Imaging of Single-Photon Orbital-Angular-Momentum Bell States

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Imaging of single-photon orbital angular momentum Bell states

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Bell states is the most fundamental resource in realizing quantum information tasks and has very unique position in the quantum mechanics. While utilize orbital angular momentum (OAM) to encode single-photon Bell states enables the realization of high-dimensional Hilbert space, which is crucial for the field of quantum information. In this paper, we design a single-photon OAM Bell states evolution device based on the Sagnac interferometer, which can make one-to-one correspondence between the input Bell states and the output states. Moreover, we also develop a single-photon single-pixel imaging (SPI) technique, which improve spatial resolution and decrease acquisition time at the meantime, to capture interference images of output states. The results show that single-photon OAM Bell states can be fully recognized by comparing the differences of interference images. We have innovatively utilized SPI technique to single-photon OAM Bell states recognition. This indicates that the SPI technique effectively promotes the study of quantum information based on OAM, while the quantum information based on OAM provides a clear application scenario for the SPI technique.

I. INTRODUCTION

Quantum entanglement was described by Einstein as spooky action at a distance [1]. It is not only a unique property of quantum mechanics, but also a core resource in quantum information [2]. The Bell states, as the largest entangled state in two-qubit system, has been extensively studied since it was proposed. The most common standard Bell states is two-qubit Bell states of two-photon in optics system [3–5]. In 2001, A. Zeilinger et al. proposed multi-dimensional entanglement is a way — in addition to multi-particle entanglement — to extend the usual two-dimensional two particle state [6]. Therefore, it has arised another form of Bell states — single-photon two-qubit Bell states, such as, single-photon polarization-path Bell states [7–10], etc.

Orbital angular momentum (OAM) [11, 12], as a brand-new photon degree of freedom (DoF), has been widely applied in the field of quantum information. For example, a number of studies have emerged that utilize OAM to encode two-qubit Bell states in recent years [13–17]. Since OAM of photons has high-dimensional characteristics [18], using OAM to encode Bell states not merely realize a high-dimensional Hilbert space, but also enhance the ability of quantum states to carry information. Importantly, the measurement of the OAM Bell states is required when the information is finally decoded. The measurement of the single-photon OAM Bell states has been achieved in the previous experiments [16, 17], which converted OAM modes to the fundamental Gaussian mode. Due to the infinite dimensions of OAM, the previous method of measuring OAM may not be able to detect the information of all OAM modes in the system, resulting in information loss. By contrast, single-photon imaging (SPI) technique can measure multiple OAM modes by measuring the spatial distribution of single-photon OAM, so as to obtain more complete information. But as the order of photonic OAM increases, the more complex the images are, the higher the spatial resolution (SR) is required. Therefore, imaging techniques with as high-SR as possible are required to realize the discrimination of high-order OAM images. The SPI [19–21], which uses dynamic illumination to obtain spatial information, is an innovative scheme and has received increasing attention recently. It is capable of imaging under low light conditions [22–26]. And its SR is limited only by the spatial light modulator, making it a potential candidate for obtaining high-SR OAM images [27]. Unfortunately, SPI requires continuous measurements, resulting in a trade-off between SR and acquisition time [25, 27, 28]. Hence, the requirement to achieve high-SR and low acquisition time at the same time is a bottleneck of SPI.

In this paper, we design a single-photon OAM Bell states evolution device based on the Sagnac interferometer, which can make one-to-one correspondence between the input single-photon OAM Bell states and the output states. Furthermore, we develop a quick-sort Hadamard SPI technique — based on compressed sensing to improve SR and decrease acquisition time at the same time — to capture interference images of output states. Then utilizing the proposed imaging technique, we experimentally demonstrate the interference images of four output states, which are shown to be different from each other. The results show that by comparing the differences of these interference images, we can realize complete Bell states recognition. In addition, we experimentally show a set of interference images with high enough SR (256×256 pixels). In our opinion, this research is at the cross field of

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quantum information and quantum imaging. The application of SPI technique in the quantum information field and the experimental research of quantum information based on SPI are helpful to promote the development of quantum information science and technology.

II. THEORY AND METHODS

We define the four single-photon two-qubit Bell states with OAM and polarization DoFs,

$$
\begin{aligned}
|\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|H,m\rangle \pm |V,-m\rangle) \\
|\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|H,-m\rangle \pm |V,m\rangle)
\end{aligned}
\tag{1}
$$

where, $|H\rangle$ and $|V\rangle$ represent the horizontal and vertical polarizations of the photons. $|\pm m\rangle$ represent the photons with OAM of $\pm m\hbar$, and m is called the topological charge. In recent years, the methods used to generate OAM include spiral phase plate (SPP) [29, 30], hologram [31, 32], q-plate [33] or vortex wave-plate(VWP) [34], etc. Here, we utilize VWP to prepare the above single-photon OAM Bell states. Fig. 1(a) is the schematic of preparing single-photon OAM Bell states using VWP. Take the preparation of the single-photon OAM Bell state $|\Psi^+\rangle$ as an example, the specific preparation details are as follows (The input is H-polarized Gaussian light, whose photons state is $|H,0\rangle$).

$$
\begin{aligned}
&|H,0\rangle \xrightarrow{\text{HWP}(45^\circ)} |V,0\rangle \xrightarrow{\text{QWP1}(0^\circ)} i|V,0\rangle = \frac{1}{\sqrt{2}}(|R,0\rangle - \\
&|L,0\rangle) \xrightarrow{VWP(m)} \frac{1}{\sqrt{2}}(|L,m\rangle - |R,-m\rangle) \xrightarrow{\text{QWP2}(-45^\circ)} = \\
&\frac{1}{\sqrt{2}}(|H,m\rangle - |iV,-m\rangle) \xrightarrow{\text{QWP3}(0^\circ)} \frac{1}{\sqrt{2}}(|H,m\rangle + |V,-m\rangle) \\
&= |\Psi^+\rangle
\end{aligned}
\tag{2}
$$

Here, the function of $m-order$ VWP is described as: $|R,m^{'}\rangle \xrightarrow{m-order} |L,m^{'}+m\rangle$; $|L,m^{'}\rangle \xrightarrow{m-order} |R,m^{'}-m\rangle$, where, $|R\rangle \propto |H\rangle + i|V\rangle$ and $|L\rangle \propto |H\rangle - i|V\rangle$ represent the right- and left-circularly polarized states, respectively. The other three OAM Bell states are prepared in the similar way, details can be found in APPENDIX A.

As shown in Fig.1(b), we design an OAM Bell states evolution device based on the Sagnac interferometer, which can make each OAM Bell state correspond to a unique output state. Its principle is as follows: Input state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|H,m\rangle + |V,-m\rangle)$, for example, after polarization beam splitter (PBS), this state will be decomposed into $\frac{1}{\sqrt{2}}|H,m\rangle$ in the transmission direction and $\frac{1}{\sqrt{2}}|V,m\rangle$ in the reflection direction. Note that the sign of topological charge in the reflection direction is changed because the reflection effect reverses the sign of topological charge [35]. Then, the transmitted part passes through Mirror 1 (M1), M2, phase controller (PC), and M3 in turn. Instead, the reflected part passes through M3, PC, M2, and M1 in turn. Specifically, $\frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\text{M1}} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\text{M2}} \frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\text{PC}} \frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle \xrightarrow{\text{M3}} \frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle$; $\frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\text{M3}} \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\text{PC}} \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\text{M2}} \frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\text{M1}} \frac{1}{\sqrt{2}}|V,-m\rangle$. It's important to point out that the phase $e^{i\theta}$ is controlled by the PC, which function is only add phase to $|H\rangle$ polarization. After the interference loop evolution, the two partial states will remeet at PBS and leave at another port, and the state is $\frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle + \frac{1}{\sqrt{2}}|V,m\rangle$. This is the output state of the state $|\Psi^+\rangle$ after the Sagnac interferometer evolution. After passing through the Sagnac interferometer, the photons are divided into two detectors by 22.5° HWP and PBS (Not plotted in Fig.1). The evolution of this step is $\frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle + \frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\text{HWP}(22.5^\circ)+\text{PBS}} \frac{1}{2}(e^{i\theta}|-m\rangle + |m\rangle)_T|H\rangle_T + \frac{1}{2}(e^{i\theta}|-m\rangle - |m\rangle)_R|V\rangle_R$. Where, the subscript *T(R)* denotes the transmitted (reflected) part of PBS. Here, we only consider the transmission part of PBS. Eventually, the state $|\Psi^+\rangle$ will correspond to a output state, i.e $|\Psi^+\rangle \to |\Psi^+\rangle^{'} = \frac{1}{2}(e^{i\theta}|-m\rangle + |m\rangle)$. The corresponding output states of the four OAM Bell states are (Details of the evolution of other three states can be found in APPENDIX B).

$$
\begin{aligned}
|\Psi^+\rangle &\to |\Psi^+\rangle^{'} = \frac{1}{2}(e^{i\theta}|-m\rangle + |m\rangle) \\
|\Psi^-\rangle &\to |\Psi^-\rangle^{'} = \frac{1}{2}(e^{i\theta}|-m\rangle - |m\rangle) \\
|\Phi^+\rangle &\to |\Phi^+\rangle^{'} = \frac{1}{2}(e^{i\theta}|m\rangle + |-m\rangle) \\
|\Phi^-\rangle &\to |\Phi^-\rangle^{'} = \frac{1}{2}(e^{i\theta}|m\rangle - |-m\rangle)
\end{aligned}
\tag{3}
$$

The interference results of above output states carrying opposite-sign ($\pm m$) OAM is a petal-like light spot, and the number of petals is $2m$ [35]. Furthermore, these output states are related to phase $e^{i\theta}$ which results in the

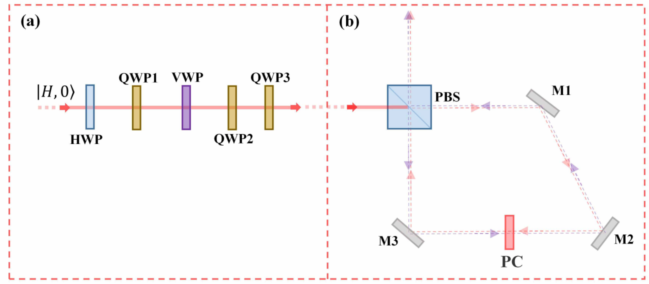

FIG. 1. (a) The schematic of preparing single-photon OAM Bell states. HWP: half-wave plate; QWP: quarter-wave plate; VWP: vortex wave-plate. (b)Single-photon OAM Bell states evolution device. The dashed lines and arrows only indicate the light propagation direction. PBS: polarization beam splitter; M: mirror; PC: phase controller.

rotation of the petal-like light spot, where $\theta \in [0, 2\pi]$ is the phase shifted by the PC. So for a given θ, we can obtain the interference images corresponding to the four output states. Because there is a one-to-one correspondence between the single-photon OAM Bell states and the output states, we can recognize which Bell state is the input based on the differences of interference images. The whole process above is what we called Bell states recognition.

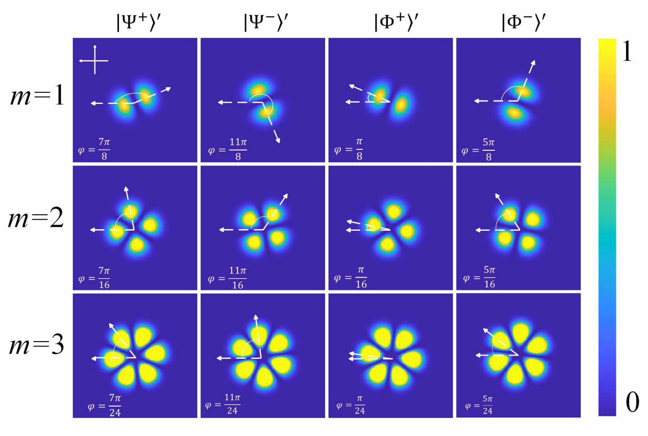

FIG. 2. The simulation images of the four output states, $\theta = \frac{7\pi}{4}$. The angles φ, which values are shown in the bottom left corner, in the figure are the angular positions of the petals that we defined in main text.

Fig.2 shows the simulation diagram of the four output states in Eq. 3. The interference images for four output states correspond to four different petal images. To illustrate how these petal images are different, we define the angular position φ of the petal of output state $|\Psi^+\rangle'$, and use it as the reference standard. For output state $|\Psi^+\rangle'$, there is a relationship between φ and θ, $\varphi = \frac{\theta}{2m}$ [35], where $\varphi_{m=1} \in [0, \pi]$, $\varphi_{m=2} \in [0, \frac{\pi}{2}]$, $\varphi_{m=3} \in [0, \frac{\pi}{3}]$. As shown in the first column of Fig.1, when $\theta = \frac{7\pi}{4}$, $\varphi_{|\Psi^+\rangle'(m=1)} = \frac{7\pi}{8}$, $\varphi_{|\Psi^+\rangle'(m=2)} = \frac{7\pi}{16}$, $\varphi_{|\Psi^+\rangle'(m=3)} = \frac{7\pi}{24}$. As sown in Fig.1, the angular positions φ of the petals in columns 2, 3, and 4 are rotated to different angles with respect to column 1, respectively. Specifically, when $m = 1$, $\varphi_{|\Psi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = \frac{4\pi}{8}$, $\varphi_{|\Phi^+\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{6\pi}{8}$, $\varphi_{|\Phi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{2\pi}{8}$; when $m = 2$, $\varphi_{|\Psi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = \frac{4\pi}{16}$, $\varphi_{|\Phi^+\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{6\pi}{16}$, $\varphi_{|\Phi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{2\pi}{16}$; when $m = 3$, $\varphi_{|\Psi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = \frac{4\pi}{24}$, $\varphi_{|\Phi^+\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{6\pi}{24}$, $\varphi_{|\Phi^-\rangle'} - \varphi_{|\Psi^+\rangle'} = -\frac{2\pi}{24}$. According to the above formulas, we can know that when m is fixed, the corresponding φ values of the four output states are different. Therefore, the differences of these images are characterized by these angular positions φ of the petals. Then, we can completely (That's 100% efficiency) recognize the OAM Bell states based on the differences in the interference images of the four output states.

In Fig.2, we only simulate the three cases $m = 1, 2, 3$. Due to the infinite dimension property of OAM, m can take an arbitrarily large integer. According to $\varphi = \frac{\theta}{2m}$, however, $\Delta\varphi = \frac{\Delta\theta}{2m}$ can obtained, where $\Delta\theta$ is phase difference, $\Delta\varphi$ is the angular positions difference of the petals. Fig. 3 shows our simulation results between m and $\Delta\varphi$ for $|\Psi^+\rangle'$. When $\Delta\theta = \frac{\pi}{4}$, $\Delta\varphi_{m=1} = \frac{\pi}{8}$, $\Delta\varphi_{m=2} = \frac{\pi}{16}$, $\Delta\varphi_{m=3} = \frac{\pi}{24}$ (This result also holds for the other three output states). It shows that when $\Delta\theta$ is fixed, the larger the values of m, the smaller the the angular positions difference of the petals $\Delta\varphi$. Unfortunately, when $\Delta\varphi$ is small enough, all petal boundaries become blurred for low spatial resolution (SR) images. This situation is not conducive to using the angular positions of the petals to distinguish the differences between the images. In contrast, images with high-SR will effectively alleviate this situation. From this point of view, high-SR is necessary for the recognition of high-order OAM Bell states.

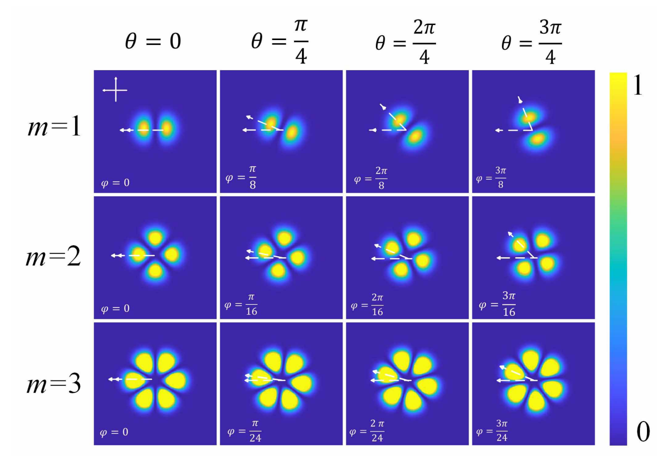

FIG. 3. Simulation results between order m and $\Delta\varphi$ for $|\Psi^+\rangle'$. For example, when $\Delta\theta = \frac{\pi}{4}$, $\Delta\varphi_{m=1} = \varphi_{(\theta=\frac{\pi}{4})} - \varphi_{(\theta=0)} = \frac{\pi}{8}$, $\Delta\varphi_{m=2} = \varphi_{(\theta=\frac{\pi}{4})} - \varphi_{(\theta=0)} = \frac{\pi}{16}$, $\Delta\varphi_{m=1} = \varphi_{(\theta=\frac{\pi}{4})} - \varphi_{(\theta=0)} = \frac{\pi}{24}$.

We intend to utilize single-photon single-pixel imaging (SPI) technique to recognize single-photon OAM Bell states. However, SPI requires continuous measurement of the target, resulting in a trade-off between SR and acquisition time. The compressed sensing techniques have been used to mitigate the trade-off. The proposed optimized methods of Hadamard SPI including Russian Dolls (RD) order[28] and Cake-Cutting (CC) order [36]. However, the RD order is limited by high computational complexity, so it can only sort patterns whose SR is less than 128×128 pixels; the CC order is limited by the calculation method, only one sorting method can be provided, and there is no clear calculation formula. In this study, we demonstrate a sort-based Hadamard SPI scheme that uses an efficient and easy-to-implement sampling method based on the Hadamard patterns to achieve fast high-SR imaging. We call it the quick-sort Hadamard SPI scheme(see APPENDIX C for details).

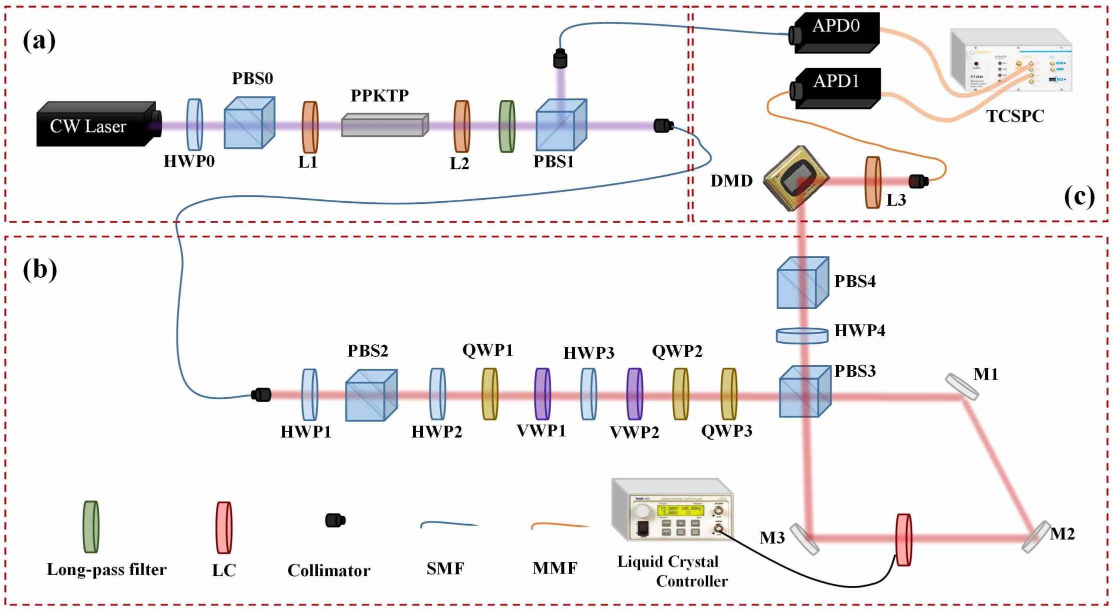

FIG. 4. Experimental setup for single-photon OAM Bell states recognition. (a) Single-photon source preparation. A 405 nm continuous wave (CW) diode laser with 20 mW power pumps a periodically poled potassium titanyl phosphate PPKTP crystal to produce photon pairs with central wavelength of 810 nm based on SPDC. (b) Single-photon OAM Bell states preparation and evolution. Here, the order of VWP1 and VWP2 are $m = 1$ and $m = 2$, respectively. HWP3 and VWP2 are only needed to prepare the OAM state with $m = 3$. (c) Single-pixel imaging device. The imaging device is composed of DMD, lens, APDs and TCSPC. Patterns are loaded on the DMD. DMD reflects photon states to APD1 for collection. Finally, the quick-sort Hadamard SPI scheme is used to reconstruct OAM Bell states images. It should be noted that our DMD is tilted 45° for convenience. MMF: Multi-mode fiber.

III. EXPERIMENTAL SETUP

We now demonstrate the experimental implementation of single-photon OAM Bell states recognition. As shown in Fig. 4, the whole experimental device consists of single-photon source preparation, single-photon OAM Bell states preparation and evolution, and SPI device. The method of preparing single-photon source is the same as Refs [37, 38]. As shown in Fig. 4(a), the pump light is a fundamental mode Gaussian continuous wave (CW) diode laser with a power of 20 mW and a central wavelength of 405 nm. It pumps type II phase-matched periodically poled potassium titanyl phosphate (PPKTP) crystal and generates an entangled photon pair at 810 nm based on spontaneous parametric down-conversion (SPDC). Then, the photon pair is split into two arms at PBS1, and the detection of a V polarization photon at avalanche photodiodes 0 (APD0) indicates a H polarization photon along the other arm. In order to characterize the performance of the single-photon source, we measure the value of the second-order correlation function $g^{(2)}$ as 0.0236.

The single-photon source is connected to the single-photon OAM Bell state preparation and evolution section using SMF. As shown in Fig. 4(b), PBS3 is used as the inlet and outlet of the device. It divides the incident light beam into horizontally polarized light beam along clockwise and vertically polarized light beam along counterclockwise, respectively. After being successively reflected by mirrors, they regroup again and exit the interferometer from the other end of PBS3. The liquid crystal (LC) is inserted into the interferometer, which function is to add a phase to the H polarization state without changing the sign of the OAM. However, the holistic setup changes the sign of the OAM because the total number of mirrors and PBS is odd for both polarizations. As shown in Fig. 4(b), when HWP4 is 22.5° and $|D\rangle \propto |H\rangle + |V\rangle$ is taken as the measurement base, we chose to conduct detection at the transmission end of PBS4.

Notably, we do not convert OAM modes to the fundamental Gaussian mode. Due to the infinite dimensions of OAM, the previous method of measuring OAM using spatial single-mode optical device has some shortcomings. The previous method only converts the OAM order concerned in the experiment into the fundamental mode for measurement, and may not be able to detect the information of all OAM modes in the system, resulting in information loss. On the contrary, we use the imaging method to measure the photon distribution in the whole space, which can be more comprehensive and intuitive to obtain all information. The evolved photons of the four single-photon OAM Bell states are emitted from the same end of PBS4 and directly received by the SPI device for imaging, as shown in Fig. 4(c). The SPI device contains digital mirror device (DMD), lens, APDs, and time-correlated single-photon counting (TCSPC). The DMD, which contains 1024×768 micro-mirrors and each pixel is $13.68\mu m$, is illuminated by photons emanating from PBS4 and operates at a refresh rate of 0.1Hz. Photons emanating from PBS4 are reflected by the Hadamard pattern displayed on DMD and collected by APD1 after passing through a lens. The role of TCSPC is to convert light intensity signals into digital signals.

IV. RESULTS AND DISCUSSION

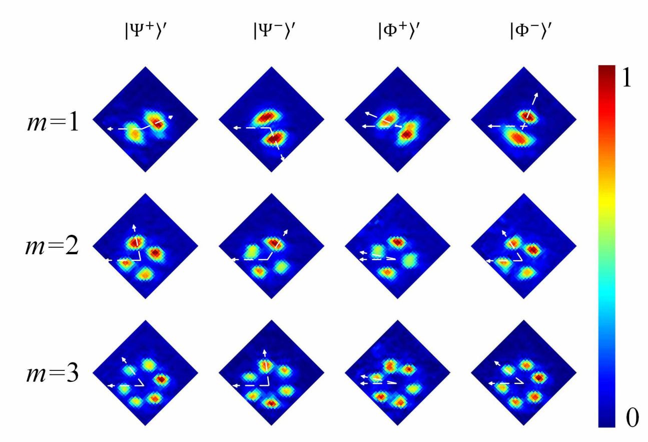

FIG. 5. The experimental results correspond to the simulation diagram of Fig.1. Patterns of 32×32 pixels are sorted that used to measure the output states which corresponding to the single-photon OAM Bell states with phase $\theta = \frac{7\pi}{4}$.

To test the performance of the quick-sort Hadamard SPI scheme, some numerical simulations are performed (see APPENDIX D for details). In addition, we have conducted experimental verification on the simulation diagram of Fig.2, and the results are shown in Fig. 5. It is necessary to state that we performed a proof-of-principle experiment. In order to qualitatively characterize the degree of agreement between experiment and theory, in Fig. 5, we have marked the angular positions φ of petals, which are theoretical values and exactly the same as in Fig.2. According to Fig. 5, the angular positions of the petals of each output state are in good agreement with the theory. One of the main reasons for the difference between experiment and theory is that the current imaging quality is not good enough, so we should develop high quality single-photon imaging technology.

The above experimental results are all low-SR images (32×32 pixels). For low-order OAM images ($m = 1, 2, 3$), it can clearly see the difference of the angular positions of petals. However, for high-order OAM, high-SR images are required to allow for accurate discrimination. As shown in Fig.6, this is the experimental results of 256×256 pixels obtained by using our proposed quick-sort Hadamard SPI scheme. Compared with the third row of Fig. 5, the petals' shape in Fig.6 is more regular, and the boundary of the petals are also clearer. This result shows that high-SR images are more suitable for the recognition of high-order OAM images than low-SR images. Although we only show high-SR images for $m = 3$, we theoretically calculate that the highest OAM order that can be resolved by 256×256 pixels is $m = 300$, while the highest OAM order that can be distinguished by 32×32 pixels is only $m = 12.5$ because of the minimum angle that can be resolved by each digital image is limited theoretically.

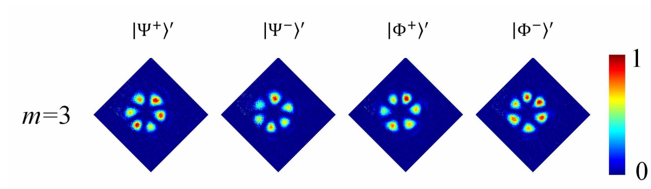

FIG. 6. High-SR images are reconstructed using patterns sorted by quick-sort Hadamard SPI. In the case of $m = 3$ and phase $\theta = \frac{\pi}{4}$, the images (256×256 pixels) of the output states which corresponding to the single-photon OAM Bell states.

V. CONCLUSIONS

In conclusion, we present a scheme for the preparation, evolution and measurement of single-photon OAM Bell states in a Hilbert space composed of polarization and OAM using a simple unitary operation of linear optics. In order to obtain images of the output states corresponding to the OAM Bell states, we develop the quick-sort Hadamard SPI scheme based on a trade-off between SR and acquisition time. Experimentally, we use the proposed SPI scheme to display interference images with different output states. By comparing the differences of these interference images, we achieve complete Bell states recognition. For traditional Bell states measurement, the input state is an arbitrary polarization state and the Bell states measurement can distinguish all four orthogonal Bell states. Our Bell states recognition technique is different with the traditional Bell states measurement. Our technique can recognize Bell states (that is the input state are Bell states), but it cannot use the discriminating information in quantum protocols. And we are still working on how to use our recognizing information in quantum protocols. By demonstrating that quick-sort Hadamard SPI can generate OAM images to achieve Bell states recognition, we combine quantum imaging technology with the most fundamental quantum information problems. This combination not only opens the way for quantum imaging scheme based on Bell states recognition, but also expands the application range of SPI.

ACKNOWLEDGEMENTS

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APPENDIX A. SINGLE-PHOTON OAM BELL STATES PREPARATION

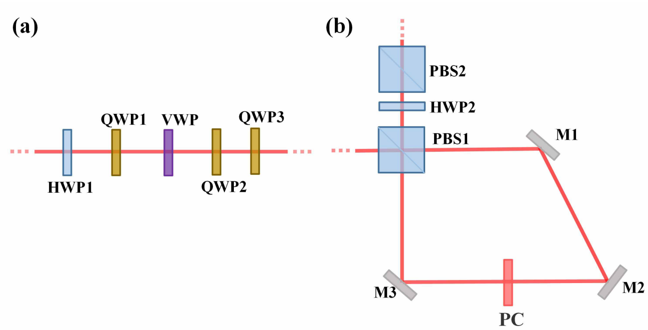

FIG. 7. (a) The schematic of preparing single-photon OAM Bell states. (b)Single-photon OAM Bell state evolution device. HWP: half-wave plate; QWP: quarter-wave plate; VWP: vortex wave-plate; PBS: polarization beam splitter; M: mirror; PC: phase controller.

As shown in Fig. 7(a), we have used vortex wave- plate (VWP) to generate Orbital angular momentum (OAM) to prepare the single-photon OAM Bell states. The VWP ($m-order$), which is used to transform the fundamental Gaussian mode state $|0\rangle$ into the OAM state $|+m\rangle$ or $|-m\rangle$. In general, the function of $m-order$ VWP is described as

$$\begin{aligned} |R,m'\rangle &\xrightarrow{m-order} |L,m'+m\rangle \\ |L,m'\rangle &\xrightarrow{m-order} |R,m'-m\rangle \end{aligned} \tag{4}$$

Here, $|R\rangle \propto |H\rangle + i|V\rangle$ and $|L\rangle \propto |H\rangle - i|V\rangle$ represent the right- and left-circularly polarized states, respectively. We have shown the preparation of the $|\Psi^+\rangle$ state in the main text, and we will focus on the preparation of the other three too below. Note that the input is H-polarized Gaussian light, whose photons state is $|H,0\rangle$. The specific process of preparing the $|\Psi^-\rangle$ state is as follows,

$$|H,0\rangle \xrightarrow{\text{HWP1}(0^\circ)} |H,0\rangle \xrightarrow{\text{QWP1}(0^\circ)} |H,0\rangle = \frac{1}{\sqrt{2}}(|R,0\rangle + |L,0\rangle) \xrightarrow{\text{VWP}(m)} \frac{1}{\sqrt{2}}(|L,m\rangle + |R,-m\rangle) \xrightarrow{\text{QWP2}(-45^\circ)} = \frac{1}{\sqrt{2}}(|H,m\rangle + |iV,-m\rangle) \xrightarrow{\text{QWP3}(0^\circ)} \frac{1}{\sqrt{2}}(|H,m\rangle - |V,-m\rangle) = \left|\Psi^-\right\rangle \tag{5}$$

The specific process of preparing the $|\Phi^+\rangle$ state is as follows,

$$|H,0\rangle \xrightarrow{\text{HWP1}(0^\circ)} |H,0\rangle \xrightarrow{\text{QWP1}(0^\circ)} |H,0\rangle = \frac{1}{\sqrt{2}}(|R,0\rangle + |L,0\rangle) \xrightarrow{\text{VWP}(m)} \frac{1}{\sqrt{2}}(|L,m\rangle + |R,-m\rangle) \xrightarrow{\text{QWP2}(45^\circ)} = \frac{1}{\sqrt{2}}(|H,-m\rangle - |iV,m\rangle) \xrightarrow{\text{QWP3}(0^\circ)} \frac{1}{\sqrt{2}}(|H,-m\rangle + |V,m\rangle) = \left|\Phi^+\right\rangle \tag{6}$$

The specific process of preparing the $|\Phi^-\rangle$ state is as follows,

$$|H,0\rangle \xrightarrow{\text{HWP1}(-45^\circ)} -|V,0\rangle \xrightarrow{\text{QWP1}(0^\circ)} -i|V,0\rangle = \frac{1}{\sqrt{2}}(-|R,0\rangle + |L,0\rangle) \xrightarrow{\text{VWP}(m)} \frac{1}{\sqrt{2}}(-|L,m\rangle + |R,-m\rangle) \xrightarrow{\text{QWP2}(45^\circ)} = \frac{1}{\sqrt{2}}(|H,-m\rangle + |iV,m\rangle) \xrightarrow{\text{QWP3}(0^\circ)} \frac{1}{\sqrt{2}}(|H,-m\rangle - |V,m\rangle) = \left|\Phi^-\right\rangle \tag{7}$$

APPENDIX B. THE EVOLUTION OF SINGLE-PHOTON OAM BELL STATES

As shown in Fig. 7(b), we design a OAM Bell states evolution device based on the Sagnac interferometer,which can make each OAM Bell state correspond to a unique output state. We have already analyzed in the main text how does the input state $|\Psi^+\rangle$ evolves in the Sagnac interference loop. Therefore, here we will only present the evolution of the remaining three OAM Bell states.

First, take $|\Psi^-\rangle$ as input state,

$$\frac{1}{\sqrt{2}}(|H,m\rangle - |V,-m\rangle) \xrightarrow{\text{PBS1}(T)} \frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\text{M1}} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\text{M2}} \frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\text{PC}} \frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle \xrightarrow{\text{M3}} \frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle; \tag{8}$$

$$\frac{1}{\sqrt{2}}(|H,m\rangle - |V,-m\rangle) \xrightarrow{\text{PBS1}(R)} -\frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\text{M3}} -\frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\text{PC}} -\frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\text{M2}} -\frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\text{M1}} -\frac{1}{\sqrt{2}}|V,-m\rangle. \tag{9}$$

where, T represents the transmission direction of PBS and R represents the reflection direction of PBS. The above two parts meet again at PBS, and the output state is $\frac{1}{\sqrt{2}}(e^{i\theta}|H,-m\rangle - |V,m\rangle)$. After HWP2

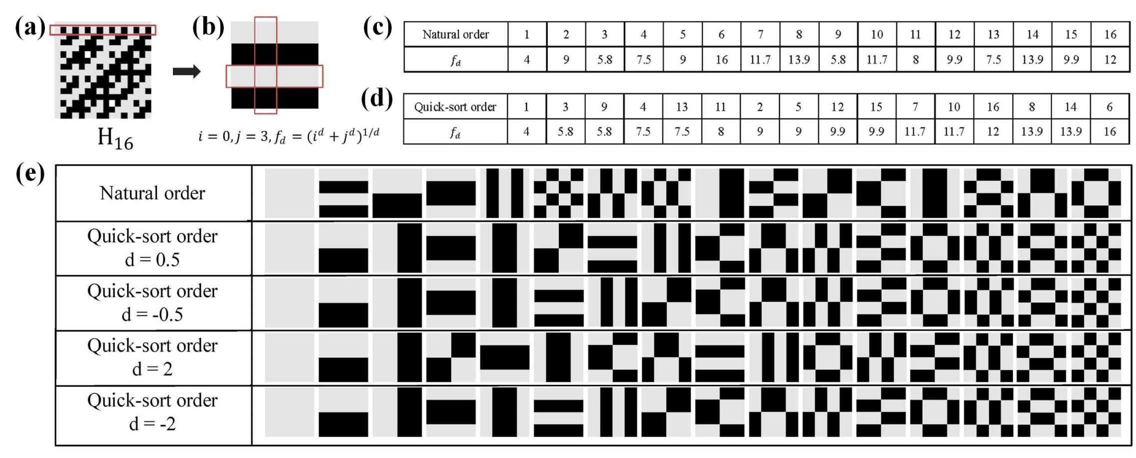

FIG. 8. An example for the quick-sort Hadamard patterns. (a) A 16×16 Hadamard matrix H_{16}. (b) A row of Hadamard matrix in (a) marked by red circles is transformed to a 2D pattern. (c) The patterns of H_{16} in the natural order, the first line is the original position of each pattern and the second line is the corresponding f_d value. (d) the patterns of H_{16} in the quick-sort order in $d = 0.5$, the first line is the new position of each pattern and the second line is the corresponding f_d value. (e) Various orders of H_{16}: natural order, quick-sort order in $d =$ 0.5, -0.5, 2, and -2.

(22.5°) and PBS2 transmission, $\frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle - \frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\mathrm{HWP}(22.5^\circ)} \frac{1}{\sqrt{2}}(e^{i\theta}|-m\rangle - |m\rangle)_T|H\rangle_T + \frac{1}{\sqrt{2}}(e^{i\theta}|-m\rangle + |m\rangle)_R|V\rangle_R \xrightarrow{\mathrm{PBS2}(T)} \frac{1}{\sqrt{2}}(e^{i\theta}|-m\rangle - |m\rangle) = |\Psi^-\rangle'$.

Second, take $|\Phi^+\rangle$ as input state,

$$\frac{1}{\sqrt{2}}(|H,-m\rangle + |V,m\rangle) \xrightarrow{\mathrm{PBS1}(T)} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\mathrm{M1}} \frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\mathrm{M2}} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\mathrm{PC}} \frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle \xrightarrow{\mathrm{M3}} \frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle; \tag{10}$$

$$\frac{1}{\sqrt{2}}(|H,-m\rangle + |V,m\rangle) \xrightarrow{\mathrm{PBS1}(R)} \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{M3}} \frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\mathrm{PC}} \frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\mathrm{M2}} \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{M1}} \frac{1}{\sqrt{2}}|V,m\rangle. \tag{11}$$

The above two parts meet again at PBS, and the output state is $\frac{1}{\sqrt{2}}(e^{i\theta}|H,m\rangle + |V,-m\rangle)$. After HWP2 (22.5°) and PBS2 transmission, $\frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle + \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{HWP}(22.5^\circ)} \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle + |-m\rangle)_T|H\rangle_T + \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle - |-m\rangle)_R|V\rangle_R \xrightarrow{\mathrm{PBS2}(T)} \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle + |-m\rangle) = |\Phi^+\rangle'$.

Third, take $|\Phi^-\rangle$ as input state,

$$\frac{1}{\sqrt{2}}(|H,-m\rangle - |V,m\rangle) \xrightarrow{\mathrm{PBS1}(T)} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\mathrm{M1}} \frac{1}{\sqrt{2}}|H,m\rangle \xrightarrow{\mathrm{M2}} \frac{1}{\sqrt{2}}|H,-m\rangle \xrightarrow{\mathrm{PC}} \frac{1}{\sqrt{2}}e^{i\theta}|H,-m\rangle \xrightarrow{\mathrm{M3}} \frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle; \tag{12}$$

$$\frac{1}{\sqrt{2}}(|H,-m\rangle - |V,m\rangle) \xrightarrow{\mathrm{PBS1}(R)} -\frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{M3}} -\frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\mathrm{PC}} -\frac{1}{\sqrt{2}}|V,m\rangle \xrightarrow{\mathrm{M2}} -\frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{M1}} -\frac{1}{\sqrt{2}}|V,m\rangle. \tag{13}$$

The above two parts meet again at PBS, and the output state is $\frac{1}{\sqrt{2}}(e^{i\theta}|H,m\rangle - |V,-m\rangle)$. After HWP2 (22.5°) and PBS2 transmission, $\frac{1}{\sqrt{2}}e^{i\theta}|H,m\rangle - \frac{1}{\sqrt{2}}|V,-m\rangle \xrightarrow{\mathrm{HWP}(22.5^\circ)} \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle - |-m\rangle)_T|H\rangle_T + \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle + |-m\rangle)_R|V\rangle_R \xrightarrow{\mathrm{PBS2}(T)} \frac{1}{\sqrt{2}}(e^{i\theta}|m\rangle - |-m\rangle) = |\Phi^-\rangle'$.

APPENDIX C. QUICK-SORT HADAMARD SPI SCHEME

The core of the proposed quick-sort Hadamard SPI scheme is based on the following equation

$$f_d = \{^{i\times j, d=0}_{(i^d+j^d)^{\frac{1}{d}}, d\neq 0} \tag{14}$$

where i and j represent the number of sign changes in any row and any column of pattern, respectively. The pattern is obtained by transforming each row of the Hadamard matrix. Constant d is a preset parameter and f_d is the corresponding value of a pattern calculated by Eq. 14. Finally, we sort the patterns in ascending order of f_d value to obtain quick-sort order. High spatial resolution can improve the distinguishing ability of OAM images. So it is necessary to sort patterns with high spatial resolution. According to our scheme, we can sort patterns of any pixel size. Further, by adjusting the preset parameter d, different orders can be obtained quickly.

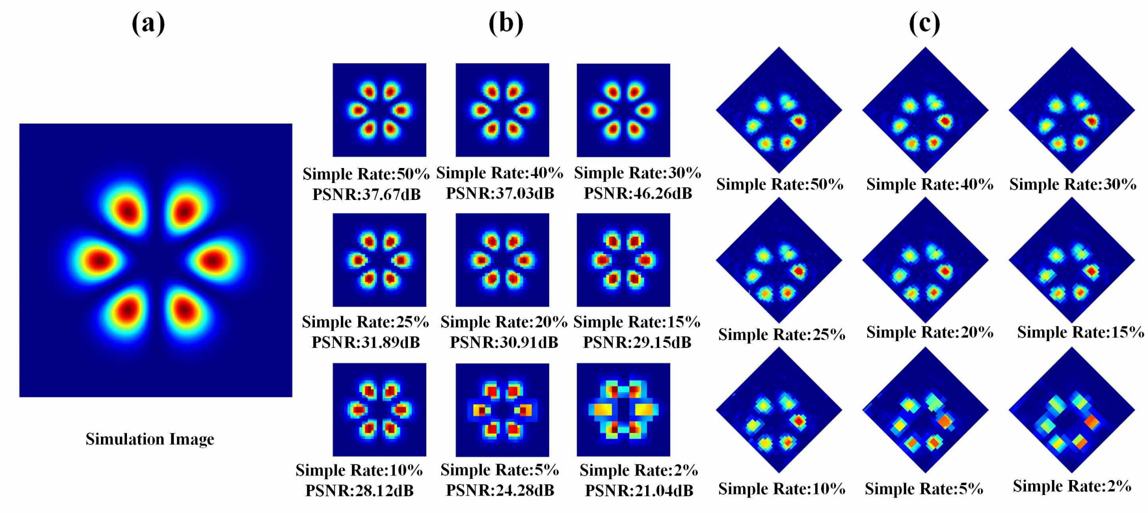

FIG. 9. The sampling ratio is determined by simulation and experiment. (a) The OAM Bell state of $m = 3$ and $\theta = 0$ generated by simulation (64×64 pixels). (b) In the simulation, the images reconstructed by TVAL3 at the sampling ratio of 50%, 40%, 30%, 25%, 20%, 15%, 10%, 5% and 2% and parameter $d = 0.5$, the sampling ratios and PSNRs are written at the bottom of each image. (c) In the experiment, the images reconstructed by TVAL3 use the same sampling ratio and d as in the simulation, the sampling ratios are written at the bottom of each image.

Here, Fig. 8 gives an example of how our quick-sort scheme works. The H_{16} matrix (Fig. 8(a)) is generated by computer, and then each row in the matrix is selected and transformed into a series of 4×4 2D patterns. Such a complete natural order 4×4 Hadamard pattern set is obtained, as shown in the first row of Fig. 8(e). According to the proposed scheme, any row of Hadamard matrix is selected and transformed into 2D pattern to count the number of sign changes, as shown in Fig. 8(b). Assuming $d = 0.5$, the f_d value is calculated by Eq. 14, and then each pattern is sorted according to the ascending order. Fig. 8(c) shows the f_d value and position of each pattern in the natural order, and Fig. 8(d) shows the f_d value and new position of each pattern after the ascending order of f_d value. The second to fifth lines of Fig. 8(e) represent the orders of different parameters d.

APPENDIX D. SIMULATIONS OF QUICK-SORT SPI SCHEME

To test the performance of the quick-sort scheme, some numerical simulations are performed. We introduce the peak signal-to-noise ratio (PSNR) as the evaluation criterion, which is defined as

$$PSNR = 10 \log 10 \frac{255^2}{MSE} \tag{15}$$

where $MSE = \frac{1}{pq} \sum_{i,j=1}^{p,q} [I_0(i,j) - \tilde{I}(i,j)]$ represents the mean square error between the original image and the reconstructed image; I_0 and $\tilde{I}$ represent the original image and the reconstructed image by the proposed scheme. p and q represent the number of pixels in the x and y directions. The higher the PSNR, the better reconstructed image quality is.

As shown in Fig. 9(a), we use the OAM Bell state $|\Psi^+\rangle$ numerical simulation image (64×64 pixels) as the original image, where $m = 3$ and $\theta = 0$. In simulation, the original image is under sampled and then the reconstructed images are obtained through TVAL3. Sampling ratio is defined as the ratio of sampling times to the total number of pixels in an image. The reconstruction results obtained from the simulation with $d = 0.5$ and sampling ratio of 50%, 40%, 30%, 25%, 20%, 15%, 10%, 5% and 2% are shown in Fig. 9(b). The corresponding reconstruction results obtained in the experiment are shown in Fig. 9(c). Based on the PSNR of the reconstructed results in simulation and the intuitive resolution performance of the results in simulation and experiment, we selected the lowest sampling rate as possible to reduce sampling time. Therefore, we selected the parameter $d = 0.5$ with a sampling rate of 10% to collect experimental data.

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