Negative Refraction in Mechanical Rotator Lattices

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Phys. Rev. Applied **18**, 064058 — Published 20 December 2022

DOI: 10.1103/PhysRevApplied.18.064058

Demonstration of Negative Refraction in Mechanical Rotator Lattices

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This paper presents numerical and experimental investigation of negative refraction at the interface between two mechanical rotator lattices, adding to the recent body of anomalous wave behavior reported for such simple lattices. Each lattice implements easily-reconfigured inter-rotator coupling which determines whether the lattice's passband is acoustic (with positive group velocity) or optic (with negative group velocity). Since the group velocities have opposite signs over the entirety of both lattices' Brillouin zones, the negative refraction at their interface is inherently broadband. The numerical study constructs a large lattice structure for full-wave simulation and quantitatively analyzes the linear and nonlinear negative refraction. The experimental study documents robust negative refraction in a smaller-scale fabricated system and serves as validation for the numerical findings. We observe frequency-dependent transmission in both simulations and experiments. A linear analysis captures the observed phenomenon at low amplitude. At larger amplitudes, numerical simulations are used to document amplitude-dependent transmission. This phenomenon is explained by a nonlinear dispersion shift and transitional evanescent waves analyzed using a perturbation method. A sensitivity test demonstrates the robustness of negative refraction in the proposed rotator lattice structure.

I. INTRODUCTION

Negative refraction describes anomalous wave phenomenon wherein the refraction of an oblique incident wave occurs on the same side of the interface normal, which can be compared to conventional materials wherein refraction occurs on the opposite side. The concept was first predicted by V. Veselago [1], who developed a theory of left-handed optical materials with simultaneous negative permittivity and permeability. Later, Pendry [2] showed that negative refractive index materials improve the imaging resolution beyond the diffraction limit. This extraordinary property motivated further exploration into sub-wavelength focusing [2–4], superlens imaging [5–7], waveguiding [8–10] and fabrication of negative index materials [11, 12].

Negative refraction is not peculiar to electromagnetic waves. Recent studies have demonstrated analogous phenomena in acoustics (to include phononics) [13, 14]. Typically, an acoustic negative refractive index material requires negative effective mass density and bulk modulus at the operating frequency [15] or metamaterial designs with the passband's group velocity anti-parallel to the wave vector. Such band structures can result from (i) resonance-based periodic structures [16–19], (ii) hyperbolic metamaterials [20, 21], (iii) space coiling metamaterials [22–26], and perforated and pillared phononic crystals [27–30]. Practical implementations of the resonating mechanism usually consider Helmholtz resonators [16] for fluid-borne sound and hollow microstructures [19] for elastic waves. Due to the nature of resonance, these materials suffer from narrow operating ranges. Alternative designs consider folded acoustic channels [22, 24] or extreme anisotropic structures [20] to generate negative refractive index passbands. These designs broaden the operating frequency range to some extent, but are restricted by intricate unit cell designs or only allow for partial focusing. In addition, the analysis of acoustic and elastic negative refraction remains largely devoted to the linear regime. The nonlinear effect of negative refraction, such as associated with amplitude-dependent dispersion, has been formulated in optics [31], yet sparsely investigated in acoustics.

In this study, we numerically and experimentally investigate negative refraction in a reconfigurable nonlinear rotator lattice. Such rotator lattice structures support a number of anomalous wave behaviors, including non-reciprocity [32] and dispersion morphing [33]. Different from other metamaterials harnessing the gyroscopic effects [34–37] and temporal modulation [38] of rotating units, the proposed 2D lattice structure leverages the in-plane rotational geometry to determine the sign of the refractive index and also yields geometric nonlinearity (Sec. II). At the interface between two lattices with opposite-sign refractive indices, we observe robust negative refraction in numerical simulations (Sec. III) and experiments (Sec. IV). Linear dispersion analysis and perturbation-based nonlinear analysis reveal the frequency and amplitude dependence of the refraction transmission, respectively, and predict an amplitude saturation effect. The experiment demonstrates the predicted negative refraction in a finite lattice. We also present a sensitivity study to quantify the robustness of the phenomenon in the experimental setup. These results improve the understanding of negative refraction in the presence of nonlinearity and suggest a promising design for future wave-based devices.

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II. LATTICE DESIGN

Figure 1a and 1b depicts two monatomic rotator lattices with alternate coupling arrangements. In both configurations, the rotators are pinned at their centers such that only angular oscillations occur. We define the counter-clockwise rotation to be positive. In Fig. 1a, each pair of adjacent rotators is connected by two linear elastic linkages mounted on their side arms. As such, we term this type of lattice the side-arm (SA) connected lattice. In Fig. 1b, however, the adjacent rotators are linked from their nearest arm by only one lightly pre-stretched elastic linkage, and this type of lattice is termed the nearest-arm (NA) connected lattice.

Figure 1c and 1d provide detailed views of the SA and NA connections, respectively. To describe the dynamics, we introduce system parameters: rotator radius r, lattice constant a, gap distance $D = a - 2r$, spring stiffness k_{SA} and k_{NA}, and undeformed length L_0^{SA} and L_0^{NA}. When the lattice is perturbed, the elastic linkages between two adjacent rotators are deformed and produce restoring torques. In Fig. 1c, the restoring torque at the rotator of interest (the left rotator) is the sum of the torques from the top and bottom elastic linkages connecting to its neighbor (the right rotator),

$$\mathbf{T}_{SA} = \mathbf{r}_{top}^{SA} \times (k_{SA}\delta\mathbf{L}_{top}) + \mathbf{r}_{bot}^{SA} \times (k_{SA}\delta\mathbf{L}_{bot}), \quad (1)$$

where the radius vectors (parallel to the connecting arm), $\mathbf{r}_{top}^{SA}$, $\mathbf{r}_{bot}^{SA}$, and the deformation vectors (parallel to the elastic linkage), $\mathbf{L}_{top}$ and $\mathbf{L}_{bot}$, are derived from the rotational geometry,

$$\begin{aligned}
\mathbf{r}_{top}^{SA} &= \begin{bmatrix} rcos(\theta_i + \frac{\pi}{2}) \\ rsin(\theta_i + \frac{\pi}{2}) \end{bmatrix}, \\
\mathbf{r}_{bot}^{SA} &= \begin{bmatrix} rcos(\theta_i - \frac{\pi}{2}) \\ rsin(\theta_i - \frac{\pi}{2}) \end{bmatrix}, \\
\delta\mathbf{L}_{top} &= (|\mathbf{L}_{top}| - L_0^{SA})\frac{\mathbf{L}_{top}}{|\mathbf{L}_{top}|}, \\
\delta\mathbf{L}_{bot} &= (|\mathbf{L}_{bot}| - L_0^{SA})\frac{\mathbf{L}_{bot}}{|\mathbf{L}_{bot}|}, \\
\mathbf{L}_{top} &= \begin{bmatrix} a + rcos(\theta_j + \frac{\pi}{2}) - rcos(\theta_i + \frac{\pi}{2}) \\ rsin(\theta_j + \frac{\pi}{2}) - rsin(\theta_i + \frac{\pi}{2}) \end{bmatrix}, \\
\mathbf{L}_{bot} &= \begin{bmatrix} a + rcos(\theta_j - \frac{\pi}{2}) - rcos(\theta_i - \frac{\pi}{2}) \\ rsin(\theta_j - \frac{\pi}{2}) - rsin(\theta_i - \frac{\pi}{2}) \end{bmatrix}, \quad (2)
\end{aligned}$$

For the NA lattice, the restoring torque arises from the spring connecting the nearest arms of a pair of neighboring rotators,

$$\mathbf{T}^{NA} = \mathbf{r}^{NA} \times (k_{NA}\delta\mathbf{L}_{mid}), \quad (3)$$

where the radius and deformation vectors are provided by,

$$\begin{aligned}
\mathbf{r}^{NA} &= \begin{bmatrix} rcos(\theta_i) \\ rsin(\theta_i) \end{bmatrix}, \\
\delta\mathbf{L}_{mid} &= (|\mathbf{L}_{mid}| - L_0^{NA})\frac{\mathbf{L}_{mid}}{|\mathbf{L}_{mid}|}, \\
\mathbf{L}_{mid} &= \begin{bmatrix} a - rcos(\theta_j) - rcos(\theta_i) \\ -rsin(\theta_j) - rsin(\theta_i) \end{bmatrix}. \quad (4)
\end{aligned}$$

Accordingly, we derive the equation of motion governing the dynamics in both SA and NA lattices. The exact equations are, however, cumbersome due to the rotational geometry. In this study, we confine our attention to small angle (wave amplitude) motions, and Taylor-expand the equations of motion around the equilibrium position of the rotators. As such, we present the equations of motions for the SA and NA lattice, respectively,

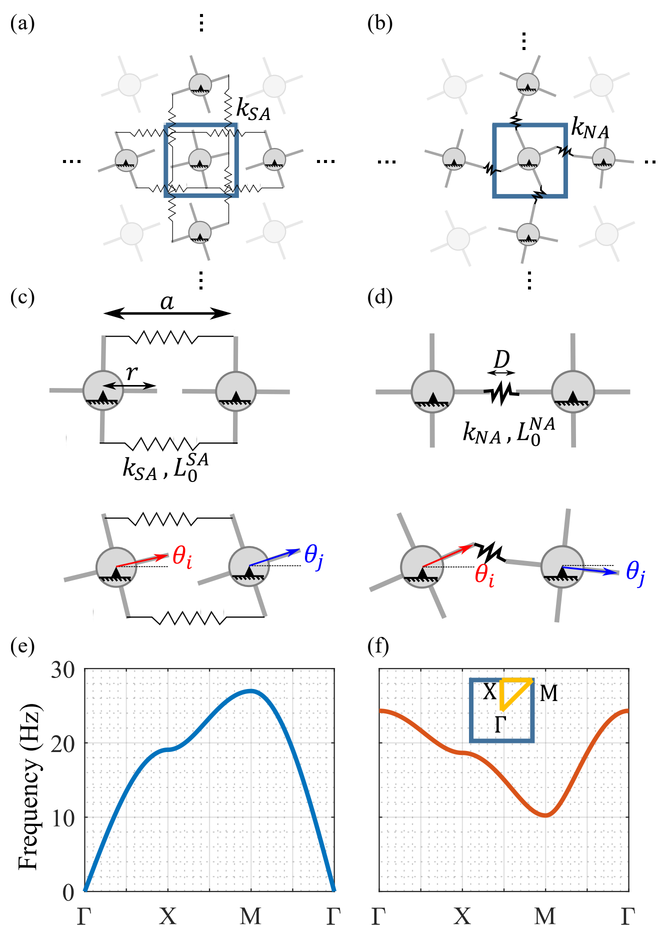

FIG. 1. Schematics of SA type (a) and NA type (b) rotator lattice structures. The blue square identifies the unit cell. (c) The detailed view of the SA inter-rotator connection. (d) The detailed view of the NA inter-rotator connection. (e) SA lattice linear dispersion relation. (f) NA lattice linear dispersion relation.

Parameter	$I_{SA}(kgm^2)$	I_{NA}	$k_1(Nm)$	k_2	k_g
Value	1.95E-5	4.05E-5	0.070	0.097	0.042
Parameter	$\gamma_+^{SA}(Nm)$	γ_-^{SA}	γ_g^{SA}		
Value	5.1E-3	-6.6E-3	-0.0406		
Parameter	$\gamma_+^{NA}(Nm)$	γ_-^{NA}	γ_g^{NA}		
Value	2.407	-0.011	1.1815		

TABLE I. Experimentally measured system parameters

$$I_{SA}\ddot{\theta}_{m,n} + \epsilon \sum_{p,q} k_1(\theta_{m,n} - \theta_{p,q}) + \epsilon^3 \sum_{p,q} (\gamma_+^{SA}(\theta_{m,n} + \theta_{p,q})^3 + \gamma_-^{SA}(\theta_{m,n} - \theta_{p,q})^3 + \gamma_g^{SA}\theta_{m,n}^3) + O(\epsilon^5) = 0, \quad (5)$$

$$I_{NA}\ddot{\theta}_{m,n} + \epsilon \sum_{p,q} (k_2(\theta_{m,n} - \theta_{p,q}) + k_g\theta_{m,n}) + \epsilon^3 \sum_{p,q} (\gamma_+^{NA}(\theta_{m,n} + \theta_{p,q})^3 + \gamma_-^{NA}(\theta_{m,n} - \theta_{p,q})^3 + \gamma_g^{NA}\theta_{m,n}^3) + O(\epsilon^5) = 0, \quad (6)$$

where I and θ denote rotational inertia and angular displacement, respectively. Subscripts $[m, n]$ define the coordinates of the rotator of interest in the 2D lattice, and $[p, q] \in \{[m-1, n], [m+1, n], [m, n-1], [m, n+1]\}$ defines the neighbors of the $[m, n]$ rotator. The linear stiffnesses k_1, k_2, k_g and nonlinear stiffness γ_+, γ_-, γ_g are functions of physical springs' stiffnesses k_{SA} and k_{NA}, as well as the rotational geometry defined by rotator radius r and lattice constant a. We provide the explicit expressions of these equivalent stiffness functions in the Appendix I. In both equations, the small parameter ϵ serves as a bookkeeping device in the series expansion and can be set to 1 in numerical evaluations. We present a set of experimentally measured parameters in Table I.

Next, we investigate linear and weakly nonlinear wave propagation. In both scenarios, the linear dispersion serves as important guidance for direct analysis and nonlinear corrections. With all nonlinear terms omitted, we present the linearized equations of motion and derive the linear dispersion relations [39],

$$I_{SA}\ddot{\theta}_{m,n} + \sum_{p,q} k_1(\theta_{m,n} - \theta_{p,q}) = 0, \quad (7)$$

$$\omega = \sqrt{\frac{k_1(4 - 2\cos(\mu_x) - 2\cos(\mu_y))}{I_{SA}}}, \quad (8)$$

$$I_{NA}\ddot{\theta}_{m,n} + \sum_{p,q} (k_2(\theta_{m,n} - \theta_{p,q}) + k_g\theta_{m,n}) = 0, \quad (9)$$

$$\omega = \sqrt{\frac{4k_g + k_2(4 + 2\cos(\mu_x) + 2\cos(\mu_y))}{I_{NA}}}, \quad (10)$$

where μ_x, μ_y and ω represent the wavenumber components along lattice directions, and the angular frequency, respectively.

For the SA lattice, Eq. (7) describes lattice dynamics identical to a conventional rectilinear 2D monatomic lattice, and Eq. (8) suggests an acoustic branch, depicted in Fig. 1e. The NA lattice, on the other hand, contains an anomalous stiffness term in Eq. (9) – the sum of adjacent rotators' displacement in contrast to the difference commonly seen in rectilinear lattices. This nuance gives rise to optical dispersion captured by Eq. (10) and depicted in Fig. 1f. The difference in group velocity sign, over the entire Brillouin zone of each lattice, leads to broadband negative refraction at the interface joining the two lattice types.

To investigate the negative refraction, we layer the SA and NA lattices as shown in Fig. 2a, and consider an oblique incident wave for the interface problem. We purposefully choose a small frequency range over which the incident and receiving media exhibit spatial beaming behavior such that visual inspection leads to clear evidence of negative refraction in a finite-sized lattice.

III. ANALYSIS AND SIMULATION

We next present linear and nonlinear analysis of the negative refraction and its transmission. The numerical study considers a 90-by-90 rotator lattice split into two domains: the incident SA lattice and receiving NA lattice. The simulations are conducted in MATLAB by directly integrating the governing equations of motion, Eq. (5) and Eq. (6), via the MATLAB function *ode*45. We assume the lattice is sufficiently large such that the reflections from boundaries do not interfere with the refraction pattern near the interface in the considered simulation time window.

A. Linear Analysis and Simulation

As shown in Fig. 2a, a 45-degree incident wave with frequency $f = 17$ Hz and small amplitude $A = 0.01$ rad transmits through the interface and refracts on the same side of the normal. At this small wave amplitude, the nonlinear effects are negligible. Since both media are anisotropic and dispersive, a powerful tool to investigate the refraction is the isofrequency contours plot derived from the linear dispersion, Eq. (8) and Eq. (10).

Fig. 2b displays the isofrequency contours, where the blue and red solid curve indicate the isofrequency contour at the signal frequency for the SA lattice (incident media) and the NA lattice (receiving media), respectively. Two lower frequency contours are also displayed

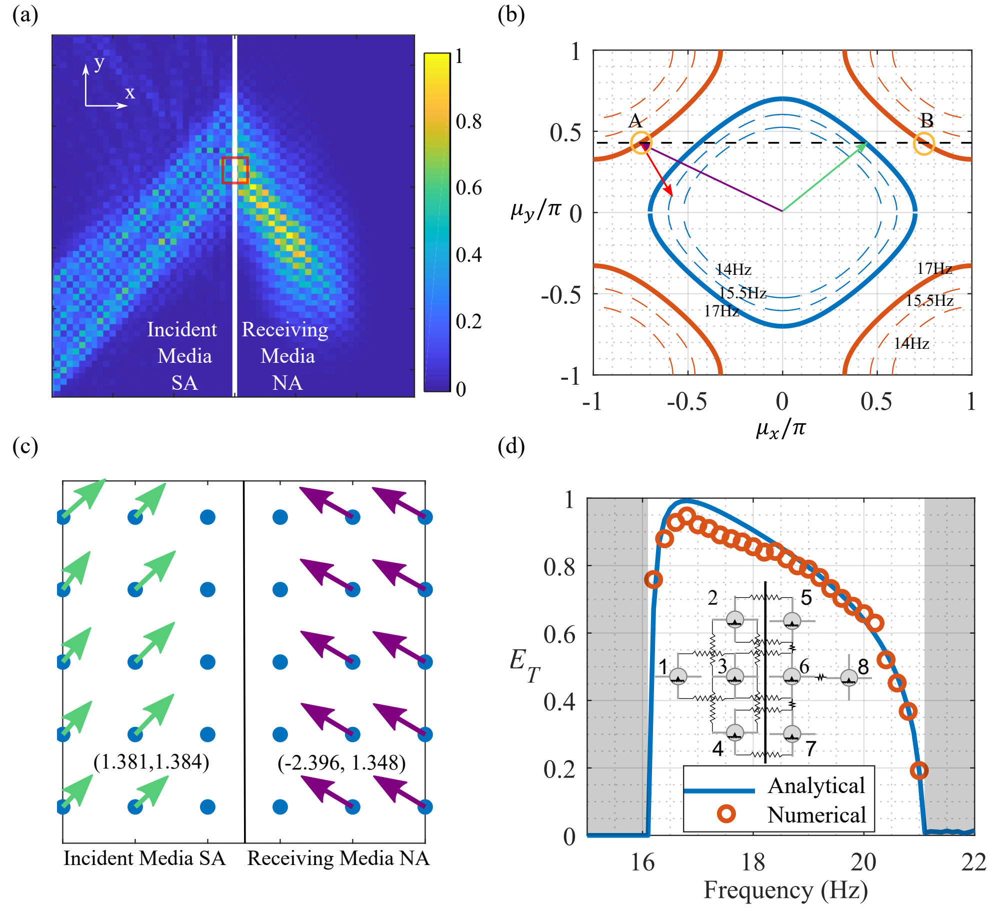

FIG. 2. Linear analysis and simulation of interfacing lattice structures. (a) Numerical demonstration of negative refraction at signal frequency $f = 17$ Hz, and amplitude $A = 0.01$ rad. Each pixel denotes the normalized energy at an individual rotator. (b) The isofrequency contours of the SA-type (blue) and NA-type (red) lattices. The contours at signal frequency ($f = 17$ Hz) are highlighted using solid lines. Dashed contours denote lower frequencies. (c) Numerically measured wave vectors at highlighted region in (a). The averaged wavenumber in each media is numerically evaluated and presented in the form of (μ_x, μ_y). (d) The analytical and numerical energy transmission. The schematic of a control volume model to study transmission is embedded in the center. Rotators $1, 2, 3, 4$ belong to the incident media, and $5, 6, 7, 8$ the receiving media.

as dashed curves, suggesting outward group velocity (gradient of the frequency contour) for the incident media (SA) and inward group velocity for the receiving media (NA). The green arrow depicts the incident wave vector. At the interface, the tangential wavenumber (μ_y) remains unchanged due to momentum conservation, and the refracted wave vector is thus constrained to fall along the black dashed line. The black dashed line intersects the receiving media's isofrequency contour at two points, A and B. To enable positive energy inlet for the receiving media, the refracted wave vector (purple arrow) directs from the origin to point A since the group velocity (red arrow) has a positive component in the horizontal direction. We note that the group velocity and wave vector at this point assume roughly opposite directions in the receiving media. This counter-intuitive result is numerically verified in Fig. 2c. The green and purple arrows depict incident and refracted wave vectors measured at each rotator (blue dot). The wavenumbers are computed via the phase-difference method [40] using adjacent rotators' responses. Note that this method assumes periodicity around the considered rotator, which does not apply to the rotators at the center two columns.

Due to the dispersive nature of the media, the transmission is frequency-dependent. We quantitatively derive this frequency-dependence by matching two plane wave solutions at the interface. A control volume straddling the interface is illustrated in Fig. 2d, where four rotators are displayed on each side of the interface. The steady-state solutions for the rotators are informed by the dispersion relations in either media and the force balance at the interface. With the dispersion relation established in Eqs. (8) and (10), we provide the force-balance equations at rotator 3 and 6 straddling the interface,

$$\begin{aligned}-\omega^2 I_{SA}\theta_3 + k_1(\theta_3-\theta_1) + k_1(\theta_3-\theta_2)\\ +k_1(\theta_3-\theta_4) + k_1(\theta_3-\theta_6) = 0,\end{aligned} \tag{11}$$

$$-\omega^2 I_{NA}\theta_6 + k_1(\theta_6 - \theta_3) + (k_g\theta_6 + k_2(\theta_6 + \theta_5))$$
$$(k_g\theta_6 + k_2(\theta_6 + \theta_7)) + (k_g\theta_6 + k_2(\theta_6 + \theta_8)) = 0. \quad (12)$$

By introducing the wave amplitude A, complex transmission T and reflection coefficient R, we can describe the steady-state response of each rotator θ_i in Table II. Substituting these assumed wave solutions back into Eq. (11) and Eq. (12), we obtain two algebraic equations with two unknowns, T and R. Note that the wave vectors for the incident and receiving media, $[\mu_x^{SA}, \mu_y^{SA}]$ and $[\mu_x^{NA}, \mu_y^{NA}]$, can be derived from the dispersion analysis presented in Fig. 2b, given incident wave angle and frequency. In Fig. 2d, we illustrate the frequency-dependent transmission from an energy perspective, $E_T = 1 - |R|^2$. The dome-shaped transmission peaks at approximately $f = 16.75$ Hz, indicating an impedance matching between two lattices. Similar to the specific acoustic impedance, the specific elastic impedance in a discrete lattice can be formulated as pressure (torque) over particle (rotator) velocity $\frac{P}{v}$ (or $\frac{T}{\omega}$). We note that the torque T is provided by either the horizontal or vertical elastic linkages. Thus the impedance in each direction needs to be considered individually. Assuming plane wave propagation $e^{i(\omega t - \mu_x x - \mu_y y)}$, with angle $\phi = atan(\frac{\mu_y}{\mu_x})$, we derive the specific elastic impedance $\mathbf{z}$ for the SA and NA lattices in each lattice direction respectively,

$$\mathbf{z}_{SA.x}(\omega,\phi) = -\frac{k_1(1 - e^{i\mu_x^{SA}(\omega,\phi)})}{i\omega},$$
$$\mathbf{z}_{SA.y}(\omega,\phi) = -\frac{k_1(1 - e^{i\mu_y^{SA}(\omega,\phi)})}{i\omega}, \quad (13)$$

$$\mathbf{z}_{NA.x}(\omega,\phi) = -\frac{k_g + k_2(1 + e^{i\mu_x^{NA}(\omega,\phi)})}{i\omega},$$
$$\mathbf{z}_{NA.y}(\omega,\phi) = -\frac{k_g + k_2(1 + e^{i\mu_y^{NA}(\omega,\phi)})}{i\omega}, \quad (14)$$

where $[\mu_x^{SA}, \mu_y^{SA}]$ and $[\mu_x^{NA}, \mu_y^{NA}]$ are dimensionless wavenumber functions of frequency ω and wave angle ϕ. Since the interface is vertically placed (two media connected horizontally), a full transmission implies horizontal impedance matching,

$$|\mathbf{z}_{SA.x}(\omega,\phi_i)| = |\mathbf{z}_{NA.x}(\omega,\phi_r)|,$$
$$|k_1(1 - e^{i\mu_x^{SA}(\omega,\phi_i)})| = |k_g + k_2(1 + e^{i\mu_x^{NA}(\omega,\phi_r)})|, \quad (15)$$

where ϕ_i and ϕ_r denote the incident wave angle and refracted wave angle respectively. For the considered 45-degrees incident wwave, this equation yields $\omega = 16.75Hz \times 2\pi$ as the impedance matching frequency, agreeing with the observation in Fig. 2d.

The gray regions in Fig. 2d ($f < 16$Hz and $f > 21$Hz) indicate complex wavenumbers in the receiving media, and trivial transmission in the far-field. From the isofrequency contour perspective, it suggests no intersection between the black dashed line and the red contours in Fig. 2b. This operation range is narrower than the common passband range of the incident and receiving media, as indicated in Fig. 1e and 1f (approximately 10Hz-24Hz). We note that the common passband in Fig. 1e and 1f accounts for wave propagation in all possible directions, while the narrower common passband in Fig. 2d corresponds to a 45-degree incident wave only. As this specific angle, the operation range is naturally narrower, as also indicated in the isofrequency contours in Fig. 2b. Finally, analytical transmission curve quantitatively agrees with the observation in numerical simulation.

Rotator	Expression
θ_1	$Ae^{-i\omega t}e^{-i\mu_x^{SA}} + R \cdot Ae^{-i\omega t}e^{i\mu_x^{SA}}$
θ_2	$Ae^{-i\omega t}e^{i\mu_y^{SA}} + R \cdot Ae^{-i\omega t}e^{i\mu_y^{SA}}$
θ_3	$Ae^{-i\omega t} + R \cdot Ae^{-i\omega t}$
θ_4	$Ae^{-i\omega t}e^{-i\mu_y^{SA}} + R \cdot Ae^{-i\omega t}e^{-i\mu_y^{SA}}$
θ_5	$T \cdot Ae^{-i\omega t}e^{i\mu_y^{NA}}$
θ_6	$T \cdot Ae^{-i\omega t}$
θ_7	$T \cdot Ae^{-i\omega t}e^{-i\mu_y^{NA}}$
θ_8	$T \cdot Ae^{-i\omega t}e^{i\mu_x^{NA}}$

TABLE II. Assumed Solutions in Interface ROM

B. Nonlinear Transmission

We next explore the refraction at higher amplitudes where nonlinear effects emerge. Figures 3a and 3b depict such a scenario – at incident wave amplitudes $A = 0.08$ rad and $A = 0.16$ rad, the increasing wave amplitude decreases the transmission. Figure 3c compares the time responses at the same rotator in the receiving field under different incident amplitudes. The results indicate the receiving amplitudes are approximately equal though their incident amplitudes differ by a factor of two.

Tracing the time responses of rotators along the transmission path, we plot the spatial evolution of the wave amplitude in Fig. 3d under different incident amplitudes. At low amplitudes, the transmitted waves maintain their amplitudes along the transmission line, agreeing with the linear theory. However, at higher amplitudes, an attenuation envelope emerges after the interface. Noteworthy, different from the attenuation envelope resultant from the linear bandgap effect, this attenuation envelope does not vanish in the far-field, but rather saturates the amplitude. This non-conventional behavior together with amplitude-dependent transmission can be explained via a perturbation analysis.

We use the Method of Multiple Scales (MMS) to quantitatively describe the observed nonlinear effect [41, 42]. Adhering to the MMS procedure [43], we introduce a small parameter ϵ as a bookkeeping device to define multiple time scales,

$$T_0 = t, T_1 = \epsilon t, T_n = \epsilon^n t. \quad (16)$$

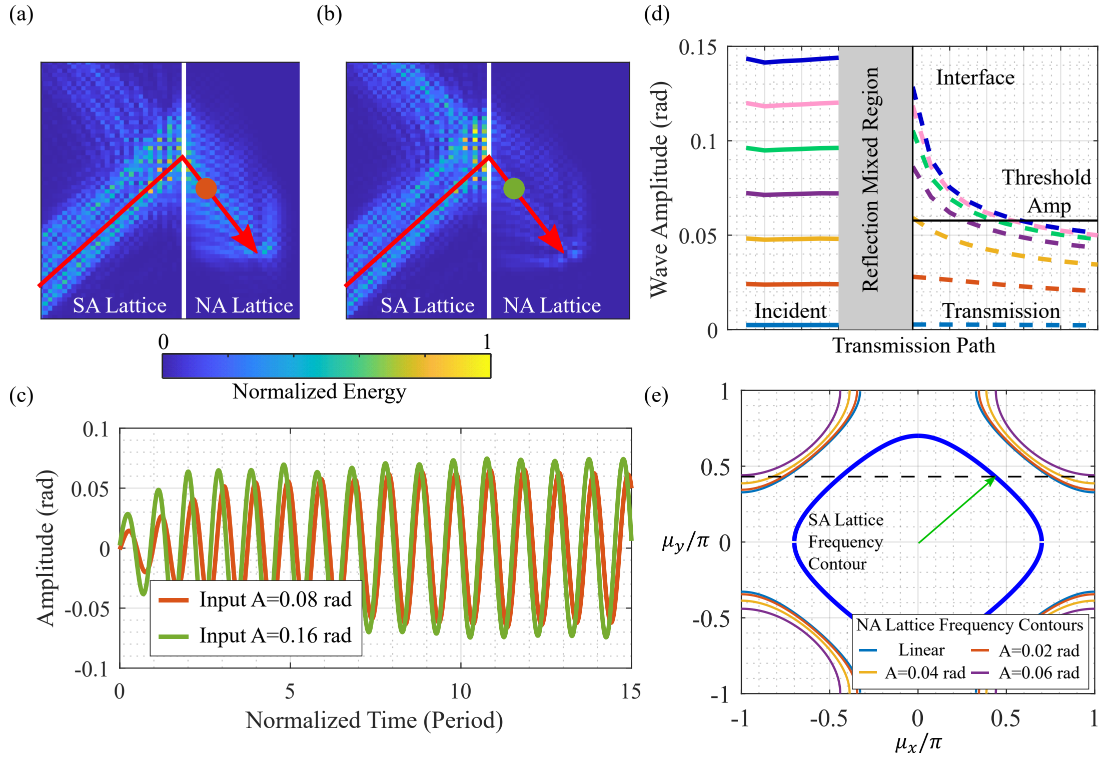

FIG. 3. (a) Wave propagation contour at amplitude $A = 0.08$ rad. The red arrow describes the considered transmission path. (b) Wave propagation contour at amplitude $A = 0.16$ rad. The pixelated energy in (a) and (b) is normalized between 0 and 1. (c) The transmitted signal at the same rotator (colored dots in (a) and (b)) of the lattice for excitation amplitudes $A = 0.08$ rad and $A = 0.16$ rad, respectively. The curve colors match the dot color in (a) and (b). (d) The amplitude envelope along the transmission path. Solid and dashed curves denote the incident and transmitted wave, respectively. (e) Amplitude-dependent isofrequency contours. The blue curve near the center refers to the isofrequency contour of the incident media at the signal frequency. The groups of curves at corners mark the isofrequency contours of the receiving media at linear (green), $A = 0$ rad (shallow blue), $A = 0.02$ rad (orange), $A = 0.04$ rad (yellow), $A = 0.06$ rad (purple). The green arrow denotes the incident wave vector. The black dashed line indicates the momentum conservation for μ_y

Accordingly, time derivatives now become,

$$\dot{(\;)} = D_0(\;) + \epsilon D_1(\;) + O(\epsilon^2),$$
$$\ddot{(\;)} = D_0^2(\;) + 2\epsilon D_0 D_1(\;) + O(\epsilon^2), \tag{17}$$

where D_n corresponds to the partial time derivative with respect to T_n. Similarly, we expand the wave response as an asymptotic series,

$$\theta_{m,n} = \theta_{m,n}^{(0)} + \epsilon\theta_{m,n}^{(1)} + O(\epsilon^2). \tag{18}$$

In Table I we notice the SA lattice has considerably smaller nonlinearity compared to the NA lattice. Hence, we deduce the observed nonlinear effects mostly arise from the NA lattice nonlinearity. For simplicity, we treat the SA lattice linear, and conduct MMS on the NA lattice.

Substituting Eq. (17) and Eq. (18) into the NA lattice equation of motion Eq. (6), we collect terms at the first two orders,

$$O(\epsilon^0):$$
$$I_{NA}D_0^2\theta_{m,n}^{(0)} + \sum_{p,q}[k_g\theta_{m,n}^{(0)} + k_2(\theta_{m,n}^{(0)} + \theta_{p,q}^{(0)})] = 0, \tag{19}$$

$$O(\epsilon^1):$$
$$I_{NA}D_0^2\theta_{m,n}^{(1)} + \sum_{p,q}[k_g\theta_{m,n}^{(1)} + k_2(\theta_{m,n}^{(1)} + \theta_{p,q}^{(1)})] =$$
$$-2ID_0D_1\theta_{m,n}^{(0)} - \sum_{p,q}[\gamma_+^{NA}(\theta_{m,n}^{(0)} + \theta_{p,q}^{(0)})^3$$
$$+\gamma_-^{NA}(\theta_{m,n}^{(0)} - \theta_{p,q}^{(0)})^3 + \gamma_g^{NA}(\theta_{m,n}^{(0)})^3]. \tag{20}$$

The solution to Eq. (19) assumes the form of a 2D plane wave,

$$\theta_{m,n}^{(0)} = \frac{1}{2}Ce^{i\omega_0 T_0}e^{-i\mu_x}e^{-i\mu_y} + c.c\;, \tag{21}$$

where amplitude C is a complex quantity, and $c.c$ denotes the complex conjugate of the preceding terms. The dispersion relation of this 0^{th}-order solution adopts the same form as in Eq. (10). In the implementation of MMS, we typically express the complex conjugate in its polar form to decouple the nonlinear effects on amplitude and phase at slower time scales,

$$\begin{aligned}\alpha &= \alpha(T_1, T_2, ...),\\ \beta &= \beta(T_1, T_2, ...),\\ C &= \alpha e^{i\beta}.\end{aligned} \tag{22}$$

Substituting the 0^{th}-order solution Eq. (21) and Eq. (22) into the 1^{st}-order expansion Eq. (20), we obtain a lengthy expression on the right-hand side (omitted here). This expression contains information to construct the 1^{st}-order solution, yet needs to be modified due to the secular terms containing $e^{i\omega T_0}e^{-i\mu_x}e^{-i\mu_y}$ and their complex conjugates. Secular terms need to be removed to avoid unbounded growth violating the asymptotic expansion in Eq. (18). By separating and removing secular terms for the real and imaginary parts, we obtain two ordinary differential equations governing slow evolution of α and β,

$$D_1\alpha = 0, \tag{23}$$

$$\begin{aligned}D_1\beta =& \frac{\alpha^2}{I_{NA}\omega_0}[\frac{3}{2}\gamma_g^{NA} + \frac{9}{2}(\gamma_-^{NA} + \gamma_+^{NA})\\ &-3(\gamma_-^{NA} - \gamma_+^{NA})(\cos\mu_x + \cos\mu_y)\\ &+\frac{3}{4}(\gamma_-^{NA} + \gamma_+^{NA})(\cos 2\mu_x + \cos 2\mu_y)] = \delta\alpha^2.\end{aligned} \tag{24}$$

These results suggest that the wave amplitude α is not a function of the slow time T_1, and phase β contributes to a frequency correction at the 1^{st}-order [41],

$$\omega = \omega_0 + \epsilon\delta\alpha^2. \tag{25}$$

We present the nonlinear dispersion curves as isofrequency contours in Fig. 3e at a variety of wave amplitudes. Similar to Fig. 2b, the blue contour at the center corresponds to the incident media. It does not deform as the wave amplitude increases. The isofrequency contours for the receiving media, however, migrate away towards the corners as the wave amplitude increases. Consider an incident signal denoted by the green arrow in Fig. 3e. The refracted wave maintains the tangential wave vector μ_y, indicated by the black dashed line, which intersects the amplitude-dependent dispersion of the receiving media at a nonlinear-shifted μ_x. As the amplitude increases, the black dashed line intersects the receiving media's isofrequency contour at a distant point, resulting in a larger difference in the phase velocity between the two media. From an impedance perspective, given unchanged inertia/density, the enlarging phase velocity difference leads to an enhanced impedance mismatch, reducing the effective transmission.

If the amplitude is sufficiently large (e.g., the purple curve at $A = 0.06$ rad in Fig. 3), the black dashed line no longer intersects the isofrequency contour. As a result, the wavenumber along the normal, μ_x, becomes complex. In contrast to the linear theory, the nonlinear evanescent wave does not decay to zero, but rather saturates the amplitude below a threshold value, as observed in Fig. 3d.

This behavior of nonlinear evanescent waves is theoretically analyzed in [44]. Typically, the nonlinear dispersion shift enables nonlinear passband extension (NPE) and/or nonlinear stopband extension (NSE), in which a propagating signal becomes a transitional evanescent wave featuring the saturation effect. In the NA-type lattice, the considered signal falls in the passband of the receiving media at low amplitudes (black dashed line intersects with the isofrequency contour), but falls in the stopband at high amplitudes (no intersection between black dashed line and the isofrequency contour), which meets the definition of a NSE. Just past the interface, the transmitted wave has high amplitude, which shifts the dispersion curves and the signal falls in the stopband. As the amplitude decreases in transmission due to the bandgap effect, the nonlinear effect also mitigates. As pertains to the isofrequency contours, the shift lessens and the contour tends to migrate from the largest-amplitude position (purple) back to a lower-amplitude position (e.g., yellow). Upon a threshold amplitude wherein the black dashed line intersects the isofrequency contour, the amplitude attenuation ends, and the wave may propagate at a lower amplitude. As such, the lattice serves as an effective amplitude saturator. The saturation threshold can be determined from Eq. (25),

$$A_{sat}(\omega, \mu_y) = \sqrt{\frac{\omega - \omega_0(\mu_x = \pi, \mu_y)}{\delta(\mu_x = \pi, \mu_y)}}. \tag{26}$$

This quantity is a function of frequency ω, and incident wavenumber μ_y. It has a numerical value of 0.058 rad in this study. Noteworthy, the saturation profile in Fig. 3d does not fully flatten in the far field due to a wavenumber clipping effect [44, 45].

IV. EXPERIMENT

This section presents an experimental study of the negative refraction at the interface of two fabricated rotator lattices. Due to the presence of dissipation and finite size, this study focuses on verifying linear negative refraction.

Figure 4a depicts the schematic of the experimental system – a matrix of 10x10 rotators ($0.66m \times 0.66m$) with the left five columns forming a SA lattice and the right five an NA lattice. In each lattice, 3D-printed rotators sit on low-friction bearings that are press-fit on fixed shafts. The distance between adjacent shafts are identical, $a = 0.068m$. The rotators adopt a multi-layer structure, allowing spring connections in both lattice directions. As depicted in the 3D illustrations in Fig. 4a, the long springs (k_1) in the SA lattice rotator (light blue rotator on the left) are pinned at different layers of the rotator such that the spring coils do not interfere. For

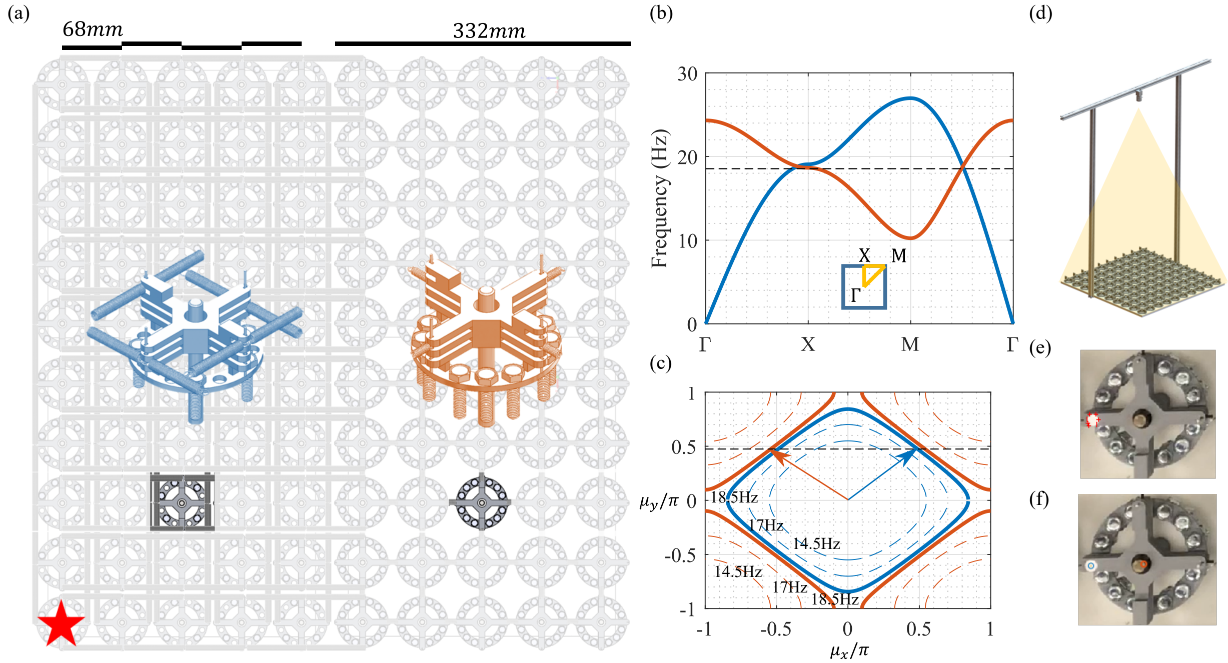

FIG. 4. (a) Schematic of the lattice interface system. The left five columns represent the SA-type connected rotator lattice (blue), where the right five columns the NA-type rotator lattice (red). The red star at the bottom left denotes the excitation position. The scale bar on the top-left marks the size of the experimental unit cell, and the top-right the depth of each media. (b) The dispersion relation of two lattices. Blue denotes the SA-type and red the NA-type. (c) The isofrequency contours of the SA-type (blue) and NA-type (red) rotator lattices near the excitation frequency 18.5 Hz, highlighted in thick solid curves. A pair of incident and refracted wave vectors are illustrated. (d) The schematic of the experimental setup with an elevated camera. (e) A frame of an individual rotator with red crosses marking the feature (reflection tape). (f) The illustration of the tracking point (blue) and the rotator center (red).

the NA lattice rotator (light red rotator on the right), the short springs (k_2) are all pinned at the same height. The rotator design also supports the installation of additional weights (screws and nuts) on the circular plate to tune the moment of inertia. In the experiment, we consider 5 pairs of screws and nuts for each SA lattice rotator and 12 screws and 48 nuts for the each NA lattice rotator. This procedure enables a common passband between the two lattices, given a limited variety of extension springs, and minimizes the damping effect on the wave propagation.

We use a single shaker (Brüel & Kjær 4810) to excite a continuous incident wave by prescribing the motion of the rotator at the bottom left corner, highlighted by the red star in Fig. 4. At a frequency of approximately 18.5 Hz (indicated as the dashed line in Fig. 4b), the isofrequency contour admits a rounded parallelogram shape (blue solid contour in Fig. 4c). Since the group velocity is always perpendicular to the isofrequency contour, the point excitation at the frequency induces a strong wave-beaming effect along 45 degrees towards the interface [39]. Due to the nature of the single point source, the incident wave contains a mixture of wave vectors despite its highly directed energy flow. The mixed wave vectors affect the refraction transmission, yet do not alter the negative refraction pattern.

Upon excitation, the wave travels to the interface and splits into refracted and reflected components. This process is recorded by a high-speed camera positioned 1.22m above the lattice, as shown in Fig. 4d. At such a height, we ensure the error from the perspective geometry is sufficiently small. The camera records at 60fps with 3840x2160 pixel resolution covering the entire 100-rotator matrix.

The data extraction involves two steps: (i) tracking the feature points and (ii) converting it into angular displacement and velocity. Fig. 4e presents a rotator image cropped from a single frame of the recording. The feature points are covered by reflection tape and marked by red crosses. We use a Kanade-Lucas-Tomasi algorithm-based MATLAB function, $PointTracker$, to track the movement of the reflection tape for each individual rotator. By relating the position of the reflection tape (blue dot) to the center of the rotator (red dot), as illustrated in Fig. 4f, we compute the angular displacement for each rotator and collectively monitor the global dynamics of the lattice. To remove small offsets on the angular displacement measurements, we present the results in terms of its time derivative (angular velocity) in Fig. 5.

Fig. 5a depicts a video frame of the global dynamics, where the size of each blue dot is shown in proportion to the rotator's angular velocity. In the receiving media, we observe most of the energy is refracted into the

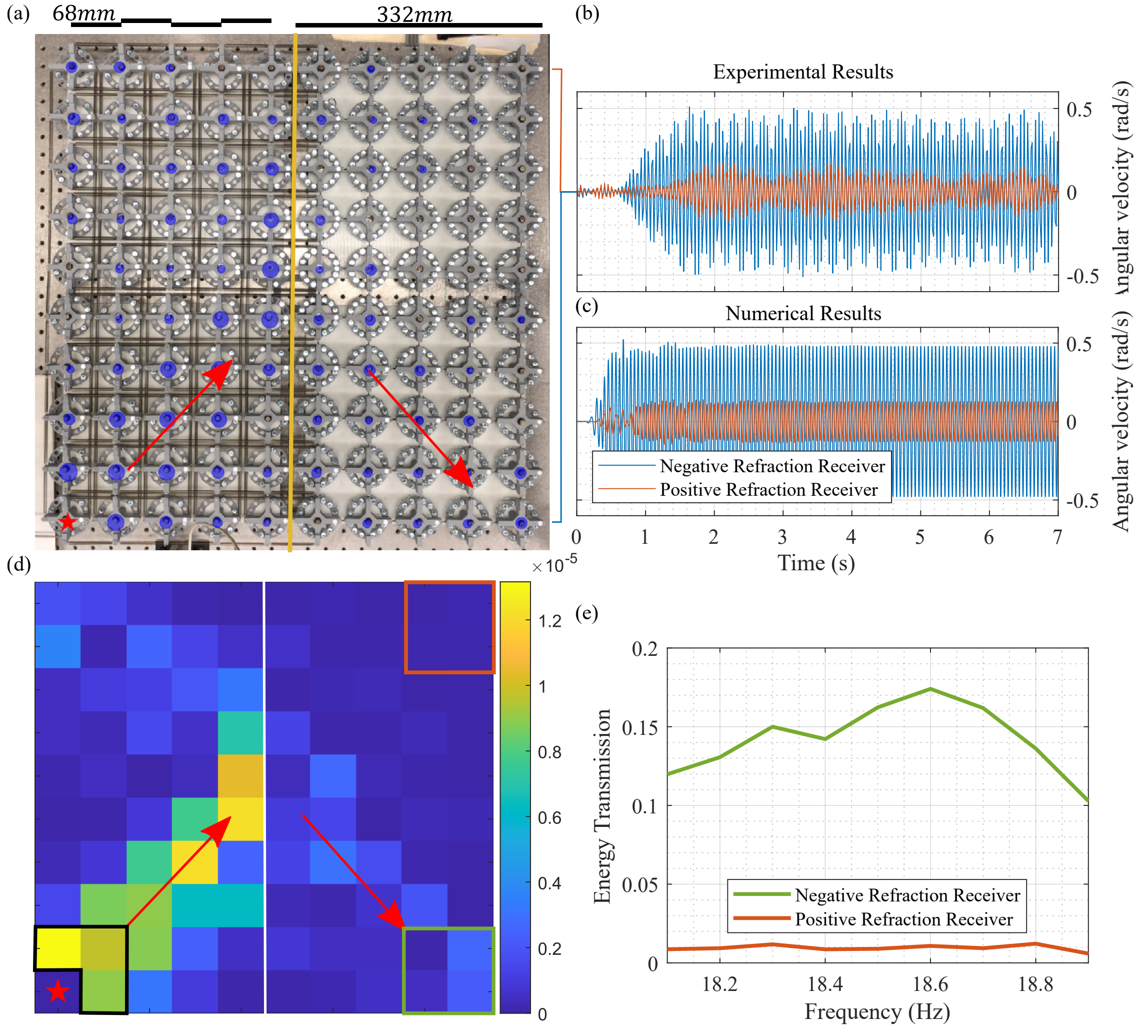

FIG. 5. (a) Single Video frame from the experiment. The blue dot depicts the angular velocity of each rotator at the current frame. The orange line denotes the interface. (b) The experimental time response of the bottom right (blue) and the top right rotator (red). (c) The numerical time response of the same two rotators as in (b). (d) The experimental average kinetic energy at each rotator. (e) The experimental frequency-transmission relation for the negative refraction receiver and the positive refraction receiver.

bottom right quarter, demonstrating negative refraction. Beyond this frame, we capture the full time history of the dynamics at the top right corner (positive refraction receiver), and bottom right corner (negative refraction receiver), and present them in Fig. 5b. The response at the negative refraction receiver is approximately three times higher than that of the positive refraction receiver, providing quantitative evidence for negative refraction. We note that the positive refraction receiver has nontrivial readings due to the vibration of the finite structure, which is not seen in the Sec. III study. Accordingly, we numerically simulate the 10x10 experiment lattice with a point source placed at the left-bottom corner and dissipation modeled as uniform viscous damping. The simulation result is illustrated in Fig. 5c, which reaches a high-degree of agreement with the experiment.

From an energy perspective, the time-averaged kinetic energy in Fig. 5d reveals a more distinct pattern for the negative refraction. We identify the negatively refracted energy and positively refracted energy in the green and red square box, respectively. By comparing them to the source energy, which can be recognized from the black contour at bottom left, we quantify the energy transmission in Fig. 5e. At the specified frequency range where directivity permits beaming incident waves, the negative refraction energy transmission is consistently higher than the positive refraction transmission. The frequency-dependent pattern deviates from the numerical prediction in Fig. 2d primarily due to the point source excitation and finite size of the lattice. The experimental transmission is lower than the theory due to dissipation and imperfect periodicity. We further evaluate the robustness of the observed negative refraction via a sensitivity study documented in Appendix II.

V. CONCLUDING REMARKS

We observe robust negative refraction between two simple rotator lattices in numerical simulations and experiments. The two proposed rotator lattices share a similar structure but have a subtle difference of spring attaching location. This variance configures the dispersion relation of the lattice to be either acoustic with positive group velocity or optic with negative group velocity. Since the group velocities have opposite signs over the entirety of both lattices' Brillouin zones, the negative refraction is inherently broadband. At low amplitude, a linear dispersion analysis illustrates the negative refraction mechanism, and demonstrates frequency-dependent transmission. The experimental results quantitatively agree with the numerical simulations. At higher amplitudes, perturbation analysis reveals amplitude-dependent transmission and a nonlinear saturation effect, verified in numerical simulations. A parameter sensitivity test indicates robust negative refraction in the rotator lattices with the presence of small imperfections.

ACKNOWLEDGMENTS

The authors would like to thank the National Science Foundation for support of this research under an Emerging Frontiers in Research and Innovation (EFRI) Grant No. 1741565. L.F. would like to thank Mr. Aryan Roy for his contributions to the rotator design.

VI. APPENDIX I

This appendix presents the explicit expressions for linear and nonlinear stiffnesses as well as a brief introduction of the experimental parameter identification process.

Both SA and NA lattices have the same lattice constant a, and rotator radius r. Rotators in the SA lattice are connected via elastic linkages with stiffness k^{SA}, and undeformed length L_0^{SA}. The associated linear and nonlinear stiffnesses are given as,

$$k_1 = 2k^{SA}r^2, \tag{27}$$

$$\gamma_+^{SA} = \frac{k^{SA}r^2L_0^{SA}}{6a}, \tag{28}$$

$$\gamma_-^{SA} = \frac{k^{SA}r^2(L_0^{SA}-2a)}{6a}, \tag{29}$$

$$\gamma_g^{SA} = -\frac{4k^{SA}r^2L_0^{SA}}{3a}. \tag{30}$$

Rotators in the NA lattice are connected via elastic linkages with stiffness k^{NA}, and undeformed length L_0^{NA}. We introduce an edge-to-edge distance between two NA-connected rotators in the equilibrium position, $D = a - 2r$, such that the stiffnesses are given as,

Parameter	$k_{SA}(N/m)$	k_{NA}	$L_0^{SA}(m)$	L_0^{NA}
Value	39	820	0.0312	0.0012
Parameter	r(m)	a(m)		
Value	0.03	0.068		

TABLE III. Experimentally identified system parameters

$$k_2 = \frac{k_{NA}r^2(D-L_0^{NA})}{D}, \tag{31}$$

$$k_g = k_{NA}r(D-L_0^{NA}), \tag{32}$$

$$\gamma_+^{NA} = \frac{k_{NA}r^2L_0^{NA}}{2D}\left(\frac{r^2}{D^2}+\frac{5r}{6D}+\frac{1}{6}\right), \tag{33}$$

$$\gamma_-^{NA} = \frac{k_{NA}r^2}{6}\left(1-\frac{rL_0^{NA}}{2D}-\frac{L_0^{NA}}{2D}\right), \tag{34}$$

$$\gamma_g^{NA} = \frac{k_{NA}r^2}{3}\left(\frac{2r^2L_0^{NA}}{D^2}-r+\frac{L_0^{NA}}{D}-\frac{D-L_0^{NA}}{2}\right). \tag{35}$$

The parameter identification process for NA and SA lattices is similar to the approach taken in our previous work [32]. By isolating each unit cell from the lattice, we experimentally measure its free oscillation response at different moments of inertia (controlled by adjusting the number of bolts and nuts) and energy levels. The collected results are fit to our analytical model, Eq. (5) and Eq. (6), via MATLAB's function *patternsearch*. In Table III, we also provide the average stiffnesses and undeformed lengths of the physical springs used in the experiments for reference.

VII. APPENDIX II

To evaluate the robustness of the results, we assess the sensitivity of the negative refraction pattern subject to imperfect periodicity. In the numerical model of the 10x10 lattice, we introduce spatial randomness to the following parameters individually: rotator inertia (I_{SA} and I_{NA}), radius (r), spring spacing (D_0), short spring undeformed length (L_0), and spring stiffnesses (k_1 and k_2). The randomness is sampled from normal distributions with standard deviation equaling 2.5% of a parameter's expected value,

$$\{P\}_{N\times 1} \leftarrow \{\mathcal{N}(\overline{P};(2.5\%\overline{P})^2)\}, \tag{36}$$

where $\{P\}$ represents the vector of an arbitrary parameter, and N denotes the number of elements containing such parameter in the 10x10 lattice structure. For example, if the parameter of interest is the SA lattice rotator inertia, we have $P = I_{SA}$, and expected value $\overline{P} = 1.95\text{E-}5\ kgm^2$. There are 50 SA rotators in the system, so $N = 50$.

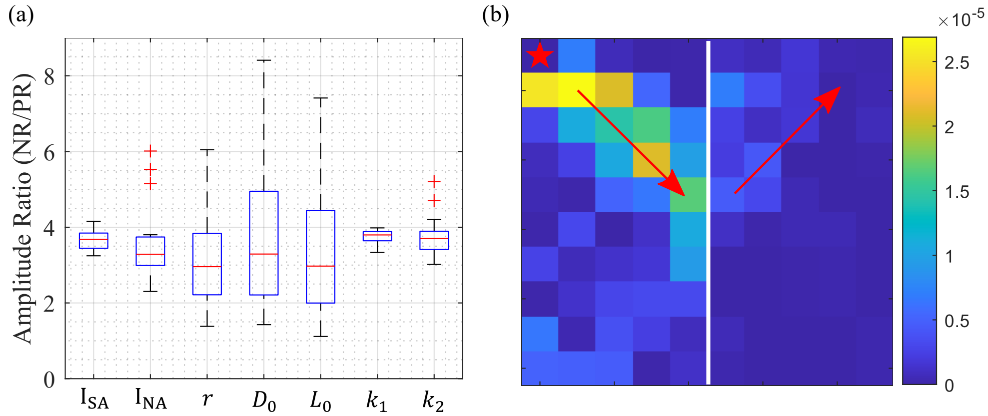

FIG. 6. (a) The sensitivity test for system parameters, shown as a boxplot. 20 numerical simulations at small signal frequency $f = 18.5$ Hz are used for each parameter study. (b) The experimental results of the system excited from the top-left corner instead of the bottom left. The average kinematic energy is plotted.

For each parameter sensitivity test, we consider 20 sets of spatial randomness sampled from Eq. (36), and conduct simulations individually. We analyze the amplitude ratio between the negative refraction receiver and positive refraction receiver. The results are illustrated in the boxplot in Fig. 6a. We observe that at the considered randomness level, the amplitude ratio is strictly larger than 1, implying a robust negative refraction. Additionally, the amplitude ratio is most sensitive to the geometric parameters r, D_0 and L_0. The variance resulting from spatial fluctuations on inertia and stiffness is small. The results connect the negative refraction performance to individual system parameters and suggest the significance of geometry resolution for designs and fabrications.

In addition to the parameter fluctuations, we provide an experimental result with varied excitation location (top left corner). Fig. 6b depicts the results in terms of time-averaged kinetic energy. We observe a negative refraction pattern with refracted energy directed towards the top right, as expected.

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