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# Stability-derived bounds on the realizable acoustic properties of active metamaterials

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Active acoustic metamaterials consisting of sensor and driver elements that produce a programmed response to impinging acoustic waves have been widely considered an appealing way to overcome the fundamental shortcomings of passive structures. However, research into active metamaterials has been hindered by their potential instability resulting from very complex interactions between cells and the lack of realistic stability models taking these interactions into account. We present a general method for assessing the stability of active metamaterials of arbitrary geometry and number of cells and show that stability imposes previously unknown limitations on the realizable acoustic properties. We validate the theory by accurately predicting the bounds on the realizable effective properties of an experimental metamaterial.

Acoustic metamaterials can exhibit effective acoustic properties beyond those of their constituent materials as a result of their subwavelength architectures [1–3]. Typically, they are passive structures, many of which leverage resonance to yield extreme properties at a select frequency, but there has been growing interest in active metamaterials that utilize powered sensor and driver components to directly generate a programmed acoustic response to impinging waves. This design strategy has great advantages in its flexibility, broadband potential, and range of achievable properties. Experimental demonstrations of active acoustic metamaterials have shown the capability to program conventional properties like the bulk modulus [4–6] and mass density [4, 5, 7–9], as well as more exotic behavior such as Willis coupling [10–14], non-reciprocity [12, 14–18], and anomalous reflection [19]. Similar sensor-driver architectures have also been demonstrated in the elastic domain [20–24]. A clear shortcoming of these demonstrations was their limitation to simple geometries, such as a single unit cell [4, 5, 7, 9, 12, 13, 16, 18], a row of a few cells with 1D operation [14, 21, 22], or a metasurface of a few cells arranged such that they do not interact with each other [10, 11, 19, 20, 25, 26].

A major obstacle to the implementation of active metamaterials of many cells and their application in practical devices is their potential instability. Each cell is independently programmed via an open-loop [4, 5, 10, 11, 14, 16, 19, 20, 23, 25] or feedback control [7–9, 13, 17, 18, 22, 26, 27] strategy, typically with the intent to realize a medium characterized by passive effective acoustic properties. Even so, discrepancies between a physical and ideal implementation will be unavoidable and can result in responses that grow exponentially. Some sources of these discrepancies include unwanted feedback between sensors and drivers within a unit cell, only marginally subwavelength unit cells, non-ideal frequency responses of the components, and causality-imposed constraints on the phase of the re-

sponse. Previous experimental demonstrations that included a stability analysis mainly focused on the feedback loop between sensors and drivers within a single cell, analyzing closed-loop transfer functions derived from first principles and lumped parameter models [4, 8, 9, 16, 27]. However, the numerous complex interactions between cells can also cause instability and are much more difficult to model. It is sometimes assumed that the stability of a single active cell implies the stability of a large metamaterial of many cells, but this is not the case, as discussed for example in Refs. [28–30] regarding active electromagnetic media and superluminal propagation. For 1D metamaterial geometries, it has been suggested to identify the system poles using a scattering matrix/transfer matrix method [4, 14, 31], though this approach does not scale well for more complicated geometries that cannot be reduced to just two ports. Another approach is to experimentally test each cell at frequencies in potentially unstable bands once the full metamaterial has already been implemented [21], but it is far preferable to have an analysis that can inform the design beforehand. In general, given a single physical active unit cell, there is no practical method to predict what effective properties a metamaterial of many such cells in a desired geometry could stably realize.

Here, we present such a method, applicable to linear time-invariant active metamaterials, which requires knowledge only of the individual unit cell frequency response. As a demonstration, we apply this method to a physical active metamaterial of programmable bulk modulus and verify the predicted range of stable gains and corresponding achievable bulk moduli. We also assess the stability consequences of varying unit cell and overall size of the metamaterial, which can be used to guide future designs.

Without loss of generality, we consider an active acoustic metamaterial model that is a two-dimensional (2D) extension of the previous works [4–24]. The metamaterial is shown embedded in the lower plate of a 2D waveguide

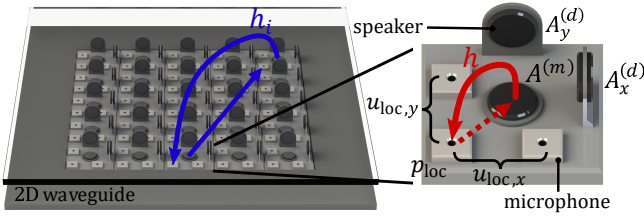


FIG. 1. Model active acoustic metamaterial, with inset of a single unit cell. The cell is shown with an example feedback loop between cells (blue) and another feedback loop (red) between sensing and driving components within the cell.

in Fig. 1. Its unit cells consist of an arrangement of three microphones, which can individually sense the local pressure  $p_{\text{loc}}$  or the particle velocity components  $u_{\text{loc},x}$ ,  $u_{\text{loc},y}$  in pairs, and three speakers that act as monopole or dipole sources with amplitudes  $A^{(m)}$ ,  $A_x^{(d)}$ ,  $A_y^{(d)}$ . Electronically programmed transfer functions between the sensed and driven quantities can be used to implement a range of effective acoustic behavior. First, we will analyze this metamaterial with cells utilizing just one pressure sensor measuring  $p_{\text{loc}}$  and one monopole driving element with transfer function  $g = A^{(m)}/p_{\text{loc}}$  and discuss later how the process can be extended to cells combining monopole and dipole responses.

Figure 1 shows two types of feedback loops that may cause instability. The first type (denoted  $h$  in the inset) is the feedback between the sensor and driver within the cell and has been well addressed by the previously mentioned works considering the stability of one cell in isolation [4, 8, 9, 16, 27]. The second type involves the interaction between any two cells in the metamaterial, i.e., one cell generates sound that is sensed by a second cell, which in turn generates sound that is sensed by the first cell. One example of this feedback loop is shown in Fig. 1 as  $h_i$ , where the index  $i$  would refer to the second cell in the above case. When a set of cells collectively provides feedback to one cell, we will denote this as  $h_{[i,j]}$ , where  $i$  and  $j$  indicate the first and last cells in the set.

A classical approach to determine the stability of an  $N$ -cell metamaterial is to treat the whole metamaterial as a single multiple-input multiple-output system, determine the transfer functions between each pair of sensors and drivers in the complex Laplace plane, and find their poles [32, 33]. However, this is an unscalable, computationally intensive approach infeasible for large values of  $N$ , which provides little insight into how stability influences the achievable effective parameters. Instead, we will employ an incremental approach in which we build our metamaterial by adding unit cells one-by-one, applying the Nyquist Stability Criterion to the response of each newly added cell until the desired geometry is reached.

We begin our analysis with a single unit cell that would ideally have a transfer function  $g$  to implement desired

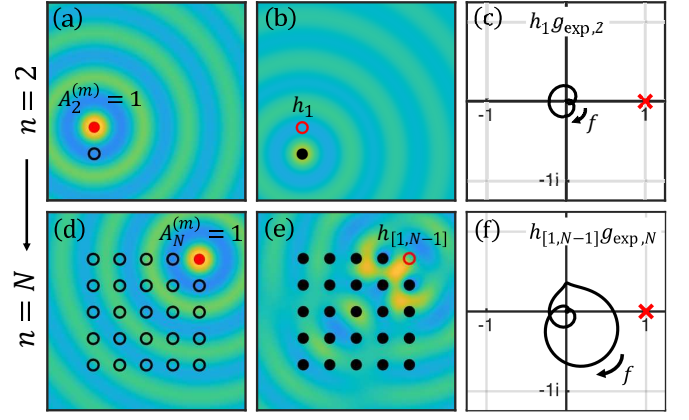


FIG. 2. The metamaterial stability is determined from the open-loop frequency responses of incrementally added unit cells, from the second  $n = 2$  (top row) to the last  $n = N$  (bottom row). (a,d) The unit amplitude output of the  $n$ -th cell results in (b,e) feedback from the previously added cells. (c,f) Example Nyquist plots where encirclement of the red  $\times$  marker would indicate instability.

effective behavior. However, discrepancies between the physical implementation and the ideal system result in the cell actually having the closed-loop transfer function  $g_{\text{exp}}$ . With the intent to use the Nyquist Stability Criterion, we will only need the frequency domain representation of  $g_{\text{exp}}$  as a steady-state frequency response. This can be found experimentally by sending a series of waves of varying frequency from a background source to impinge on the active cell and measuring the scattered pressure with an external microphone.

Given the first cell is stable in isolation, we can consider the addition of a second individually stable unit cell with frequency response  $g_{\text{exp},2}$  to our metamaterial. When the field of an arbitrary background source of amplitude  $A_b$  impinges on the metamaterial, the amplitude  $A_2^{(m)}$  of the newly added unit cell can be written as

$$\frac{A_2^{(m)}}{A_b} = \frac{g_{\text{exp},2} (q_{b2} + q_{b1} g_{\text{exp},1} q_{12})}{1 - g_{\text{exp},2} (q_{21} g_{\text{exp},1} q_{12})}. \quad (1)$$

Here,  $q_{ij}$  is the propagation of the acoustic field from the location of source/driver  $i$  to that of sensor  $j$  (with 1 corresponding to the first unit cell, 2 to the second). In the frequency domain,  $q_{ij}$  can be identified through experiment, simulation, or most conveniently, analytical modeling using simple sources [34, 35]. The terms in Eq. (1) can be rearranged and combined to write

$$\frac{A_2^{(m)}}{p_b} = \frac{g_{\text{exp},2}}{1 - g_{\text{exp},2} h_1}, \quad (2)$$

with  $p_b = A_b (q_{b2} + q_{b1} g_{\text{exp},1} q_{12})$  and  $h_1 = q_{21} g_{\text{exp},1} q_{12}$ . The response of the second cell has been reduced to a closed-loop form. Here,  $p_b$  is the pressure at the cell 2

location before cell 2 was added, generated by the stable system consisting of just cell 1 and the background source. The transfer function  $h_1$  represents the ratio between the field scattered by cell 1 (due to the cell 2 driver) and  $A_2^{(m)}$ , i.e., the cell 2 driver amplitude. The scattered field is evaluated at the cell 2 location. Example pressure fields of this feedback loop at one frequency are shown in Fig. 2(a,b), where the newly added cell is indicated by a red marker, previously added cell by a black marker, driving cells by filled markers, and sensing cells by open markers.

To determine the stability of the closed-loop response given by Eq. (2), we can employ the Nyquist Stability Criterion  $Z = P + N$ , where  $Z$  is the number of unstable closed-loop poles,  $P$  is the number of unstable open-loop poles, and  $N$  is the number of encirclements of the point  $(+1, 0)$  on the Nyquist plot of the open-loop system [36]. Given that the open-loop response is stable, which we know is true for  $g_{\text{exp},2}h_1$  because both individual unit cells are stable, then  $P = 0$  and our closed-loop stability criterion becomes  $N = 0$ , no encirclements of  $(+1, 0)$ . An example Nyquist plot for the second cell is shown in Fig. 2(c). It should be noted that the background pressure  $p_b$  does not affect the determination of stability given that it is finite.

We can extend this procedure to determine the stability of an  $N$ -cell metamaterial. Adapting Eq. (2), the frequency response for the  $N$ -th cell in closed-loop form is derived to be (see the Supplemental Material [37])

$$\frac{A_N^{(m)}}{p_b} = \frac{g_{\text{exp},N}}{1 - g_{\text{exp},N}h_{[1,N-1]}}, \quad (3)$$

where  $p_b$  represents the pressure due to the background source and the  $(N - 1)$ -cell metamaterial at the cell  $N$  location before the addition of cell  $N$  and  $h_{[1,N-1]}$  is the transfer function quantifying how the  $(N - 1)$ -cell metamaterial scatters the field generated by the cell  $N$  driver, i.e., it is the extension of  $h_1$  to the  $(N - 1)$ -cell metamaterial. The transfer function  $h_{[1,N-1]}$  is shown in Fig. 2(d,e) and can now be evaluated as a sum of the responses of the individual cells that make up the  $(N - 1)$ -cell metamaterial. The derivation of  $h_{[1,N-1]}$  is provided in the Supplemental Material [37].

We are now able to calculate the open-loop response of Eq. (3) for the Nyquist plot, an example of which is shown in Fig. 2(f) for a nine-cell metamaterial. However, this is not sufficient to immediately determine the stability of a metamaterial of  $N$  cells. Our use of the stability criterion  $N = 0$  requires open-loop stability and a finite  $p_b$ . This means that we need to first prove the stability of the metamaterial of  $N - 1$  cells, which would require the stability of the metamaterial of  $N - 2$  cells, and so on. Therefore, we must add new cells one at a time, checking stability for the addition of the  $n$ -th cell at every step until the full metamaterial is reached. The order in which

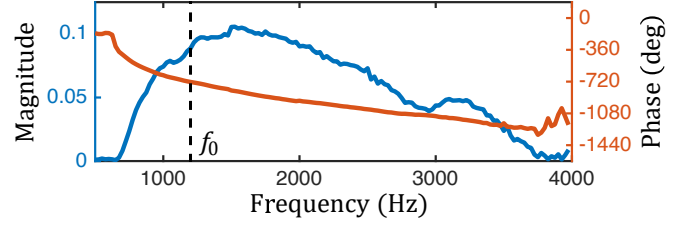


FIG. 3. Experimental frequency response  $g_{\text{exp}}^{(m)}$  of the monopole component of a unit cell.

the cells are added to the metamaterial is arbitrary, so only one path to the full metamaterial must be stable at every addition.

As a demonstration, we will apply this stability analysis to an example physical active acoustic metamaterial with the same unit cell architecture. The metamaterial consists of a  $3 \times 3$  arrangement of  $N = 9$  unit cells, which we will see is sufficiently many to clearly demonstrate the stability constraints that arise from cell interactions. The metamaterial was designed to operate at the frequency  $f_0 = 1200$  Hz, with six cells per wavelength  $\lambda_0$  and overall size of  $\lambda_0/2 \times \lambda_0/2$ . Each cell is programmed using an open-loop strategy according to a polarized source model [35]. The open-loop frequency response  $g^{(m)}$  necessary for the desired effective bulk modulus  $\kappa_{\text{eff}}$  relative to the background medium (i.e.  $\kappa_{\text{eff}} = 1$  when the metamaterial is off) is

$$g^{(m)} = \frac{A^{(m)}}{p_{\text{loc}}} = (ka/2)^2 \left(1 - \frac{1}{\kappa_{\text{eff}}}\right), \quad (4)$$

where  $k$  is the background wavenumber and  $a$  is the unit cell width. To implement a passive medium with real  $\kappa_{\text{eff}}$ , one that does not add or remove energy from the impinging acoustic field,  $g^{(m)}$  must have a phase of  $0^\circ$  ( $\kappa_{\text{eff}} > 1$ ) or  $180^\circ$  ( $\kappa_{\text{eff}} < 1$ ) that is constant with frequency. The magnitude of  $g^{(m)}$  required for a  $\kappa_{\text{eff}}$  that is constant with frequency increases with the square of the frequency. The magnitude also increases as the desired  $\kappa_{\text{eff}}$  becomes further from that of the background medium. More detail on the operation of the physical metamaterial and its experimental testing can be found in the related work Ref. [6].

We begin our stability analysis by determining the experimental frequency response  $g_{\text{exp}}^{(m)}$  of a single physical unit cell. A microcontroller or other programmable circuitry can be used to adjust the gain between the speaker and microphone, but they will also have their own intrinsic transfer functions. Additionally, the speaker and microphone are next to each other within the unit cell, resulting in unwanted feedback. The closed-loop  $g_{\text{exp}}^{(m)}$  can be written as

$$g_{\text{exp}}^{(m)} = \frac{A^{(m)}}{p_{\text{loc}}} = \frac{g_{\text{adj}}g_{\text{fix}}}{1 - g_{\text{adj}}g_{\text{fix}}h}. \quad (5)$$

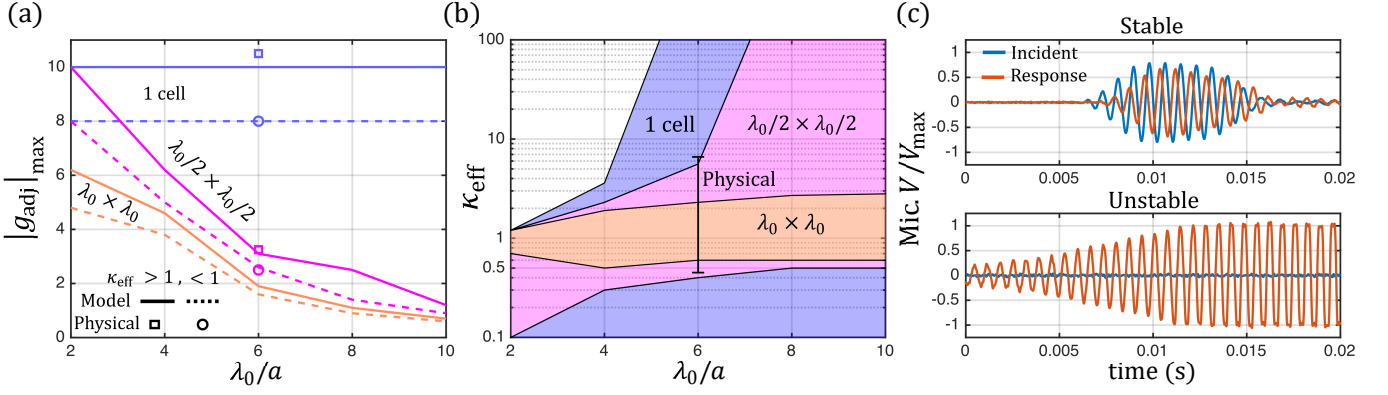


FIG. 4. (a) Maximum stable gains and (b) the corresponding regions of stably realizable effective properties for metamaterials by overall size and cell size. (c) Experimentally obtained stable ( $g_{\text{adj}} = 2.5$ ) and unstable ( $g_{\text{adj}} = 5$ ) time domain responses of the physical  $\lambda_0/2 \times \lambda_0/2$  metamaterial with  $3 \times 3$  arrangement of cells.

The feedforward is composed of an adjustable portion  $g_{\text{adj}}$ , which is a gain and unit delay, and a fixed portion  $g_{\text{fix}}$ , which includes the physical component frequency responses and a digitally implemented bandpass filter. As before, the feedback  $h$  captures the wave propagation from the speaker to the microphone within the cell. To achieve a stable unit cell, the bandpass filter must be designed to exclude frequencies with high feedback and the overall gain limited through  $g_{\text{adj}}$ . As a result, the achievable bandwidth and range of effective properties will be restricted. The experimentally measured  $g_{\text{exp}}^{(m)}$  with  $|g_{\text{adj}}| = 1$  is plotted in Fig. 3. At the target frequency  $f_0$ , the metamaterial is programmed to produce the same scattered field as a passive continuous medium with a desired  $\kappa_{\text{eff}}$ . While the metamaterial will exhibit this passive, and thereby stable, behavior at  $f_0$ , this performance cannot be maintained at the other frequencies in the operating band because causality constrains the phase of  $g_{\text{exp}}^{(m)}$ . For example, the phase must be a decreasing function of frequency in low dispersion, broadband metamaterials (see Fig. 3) leading to loss or gain behavior where the phase is not a multiple of  $180^\circ$ . There is then a trade-off in the bandpass filter design, where reducing the size of the passband may enable stable behavior at higher  $|g_{\text{exp}}^{(m)}|$ .

To determine the stability of a single cell, we apply the Nyquist Stability Criterion to Eq. (5), increasing  $|g_{\text{adj}}|$  until the cell is no longer stable. Using the delay of  $g_{\text{adj}}$ , we do this for both a  $0^\circ$  and  $180^\circ$  response at  $f_0$ , representing a  $\kappa_{\text{eff}}$  greater and less than the background respectively. We then repeat the process, but using our multiple cell incremental stability analysis, to predict the maximum stable gains for the physical metamaterial with the  $3 \times 3$  arrangement of cells as well as theoretical metamaterials with the same cell transfer function  $g_{\text{exp}}^{(m)}$ , but various cell and overall sizes. The unit cell speakers are modeled as a monopole sources with the Hankel function  $jH_0^{(2)}(kr)$  as the 2D propagation frequency response  $q_{ij}$ ,

where  $r$  is the distance between the driving cell  $i$  and the sensing cell  $j$ . For each metamaterial configuration of interest, the cells are added to the system being considered one-by-one and their interactions are modelled analytically. Stability is checked through the Nyquist Criterion at each step, until the full metamaterial has been constructed and its stability assessed.

The predictions for the maximum stable  $|g_{\text{adj}}|$  are plotted in Fig. 4(a) as lines, along with the actual experimental results for the physical metamaterial as markers. The accuracy of the predictions for the physical metamaterial with the  $3 \times 3$  arrangement of cells ( $\lambda_0/2 \times \lambda_0/2$  overall size and  $\lambda_0/a = 6$  unit cell size) indicates that we were able to effectively model the cell interactions with our multi-cell stability analysis. Additionally, it is clear that the interactions between cells, which are strengthened by the proximity and number of cells, reduce the stable  $|g_{\text{adj}}|$  compared to the single cell. The predicted maximum stable gains are translated to boundaries of regions of  $\kappa_{\text{eff}}$  that the metamaterial can realize at the design frequency with stable performance in Fig. 4(b). The regions are overlapping, such that all  $\kappa_{\text{eff}}$  accessible for larger geometries are also accessible at smaller geometries. The  $\kappa_{\text{eff}}$  range corresponding to the results for the physical  $3 \times 3$  metamaterial is shown with a vertical span. Though smaller unit cells are seen to have a lower gain ceiling, the required  $g_{\text{exp}}^{(m)}$  for the same  $\kappa_{\text{eff}}$  is also smaller as seen in Eq. (4). Example stable and unstable time domain responses (cells programmed with  $g_{\text{adj}} = 2.5$  and  $g_{\text{adj}} = 5$  respectively) that were experimentally acquired with an external microphone over the center of the physical metamaterial are plotted in Fig. 4(c). The stable response is a bounded replication of the incident waveform, while the unstable response exponentially grows, even without an incident wave, saturating the external measuring microphone and the metamaterial speakers.

In our analysis and physical example, we considered the stability of an active metamaterial with unit cells

utilizing just a single input and single output, but our approach can easily be extended to multiple-input multiple-output programming. Rather than characterizing the behavior of the  $n$ -th unit cell with a single frequency response, a matrix of frequency responses will be used. To implement a metamaterial of both programmable bulk modulus and mass density, two more gains  $g_x^{(d)}$  and  $g_y^{(d)}$  are required, such that this matrix is diagonal:

$$\begin{bmatrix} A^{(m)} \\ A_x^{(d)} \\ A_y^{(d)} \end{bmatrix} = \begin{bmatrix} g^{(m)} & 0 & 0 \\ 0 & g_x^{(d)} & 0 \\ 0 & 0 & g_y^{(d)} \end{bmatrix} \begin{bmatrix} p_{\text{loc}} \\ u_{\text{loc},x} \\ u_{\text{loc},y} \end{bmatrix}. \quad (6)$$

An active metamaterial with a non-diagonal mass density tensor and Willis coupling could be implemented with a fully populated matrix. Given that one of these cells is stable in isolation, we can apply our incremental method to determine the stability of a metamaterial of many such cells. For the addition of the  $n$ -th cell, each of the matrix elements must be individually substituted as  $g_{\text{exp},n}$  and the resultant closed-loop transfer functions all found to be stable before the next cell is added. The feedback  $h_{[1,n-1]}$  will always include the full response of the previously added cells. If all cell additions are stable, then the overall metamaterial will also be stable.

In conclusion, we have presented an effective method for determining the stability and resultant realizable effective property constraints of an active acoustic metamaterial composed of an arbitrary number of unit cells. We have shown that these constraints depend on the geometry of the metamaterial and the way in which its constituent cells interact. The method, illustrated in detail in an example, can be applied to a variety of architectures and programming schemes of interest. This includes more complex designs, such as those in three-dimensions, with multiple programmed transfer functions, irregular cell arrangements, and spatially-varying cell programming. The ability to predict stability based on parameters such as individual cell and overall metamaterial size is critical to the application of active metamaterials in actual devices.

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  - [37] See Supplemental Material at URL-will-be-inserted-by-publisher for the derivation of the frequency response for the  $N$ -th cell in closed-loop form.