the circuit rotated. Subsequent researchers have explored additional permutations in the configurations of Faraday's experiments [7].

In modern terms, the conducting magnet rotates at velocity $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$ (for angular velocity $\boldsymbol{\omega}$ and cylindrical radius $\boldsymbol{\rho}$) through its own magnetic field $\mathbf{H}$ (or equivalently, through its own magnetic flux density $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability), generating a $\mathbf{v} \times \mathbf{B}$ Lorentz force that drives the current.

The electromotive force (emf) around a path C with line element $\mathbf{dl}$ is given by [8–10]:

$$\text{emf} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} = \int_S [-\partial \mathbf{B}/\partial t + \nabla \times (\mathbf{v} \times \mathbf{B})] \cdot \mathbf{da}, \tag{1}$$

where $\mathbf{E}$ is the electric field, and the area element $\mathbf{da}$ is right-hand normal to the surface S bounded by C. The second equality in Eq. (1) holds via Stokes' theorem and the Faraday's law Maxwell equation provided there is no jump discontinuity on S [11]. This condition will be met in our work below, and we will calculate the emf using Eq. (1), which we take as the definition of the term.

For the Faraday disk, for which $\mathbf{B}$ is spatially constant and $\partial \mathbf{B}/\partial t = 0$, only the $\mathbf{v} \times \mathbf{B}$ term contributes to the integral. Were the entire circuit rotating at constant $\boldsymbol{\omega}$, the curl of $\mathbf{v} \times \mathbf{B}$ would be zero and we would have emf $= 0$. But because the galvanometer circuit is stationary while the disk rotates in the laboratory frame, the line integral of $\mathbf{v} \times \mathbf{B}$ around C is nonzero. In any frame at least part of C is in motion. A Poynting theorem analysis of the Faraday disk shows that the energy for the electric current flowing between axle and rim in the disk comes from the disk's kinetic energy of rotation [12]. Taking into account the small magnetic perturbations to the applied $\mathbf{B}$ due to the current that flows in C does not change these conclusions [12].

The emf in Eq. (1) is often identical to an electromotive force defined by the "flux rule":

$$\text{emf}_\Phi = -d\Phi/dt, \tag{2}$$

where magnetic flux $\Phi = \int_S \mathbf{B} \cdot \mathbf{da}$. Inequality between emf and emf$_\Phi$ in Eqs. (1) and (2) in certain circumstances give rise to so-called Faraday paradoxes. Auchmann et al. [11], consistent with some earlier discussions [13], show that equality requires the path velocity of the moving surface S (and its boundary C) to be equal to the material velocity of the conducting medium in which S is embedded. Our applications below meet this requirement.

Faraday concluded from his experiments that magnetic field lines do not rotate with a magnet when the magnet rotates around its axis of symmetry [3, 4, 6]. But Preston [2] showed in 1885 that Faraday's results are equally explained if the magnetic field does rotate with the magnet, producing a $\mathbf{v} \times \mathbf{B}$ force on the stationary part of C, giving an emf identical to that of a non-rotating field with the rotating disk. The idea of the field "rotating with the magnet" was understood [2, 14, 15] to mean that a force $q\mathbf{v} \times \mathbf{B}$ would be experienced by an electric charge q if q had a velocity $\mathbf{v}$ *relative to axes fixed in (so co-rotating with) the* *rotating magnet.* This differs from the current understanding of the $q\mathbf{v} \times \mathbf{B}$ force, in which $\mathbf{v}$ is the velocity of q in the frame in which the magnetic flux density is $\mathbf{B}$ [13, 16].

Poincaré [17] and others [18] asserted that since both the rotating and non-rotating pictures appeared to give identical results, the distinction between them was meaningless. But Barnett's experiments in 1912 [1], reproduced by Kennard [15] and improved upon by Pegram in 1917 [19], demonstrated a difference and resolved the question for the magnetic fields of electromagnets by using an open circuit. Barnett placed a cylindrical capacitor axially in the field of a solenoid (or, in analogous experiments, between two large iron flat-pole electromagnets); a thin wire connected the two concentric cylinders of the capacitor. Co-rotation of the cylinders and their connecting wire while holding the solenoid stationary charged the capacitor (due to the $\mathbf{v} \times \mathbf{B}$ force on the wire). After charging, the connecting wire was disconnected, the system despun, and opposite charges on the cylinders were measured by electrometer. But rotating the solenoid (or flat-pole electromagnets) while holding the cylindrical capacitor and connecting wire stationary generated no charge. Co-rotation of the capacitor and connecting wire together with the solenoid charged the capacitor [19]. Barnett and his contemporaries thereby proved that the field of a rotating axially symmetric electromagnet does not itself rotate [1, 5, 15, 18, 19].

III. NON-ROTATION OF EARTH'S AXISYMMETRIC FIELD

The Barnett [1], Kennard [15], and Pegram [19] experiments with electromagnets suggest that those components of Earth's magnetic field that are axisymmetric about Earth's rotation axis will be stationary with respect to (do not rotate with) the rotating Earth, understood in the sense described for rotating electromagnets in the previous section [20–22]. These would be, for example, the axially symmetric dipole, quadrupole, and octopole components, with coefficients in the usual Schmidt-normalized Legendre-function expansion of g_1^0, g_2^0 and g_3^0, respectively [23]. Components with coefficients g_n^m or h_n^m where $m \neq 0$ depend on azimuthal angle φ like $\cos(m\varphi)$ and $\sin(m\varphi)$, respectively, and therefore rotate with Earth.

To our knowledge, no experiment has been performed that demonstrates that Earth's axisymmetric field does not rotate with Earth. Non-rotation of Earth's axisymmetric field is the conservative expectation given the experimental results for electromagnets [1, 5, 15, 19], and this has been taken to be the case by many authors [20–22, 24]. Were the effect we predict in this paper to be demonstrated, an ancillary consequence would be an experimental demonstration of the non-rotation of Earth's axisymmetric field.

Earth's rotation carries its surface through the non-rotating component of Earth's magnetic flux density $\mathbf{B}$ with an azimuthal speed $\mathrm{v} = 465 \sin\theta$ m s^{-1} at colatitude θ. The resulting $\mathbf{v} \times \mathbf{B}$ force generates position-dependent volume charge densities (of order 1 e^{-} m^{-3} [22]) whose electric field perfectly cancels $\mathbf{v} \times \mathbf{B}$ [20, 24, 25]. A resulting latitude-dependent surface charge density maintains overall charge balance, with a corresponding electric potential at Earth's surface. Any additional motion of individual conductors or conducting fluids leads to continuous extremely rapid charge redistribution with resulting perfect cancellation of fields.

However, Earth is surrounded by a conducting ionosphere co-rotating with Earth. Does this external conducting spherical shell mean that Earth's axisymmetric magnetic field is somehow "dragged" into co-rotation with Earth? One might imagine that this is an implication of Alfvén's "frozen-flux" theorem [26], which considers Ohm's law (for current density $\mathbf{J}$) for a moving conductor

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{J}/\sigma \tag{3}$$

in the limit $\sigma \to \infty$ (a so-called perfect conductor), so that $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$. Then Eqs. (1) and (2) imply that the magnetic flux Φ cannot change through the surface S as C moves along — in the usual picturesque language, the flux is "frozen in."

But consider an axially symmetric conductor rotating with angular speed ω about the axis of Earth's axisymmetric magnetic field. Clearly $\partial\mathbf{B}/\partial t = 0$ in such a case. This is the first term in the integrand of the surface integral in Eq. (1). In spherical polar coordinates (r, θ, φ) we have $\mathbf{v} = \omega r \sin\theta \hat{\varphi}$ and find

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = -\omega \partial\mathbf{B}/\partial\varphi, \tag{4}$$

using $\nabla \times (\mathbf{v} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{B} + \mathbf{v}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{v})$ and $\nabla \cdot \mathbf{B} = 0$. Therefore the second term in the integrand in Eq. (1) is also zero due to axisymmetry, and by Eq. (2) this means $d\Phi/dt = 0$. But this has nothing to do with $\sigma \to \infty$; it would be just as true for a very poor conductor as for a perfect conductor. It is therefore not a consequence of the "frozen-flux" theorem. It is simply a consequence of the symmetry involved, and there is no reason to view the field as somehow being dragged around with the ionosphere.

In fact, there are well-known examples where translating or rotating conductors do not "drag" magnetic fields at all, and cause no distortions in the magnetic fields through which they are moving [22, 27, 28]. This is because the background field is only modified if a current density $\mathbf{J}$ is induced in the conductor; $\mathbf{J}$ then induces a magnetic field of its own via Ampère's law and it is this induced field that distorts the shape of the overall $\mathbf{B}$ field away from the background field. This distortion often, though not always [27], leads to field lines that have the appearance of being dragged by the moving conductor. But Van Bladel has proven that it is impossible to induce a non-zero $\mathbf{J}$ for any axially symmetric conductor rotating in an axially symmetric field [29]. By this theorem a conducting ionosphere rotating about Earth's axially symmetric field components cannot induce a $\mathbf{J}$, and therefore cannot distort ("drag" along with it) Earth's axially symmetric field. To the contrary, Appendix A describes how it is Earth's non-rotating axially symmetric field that brings charged particles in a conducting plasma around Earth into co-rotation [24].

IV. A PROOF THAT ELECTRIC POWER GENERATION IS IMPOSSIBLE

Could we construct a circuit C in the lab whose rotation along with Earth's surface through Earth's axially symmetric field would generate a continuous electric current via the $\mathbf{v} \times \mathbf{B}$ force? The emf around any path C is given by Eq. (1). The $\mathbf{v} \times \mathbf{B}$ force experienced as the conductor C rotates through Earth's magnetic field drives electron redistribution until the resulting electrostatic field $\mathbf{E}$ perfectly cancels the $\mathbf{v} \times \mathbf{B}$ field: $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ everywhere within C [5, 22, 30]. Redistribution of charge occurs extremely rapidly, on a classical charge relaxation timescale $\tau_e \sim \epsilon_0/\sigma \approx 10^{-11}$ $(1 \text{ S m}^{-1}/\sigma)$ s [31]. For very good conductors such as typical metals for which $\sigma \sim 10^7$ S m^{-1}, the relaxation time is given by the electron collision timescale $\tau_c \sim 10^7 \tau_e$, or $\sim 10^{-11}$ s [32]. Since charge redistributes rapidly and continuously to maintain $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, emf $= 0$ by Eq. (1) always. Electric power generation therefore appears impossible for uniform rotation about an axially symmetric field.

However, this argument contains hidden assumptions. The electric field of a static charge distribution may always be written as a potential of a scalar field: $\mathbf{E} = -\nabla V$. But since $\nabla \times \nabla V = 0$ always, the equation $\mathbf{E} = -\nabla V = -\mathbf{v} \times \mathbf{B}$ can hold only if $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$. We will use magnetically permeable materials to violate this requirement, providing a necessary, but not sufficient, condition for generating a non-zero emf.

Of course, one can always choose to transform to a gauge in which the transformed scalar potential $\tilde{V} = 0$. But then the vector potential transforms to $\tilde{\mathbf{A}} = (\nabla V)t$, and $\mathbf{E} = -\partial\tilde{\mathbf{A}}/\partial t - \nabla\tilde{V} = -\nabla V$, as before [33]. So once again $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ can only hold if $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$.

V. THE LOOPHOLE IN THE PROOF

Magnetically permeable materials channel magnetic flux, and can be used to alter $\mathbf{B}$ to give $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq 0$. This guarantees that the electrons in such a conductor cannot rearrange themselves to generate an electrostatic field $\mathbf{E} = -\nabla V$ that satisfies $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ in Eq. (1). This is the first of our two necessary conditions for electric power generation. But one could still have $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ if $\mathbf{E}$ were no longer purely electrostatic, i.e. if one had $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla V = -\mathbf{v} \times \mathbf{B}$, where $\mathbf{A}$ is the magnetic vector potential. If this equality were always to hold for our system, power generation would still be impossible. Are there circumstances where this equality can be circumvented? This may be answered using the advection-diffusion equation for $\mathbf{A}$, to which we now turn.

Consider two inertial frames. In frame K at infinity there is a constant background magnetic flux density ($\mathbf{B}_\infty$), and no electric field ($\mathbf{E}_\infty = \mathbf{0}$). A conductor is moving at constant velocity $\mathbf{v} = v\hat{\mathbf{y}}$ in K. Frame K' is the frame co-moving with the conductor. Frame K' approximates our frame on Earth's surface (the laboratory frame), translating through the non-rotating component of Earth's field. Frame K approximates a non-rotating frame fixed at Earth's center and moving with Earth in its orbit.

Frames K and K' are not exactly related by a Lorentz boost because of Earth's rotation. In K', Maxwell's equations incorporate rotation via the metric tensor $g_{\mu\nu}$, introducing factors $\sqrt{g_{00}} \approx 1 - \frac{1}{2}(v/c)^2$ when $(v/c) \ll 1$ [29]. For v = 465 m s^{-1}, $(v/c)^2 \approx 10^{-12}$. We show below that these corrections are negligible compared to the effects of interest. We assume $(v/c)^2 \ll 1$ throughout. We may therefore approximate K and K' as two inertial frames in relative linear motion.

Coordinates in the two frames are then related by $t' = t$, $x' = x$, $y' = y - vt$, and $z' = z$. We have $\partial x^\mu/\partial x^\nu = \delta^\mu_\nu$ and $\partial x'^\mu/\partial x'^\nu = \delta^\mu_\nu$, $\partial t/\partial t' = 1$, $v = \partial y/\partial t'$, $\partial/\partial t' = \partial/\partial t + v\partial/\partial y$, and $\partial/\partial x' = \partial/\partial x$, $\partial/\partial y' = \partial/\partial y$, $\partial/\partial z' = \partial/\partial z$, so $\nabla'^{\mathbf{2}} = \nabla^{\mathbf{2}}$. The fields are related by

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{5}$$

$\mathbf{B}' = \mathbf{B}$, and $\mathbf{A}' = \mathbf{A}$. For our system, the curl of Eq. (5) yields $\partial\mathbf{B}/\partial t' = \partial\mathbf{B}/\partial t + v\partial\mathbf{B}/\partial y$, i.e. the curl of the field transformation for $\mathbf{E}'$ is just the advective derivative for $\mathbf{B}$. While $\mathbf{B} = \mathbf{B}'$, Eq. (5) means that $\mathbf{E} = \mathbf{0}$ in K implies $\mathbf{E}' = \mathbf{v} \times \mathbf{B}$ in K'.

We begin with an analysis in frame K, and examine results in K' in Section XI. Ohm's law in K' is $\mathbf{E}' = \mathbf{J}'/\sigma$, so by Eqs. (3) and (5), $\mathbf{J}' = \mathbf{J}$. We have $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{B}' = \mu\mathbf{H}'$ [29]. Using $\mathbf{E} = -\nabla V - \partial\mathbf{A}/\partial t$ and $\mathbf{J} = \nabla \times \mathbf{H}$, Eq. (3) yields the advection-diffusion equation for $\mathbf{A}$ in K:

$$-\nabla V - \partial\mathbf{A}/\partial t + \mathbf{v} \times (\nabla \times \mathbf{A}) = \eta\nabla \times \nabla \times \mathbf{A}, \tag{6}$$

where $\eta = (\sigma\mu)^{-1}$ is the magnetic diffusivity, here assumed constant. The displacement current does not appear in Ampère's law because $\mid \epsilon_0\partial\mathbf{E}/\partial t \mid \ll \mid \mathbf{J} \mid$ (ϵ_0 is the vacuum permittivity) for timescales $t \gg \tau_e$ [10, 34]. Ohm's law in K' also yields Eq. (6) because of the field transformation Eq. (5).

The curl of Eq. (6) yields the advection-diffusion equation for $\mathbf{B}$, or "induction equation":

$$-\partial\mathbf{B}/\partial t + \nabla \times (\mathbf{v} \times \mathbf{B}) = -\eta\nabla^2\mathbf{B}. \tag{7}$$

Integrated over S, Eq. (7) is identical to Eq. (1). Therefore

$$\text{emf} = -\eta \int_S \nabla^2\mathbf{B} \cdot \mathbf{da} = \eta \oint_C (\nabla \times \mathbf{B}) \cdot \mathbf{dl}. \tag{8}$$

Whether $\eta\nabla^2\mathbf{B}$ is negligible in Eq. (7) depends on the magnetic Reynolds number $R_m = \tau_D/\tau_v = \sigma\mu v\xi$, where $\tau_D = \xi^2/\eta$ is the magnetic diffusion time, and $\tau_v = \xi/v$ the transport time, for a system that varies over a characteristic length scale ξ. Then $\mid \eta\nabla^2\mathbf{B} \mid \sim \eta B/\xi^2$ and $\mid \nabla \times (\mathbf{v} \times \mathbf{B}) \mid \sim vB/\xi$ so $R_m = \mid \nabla \times (\mathbf{v} \times \mathbf{B}) \mid / \mid \eta\nabla^2\mathbf{B} \mid$ [35, 36]. If $R_m \gg 1$, $\eta\nabla^2\mathbf{B}$ is negligible in Eq. (7), so emf $= 0$. If $R_m \ll 1$ however, we may have emf $\neq 0$. $R_m \ll 1$ is the second of our two necessary conditions for electric power generation.

In Eq. (1) consider a path C lying within a conducting slab, made say of aluminum for which $\sigma = 4 \times 10^7$ S m^{-1} and relative permeability $\mu_r = 1$ ($\mu_r = \mu/\mu_0$ where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}) [37]. Then for $v = 465$ m s^{-1}, $R_m \gg 1$ if $\xi > 1$ mm so emf $= 0$. We instead explore a system satisfying Ohm's law with $R_m \ll 1$ and $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq 0$. As we show below, one realization is a path C lying within a long cylindrical shell made of an appropriate magnetically permeable MnZn ferrite [38]. We first consider this system at rest in a frame K in which $\mathbf{B}_\infty$ is constant and there is no background electric field ($\mathbf{E}_\infty = 0$), in which case emf $= 0$. We then give this system a velocity $\mathbf{v}$ and show that a nonzero emf will be generated. Such a system at rest in a laboratory on Earth's surface would therefore generate electrical power as Earth rotates.

VI. MAGNETICALLY PERMEABLE CYLINDRICAL SHELL

Consider an infinitely long magnetically permeable conducting cylindrical shell with axis of symmetry along the z-axis, and inner and outer radii a and b, respectively. The background fields at infinity are $\mathbf{B}_\infty = B_\infty \hat{\mathbf{x}}$ and $\mathbf{E}_\infty = \mathbf{0}$ in a frame in which the shell has $\mathbf{v} = \mathbf{0}$. Of course $\mathbf{v} \times \mathbf{B} = \mathbf{0}$ and with $\mathbf{E}_\infty = \mathbf{0}$ we must have by Eq. (3) $\mathbf{J} = \mathbf{0}$. Therefore $\nabla \times \mathbf{H} = \mathbf{J} = \mathbf{0}$, so $\mathbf{H} = -\nabla W$, where W is a magnetic potential. We designate $\mathbf{H}$ when $\mathbf{v} = \mathbf{0}$ as $\mathbf{H_0}$ (and define $\mathbf{B_0} = \mu \mathbf{H_0}$) so $\nabla \cdot \mathbf{H_0} = \nabla^2 W = 0$, whose solution for a magnetically permeable cylindrical shell for all space is well known in cylindrical (ρ, ϕ, z) coordinates [39, 40]. In Cartesian coordinates, the resulting magnetic flux densities exterior to the shell, within its conducting body, and within its hollow interior are:

$$B_{0x}(\rho > b) = B_\infty + \beta_3 (b/\rho)^2 \cos 2\phi, \tag{9a}$$

$$B_{0y}(\rho > b) = \beta_3 (b/\rho)^2 \sin 2\phi; \tag{9b}$$

$$B_{0x}(a \leq \rho \leq b) = \beta_1 - \beta_2 (a/\rho)^2 \cos 2\phi, \tag{10a}$$

$$B_{0y}(a \leq \rho \leq b) = -\beta_2 (a/\rho)^2 \sin 2\phi; \tag{10b}$$

and

$$B_{0x}(\rho < a) = 2\beta_1 (\mu_r + 1)^{-1}, \tag{11a}$$

$$B_{0y}(\rho < a) = 0. \tag{11b}$$

Here

$$\beta_1 = 2B_\infty \mu_r (\mu_r + 1)\zeta, \tag{12}$$

$$\beta_2 = 2B_\infty \mu_r (\mu_r - 1)\zeta, \tag{13}$$

$$\beta_3 = B_\infty [1 - (a/b)^2](\mu_r^2 - 1)\zeta, \tag{14}$$

and

$$\zeta = [(\mu_r + 1)^2 - (a/b)^2 (\mu_r - 1)^2]^{-1}. \tag{15}$$

If $a = 0$, Eq. (10) collapses to that for a solid magnetically permeable cylinder:

$$\mathbf{B}(\rho \leq b) = \beta_1 (a = 0)\hat{\mathbf{x}} = 2\mu_r (\mu_r + 1)^{-1} B_\infty \hat{\mathbf{x}}, \tag{16}$$

for which the magnetic field is constant in the interior, although of course the exterior ($\rho > b$) field is distorted according to Eq. (9) with $a = 0$.

Because $B_z = 0$, only the z-component of $\mathbf{A}$ is non-zero [9, 27], so

$$B_x = \partial A_z/\partial y, \tag{17a}$$

and

$$B_y = -\partial A_z/\partial x. \tag{17b}$$

Eqs. (9) to (11) then correspond to the vector potential $\mathbf{A_0} = A_0\hat{\mathbf{z}}$, with

$$A_0(\rho > b) = B_\infty y + \beta_3(b^2/\rho)\sin\phi, \tag{18}$$

$$A_0(a \leq \rho \leq b) = \beta_1 y - \beta_2(a^2/\rho)\sin\phi, \tag{19}$$

and

$$A_0(\rho < a) = 2\beta_1(\mu_r + 1)^{-1}\rho\sin\phi, \tag{20}$$

with the usual gauge ambiguity allowing the addition of a gradient of a single-valued function. Moreover, because of Eq. (17), any function of z alone may be added to $\mathbf{A_0}$ without affecting $\mathbf{B_0}$ (or $\mathbf{E}$). $\mathbf{A_0}$ must be continuous across the boundaries at $\rho = a$ and $\rho = b$; this is easy to verify for Eqs. (18) to (20). This requirement means that a choice of gauge on one side of a boundary restricts the choice of gauge on the other [41]: one cannot arbitrarily assign different gradient terms (or functions of z) to A_0 in each of Eqs. (18) to (20), and this will prove important below.

From Eq. (19) we see that a solid permeable cylinder has $A_0(\rho \leq b) = \beta_1 y$ in its interior. That is, the first term on the right in Eq. (19) is that for a solid cylinder; when $a \neq 0$ a second term enters as a modification of this first $a = 0$ term.

For the region within the body of the cylindrical shell ($a \leq \rho \leq b$), we find by Eq. (10):

$$\nabla \times (\mathbf{v} \times \mathbf{B_0}) = 2v\beta_2 a^2\rho^{-3}[(3\sin\phi - 4\sin^3\phi)\hat{\mathbf{x}} + (3\cos\phi - 4\cos^3\phi)\hat{\mathbf{y}})] \neq 0, \tag{21}$$

using $\partial\rho/\partial x = \cos\phi$, $\partial\phi/\partial x = -\rho^{-1}\sin\phi$, $\partial\rho/\partial y = \sin\phi$, and $\partial\phi/\partial y = \rho^{-1}\cos\phi$. Such a translating shell, if made out of conducting material satisfying $R_m \ll 1$, would therefore satisfy our two necessary criteria for electric power generation. We show below that in this case these conditions are also sufficient.

Instead of an infinite shell, consider a finite shell lying along the z axis from $-L/2$ to $L/2$, with $L \gg 2b$. The magnetic field in the interior ($\rho < a$) of a finite permeable cylindrical shell may be written as the sum of two contributions: the field corresponding to the shielded interior of an infinitely long shell, plus a contribution from the field penetrating in from the openings [42]. For $\rho < a$ near $z = \pm L/2$, $\mathbf{B}$ deviates from $\mathbf{B_0}$, but moving inward this deviation falls off rapidly like $\exp(-3.83z/a)$ [42], so in the interior the result for an infinite shell should hold for a finite shell provided $|z| \lesssim L/2 - a$. For this result to hold for $\rho < a$, the field in the region $a \leq \rho \leq b$ must be similarly undisturbed, so we take the result for a finite shell in this region to correspond to those for an infinite shell provided $|z| \lesssim L/2 - a$.

VII. TIME-DEPENDENT SOLUTION FOR $\mathbf{v} \neq \mathbf{0}$ AND $R_m \ll 1$

It is clear that $\mathbf{B_0}$ in Eq. (10) can no longer be a solution for our system once $\mathbf{v} \neq 0$, since $\mathbf{B_0}$ could only solve Eq. (7) were $\nabla \times (\mathbf{v} \times \mathbf{B_0}) \neq \mathbf{0}$, in contradiction to Eq. (21). We show in Appendix B that $\mathbf{B_0}(x, y') = \mathbf{B_0}(x, y - vt)$, i.e. the advecting version of Eq. (10), also cannot be a general solution when $\mathbf{v} \neq \mathbf{0}$. Any traveling wave solution of the form $\mathbf{B}(x, y - vt)$ solves the transport equation, so will solve Eq. (7) in the limit $R_m \gg 1$. We wish to solve Eq. (7) for smaller R_m, when the diffusion term is not negligible.

It is easiest first to solve for $\mathbf{A}$. We therefore solve Eq. (6) explicitly to find $\mathbf{A}$ (and so $\mathbf{B}$) for the magnetically permeable cylindrical shell in the case $\mathbf{v} \neq \mathbf{0}$ (Fig. 1) with $R_m \ll 1$. We impose the requirement that in the limit $\mathbf{v} \to \mathbf{0}$ we must have $\mathbf{A} \to \mathbf{A_0}$ and $\mathbf{B} \to \mathbf{B_0}$. We first work in K, and examine the picture in K' in Sec. XII.

Our calculations can be facilitated by a choice of gauge to simplify Eq. (6). We choose a gauge sometimes used in eddy current [43] or magnetohydrodynamic (MHD) [44] applications that relates the potentials by the gauge condition:

$$\nabla \cdot \mathbf{A} = -V/\eta. \tag{22}$$

Because Eq. (22) is less familiar than the more commonly used Lorenz ($\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \partial V/\partial t$) or Coulomb ($\nabla \cdot \mathbf{A} = \mathbf{0}$) gauges [45], we discuss it further in Appendix C. In the gauge of Eq. (22), Eq. (6) simplifies to

$$-\partial \mathbf{A}/\partial t + \mathbf{v} \times (\nabla \times \mathbf{A}) = -\eta \nabla^2 \mathbf{A} \tag{23}$$

using the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^{\mathbf{2}}\mathbf{A}. \tag{24}$$

A complete solution to the system is given by Eqs. (22) and (23) together. We have $\mathbf{A} = A_z\hat{\mathbf{z}}$ and $\nabla \cdot \mathbf{A} = \partial A_z/\partial z$. Because $\mathbf{v} \times (\nabla \times \mathbf{A}) = -v\partial A_z/\partial y\ \hat{\mathbf{z}}$, Eq. (23) reduces to a single non-trivial equation:

$$\partial A_z/\partial t + v\partial A_z/\partial y = \eta\nabla^2 A_z. \tag{25}$$

By Eq. (17), a function $f(z)$ may be added to A_z without altering $\mathbf{B}$, so $f(z)$ may be chosen to yield in Eq. (22) the appropriate V expected by physical arguments. However, by Eq. (1), the emf around C is independent of V, so for the emf it is enough to solve Eq. (23).

If $R_m \gg 1$, $\mid \eta\nabla^2 A_z \mid \ll \mid v\partial A_z/\partial y|$ and Eq. (25) collapses to a transport equation whose solution is a function of the form $A_z(x, y - vt)$. We are interested in the case $R_m \ll 1$, for which we expect diffusion to be important. The advection term $v\partial A_z/\partial t$ in Eq. (25) cannot be neglected even with $R_m \ll 1$ because $\eta\nabla^2 A_0 = 0$ and by analogy to MHD [10, 34, 41], we expect (or at least must not exclude *ab initio*) $\mid v\partial A_0/\partial t \mid \sim \mid \eta\nabla^2 A_1 \mid$, where A_1 is a small perturbation term satisfying $\mid A_1 \mid \sim R_m \mid A_0 \mid$. We seek a solution A_z to Eq. (25) that holds when $R_m \ll 1$ for any $\mathbf{v} = v\hat{\mathbf{y}}$ with the requirement $A_z \to A_0$ as $\mathbf{v} \to \mathbf{0}$. Henceforth we set $\xi = b$ as the relevant diffusion length scale, so put $R_m = \mu\sigma v b$, with advection timescale $\tau_v = b/v$, and diffusion timescale $\tau_D = b^2/\eta = R_m\tau_v$.

We solve Eq. (25) exactly using cylindrical coordinates in K with the origin centered in the shell at some particular instant; such a solution will hold only for a time short compared with τ_v, after which the shell will have moved sufficiently far from the origin that the cylindrical symmetry assumed in Eqs. (9) to (11) is broken. However, we will see that the system reaches steady-state extremely rapidly with $\tau_D \ll \tau_v$, meaning that $\mathbf{B}$ extremely rapidly adapts itself via diffusion to the shell's motion [34, 41]. For any location of the translating shell, we may choose the origin in K to coincide with the center of the shell at that instant. Since there was nothing special about the instant chosen, this should represent the steady-state for the system.

The solution to Eq. (25) may in general be written

$$A_z = A_s(\rho, \phi) + A_t(\rho, \phi, t), \tag{26}$$

where $A_s(\rho, \phi)$ solves the steady-state equation

$$v \partial A_s / \partial y = \eta \nabla^2 A_s, \tag{27}$$

and $A_t(\rho, \phi, t)$ solves the time-dependent equation

$$\partial A_t / \partial t = -v \partial A_t / \partial y + \eta \nabla^2 A_t. \tag{28}$$

When $R_m \ll 1$, naive inspection of Eq. (28) suggests A_t will exponentially decay away on a timescale $\sim \tau_D$ [9, 44]. We explicitly solve Eq. (28) by separation of variables using $A_t = G(\rho, \phi) W(t)$. With separation constant $-\alpha^2$, this gives

$$\eta^{-1} \partial W(t) / \partial t = -\alpha^2 W(t), \tag{29}$$

and

$$\nabla^2 G - (v/\eta) \partial G / \partial y + \alpha^2 G = 0. \tag{30}$$

By Eq. (29),

$$W(t) = C_0 e^{-\eta \alpha^2 t}; \tag{31}$$

all C_i are constants. The alternative choice of separation constant $+\alpha^2$ yields an A_t (hence $\mathbf{B}$) exponentially growing with time, so we exclude this solution on physical grounds. Were $R_m \gg 1$, Eq. (28) would become the transport equation for which separation of variables yields a traveling wave solution.

Putting $G = g(\rho, \phi) e^{ky}$ (a standard technique from MHD [10, 27]) in Eq. (30) yields

$$\nabla^2 g + \lambda^2 g = 0, \tag{32}$$

with

$$k = v / 2\eta \tag{33}$$

and $\lambda^2 = \alpha^2 - k^2$. Therefore Eq. (31) becomes

$$W(t) = C_0 e^{-\eta (k^2 + \lambda^2) t}. \tag{34}$$

Solving Eq. (32) by putting $g = m(\rho) n(\phi)$ with separation constant ν^2 yields

$$m(\rho) = C_1 J_\nu(\lambda \rho) + C_2 Y_\nu(\lambda \rho) \tag{35}$$

and

$$n(\phi) = C_3 \cos(\nu \phi) + C_4 \sin(\nu \phi), \tag{36}$$

where the J_ν and Y_ν are Bessel functions of the first and second kinds of order ν. Therefore

$$A_t = C_0 m(\rho) n(\phi) e^{k\rho \sin\phi} e^{-\eta(k^2+\lambda^2)t}. \tag{37}$$

Since $\eta(k^2 + \lambda^2) > 0$ always, in Eq. (37) A_t decays exponentially and the system over time goes to the steady-state solution $A_s(\rho, \phi)$ in Eq. (26). We therefore cannot choose the trivial solution $A_s = 0$ in Eqs. (26) and (27) since for $v = 0$ we must have $A_z = A_0$ with A_0 given by Eq. (19). Therefore $A_s(v = 0) = A_0$. The condition $R_m \gg 1$ requires $v \neq 0$ so solutions for $R_m \gg 1$ need not satisfy this constraint.

We use boundary conditions and Eqs. (35) to (37) to solve for λ. This allows us to show explicitly that the exponential in Eq. (37) does indeed decay on a timescale (even faster than) $\sim \tau_D$, consistent with more general arguments [9, 44]. In Eq. (35), we set $C_2 = 0$ so that our solutions remain bounded in the case $a \to 0$. By Gauss' law, we know that B_ρ must be continuous across the boundary of the cylindrical shell at $\rho = a$. In the case of a static external transverse magnetic field, $B_\rho(\rho < a) \sim 10^{-3} B_\infty$ for $\mu_r \sim 5 \times 10^3$, a value typical of the magnetically permeable materials we will discuss here. That is, the shell acts as a magnetic shield for its hollow interior [39, 40, 42, 46]. For time-varying fields, the shielding is as good or better than it is for the static field case [46–48]. We may then take as a boundary condition for the time-dependent part $\mathbf{B_t}$ of $\mathbf{B}$ that $B_{t\rho}(\rho = a) \to 0$ for all ϕ as one approaches the boundary from $\rho > a$ within the shell. Since $B_{t\rho} = \rho^{-1}\partial A_t/\partial\phi$ and A_t evolves independently of A_s, this boundary condition then implies that, for all ϕ,

$$\left.\frac{1}{\rho}\frac{\partial A_t}{\partial \phi}\right|_{\rho=a} = 0. \tag{38}$$

One could try to satisfy Eq. (38) for all ϕ by setting the product of constants in Eq. (37), either $C_0 C_1 C_3$ or $C_0 C_1 C_4$, to be proportional to some negative power of μ_r that goes to zero for large μ_r. But this is an unphysical choice, since its effect is to force A_t to 0 for all ρ within $a \leq \rho \leq b$, meaning that the cylindrical shell magnetically shields itself throughout its entire volume, as well as its hollow interior. If this unphysical choice were made nonetheless, it would render A_t negligible so that only the steady-state solution A_s would remain. This conclusion would be the same as that we obtain below, but below it will be for the reason that A_t decays away extremely quickly.

Using Eqs. (35) with $C_2 = 0$ and Eq. (37), requiring that Eq. (38) be true for all ϕ in turn requires

$$J_\nu(\lambda a) = 0 \tag{39}$$

for all ν. Choosing $\nu = 0$, the first zero of the Bessel function gives $\lambda = 2.40/a$, and Eq. (34) becomes

$$W(t) = C_0 e^{-(R_m/4)t/\tau_v} e^{-(2.4b/a)^2 t/\tau_D}, \tag{40}$$

using the definitions of R_m, τ_v, and τ_D. The first exponential in Eq. (40) is a decay that is slow with respect to the translation timescale τ_v. The second exponential is a decay that is much faster than the diffusion timescale τ_D. Since $\tau_D = R_m \tau_v$, this decay for $R_m \ll 1$ is extremely fast with respect to τ_v. Choosing any higher value of ν (or values, were one to make the solution a series of terms in ν) would yield larger values for λ, leading to even faster exponential decays in Eq. (40). In the special case $a = 0$ in Eq. (40), $W(t) = 0$ so $A_t = 0$ and the full solution is is just the steady-state solution $A_s(a = 0)$ from Eq. (26).

VIII. TIME-INDEPENDENT SOLUTION FOR $\mathbf{v} \neq \mathbf{0}$, $R_m \ll 1$

Clearly $W(t) \to 0$ as $t \to \infty$ in Eq. (40), with A_s in Eq. (26) the steady-state solution that remains. When $R_m \ll 1$, $W(t) \to 0$ on a timescale $< \tau_D \ll \tau_v$, so A_t in Eq. (37) decays rapidly away in the time that it takes the shell to move a distance $< v\tau_D$, where $v\tau_D \ll v\tau_v = b$. That is, we are — as expected — in a quasi-stationary situation where at any point in the shell's translation, $A_z = A_s(\rho, \phi)$ to a good approximation, with A_s given by Eq. (27). We now solve Eq. (27).

First consider the special case where our cylinder is solid ($a = 0$) and translating at $\mathbf{v} = v\hat{\mathbf{y}}$ through the background field $\mathbf{B}_\infty = B_\infty \hat{\mathbf{x}}$. Then $A_0(a = 0)$ must be given by Eq. (19) with $a = 0$, i.e $A_0 = \beta_1 y + h(z)$, where $h(z)$ is an arbitrary function of z. This satisfies Eq. (27) provided $h(z) = k\beta_1 z^2$ with k given by Eq. (33), so that

$$A_0(a = 0) = k\beta_1 z^2 + \beta_1 y. \tag{41}$$

By the gauge condition Eq. (22), we then have

$$V(a = 0) = -v\beta_1 z. \tag{42}$$

For a finite solid cylinder, charges in the cylinder experience a $\mathbf{v} \times \mathbf{B}$ force and flow in response, redistributing on an extremely short timescale τ_e until an electric field $\mathbf{E} = -\nabla V$ is established that perfectly cancels $\mathbf{v} \times \mathbf{B}$. In particular, we see that Eq. (42) gives the physically correct answer for the special case of a translating finite solid cylinder.

Now what happens when our cylinder becomes a cylindrical shell with $a \neq 0$? We anticipate from Eq. (19) that $A_0(a \neq 0)$ will be given by $A_0(a = 0)$ plus additional terms. We solve Eq. (27) for the general $(a \neq 0, \mathbf{v} \neq \mathbf{0})$ case, with the requirements that we recover Eq. (41) when $a = 0$ and Eq. (19) when $\mathbf{v} = \mathbf{0}$.

Eq. (27) is solved by $f(\rho, \phi)e^{ky}$, where the function f satisfies

$$\nabla^2 f - k^2 f = 0 \tag{43}$$

with k given by Eq. (33), so

$$f(\rho, \phi) = [C_5 \cos(\nu\phi) + C_6 \sin(\nu\phi)][C_7 I_\nu(k\rho) + C_8 K_\nu(k\rho)], \tag{44}$$

where the separation constant is ν^2 and I_ν and K_ν are modified Bessel functions of order ν of the first and second kind. We therefore write the general solution as

$$A_s = k\beta_1 z^2 + \beta_1 y + f(\rho, \phi)e^{ky}, \tag{45}$$

where the first two terms provide the solution to Eq. (27) for the case $a = 0$ and the final term modifies that solution, analogously to Eq. (19), for the case $a \neq 0$. Eq. (45) must go to Eq. (19) in the $v = 0$ limit. Noting that as $k\rho \to 0$ [49]:

$$K_1(k\rho) = (k\rho)^{-1} + k\rho(2\gamma - 1) + (k\rho/2)\ln(k\rho/2) + O(k\rho)^2, \tag{46}$$

where $\gamma = 0.5772...$ is the Euler constant;

$$I_\nu(k\rho) = (k\rho)^\nu/(2^\nu \nu!) + O(k\rho)^{\nu+2}; \tag{47}$$

and

$$e^{ky} = 1 + k\rho\sin\phi + (1/2)(k\rho)^2 \sin^2\phi + O(k\rho)^3, \tag{48}$$

requiring Eq. (45) to equal Eq. (19) for $v = 0$ fixes in Eq. (44) $C_5 = 0 = C_7$ and $\nu = 1$, with $C_6 = 1$ and $C_8 = -\beta_2 k a^2$. Then the solution to Eq. (27) is

$$A_z(a \leq \rho \leq b) = A_s = k\beta_1 z^2 + \beta_1 y - \beta_2 k a^2 K_1(k\rho)e^{ky}\sin\phi. \tag{49}$$

For $k\rho \to 0$, Eq. (49) becomes

$$A_z(a \leq \rho \leq b) = A_s = A_0 + A_1 + O(R_m)^2 \tag{50}$$

where A_0 is given by Eq. (19),

$$A_1 = -(R_m/2)b^{-1}\beta_2 a^2 \sin^2\phi, \tag{51}$$

and $R_m = 2kb = \mu\sigma v b$. That is, when $R_m \ll 1$, A_z for $v \neq 0$ is perturbed away from the $v = 0$ solution (A_0) by a series whose terms are scaled by powers of R_m.

Finally, applying the gauge condition Eq. (22) to $\mathbf{A} = A_z\hat{\mathbf{z}}$ with Eq. (49), we find

$$V(a \leq \rho \leq b) = -v\beta_1 z, \tag{52}$$

so that even when $a \neq 0$,

$$\nabla V = -v\beta_1\hat{\mathbf{z}}. \tag{53}$$

IX. GENERATION OF AN EMF

Eqs. (17a) and (49) yield

$$B_x(a \leq \rho \leq b) = \beta_1 - \beta_2(a/\rho)^2 e^{ky}\{[k\rho\cos 2\phi + (k\rho)^2\sin\phi]K_1(k\rho) - (k\rho)^2\sin^2\phi K_0(k\rho)\}, \tag{54}$$

using the identities [50]

$$\partial K_1(k\rho)/\partial(k\rho) = -[K_0(k\rho) + K_2(k\rho)]/2 \tag{55}$$

and

$$K_0(k\rho) - K_2(k\rho) = -(2/k\rho)K_1(k\rho), \tag{56}$$

so that

$$\partial K_1(k\rho)/\partial(k\rho) = -K_0(k\rho) - (k\rho)^{-1}K_1(k\rho). \tag{57}$$

Noting that [49]

$$K_0(k\rho) = -\gamma - \ln(k\rho/2) + O(k\rho), \tag{58}$$

as $v \to 0$ we have by Eq. (54) for $R_m \ll 1$:

$$B_x(a \leq \rho \leq b) = B_{0x} + B_{1x} + O(R_m)^2, \tag{59}$$

with

$$B_{1x} = -R_m b^{-1}\beta_2 a^2\rho^{-1}\sin\phi\cos^2\phi. \tag{60}$$

Fig. 2 shows for a particular case how B_x differs from B_{0x} to $O(R_m)$.

Similarly, Eqs. (17b) and (54) yield

$$B_y(a \leq \rho \leq b) = B_{0y} + B_{1y} + O(R_m)^2, \tag{61}$$

with

$$B_{1y} = -R_m b^{-1} \beta_2 a^2 \rho^{-1} \sin^2\phi \cos\phi. \tag{62}$$

Fig. 3 shows for a particular case how B_y differs from B_{0y} to $O(R_m)$.

We see that $B_{1x} = \partial A_1/\partial y$ and $B_{1y} = -\partial A_1/\partial x$. Eqs. (60) and (62) are readily checked to verify that they do solve Eq. (7); e.g. $v\partial B_{0x}/\partial y = \eta\nabla^2 B_{1x}$ as required. Eqs. (59) and (61) show that the effect of $v \neq 0$ for $R_m \ll 1$ is to perturb $\mathbf{B}$ away from $\mathbf{B_0}$ by a series scaled by successive powers of R_m. The asymmetry of B_x about $y = 0$ leads to the continuous generation of an emf within the cylindrical shell. Consider the current path C in Fig. 1 for $x_0 = b\cos\phi_0$, and $y_0 = b\sin\phi_0$. Then Eq. (8) for this path gives:

$$\mathrm{emf}(x_0, y_0) = -\eta \oint_C \nabla^2 A_z \hat{\mathbf{z}} \cdot \mathbf{dl} = -2R_m v\beta_2 l(a/b)^2 \sin\phi_0 \cos^2\phi_0 + O(R_m)^2, \tag{63}$$

using

$$\eta\nabla \times \mathbf{B} = -\nabla V - \eta\nabla^2 \mathbf{A} \tag{64}$$

from Eqs. (22) and (24), $A_z = A_s$, and Eqs. (27), (17), and (59). Eq. (63) is only valid for $R_m \ll 1$; if $R_m \gg 1$, emf $= 0$ by Eq. (7). Even for $R_m \ll 1$, emf $= 0$ in Eq. (63) if $v = 0$, or $a = 0$, or $\mu_r = 1$. The emf in K' is the same as that in K provided $(v/c)^2 \ll 1$ [13].

The result in Eq. (63) is for one designated current path C. For an arbitrary C with segments parallel to the z axis, the integration underlying Eq. (63) leads to emf $\propto$ $[B_x(x_1, y_1) - B_x(x_2, y_2)]$, where the coordinates designate the (x, y) coordinates of the two segments of C parallel to the z axis. (Arbitrary current paths can then be built up by fusing such rectangular sub-paths.) Because of the symmetry in ϕ of B_{0x}, Eq. (10a), it is clear that for every such circuit with $0 < \phi < \pi$ there is a corresponding circuit with $\pi < \phi < 2\pi$ that yields an emf of opposite sign with respect to B_{0x}. This is because the scalar product $(\mathbf{v} \times \mathbf{B_0}) \cdot \mathbf{dl} = vB_{0x}\hat{\mathbf{z}} \cdot \mathbf{dl}$ has the opposite sign in the two cases, since the circuits are mirror reflections across the $y = 0$ plane and in each case C is traversed using a right-hand rule. Over the entire shell, these therefore average to zero. It is the component of B_x of $O(R_m)$ that makes a non-zero contribution, because of the asymmetry in ϕ of B_{1x}, with B_{1x} switching sign at the $y = 0$ plane (Fig. 2). With respect to B_{1x}, for every current path C with $0 < \phi < \pi$, there is a corresponding path with $\pi < \phi < 2\pi$ that yields an emf of identical sign, so the two do not cancel. To $O(R_m)$ therefore, the average emf around the shell cannot be zero. We will see in Sec. XII that, consistent with a non-zero emf, there is a net absorption of power by the shell from Poynting vector inflow.

An infinite solid conducting bar moving through a background magnetic field will (in principle) generate a current due to $\mathbf{v} \times \mathbf{B}$ since its infinite extent prevents the accumulation of the charges at its ends that for a finite bar generates the electric field $\mathbf{E} = -\nabla V = -\mathbf{v} \times \mathbf{B}$. However, even for the infinite solid bar, $\nabla \times (\mathbf{v} \times \mathbf{B}) = 0$ so there will be emf $= 0$ about any closed path C lying within the bar. The nonzero emf in Eq. (63) is not, therefore, attributable to the fact that our formalism began with an infinite cylindrical shell.

X. INTUITIVE PHYSICAL PICTURE

A simple physical picture offers insight into why a magnetically permeable cylindrical shell moving at velocity $\mathbf{v}$ and satisfying $R_m \ll 1$ would be expected to generate an emf according to Eq. (8). In frame K, picture a finite cylindrical shell moving transversely to its long axis (Fig. 1). Assume $a \neq 0$. By Eq. (21), we know that it is impossible for the shell's electrons to establish a configuration such that $-\nabla V = -\mathbf{v} \times \mathbf{B}$.

Imagine beginning with the cylindrical shell at rest and then placing it into motion at velocity $\mathbf{v}$. The magnetic flux density within the shell itself is initially $\mathbf{B_0}$, given by Eq. (10). $\mathbf{B_0}$ results from Maxwell's equations requiring the continuity of the normal component of $\mathbf{B}$ and tangential component of $\mathbf{H}$ at the surfaces $\rho = a$ and $\rho = b$. As the shell moves it attempts, so to speak, to enforce $\mathbf{B} = \mathbf{B_0}$ throughout $a \leq \rho \leq b$. If $\tau_D/\tau_v = R_m \gg 1$, the diffusion timescale τ_D for the magnetic flux density is much longer than the advection timescale τ_v. That is, diffusion is negligible compared to advection and the field $\mathbf{B}(a \leq \rho \leq b)$ advects along with the shell, so that $\mathbf{B} = \mathbf{B_0}$ to very high precision. (Alfvén's frozen-flux theorem [26] holds.) Since $\nabla \times \mathbf{B_0} = 0$, by Eq. (8) we must have emf $= 0$ when $R_m \gg 1$.

Contrast this with the case $\tau_D/\tau_v = R_m \ll 1$. Now the timescale τ_D for diffusion is much shorter than τ_v. That is, as the shell moves, the field's adjustment is dominated not by advection but by diffusion, toward a field configuration at which diffusion would stop, i.e. toward $\mathbf{B_0}(a \leq \rho \leq b)$, where the "destination" $\mathbf{B_0}$ is the value that would apply for a stationary shell at the location to which the shell has just moved. The field can never reach this end point since the shell keeps moving even as the field diffuses, so a steady-state is reached in which the diffusing field differs slightly from $\mathbf{B_0}$. The field within the shell does not adjust instantaneously to the shell's motion, so never (unless the shell is brought to rest in K) fully "catches up" to that motion. A closed path C moving with the shell constantly experiences a field that is diffusing across its boundaries, and Eq. (8) in general is nonzero.

XI. ANALYSIS IN THE LABORATORY FRAME

We now consider our system in the laboratory frame K', where $\mathbf{v} = \mathbf{0}$ so there is no magnetic $\mathbf{v} \times \mathbf{B}$ force, but there is instead an electric field given by the field transformation Eq. (5): $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. Ohm's law in K' is simply $\mathbf{E}' = \mathbf{J}'/\sigma$, which leads to the induction equation in K':

$$\partial \mathbf{B}/\partial t' = \eta \nabla^2 \mathbf{B}. \tag{65}$$

Since $\mathbf{J}' = \mathbf{J}$ and $\mathbf{B}' = \mathbf{B}$ when $(v/c)^2 \ll 1$, the emf is given by

$$\text{emf}' = \oint_C \mathbf{E}' \cdot \mathbf{dl}' = \sigma^{-1} \oint_C \mathbf{J} \cdot \mathbf{dl} = \eta \oint_C (\nabla \times \mathbf{B}) \cdot \mathbf{dl}. \tag{66}$$

We have used the fact that $\mathbf{E}'$ and therefore $\mathbf{J}$ must be parallel to $\hat{\mathbf{z}}$, so the relevant part of $\mathbf{dl}$ is perpendicular to y and therefore we can put $\mathbf{dl}' = \mathbf{dl}$. Eq. (66) is identical to Eq. (8) and therefore to Eq. (63), so $\text{emf}' = \text{emf}$, as expected [13]. Eq. (66) is nonzero in K' provided the same conditions hold as those needed for Eq. (63) to give $\text{emf} \neq 0$.

It might nevertheless seem puzzling that an emf could be generated in K'. We intuitively expect $\partial \mathbf{B}/\partial t' = 0$ in steady state, so that by Eqs. (8) and (65), $\text{emf}' = 0$. But care must be taken with $\mathbf{B}$ in Eq. (65): because $\mathbf{B}$ is not rotating with Earth, it cannot be treated implicitly as $\mathbf{B}(x, y')$, where $y' = y - vt$ relates the coordinates in K' and K. If $\mathbf{B}$ *were* rotating with Earth, then in K' we would simply have $\mathbf{B} = \mathbf{B}(x, y')$ and $\partial \mathbf{B}(x, y')/\partial t' = (\partial \mathbf{B}/\partial y')(\partial y'/\partial t') = \mathbf{0}$, using the chain rule and $\partial y'/\partial t' = 0$.

But treating $\mathbf{B}$ as rotating with Earth is inconsistent with the clear expectation from the results of the Barnett [1], Kennard [15], and Pegram [19] experiments. Rather, in K' we must treat $\mathbf{B}$ as advecting through the cylindrical shell at velocity $\mathbf{v} = -v\hat{\mathbf{y}}$, which we capture by writing $\mathbf{B} = \mathbf{B}(x, y)$ with $y = y' + vt$. The time-dependence of $\mathbf{B}(x, y)$ is driven by the advection of $\mathbf{B}$ through the shell; this dependence is included implicitly by

$y = y(y', t) = y' + vt$. Then by the chain rule and $\partial y/\partial t' = v$,

$$\partial \mathbf{B}/\partial t' = \partial \mathbf{B}(x, y(y', t))/\partial t' = (\partial \mathbf{B}/\partial y)(\partial y/\partial t') = v\partial \mathbf{B}/\partial y. \tag{67}$$

That is, we *do* have $\partial \mathbf{B}(x, y')/\partial t' = \mathbf{0}$, but we also have $\partial \mathbf{B}(x, y)/\partial t' = v\partial \mathbf{B}/\partial y \neq \mathbf{0}$, and we must distinguish between the two. Only the second representation for $\mathbf{B}$ in K' is consistent with experiment.

Substituting Eq. (67) into (65) gives a time-independent equation for $\mathbf{B}$:

$$v\partial \mathbf{B}/\partial y' = \eta \nabla^2 \mathbf{B}. \tag{68}$$

Recalling $\partial/\partial y' = \partial/\partial y$, Eq. (68) yields Eq. (27) for $\mathbf{A}$ given the gauge choice Eq. (22). Physically, the induction (advection-diffusion) equation concerns the steady-state that is reached in $\mathbf{B}$ as it advects through the $R_m \ll 1$ cylindrical shell and undergoes concomitant diffusion; as a result $\mathbf{B}$ (as we know from Eqs. (59) to (62)) is slightly perturbed away from $\mathbf{B_0}$. Were $\mathbf{B}$ instead advecting along with the shell, there would be no emf.

A Poynting vector and flux transport analysis [41, 51, 52] in K' make it clear that energy is flowing into our $R_m \ll 1$ cylindrical shell, providing the power required to sustain emf$' \neq 0$. The Poynting vector in K' is

$$\mathbf{S}' = \mu^{-\mathbf{1}}(\mathbf{E}' \times \mathbf{B}) = \mu^{-\mathbf{1}}\eta(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{69}$$

where we have used $\mathbf{E}' = \mathbf{J}/\sigma$ and Ampère's law. Were it the case that $\mathbf{B} = \mathbf{B_0}$, we would have $\mathbf{S}' = 0$ by $\nabla \times \mathbf{B_0} = 0$ in Eq. (69) and there would be no energy input to the cylindrical shell. However, $\nabla \times \mathbf{B_1} \neq 0$ and using Eqs. (24), (23), (53), (27), and (17), we find

$$\eta \nabla \times \mathbf{B} = v(\beta_1 - B_x)\hat{\mathbf{z}}, \tag{70}$$

giving

$$\mathbf{S}' = v\mu^{-1}(\beta_1 - B_x)(B_x\hat{\mathbf{y}} - B_y\hat{\mathbf{x}}). \tag{71}$$

We perform our calculations at the instant at which the origins of the K' and K frames coincide. The net energy flux P'_S into the shell's surface within $l/2 \leq z \leq l/2$ (where $l/2$ is chosen to be sufficiently far in from the shell's edge at $L/2$) is given by:

$$P'_S = \int_0^{2\pi} \int_{-l/2}^{l/2} \mathbf{S}' \cdot \hat{\rho} \; \rho d\phi dz, \tag{72}$$

where $\hat{\rho} = \cos\phi\ \hat{\mathbf{x}} + \sin\phi\ \hat{\mathbf{y}}$ and the boundaries at both $\rho = b$ and $\rho = a$ must be taken into account by summing the contributions from evaluating Eq. (72) at $\rho = b$ and at $\rho = a$. The boundary at $\rho = a$ enters with a negative sign, opposite from that at $\rho = b$. The calculation is simplified by noting

$$(B_x\hat{\mathbf{y}} - B_y\hat{\mathbf{x}}) \cdot \hat{\rho} = -B_y \cos\phi + B_x \sin\phi = [\beta_1 + \beta_2(a/\rho)^2]\sin\phi + O(R_m)^2, \tag{73}$$

i.e. the $O(R_m)^1$ terms cancel in Eq. (73). In Eq. (72), nearly all terms integrate to zero, and

$$P'_S = (\pi/4)\sigma v^2 \beta_2^2 a^2 [1 - (a/b)^2] l + O(R_m)^2. \tag{74}$$

$P'_S = 0$ if $v = 0$, or $\mu_r = 1$ (because then $\beta_2 = 0$), or $a = 0$. Otherwise, the Poynting vector $\mathbf{S}'$ in K' gives a net energy flow into the cylindrical shell that sustains the emf$'$.

It is interesting to ask which terms within $\mathbf{S}'$ provide this energy. Nearly every term in Eq. (71) makes zero contribution to Eq. (72), either because it cancels an identical term of opposite sign, or because it integrates to zero over ϕ: the energy from most terms simply flows through the shell, with as much energy leaving as entering. The only term in $\mathbf{S}'$ that makes a nonzero contribution is

$$\mathbf{S}'_{\mathbf{x}} = 2\sigma v^2 \beta_2^2 a^4 \rho^{-3} \sin^2\phi \cos\phi (4\cos^2\phi - 1)\ \hat{\mathbf{x}}. \tag{75}$$

Eq. (65) can be written [41, 52]:

$$\partial\mathbf{B}/\partial t' = \nabla \times (\mathbf{w} \times \mathbf{B}) \tag{76}$$

for an appropriate velocity $\mathbf{w}$. By Ampère's and Ohm's law in K', we have

$$\mathbf{E}' = \eta \nabla \times \mathbf{B}, \tag{77}$$

so $\mathbf{E}' \cdot \mathbf{B} = 0$ since $\mathbf{B} = (B_x, B_y, 0)$. When $\mathbf{E}' \cdot \mathbf{B} = 0$, $\mathbf{w}$ in Eq. (76) is [41, 52]:

$$\mathbf{w} = (\mathbf{E}' \times \mathbf{B})/B^2, \tag{78}$$

which is called the "flux transporting velocity" [52], meaning that $\mathbf{w}$ for the case $\sigma \neq \infty$ preserves flux, because it satisfies Eq. (76) in the same way that $\mathbf{v}$ satisfies Eq. (7) for the case $\sigma = \infty$. A contour C in the cylindrical shell will therefore have $d\Phi/dt = 0$ through its corresponding surface S if C is moving at $\mathbf{w}$, where $\mathbf{w}$ may vary point-to-point along C.

By Eqs. (68) and (78),

$$\mathbf{w} = \mu \mathbf{S}'/B^2. \tag{79}$$

Direct calculation using Eqs. (79) and (71) reveals $\mathbf{w}$ to be algebraically complicated with $\mathbf{w} \neq \mathbf{v}$, even while satisfying $\nabla \times (\mathbf{w} \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B})$, as it must. When $\mathbf{w} \neq \mathbf{v}$, then if C is simply being transported with the conductor at $\mathbf{v} \neq \mathbf{0}$, there must be an emf around C. This is the case, for example, for the contour C in Fig. 1. By Eqs. (78), the zero-velocity solution $\mathbf{B_0}$ in K' is not transported through the shell — because in this case $\mathbf{E}' = 0$ by Eq. (77), so $\mathbf{w} = 0$ in Eq. (78). Of course, $\mathbf{B_0}$ does not diffuse: $\nabla^2 \mathbf{B_0} = 0$. Only the perturbations $\mathbf{B_1}$ and higher orders will have $\mathbf{w} \neq 0$.

Eq. (79) means that magnetic flux is transported proportionally to the transport of energy defined by the Poynting vector. Since $\mathbf{S}'$ integrated over the cylindrical shell is non-zero, there is a corresponding net flow of magnetic flux into the shell. We have a picture in K' in which by Eq. (75), near the $x = 0$ plane for $\pi/3 < \phi < 2\pi/3$ and again for $4\pi/3 < \phi < 5\pi/3$, magnetic field lines are diffusing (transported at velocity $\mathbf{w}$) in the x direction vertically toward the $x = 0$ plane from above and below. These lines annihilate [9, 52, 53] in the $x = 0$ plane, providing energy that drives the current flow in C. The cancellation (annihilation) of the magnetic filed lines in the $x = 0$ plane preserves the gradient, which in turn maintains the continuing inward diffusion of the field.

We note an analogy to the homopolar generator. By Eqs. (67) and (76),

$$\text{emf}' = \oint_C (\mathbf{w} \times \mathbf{B}) \cdot \mathbf{dl}, \tag{80}$$

where $\mathbf{w}$ is given by Eq. (78). In the homopolar generator, the analog to Eq. (80) gives $\text{emf}' \neq 0$ because only part of C is rotating, so $\mathbf{v}$ varies (stepwise) around C and $\mathbf{v} \times \mathbf{B}$ does not integrate to 0 around C. In the $R_m \ll 1$ cylindrical shell of Fig. 1, $\text{emf}' \neq 0$ because $\mathbf{w}$ varies around C according to Eq. (79), so $\mathbf{w} \times \mathbf{B}$ does not integrate to 0 around C.

XII. POYNTING'S THEOREM AND MAGNETIC BRAKING

When $R_m \ll 1$, in the steady-state (e.g. Eq. (45)) we have $\partial \mathbf{A}/\partial t = \mathbf{0}$, so $\mathbf{E} = -\nabla V = v\beta_1 \hat{\mathbf{z}}$ by Eq. (53). Together with Ampère's law and Eq. (69), we have

$$\mathbf{E} \cdot \mathbf{J} = \sigma v^2 \beta_1 (\beta_1 - B_x). \tag{81}$$

By Eq. (3) we also have

$$\mathbf{E}\cdot\mathbf{J} = \sigma^{-\mathbf{1}}J^2 + (\mathbf{J}\times\mathbf{B})\cdot\mathbf{v}. \tag{82}$$

Integrating Eq. (81) over the volume $dV = \rho d\phi d\rho dz$ gives zero, so Eq. (82) implies

$$\sigma^{-1}\int_V J^2\ dV = -\int_V (\mathbf{J}\times\mathbf{B})\cdot\mathbf{v}\ dV, \tag{83}$$

where the integral on the right-hand side is the familiar expression for magnetic braking. In K, Joule heating therefore derives from the energy made available by magnetic braking of the cylindrical shell. Since the shell is being carried by Earth, it is clear that electrical power in our system derives ultimately from the kinetic energy of Earth's rotation. This is analogous to the Poynting theorem analysis of the homopolar generator [12].

Explicitly integrating $(\mathbf{J}\times\mathbf{B})\cdot\mathbf{v}$ in Eq. (83) over the volume V of the shell with $\mathbf{J} = \mu^{-1}\nabla\times\mathbf{B}$ shows the power removed from Earth's rotational kinetic energy to be:

$$P_k = -\sigma v^2 l\int_a^b\int_0^{2\pi} B_x(\beta_1 - B_x)\rho d\rho d\phi = (\pi/2)\sigma v^2\beta_2^2 l a^2[1-(a/b)^2] + O(R_m)^2. \tag{84}$$

Were $\mathbf{B} = \mathbf{B_0}$, we would have $P_k = 0$ since $\nabla\times\mathbf{B_0} = 0$. If $v = 0$ or $\mu_r = 1$ or $a = 0$, then $P_k = 0$. The power in K must equal that in the laboratory frame K' to $O(v/c)^2$ [13].

The manner in which this power arises in K' is of interest. Poynting's theorem [54] states that the rate at which work is done on the electrical charges within a volume V of surface area Σ is equal to the decrease in energy stored in the electric and magnetic fields, minus the energy that flowed out through the surface bounding the volume. In K', Poynting's theorem is:

$$\int_V \mathbf{E}'\cdot\mathbf{J}\ dV = -\mu^{-1}\int_V \mathbf{B}\cdot\partial\mathbf{B}/\partial t'\ dV - \int_\Sigma \mathbf{S}'\cdot\mathbf{d\Sigma}, \tag{85}$$

where as before the displacement current is negligible. By Eq. (65), $\mathbf{E}'\cdot\mathbf{J} = \sigma^{-1}J^2$. The second term on the right of Eq. (85) is just Eq. (72). The first term on the right may be evaluated using Eq. (76), a calculation most easily performed with $\mathbf{B}$ in cylindrical coordinates (Appendix D). The result is identical to Eq. (74).

Therefore in K', Eq. (85) gives the power P'_P provided to the shell, to be

$$P'_P = \sigma^{-1}\int_V J^2\ dV = (\pi/2)\sigma v^2\beta_2^2 a^2[1-(a/b)^2]l + O(R_m)^2, \tag{86}$$

with the energy for electrical power being provided in Poynting's theorem coming equally from Poynting vector inflow and the $\partial\mathbf{B}/\partial t'$ term. This is indeed equal to the expression found in Eq. (84) by calculating in the K frame: $P_k = P'_P$. For a given choice of b, Eqs. (84) and (86) reach their maximum values for $a = b/\sqrt{2}$.

An important question is whether the slight de-spinning of Earth caused by the magnetic braking found here is consistent with angular momentum conservation. Mechanical systems can come in or out of rotation solely via the transfer of angular momentum between mechanical rotation and the electromagnetic field. (For the fully calculated example of a charged magnetized sphere exhibiting this behavior, see [55, 56].) The angular momentum of the electromagnetic field is proportional to $\mathbf{r} \times \mathbf{S}$ (with $\mathbf{r}$ the usual radial component in a spherical coordinate system). For the system in our thought experiment, the analogous issue is linear momentum conservation. In K, the mechanical momentum of the cylindrical shell lies in the $\hat{\mathbf{y}}$ direction, and the braking force per unit volume, given by $\mathbf{J} \times \mathbf{B}$, acts in the $-\hat{\mathbf{y}}$ direction. The momentum (per unit volume) of the electromagnetic field associated with this system is $\mathbf{p} = \varepsilon_{\mathbf{0}}\mu_{\mathbf{0}}\mathbf{S}$, with $\mathbf{S} = \mu^{-\mathbf{1}}\mathbf{E} \times \mathbf{B} = (\mu\sigma)^{-\mathbf{1}}\mathbf{J} \times \mathbf{B}$, i.e. $\mathbf{p}$ is proportional to, and lies in the direction of, the magnetic braking vector. Therefore, as the system is braked, positive mechanical linear momentum is lost from the cylindrical shell while negative linear momentum is lost from the electromagnetic field, and momentum conservation is possible.

XIII. EXPERIMENTAL PREDICTIONS

Evaluating the emf expected to be measured in a laboratory test of these claims requires that part $\mathbf{B}_\infty$ of Earth's total field that is axially symmetric about the planet's rotation axis. This is well approximated by summing the axisymmetric dipole, quadrupole, and octupole components of the total field to yield the northward (X) and downward (Z) components of $\mathbf{B}_\infty$ at a point on the surface of the Earth. These are [23], at colatitude θ,

$$X = -g_1^0 \sin\theta - 3g_2^0 \sin\theta\cos\theta - (3/2)g_3^0 \sin\theta(5\cos^2\theta - 1) \tag{87a}$$

and

$$Z = -2g_1^0 \cos\theta - (3/2)g_2^0(3\cos^2\theta - 1) - 2g_3^0 \cos\theta(5\cos^2\theta - 3), \tag{87b}$$

giving

$$B_\infty = (X^2 + Z^2)^{1/2}, \tag{87c}$$

where the Gauss coefficients g_1^0 = -29496.5 nT, g_2^0 = -2396.6 nT, and g_3^0 = 1339.7 nT [57]. Then for, say, Princeton's colatitude $\theta = 49°39'$ (for which v = 354 m s^{-1}) $B_\infty = 45\ \mu T$, pointed downward into Earth's surface at an angle (from the horizontal when facing the north geographic pole) $\tan^{-1}(Z/X) = 57.5°$.

Suppose that our cylindrical shell had dimensions $l = 20$ cm, $b = 1$ cm and $a = b/\sqrt{2}$, and was made of MN60 MnZn ferrite, with data sheet values given to be $\mu_r = 6,500 \pm 3,000$ and $\sigma \approx 0.5$ S m^{-1} [58]. Then $R_m = 1.4 \times 10^{-2} \ll 1$, while $R_m \gg (v/c)^2$ ensures that $\sqrt{g_{00}}$ effects are small compared to first-order perturbations scaled by R_m. For $\phi_0 = 45°$, Eq. (63) gives emf $= 65\ \mu V$.

By inspection of the integral in Eq. (63), the emf should reverse sign when the shell (together with the attached measuring apparatus, a digital voltmeter; see Fig. 1) is rotated by 180°. This is a striking prediction that should separate an emf generated by the effect predicted here from other types of emf generation. Our derivation is only valid for **v** transverse to the shell, but the emf must pass through zero between the two transverse orientations that are separated by 180°. A voltmeter across d and f in Fig. 1 would measure half the emf around C in Eq. (63). We caution that C may "choose itself" under rotation, and experiment will show whether a voltage measurement actually somehow averages over many possible current paths. If so, we may approximate the expected emf by averaging over ρ and ϕ in the calculation leading to Eq. (63):

$$< \text{emf} >= -\frac{1}{\pi(b-a)} \int_a^b \int_0^\pi v B_x l \; d\rho d\phi = -(4/3\pi) R_m v \beta_2 l (a/b)^2 (1 - a/b)^{-1} \ln(b/a), \quad (88)$$

which for the identical parameter values as above gives $< \text{emf} >= 46\ \mu$V. Once again, emf measured as in Fig. 1 yields half this value, and the sign reverses under 180° rotation.

XIV. SCALING AND CONCLUSIONS

The cylindrical shell was chosen as an especially simple realization of a conductor with $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq \mathbf{0}$, and MN60 material was chosen to provide $R_m \ll 1$ on a laboratory scale. For the MN60 device considered above, $P_k \approx 16$ nW by Eq. (84). By the maximum power transfer theorem at most half of this power can be transferred to the load [59]. To be useful, the effect must be scaled up greatly in voltage and power. One way might be to maintain $R_m \ll 1$ while increasing σ, by decreasing μ_r, b, and therefore a. Carbon nanotubes can be coated with materials such as iron [60, 61], so very small low-R_m magnetically permeable tubes seem plausible. One must also consider resistance and ohmic loss.

It should be possible to separate the magnetic shield producing $\nabla \times (\mathbf{v} \times \mathbf{B}) \neq \mathbf{0}$ from the conductor providing $R_m \ll 1$. For example, note that the functional form of Eq. (9), for the magnetic flux density outside a magnetically permeable shell, is identical to that of Eq. (10) for the flux density in the shell's interior. This guarantees that Eq. (21) holds outside the shell with the substitutions $\beta_2 \to -\beta_3$ and $a^2 \to b^2$. Therefore we should be able to realize the effect using a magnetically permeable cylinder surrounded by an insulated concentric cylindrical shell of a non-permeable low-R_m material, and find results analogous to those found above. Graphite has $\sigma = 7.3 \times 10^4$ S m^{-1} [37], giving $R_m \approx 2 \times 10^{-2}$ for $b = 1$ mm, so we can hope to realize the effect for a mu-metal or ferrous cylindrical core surrounded by a thin insulator with an overlying shell of graphite. Decreasing b to 5 μm would allow copper ($\sigma = 6.0 \times 10^7$ S m^{-1} [37]) or other common metals to be used for the outer layer, with obvious advantages. Altogether different topologies and materials are possible.

The effect predicted here would be available nearly globally and with no intermittency, but requires testing then further examination to see if it or some other configuration based on broadly similar principles could be scaled to practical emission–free power generation. Devices could have important practical implications even if only voltages of ~ 1 volt could be achieved. Such a device would represent a small-application power supply whose lifetime would be limited only by material degradation. At the other, extreme end of speculations regarding generated power, we note that global installed power generation capacity is projected to grow to 10,700 GW by 2040 [62]. Imagine as an upper limit that human civilization generated this power entirely from Earth's rotation through its magnetic field. Over a century, the resulting kinetic energy loss would increase Earth's rotation period by 7 ms. This may be compared to fluctuations in the length of Earth's day of 10 ms over time intervals of several decades [63], and an observed long-term increase (dominated by lunar tidal recession) of 2.5 ms per century [64].

ACKNOWLEDGMENTS

We thank three anonymous referees for their reviews, and are grateful for helpful discussions with T. H. Chyba, R. L. Garwin, M. J. Rees, P. J. Thomas, E. L. Turner, and three anonymous colleagues. We thank G. Z. McDermott and G. Cooper for administrative support, B. A. Lin for reference assistance, and M. Northrup of National Magnetics Group Inc.

for help with materials. CFC acknowledges research funds from the Woodrow Wilson School and the Department of Astrophysical Sciences at Princeton University. KPH acknowledges support through the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration, and through the NASA Exobiology Program (NNH09ZDA001N).

APPENDIX A: CO-ROTATION OF THE IONOSPHERE

What happens to charged particles in a conducting plasma around Earth in the presence of Earth's non-rotating axially symmetric field? Hones and Bergeson [24], building on Davis [20, 25] and Backus [30] examined this question for the complicated general case of magnetic fields with both axisymmetric and non-axisymmetric components, treating the purely axisymmetric case as a special case of their more general result. Here we follow their overall logic but present the simpler calculation for the axisymmetric component only: the special case of a magnetic dipole aligned antiparallel to Earth's rotation axis.

An observer in a non-rotating frame sees a (non-rotating) dipole field anti-aligned with Earth's axis, given by $B_r = -(2M/r^3)\cos\theta$, $B_\theta = -(M/r^3)\sin\theta$, and $B_\varphi = 0$, where M is a constant proportional to the magnetic dipole moment. In general the electric field seen in this frame is given by $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla V$. However, for our non-rotating dipole we can put $\partial\mathbf{A}/\partial t = 0$ so $\mathbf{E} = -\nabla V$. Earth's rotation through its own dipole field leads to an electrostatic field within the Earth that balances the resulting $\mathbf{v}\times\mathbf{B}$ force: $\mathbf{E} = -\nabla\mathbf{V} = -\mathbf{v}\times\mathbf{B}$, which gives $V = -(M\omega/r)\sin^2\theta$, and a surface potential for Earth (of radius $R_\oplus$) of $V(r = R_\oplus) = -(M\omega/R_\oplus)\sin^2\theta$. Fields $\mathbf{E}$ and $\mathbf{B}$ in the plasma must satisfy $\mathbf{E}\cdot\mathbf{B} = 0$, due to the plasma's near-infinite conductivity parallel to the magnetic field lines. This condition gives $2\cos\theta\,\partial V/\partial r = -(1/r)\sin\theta\,\partial V/\partial\theta$. The solution consistent with the boundary condition at $r = R_\oplus$ is $V = -(M\omega/r)\sin^2\theta$, so that $E_r = (M\omega/r^2)\sin^2\theta$, $E_\theta = -(2M\omega/r^2)\sin\theta\cos\theta$, and $E_\varphi = 0$. (Note that this satisfies $\mathbf{E}_\infty = 0$.) Charged particles in this plasma drift azimuthally at a velocity $\mathbf{v} = (\mathbf{E}\times\mathbf{B})/B^2$; direct calculation gives $\mathbf{v} = -\omega r\sin\theta\,\hat{\phi}$ for particles of any charge or mass. That is, the ionosphere comes into co-rotation with Earth because the charged particles composing it acquire exactly the necessary co-rotation velocity from their interactions with Earth's non-rotating axially symmetric field together with the electric field induced in the ionosphere. Earth's rotation through the non-rotating axisymmetric component of its magnetic field drives ionospheric co-rotation.

The non-axisymmetric components — those components that give Earth's magnetic field its tilt away from Earth's rotation axis — of course do rotate with Earth. Since magnetic field lines are defined as lines everywhere tangent to the magnetic field, an observer well away from Earth who could somehow see field lines would see Earth's tilted dipole lines rotating with Earth — the rotating lines being the vector sum of a non-rotating azimuthally constant component plus a rotating azimuthally varying component.

There is a standard result that magnetic lines of force in a perfectly conducting fluid move with the fluid — the fluid is "line-preserving." [9, 51, 65]. (However, magnetic field lines are not relativistically covariant [51], and their reality must be treated with care [21, 66, 67].) When we calculate the equations for the magnetic field lines of a tilted dipole, we find that these lines are described by axisymmetric time-independent terms (from the non-rotating axisymmetric dipole) plus terms sinusoidal in ωt, i.e. terms that rotate with Earth. The field lines do indeed vary sinusoidally with ωt, due to the superposition of a rotating component on top of an underlying axially symmetric component.

Magnetic field lines must satisfy $\mathbf{dl} \times \mathbf{B} = 0$, where $\mathbf{dl}$ is the arc length. This leads to the usual condition

$$dr/B_r = r d\theta / B_\theta. \tag{A1}$$

Earth's magnetic potential U, taking into account only the lowest-order terms for the axisymmetric dipole (g_1^0) and inclined dipole (g_1^1 and h_1^1) terms, is [23]:

$$U = g_1^0 (a^3/r^2) \cos\theta + (a^3/r^2)(g_1^1 \cos\varphi + h_1^1 \sin\varphi) \sin\theta, \tag{A2}$$

where $g_1^0 = -29496.5$ nT, $g_1^1 = -1585.9$ nT, and $h_1^1 = 4945.1$ nT [57]. Because of Earth's rotation, a non-rotating observer co-orbiting with Earth would see $\varphi = \omega t$ where ω is Earth's angular speed. Using Eq. (A1) with $B_r = -\partial U/\partial r$ and $B_\theta = -r^{-1}\partial U/\partial\theta$, we find

$$B_r = 2 g_1^0 (a/r)^3 \cos\theta + 2(a/r)^3 (g_1^1 \cos\varphi + h_1^1 \sin\varphi) \sin\theta, \tag{A3}$$

$$B_\theta = g_1^0 (a/r)^3 \sin\theta - (a/r)^3 (g_1^1 \cos\varphi + h_1^1 \sin\varphi) \cos\theta, \tag{A4}$$

and

$$\frac{dr}{r} = 2 d\theta \; \frac{g_1^0 \cos\theta + (g_1^1 \cos\varphi + h_1^1 \sin\varphi)\sin\theta}{g_1^0 \sin\theta - (g_1^1 \cos\varphi + h_1^1 \sin\varphi)\cos\theta}. \tag{A5}$$

Now since $g_1^1/g_1^0 \approx 0.05$ and $h_1^1/g_1^0 \approx 0.17$, we may roughly approximate this as

$$\frac{dr}{r} \approx 2d\theta\{\cot\theta + [(g_1^1/g_1^0)\cos\phi + (h_1^1/g_1^0)\sin\phi]\csc^2\theta\}. \tag{A6}$$

Integrating, then exponentiating both sides and using a Taylor expansion yields

$$r \approx r_0 \sin^2\theta - r_0[(g_1^1/g_1^0)\cos\varphi + (h_1^1/g_1^0)\sin\varphi]\sin 2\theta, \tag{A7}$$

where r_0 is a constant of integration and $\varphi = \omega t$. The first term in Eq. (A7) is identical to the usual equation for the field lines of an axisymmetric dipole field [9]. The next term gives the inclined dipole and its rotation with Earth. An observer rotating with Earth at a particular φ could interpret what he or she sees as co-rotating field lines with a shape specific to that value of φ. An observer looking back at Earth who could see field lines would see an inclined dipole rotating with Earth.

APPENDIX B: FAILURE OF THE $\mathbf{v} = \mathbf{0}$ SOLUTION

We demonstrate that the $\mathbf{v} = \mathbf{0}$ solution $\mathbf{B_0}$ (Eq. (10)) is no longer a solution for the magnetically permeable cylindrical shell once the shell is moving with $\mathbf{v} = v\hat{\mathbf{y}}$ in K (Fig. 1). We assume $\mathbf{B_0}$ (or equivalently, $\mathbf{A_0}$ with allowance for gauge ambiguity) remains the solution even though $\mathbf{v} \neq \mathbf{0}$, and show that this leads to a contradiction.

When $\mathbf{v} = \mathbf{0}$, we have $\mathbf{B}_\infty(\rho \gg b) = B_\infty\hat{\mathbf{x}}$ and $\mathbf{E}_\infty(\rho \gg b) = \mathbf{0}$ in K. These must continue to hold once $\mathbf{v} \neq \mathbf{0}$, since the shell's distortion of the fields must go to zero at infinity.

First assume $\mathbf{B_0}(x, y, z, t)$ to be a solution for the $a \neq 0$ cylindrical shell for $\mathbf{v} \neq 0$. Inserting Eq. (10) into Eq. (7) requires $\nabla \times (\mathbf{v} \times \mathbf{B_0}) = 0$. But we know by Eq. (21) that this is false in general. Therefore $\mathbf{B_0}(x, y, z, t)$ cannot be a solution when $\mathbf{v} \neq \mathbf{0}$.

Rather than assuming a solution $\mathbf{B_0}(x, y, z, t)$, we instead treat the disturbance in the background field $\mathbf{B}_\infty$ as moving together with the cylindrical shell at $\mathbf{v}$. We implement this in Eqs. (9) to (11) by referring the coordinates of $\mathbf{B_0}$ to the K' system $(x', y', z', t') = (x, y - vt, z, t)$. For example when $\mathbf{v} \neq \mathbf{0}$, Eq. (9a) would be

$$B'_{0x}(\rho' > b) = B_\infty + \beta_3(b/\rho')^2\cos 2\phi', \tag{B1}$$

where $\rho' = (x^2 + y'^2)^{1/2}$, $\phi' = \tan^{-1}(y'/x)$, and of course $y' = y - vt$. Correspondingly, Eq. (18) becomes

$$A'_0(\rho' > b) = B_\infty y' + \beta_3(b^2/\rho')\sin\phi'. \tag{B2}$$

Henceforth in this discussion primed field quantities are understood to be written in terms of the coordinate y'. In the limit $\mathbf{v} \to \mathbf{0}$, Eqs. (B1), (B2), and their analogs go to Eqs. (9) to (11) and (18) to (20), as required.

In this appendix only we make the following simplifying choice of gauge [33, 41]:

$$\mathbf{A}' \to \tilde{\mathbf{A}}' = \mathbf{A}' + \nabla' \int V' dt', \tag{B3a}$$

so that

$$V' \to \tilde{V}' = V' - \frac{\partial}{\partial t'} \int V' dt' = 0. \tag{B3b}$$

I.e. the corresponding gauge condition is $V' = 0$. Because $V' = V - vA_y = V$ since only A_z is nonzero, we have $V' = V = 0$. Henceforth dropping the tilde on $\mathbf{A}'$, we have

$$\mathbf{E}' = -\partial \mathbf{A}'/\partial t'. \tag{B4}$$

Eqs. (B2) and (B4) give $\mathbf{E}'(\rho' > b) = 0$, so by Eq. (5), $\mathbf{E}(\rho > b) = vB_{0x}\hat{\mathbf{z}}$, where B_{0x} is given by Eq. (9a). But by taking $\rho \to \infty$ in Eq. (9a), this means $\mathbf{E}_\infty = vB_\infty\hat{\mathbf{z}}$, which contradicts our premise that $\mathbf{E}_\infty = \mathbf{0}$ in K. Therefore Eq. (B2) cannot be a solution for the magnetically permeable cylindrical shell once $\mathbf{v} \neq \mathbf{0}$.

But perhaps we can add a piece to A_0' that preserves $\mathbf{B_0'}$ while giving $\mathbf{E}_\infty = \mathbf{0}$? (This would not be a gauge transformation, as we would be explicitly attempting to alter the field quantity $\mathbf{E}$ while preserving $\mathbf{B_0'}$.) We now show that this is impossible, so that there is no modification of Eqs. (18) to (20) that both maintains $\mathbf{B}' = \mathbf{B_0'}$ and is consistent with the requirement that $\mathbf{B}_\infty(\rho \gg b) = B_\infty\hat{\mathbf{x}}$ and $\mathbf{E}_\infty(\rho \gg b) = 0$. Whatever term is added to Eq. (B2) cannot vary with x or y', or else $\mathbf{B_0'}$ will be changed, in contradiction to our premise. If we tried instead to add a spatially constant term $vB_\infty t'$ to Eq. (B2) to alter $\mathbf{E}'$ and thereby $\mathbf{E}_\infty$, by Eq. (17) we would still change B_{0x}' by $vB_\infty(\partial t'/\partial y) = B_\infty$, again contradicting a premise. We have therefore shown that $\mathbf{B_0'}(x', y', z', t') = \mathbf{B_0'}(x, y - vt, z, t)$ cannot be a solution when $\mathbf{v} \neq \mathbf{0}$. In effect, the "solution" $\mathbf{B_0'}(x, y - vt, z, t)$ is incompatible with the premise that $\mathbf{B}_\infty$ does not rotate together with the frame K'.

APPENDIX C: CHOICE OF GAUGE

While the gauge condition Eq. (22) is cited in the literature [43, 44], it is not included in lists of standard electrodynamics gauges [68]. We therefore discuss it further here, and show that it satisfies the requirements of gauge invariance. A gauge transformation leaves $\mathbf{B}$ and $\mathbf{E}$ unchanged provided the transformed vector and scalar potentials satisfy

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\chi \tag{C1}$$

and

$$\tilde{V} = V - \partial\chi/\partial t. \tag{C2}$$

Now take the divergence of Eq. (C1), multiply it by η, and add to this Eq. (C2), to obtain:

$$\tilde{V} + \eta\nabla\cdot\tilde{\mathbf{A}} = V + \eta\nabla\cdot\mathbf{A} + (-\partial\chi/\partial t + \eta\nabla^2\chi). \tag{C3}$$

The gauge condition Eq. (22) therefore holds both before and after the gauge transformation Eqs. (C1) and (C2) provided χ satisfies the diffusion equation

$$\partial\chi/\partial t = \eta\nabla^2\chi. \tag{C4}$$

That is, gauge invariance with χ satisfying Eq. (C4) leads to the gauge condition Eq. (22).

APPENDIX D: CYLINDRICAL COORDINATE REPRESENTATION

Some calculations are most easily performed with $\mathbf{B_0}$ and $\mathbf{B_1}$ in cylindrical coordinates. For convenience, we give this representation here. We have

$$B_\rho = B_x\cos\phi + B_y\sin\phi, \tag{D1a}$$

$$B_\phi = -B_x\sin\phi + B_y\cos\phi, \tag{D1b}$$

so that, from Eq. (10):

$$B_{0\rho}(a\le\rho\le b) = [\beta_1 - \beta_2(a/\rho)^2]\cos\phi; \tag{D2a}$$

and

$$B_{0\phi}(a\le\rho\le b) = [-\beta_1 - \beta_2(a/\rho)^2]\sin\phi. \tag{D2b}$$

Using Eq. (D2) and $\mathbf{v} = v\hat{\mathbf{y}} = v\sin\phi\ \hat{\rho} + v\cos\phi\ \hat{\phi}$ yields a simpler expression for Eq. (21):

$$\nabla\times(\mathbf{v}\times\mathbf{B_0}) = 2v\beta_2a^2\rho^{-3}[\sin 2\phi\ \hat{\rho} + \cos 2\phi\ \hat{\phi}]. \tag{D3}$$

We also have, from Eqs. (60) and (62):

$$B_{1\rho}(a\le\rho\le b) = -(1/2)R_mb^{-1}\beta_2a^2\rho^{-1}\sin 2\phi; \tag{D4a}$$

and

$$B_{1\phi}(a \leq \rho \leq b) = 0. \tag{D4b}$$

The first term on the right hand side of Eq. (85) may then be evaluated via Eq. (65) and the vector Laplacian in cylindrical coordinates. The calculation is tedious but straightforward.

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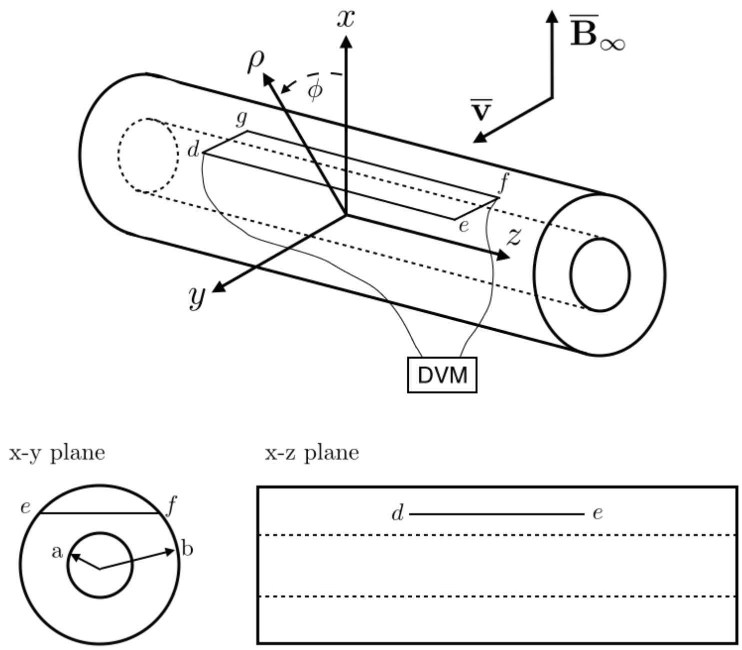

FIG. 1. A magnetically permeable, low-R_m cylindrical shell with inner and outer radii a and b and length L is moving at velocity $\mathbf{v} = v\hat{\mathbf{y}}$ through background fields $\mathbf{B}_\infty = B_\infty\hat{\mathbf{x}}$ and $\mathbf{E}_\infty = 0$. A rectangular current path C with vertices d, e, f, g is embedded in, and translating with, the shell. An emf is generated around C according to Eq. (63). A digital voltmeter (DVM) measures half this emf between d and f. C lies in the plane $x = b\cos\phi_0 \equiv x_0$, with $|x_0| \geq a$. It has right-angle vertices at $d = (x_0, y_0, -l/2), e = (x_0, y_0, l/2), f = (x_0, -y_0, l/2)$ and $g = (x_0, -y_0, -l/2)$, where $y_0 = b\sin\phi_0$ and $l/2 < L/2 - a$.

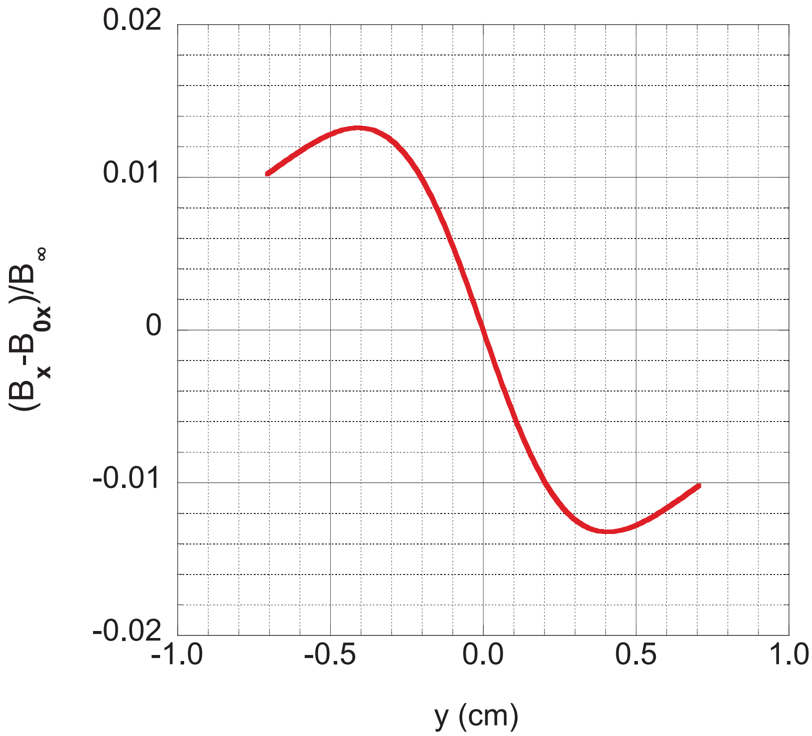

FIG. 2. Deviations in the x component B_x of the magnetic flux density (Eq. 59) for the moving cylindrical shell from that for the stationary shell (Eq. 10), relative to the flux density at infinity, for a shell made of MN60 material (see text) with $b = 1$ cm. These results are for the $x_0 = b/\sqrt{2}$ plane, with $(a/b) = 1/\sqrt{2}$.

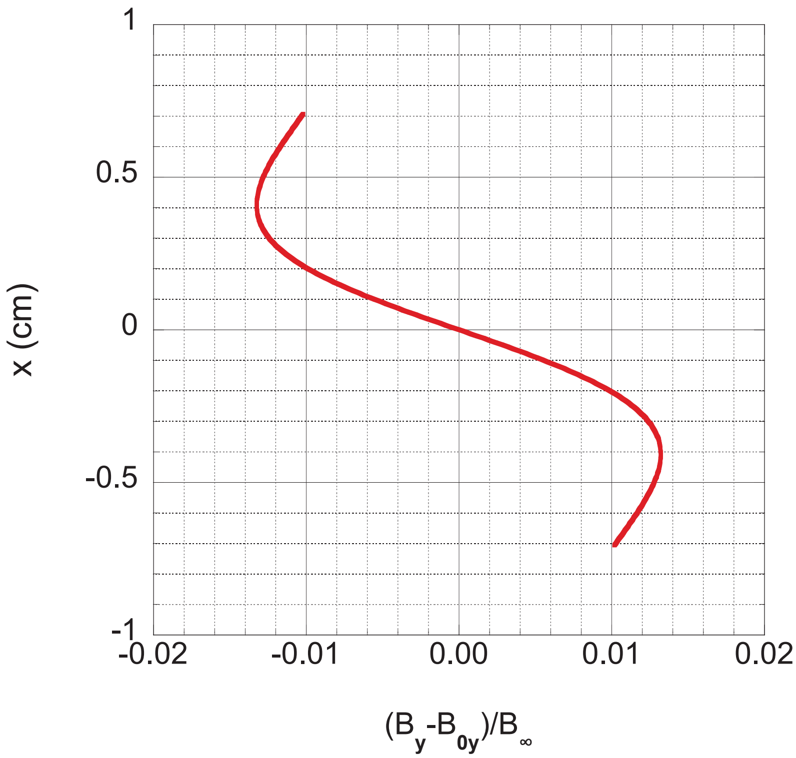

FIG. 3. Deviations in the y component B_y of the magnetic flux density (Eq. 61) for the moving cylindrical shell from that for the stationary shell (Eq. 10), relative to the flux density at infinity, for a shell made of MN60 material (see text) with $b = 1$ cm. These results are for the $y_0 = b/\sqrt{2}$ plane, with $(a/b) = 1/\sqrt{2}$.