

This is the accepted manuscript made available via CHORUS. The article has been published as:

## Proposal for a Spin-Torque-Oscillator Maser Enabled by Microwave Photon-Spin Coupling

Justin T. Hou, Pengxiang Zhang, and Luqiao Liu

Phys. Rev. Applied **16**, 034034 — Published 20 September 2021

DOI: [10.1103/PhysRevApplied.16.034034](https://doi.org/10.1103/PhysRevApplied.16.034034)

# Proposal for a Spin Torque Oscillator Maser Enabled by Microwave Photon-Spin Coupling

Justin T. Hou,<sup>1</sup> Pengxiang Zhang,<sup>1</sup> and Luqiao Liu<sup>1</sup>

<sup>1</sup>*Department of Electrical Engineering and Computer Science,  
Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

(Dated: August 30, 2021)

We study a direct-current driven maser device enabled by spin-photon coupling, where coherent magnetic self-oscillation can be realized in a large area ferromagnetic thin film under the excitation of spin orbit torques. We show via both micromagnetic simulation and analytical derivation that above a critical value of the coupling strength between spin torque oscillator and microwave resonator, magnetic oscillation develops macroscopic phase coherence, narrow linewidth, and gets phase-locked with the photon mode. The threshold coupling strength for synchronizing individual spins reduces as the sample dimension increases, suggesting that the spin torque oscillator maser can be readily realized using large area thin film ferromagnets without relying on dimension confinement. Moreover, the photon mode can directly provide microwave emission which exhibits enhanced power and reduced linewidth with increasing number of spins, leading to a useful approach for developing highly coherent on-chip microwave sources.

## I. INTRODUCTION

Spin torque oscillators (STOs) [1, 2] have been extensively studied for realizing on-chip microwave sources [3–5], detectors [6], and components for neuromorphic applications [7, 8]. STOs were first realized with the spin transfer torque effect in quasi-zero-dimensional magnetic systems, such as magnetic tunnel junctions [9], nanopillar spin valves [3, 10, 11], and nanocontacts [4]. Nanoscale, quasi single domain structures are generally required in these experiments since the needed excitation current scales quickly with the magnetic area. The discovery of the spin orbit torque effect opened up the possibility of exciting magnetic oscillations on a larger area, as a charge current flowing across a very small cross section can now inject spins into a magnetic film with a much larger area [12–15]. However, coherent magnetic self-oscillation over an extended ferromagnetic thin film has proven to be hard due to the existence of a continuous magnon band, under which magnon excitations are quickly scattered into different modes, losing the global phase coherence [16, 17]. To restrain this decoherence process, dimension confinement has been introduced to discretize magnon bands and to enforce coherent oscillation of spins at different regions of the oscillator [15, 18]. Under the size limit imposed by the coherence requirement, various methods have been utilized to overcome the power bottleneck from a single nanoscale STO, including synchronizing different STOs through electrical connections [19, 20], spin wave interactions [21–24], and dipolar interactions [25, 26], which bring in complexities in circuit design and device fabrication. On the other hand, the small volume of STOs renders them susceptible to thermal fluctuations [1, 27]. Therefore, different approaches have been pursued to achieve linewidth reduction with the aid of external circuits and signals [28–32].

In this paper, we propose an approach for realiz-

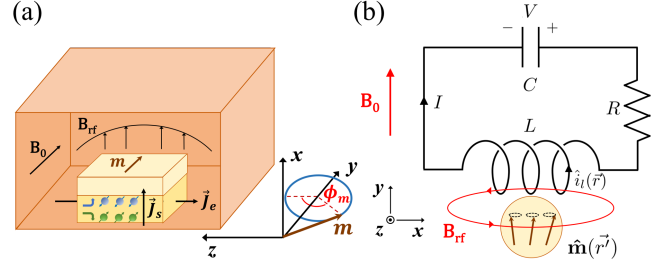


FIG. 1. (a) Schematic of the considered device structure. A magnetic oscillator driven by spin orbit torque is placed inside a microwave cavity and inductively coupled to the cavity photon mode. (b) Effective circuit diagram. The microwave cavity is modeled as a serial LCR circuit. The magnetic thin film is coupled with the LCR circuit through the rf magnetic field  $\mathbf{B}_{rf}$  generated by the inductor current. No external signals are used to drive the LCR circuit.

ing magnetic self-oscillation in a large area ferromagnetic thin film with high emission power and narrow linewidth, by exploiting spin-photon coupling. Hybrid magnon-photon systems have recently attracted great attention for reaching coherent information processing and transduction [33–40]. On-chip architecture with large magnon-photon coupling strength has been demonstrated for potentially scalable device applications [41, 42]. The integration of an active spintronic device with the photon mode in a microwave cavity is therefore highly attractive for bringing new functions to these hybrid systems. Here, with micromagnetic simulations and analytical derivation, we study the magnetic dynamics of an STO located within a microwave cavity, driven by anti-damping spin orbit torque from a direct current (DC). We find that under a strong enough coupling between an STO and a cavity, spontaneous magnetic oscillations with macroscopic phase coherence can be achieved without any externally applied locking signals, even in large size ferromagnetic thin films. Moreover, with the

increase of number of spins in the magnetic film  $N_s$ , the coupling strength needed for overcoming the synchronization threshold decreases, the microwave output power increases, and the oscillation linewidth becomes narrower. The idea of using cavity to realize highly coherent emission is rooted in the design principles of laser or maser [43–45], where the phase coherence at different regions of the nonlinear gain medium is enforced by a global coupling mechanism – cavity photons, rather than short-range forces such as dipolar or exchange interactions in conventional STOs.

## II. RESULTS

### A. Micromagnetic Equations

Fig. 1(a) shows a schematic of our proposed device. A spin current density  $J_s$  for generating damping-like torque is injected into a ferromagnetic thin film, whose dynamics is coupled with the cavity photon mode. This microwave cavity can be experimentally realized with the usage of on-chip two dimensional resonators for the small mode volume and the high spin-photon coupling strength [41, 42]. The system can be represented as the circuit diagram shown in Fig. 1(b). We assume that the frequencies of higher order modes of the resonator are far away from its fundamental mode and the dimension of the ferromagnetic device is small compared with this fundamental mode's wavelength. Therefore, for the interested spectrum region, the microwave resonator can be modeled as a serial LCR circuit. In the presence of resonator current, each spin experiences an oscillating magnetic field  $\mathbf{B}_{\text{rf}}$  generated by the inductor. For an on-chip resonator design,  $\mathbf{B}_{\text{rf}}$  is uniformly polarized across the magnetic device in one specific direction. For the configuration shown in Fig. 1(b), to induce maximal spin-photon coupling, we set  $\mathbf{B}_{\text{rf}} = b_{\text{rf}}I\hat{\mathbf{x}}$  along  $x$ -direction which is perpendicular to external biasing field  $\mathbf{B}_0 = B_0\hat{\mathbf{y}}$ . Here,  $b_{\text{rf}}$  is the rf field generated per unit inductor current  $I$ . For the circuit shown in Fig. 1(b),  $\mathbf{B}_{\text{rf}}$  generated by the inductor loop at the magnet location  $\vec{r}$  can be written as:  $\mathbf{B}_{\text{rf}}(\vec{r}) = \mathbf{b}_{\text{rf}}(\vec{r})I$ , where  $\mathbf{b}_{\text{rf}}(\vec{r}) = \oint_L \frac{\mu_0}{4\pi} \frac{(\vec{r}-\vec{r}') \times \hat{\mathbf{i}}_l(\vec{r}')}{|\vec{r}-\vec{r}'|^3} dl = b_{\text{rf}}\hat{\mathbf{x}}$  with the integral taken along  $\vec{r}'$  on the inductor loop. Here  $\hat{\mathbf{i}}_l(\vec{r}')$  is the unit vector along the current flowing direction in the inductor. The dynamics of the hybrid system can be modeled with equations:

$$\frac{d\hat{\mathbf{m}}_i}{dt} = \boldsymbol{\tau}_i - \gamma\hat{\mathbf{m}}_i \times \mathbf{b}_{\text{rf}}I \quad (1)$$

$$\frac{dI}{dt} = \frac{V}{L} - \frac{R}{L}I - \frac{b_{\text{rf}}M_sV_c}{L} \sum_i \frac{d\hat{m}_{xi}}{dt} + f_I(t) \quad (2)$$

$$\frac{dV}{dt} = -\frac{I}{C} + f_V(t). \quad (3)$$

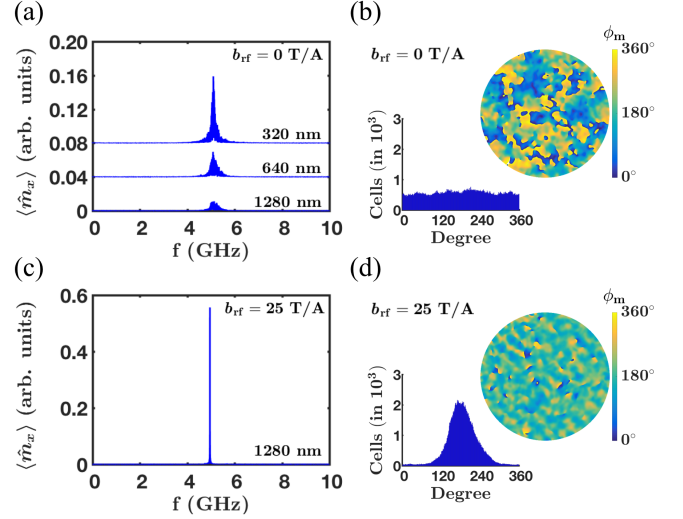


FIG. 2. Micromagnetic simulation results of STOs without [(a) and (b)] and with [(c) and (d)] coupling onto a microwave resonator. (a)(c) Fourier Transform of  $\langle \hat{m}_x \rangle$ . (b)(d) Real-space distribution of magnetization azimuthal angle  $\phi_m$  in  $xz$ -plane with  $D = 1280$  nm. The histograms in insets show the number of cells as a function of  $\phi_m$ , with the bins chosen to be  $1^\circ$ .  $T$  is set to be 0 K in the simulations of this figure.

Here we employ a micromagnetic approach by considering a magnetic free layer in an STO consisting of  $N_c$  cells.  $\gamma$  is the gyromagnetic ratio.  $\hat{\mathbf{m}}_i$  is the unit vector representing the magnetic moment direction of cell  $i$ .  $\boldsymbol{\tau}_i$  is the total torque acting on  $\hat{\mathbf{m}}_i$  considered in standard micromagnetic simulations, including contributions from Zeeman field, anisotropy field, exchange field, dipolar field, as well as Gilbert damping and spin torque [46]. The second torque term on the right hand side of Eq. (1) originates from the Oersted field from the LCR resonator. On the other hand, because of Faraday's law, the oscillating magnetic moment leads to an electromotive force onto the inductor, which is proportional to the time change of the magnetic flux  $\lambda$  through the inductor loop:  $\lambda = \sum_i \oint_L \vec{A}_i(\vec{r}) \cdot \hat{\mathbf{i}}_l(\vec{r}') dl$ , where  $\vec{A}_i(\vec{r}) = \frac{\mu_0}{4\pi} \frac{M_s V_c \hat{\mathbf{m}}_i \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$  is the vector potential at position  $\vec{r}$  generated by magnetic moment  $M_s V_c \hat{\mathbf{m}}_i$  of cell  $i$  at location  $\vec{r}'$ , with  $V_c$  being cell volume and  $M_s$  being saturation magnetization. Using the previous expression of  $\mathbf{b}_{\text{rf}}(\vec{r})$ , we can rewrite  $\lambda = M_s V_c \sum_i \hat{\mathbf{m}}_i \cdot \mathbf{b}_{\text{rf}}(\vec{r})$ . Therefore, the electromotive force is given by  $-b_{\text{rf}} M_s V_c \sum_i \frac{d\hat{m}_{xi}}{dt}$  in Eq. (2), with  $\hat{m}_{xi}$  being the projection of the  $\hat{\mathbf{m}}_i$  vector along the  $x$ -direction. Eq. (2) and (3) therefore describe dynamics of current  $I$  and voltage  $V$  of the LCR resonator in the presence of magnetic oscillations. The only external drive of the whole system is the DC spin torque included in  $\tau_i$ . To account for the thermal fluctuation at finite temperature  $T$ , we include the torque from the thermal magnetic field onto the dynamics of  $\hat{\mathbf{m}}_i$  in Eq. (1) [46], as well as a white Gaussian thermal noise  $f_I(t)$  and  $f_V(t)$  in Eq. (2) and (3). The latter

two noise terms satisfy fluctuation-dissipation relation of LCR circuit with  $\langle f_I(t)f_I(t') \rangle = Rk_B T/L^2 \delta(t-t')$  and  $\langle f_V(t)f_V(t') \rangle = Rk_B T/LC \delta(t-t')$ , respectively [1, 27] (See Appendix A for derivation). Here  $k_B$  is the Boltzmann constant. To numerically solve the  $N_c + 2$  coupled dynamical equations, we augment the ordinary differential equation solver for Landau Lifshitz Gilbert equations with additional torque term in Eq. (1) and introduce new time-dependent quantities  $I$  and  $V$  by modifying the micromagnetic package Mumax3 [46].  $f_I$  and  $f_V$  are implemented according to  $f_I(t_n) = \eta_I(t_n) \sqrt{Rk_B T/(L^2 \Delta t)}$  and  $f_V(t_n) = \eta_V(t_n) \sqrt{Rk_B T/(LC \Delta t)}$ , where  $\eta_I(t_n)$  and  $\eta_V(t_n)$  are random numbers from standard normal distribution whose values are changed after every time step with step size  $\Delta t = t_n - t_{n-1}$  [46]. Our modified simulation codes are available online [47].

### B. Micromagnetic simulation results

We compare the magnetic dynamics of an STO with and without being coupled onto a microwave resonator. For micromagnetic simulation, we set the ferromagnetic film to be a circular disk with diameter  $D$  for simplicity. We have verified that the observations in the following apply to other geometries such as rectangles or long wires. The film thickness is 5 nm and each magnetic cell is a cuboid with size  $2.5\text{nm} \times 2.5\text{nm} \times 5\text{nm}$ . We set saturation magnetization  $M_s = 5.5 \times 10^5 \text{A/m}$ , exchange stiffness  $A_{\text{ex}} = 9 \text{pJ/m}$ , and Gilbert damping coefficient  $\alpha = 0.04$ . To reduce the threshold current and minimize nonlinear damping [48], we introduce a perpendicular anisotropy field of  $B_a = 0.66 \text{T}$  to partially cancel the demagnetization field. The ferromagnet is biased with a DC field along  $y$ -axis  $B_0 = 0.18 \text{T}$  and a DC spin current  $J_s = 1.0 \times 10^{11} \text{A/m}^2$  which is larger than the oscillation threshold current for all of the cases considered below. Except for the spin current, no external DC or AC drives are assumed on the STO-resonator system. The parameters of the serial LCR resonator are chosen as  $L = 1.56 \text{nH}$ ,  $C = 0.637 \text{pF}$ , and  $R = 0.05 \Omega$ , which lead to characteristic impedance  $Z = \sqrt{L/C} = 50 \Omega$ , quality factor of  $Q = Z/R = 1000$ , and the resonant frequency  $\omega_r = 1/\sqrt{LC} = 2\pi \times 5 \text{GHz}$ , close to the uniform ferromagnetic resonance (FMR) frequency  $\omega_0 = \gamma \sqrt{B_0(B_0 + \mu_0 M_{\text{eff}})} \approx 2\pi \times 5.44 \text{GHz}$  of the ferromagnetic film. Fig. 2(a) shows the simulation results in the absence of an LCR resonator ( $b_{\text{rf}} = 0 \text{T/A}$ ), where the Fourier transform of  $x$ -component of magnetic oscillation  $\langle \hat{m}_x \rangle$  is illustrated for  $D = 320, 640$ , and  $1280 \text{nm}$  at  $T = 0 \text{K}$ . With the increase of device size, the current induced magnetic oscillation loses coherence, where the peak power decreases and linewidth increases, consistent with previous experimental observations [16, 17]. An inspection on the real-space distribution of magnetization azimuthal angle  $\phi_m$  [see Fig. 1(a) for definition] shows that the lack of phase coherence of magnetic dynamics

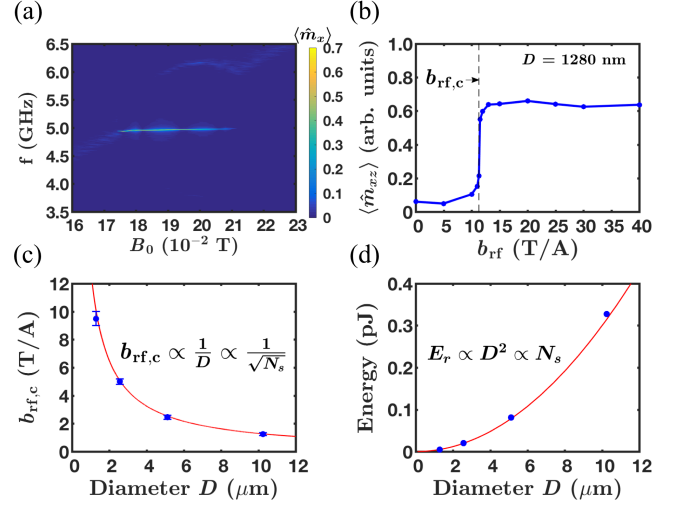


FIG. 3. (a) Fourier Transform of  $\langle \hat{m}_x \rangle$  as a function of  $B_0$ . In this simulation,  $b_{\text{rf}} = 25 \text{T/A}$  and  $D = 1280 \text{nm}$ . (b) The average magnetization projected onto  $xz$ -plane,  $\langle \hat{m}_{xz} \rangle$ , as a function of spin-photon coupling coefficient  $b_{\text{rf}}$ . The incoherence-to-coherence transition happens at  $b_{\text{rf},c} \approx 11.3 \text{T/A}$ . (c)  $b_{\text{rf},c}$  as a function of ferromagnetic disk diameter  $D$ . The red curve shows the result from analytical modeling.  $N_s$  is the number of spins in the disks. (d) Energy stored in the resonator  $E_r$  as a function of diameter  $D$ .  $b_{\text{rf}}D$  is kept constant in simulations with different sample sizes, with  $b_{\text{rf}} = 15 \text{T/A}$  for  $D = 1280 \text{nm}$ . The red curve represents the analytical results. Simulation cell size is  $2.5\text{nm} \times 2.5\text{nm} \times 5\text{nm}$  for (a)(b) and  $10\text{nm} \times 10\text{nm} \times 5\text{nm}$  for (c)(d).  $T$  is set to be  $0 \text{K}$  in simulations of this figure. In (c)(d),  $B_0$  and  $J_s$  are fixed in simulations with different diameters.

in different cells accounts for the small signal and broad linewidth [Fig. 2(b)]. The simulation results with finite spin-photon coupling ( $b_{\text{rf}} = 25 \text{T/A}$ ) are shown in Fig. 2(c) for the  $D = 1280 \text{nm}$  sample. In contrast to the uncoupled case,  $\langle \hat{m}_x \rangle$  exhibits a much larger oscillation amplitude and a much narrower linewidth. Moreover, real-space distribution of  $\phi_m$  shows that inhomogeneities are greatly suppressed, indicating the realization of macroscopically coherent magnetic self-oscillation [Fig. 2(d)].

To understand the required condition for reaching coherent magnetic oscillations, we carry out simulations for the  $D = 1280 \text{nm}$  sample under different external field  $B_0$ . As shown in Fig. 3(a), when the detuning between the STO free-running frequency  $\omega_S$  and the resonator frequency  $\omega_r$  is small ( $0.176 \text{T} < B_0 < 0.208 \text{T}$ ),  $\omega_S$  is locked onto  $\omega_r$ . We further verify that within the frequency-locking region, the relative phase difference between resonator and STO remains nearly constant as a function of time, indicating phase-locking. Next, we fix  $B_0 = 0.18 \text{T}$  and study the STO dynamics as a function of coupling coefficient  $b_{\text{rf}}$  [Fig. 3(b)]. We note that below a critical value  $b_{\text{rf},c}$ , the oscillation amplitude on the  $xz$ -plane  $\langle \hat{m}_{xz} \rangle$  remains almost zero, while above  $b_{\text{rf},c}$ ,  $\langle \hat{m}_{xz} \rangle$  jumps to a much higher value, indicating a transition from incoherent to coherent oscillation. In Fig. 3(c),

we plot the dependence of  $b_{\text{rf},c}$  on spin number  $N_s$  by carrying out simulations on samples with different sizes. We find that the threshold coupling coefficient decreases as  $N_s$  increases, satisfying a relationship of  $b_{\text{rf},c} \propto 1/\sqrt{N_s}$ , a result which can be further verified through analytical derivation (see Section II C). The scaling relation of  $b_{\text{rf},c} \propto 1/\sqrt{N_s}$  implies that it is easier to achieve coherent oscillations in samples with large  $N_s$ , e.g., with an extended thin film.  $b_{\text{rf}}$  is a factor that depends on the LCR resonator geometry and the proximity of the magnetic material to the inductor. For a two dimensional on-chip resonator design with a simple inductive wire [42], one has the relation of  $b_{\text{rf}} = \mu_0/2w$ , where  $w$  is the inductive wire width. Previously, it has been shown that with lithographically defined superconducting resonators,  $b_{\text{rf}}$  on the order of 0.1 T/A can be achieved [42]. By extrapolating results in Fig. 3(c), we see that with this  $b_{\text{rf}}$ , synchronized oscillations can be reached in samples with  $N_s > 10^{13}$ , corresponding to a lateral dimension of  $\approx 200 \mu\text{m}$ . Besides allowing easier synchronization with the resonator mode, a larger magnetic volume is also beneficial for increasing the emission power. In Fig. 3(d), we studied the energy stored in the resonator  $E_r$  as a function of the sample size, which shows a linear increase with  $N_s$ , providing a promising approach to overcome the power bottleneck encountered in classical STOs. Moreover, different from previous STO designs where the magnetic oscillation needs to be converted to output signal through certain magnetoresistance effects [2], the resonator can directly provide microwave emission to the external circuit, whose strength depends on the stored energy as well as the coupling with the external circuit [49].

### C. Analytical theory for oscillator synchronization

The cavity-assisted coherent magnetic self-oscillation can be further verified and extended with analytical studies. We rewrite Eq. (1-3) in complex oscillator representation following Slavin *et al.* [1, 50] (See Appendix BC):

$$\frac{dc_i}{dt} + i\omega_i(p_i)c_i + \Gamma_i(p_i)c_i = F_{mi}(c_r) + \sum_{i \neq j} G_{ji}(c_i, c_j) \quad (4)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = \sum_i F_{ri}(c_i). \quad (5)$$

Here  $c_i$ , defined as a superposition of  $\hat{m}_{xi}$  and  $\hat{m}_{zi}$  by taking into account the elliptical oscillation orbit [1, 50], is the dimensionless complex oscillation amplitude for magnetic cell  $i$  with power  $p_i = |c_i|^2$  and phase  $\phi_i = \arg(c_i)$ . In our simulations with nearly compensated magnetic anisotropy, we have  $c_i \approx (\hat{m}_{zi} - j\hat{m}_{xi})/\sqrt{2(1 + \hat{m}_{yi})}$  where  $j = \sqrt{-1}$ .  $\omega_i(p_i)$  is the power-dependent self-oscillation frequency which is related to the effective field experienced by each cell. Under first order approximation,  $\omega_i(p_i) = \omega_{0i} + K_i p_i$  with  $K_i = \frac{d\omega_i}{dp_i}$ .  $\Gamma_i(p_i) =$

$\Gamma_{i,+}(p_i) - \Gamma_{i,-}(p_i)$  is the nonlinear damping coefficient which includes both positive Gilbert damping  $\Gamma_{i,+}(p_i)$  and negative effective damping  $\Gamma_{i,-}(p_i)$  from spin torque.  $c_r = I + iV/Z = \sqrt{p_r}e^{i\phi_r}$  is the complex amplitude of LCR resonator with intrinsic frequency  $\omega_r$  and dissipation rate  $\omega_r/2Q$ .  $F_{ri}(c_i) = -g_{ri}c_i$  and  $F_{mi}(c_r) = g_m c_r$  reflect the mutual coupling between the  $i$ th magnetic cell and the LCR resonator, with  $g_{ri} = \omega_{0i}b_{\text{rf}}M_s V_c/L$  and  $g_m = \gamma b_{\text{rf}}/4$  in the case of nearly compensated magnetic anisotropy. Meanwhile, magnetic cells are also coupled with each other through  $G_{ji}$ , which includes exchange and dynamic dipolar interactions between cell  $i$  and cell  $j$ . Without  $G_{ji}$  and  $F_{mi}$ , magnetization in each cell  $i$  self-oscillates independently at frequency  $\omega_{gi} = \omega_i(p_{i0})$  with the equilibrium power  $p_{i0}$  satisfying  $\Gamma_i(p_{i0}) = 0$ , and no global coherence is expected. The existence of short range interaction  $G_{ji}$ , on the other hand, allows coherent oscillations from small, quasi-single domain samples, as shown in previous experiments [2]. However, it has been demonstrated that the short range interactions become less efficient and they cannot lead to global phase synchronization when the size of the sample goes to infinity [51, 52]. For this reason and also to get a closed mathematical form, we only consider the interactions between magnetic cells and the LCR resonator,  $F_{mi}$  and  $F_{ri}$ , in the following analytical derivation and rely on the full micromagnetic solution to check the extra influences brought by the short range interactions  $G_{ji}$ .

Isolating  $c_i$  into its phase and power parts, the coupled dynamics of the system is captured by the equations on magnetic cell phase  $\phi_i$  and resonator amplitude  $c_r$ :

$$\frac{d\phi_i}{dt} + \omega_{gi} = g_m \sqrt{1 + \nu_i^2} \sqrt{\frac{p_r}{p_{i0}}} \sin(\phi_r - \phi_i - \beta_i) \quad (6)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = \sum_i -g_{ri} \sqrt{p_{i0}} e^{i\phi_i} \quad (7)$$

where  $\nu_i = K_i/(G_{+,i} - G_{-,i})$  is a parameter quantifying the frequency nonlinearity of STO [1], with  $G_{+,i} = \frac{d\Gamma_{+,i}}{dp_i}|_{p_{i0}}$ ,  $G_{-,i} = \frac{d\Gamma_{-,i}}{dp_i}|_{p_{i0}}$ , and  $\beta_i = \arctan(\nu_i)$ . To capture the main physics of coupled phase oscillators, we set all parameters except  $\omega_{gi}$  to their average value  $\nu_i = \nu, p_{i0} = p_0, \beta_i = \beta_m, g_{ri} = g_r$ . In the limit of large  $N_c$ , we assume the frequency and phase distribution of magnetic cells satisfy a probability density function  $f(\phi, \omega, t)$  where  $f(\phi, \omega, t)d\phi d\omega$  describes the fraction of cells in phase  $(\phi, \phi + d\phi)$  and frequency  $(\omega, \omega + d\omega)$  [53]. To describe the coherence of the STO, we define the phase order parameter:

$$\Psi = \frac{1}{N_c} \sum_i e^{i\phi_i} = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f(\phi, \omega, t) e^{i\phi} d\phi \quad (8)$$

and we can rewrite Eq. (6) and (7) as:

$$\frac{d\phi}{dt} + \omega = g_m \sqrt{1 + \nu^2} \sqrt{\frac{p_r}{p_0}} \sin(\phi_r - \phi - \beta_m) \quad (9)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = -g_r \sqrt{p_0} N_c \Psi. \quad (10)$$



According to probability conservation, the probability function  $f(\phi, \omega, t)$  satisfies a continuity equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \phi} \left( f \frac{d\phi}{dt} \right) = 0 \quad (11)$$

where  $\frac{d\phi}{dt}$  is given by Eq. (9).

Eq. (9)-(11) give a description on the time evolution of the phase distribution of the coupled magnetic cells. To solve this equation set, we follow Ott-Antonsen ansatz [53, 54] and have a tentative solution  $f(\phi, \omega, t)$  expressed in Fourier series:

$$f(\phi, \omega, t) = \frac{h(\omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} [\alpha(\omega, t)]^n e^{in\phi} + c.c. \right] \quad (12)$$

where the Fourier coefficients take functional form of  $[\alpha(\omega, t)]^n$ . This type of solutions have been proven to be useful for describing systems with large number of oscillators. In Eq. (12),  $h(\omega) = \int_0^{2\pi} f(\phi, \omega, t) d\phi$  is the distribution of generation frequency of magnetic cells. The coupled Eq. (9)-(11) are solvable when the form of the frequency distribution  $h(\omega)$  is specified. Here, to obtain a closed-form analytical solution, we assume that frequency distribution follows Lorentzian distribution  $h(\omega) = \frac{\Delta_m}{\pi} \frac{1}{(\omega - \bar{\omega})^2 + \Delta_m^2}$  with a central frequency of  $\bar{\omega}$  and linewidth  $\Delta_m$ . Under this choice of frequency distribution, the phase order parameter has a simple expression  $\Psi = [\alpha(\bar{\omega} + i\Delta_m, t)]^*$ . Substituting Eq. (12) and Eq. (9) into Eq. (11), we find the order parameter obeys the following dynamics equation:

$$\frac{d\Psi}{dt} + i(\bar{\omega} - i\Delta_m)\Psi = \frac{g_m \sqrt{1 + \nu^2}}{2\sqrt{p_0}} [c_r e^{-i\beta_m} - \Psi^2 c_r^* e^{i\beta_m}] \quad (13)$$

which together with Eq. (10) describes the dynamics of the system.

Eq. (10) and (13) are two coupled differential equations with respect to  $\Psi$  and  $c_r$ . The bifurcation point of the dynamical system described by these two equations can be obtained via linear stability analysis. By assuming solutions of  $\Psi = \Psi_0 e^{(\lambda - i\Omega)t}$  and  $c_r = c_{r0} e^{(\lambda - i\Omega)t}$  and substituting them into Eq. (10) and (13), we find that non-trivial solution of  $|\Psi| \neq 0$  becomes the stable one when the coupling coefficient  $b_{rf}$  exceeds a threshold value of  $b_{rf,c}$ , which has the following simple expression:

$$b_{rf,c} = \sqrt{\frac{4Z\Delta_m}{\gamma\omega_0 Q M_s V_c N_c}} \propto \frac{1}{\sqrt{N_s}}, \quad (14)$$

when there is zero frequency nonlinearity in auto-oscillators ( $\nu = \beta_m = 0$ ) and zero detuning between central frequency of magnetic cells and the LCR resonator ( $\bar{\omega} = \omega_r$ ). In Appendix C, we further verify that  $b_{rf,c} \propto 1/\sqrt{N_s}$  holds for the more general case where the nonlinear frequency shift of the magnetic cells and a finite detuning are taken into account ( $d\omega_i/dp_i \neq 0$ ,  $\bar{\omega} \neq \omega_r$ ). Choosing parameters ( $\Delta_m$ ,  $\delta\omega = \bar{\omega} - \omega_r$ , and  $\beta_m$ ) that

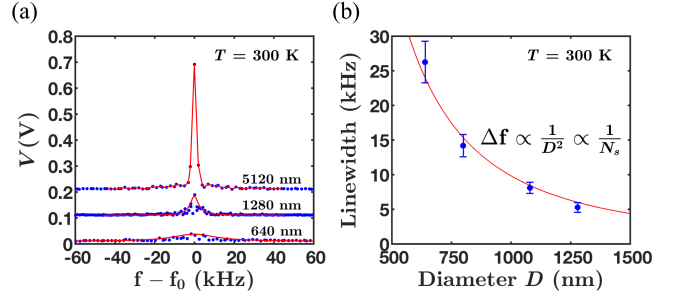


FIG. 4. (a) Fourier transform of resonator voltage  $V$  for samples with  $D = 640$ ,  $1280$ , and  $5120$  nm. Red curves for  $D = 640$  and  $1280$  nm samples illustrate Lorentzian fittings. (b) Voltage FWHM linewidth as a function of diameter  $D$ . To ensure fair comparisons,  $b_{rf}D$  is kept constant in simulating samples with different sizes, with  $b_{rf} = 15$  T/A for  $D = 1280$  nm. In this simulation,  $T = 300$  K and the lateral size of cells is chosen to be  $10$  nm.

are consistent with our simulated materials (Appendix D), we calculate  $b_{rf,c}$  using the full analytical model and get the red curve in Fig. 3(c), which agrees well with the micromagnetic simulation results (solid dots).

Besides the threshold coupling strength, the amplitude of the LCR resonator  $c_r$  in the synchronized state can also be derived analytically (Appendix C). With the same parameter set which determines  $b_{rf,c}$ , we calculate the stored energy in the LCR resonator  $E_r = L|c_r|^2/2$ , as represented by the red curve in Fig. 3(d), consistent with the numerical simulation results, where  $E_r$  scales with  $N_s$ .

#### D. Simulation results on oscillation linewidth

One of the key benefits of coupling auto-oscillators with a cavity is to get very narrow linewidths. To evaluate the generation linewidth of our proposed device at finite temperature, we carry out micromagnetic simulation at  $T = 300$  K. We note that compared with the  $T = 0$  K case shown in Fig. 3(c), the required coupling coefficient  $b_{rf,c}$  increases by 20%, but the synchronization and phase-locking behaviors remain. The resolution of the oscillation linewidth is determined by the simulated evolution time for magnetization, which is further limited by the hardware and software efficiency. For a sample with  $D = 5120$  nm, an upper bound of 4 kHz is determined for the linewidth of voltage (full-width-at-half-maximum, or FWHM) after 35 days' simulation, as shown in Fig. 4(a). The quality of generation signal ( $Q_g > 10^6$ ) is much higher than intrinsic quality factor of the LCR resonator ( $Q = 10^3$ ), suggesting that the narrow linewidth does not originate from the simple filtering effect of the resonator. To get a better understanding on generation linewidth as a function of device size, we perform simulations on a series of samples and the results are summarized in Fig. 4(b). It can be seen that

the linewidth  $\Delta f$  reduces as  $N_s$  increases, satisfying a relationship  $\Delta f \propto 1/N_s$  (red curve), which suggests that even narrower linewidth than shown in Fig. 4 can be potentially achieved in real samples with larger dimensions. This scaling trend is consistent with the theory on thermal phase noise induced auto-oscillator linewidth broadening [26, 27], where it was shown  $\Delta f \propto k_B T/E$ , with  $E$  being the oscillation energy of the system. In our system, the total energy contains both resonator energy and the energy of the magnetization oscillation [26, 27], both of which are proportional to  $N_s$ . The analytical understanding of the linewidth and the simulations beyond white noise, such as  $1/f$  noise in the coupled system [26, 55], are important future research directions.

### III. DISCUSSION AND CONCLUSIONS

The proposed device structure can not only be used as a general purpose, on-chip microwave source, but also provides important functionalities in emerging fields such as quantum electronics. In the field of quantum engineering, a scalable approach for interconnecting quantum bits with high quality microwave sources is required for realizing large scale quantum processors [56]. The STO maser discussed in this paper is a compact, highly coherent microwave source which can be realized using scalable photolithography process [41, 42]. The oscillator is already coupled with photon mode of the cavity, which can provide direct output to other parts of the quantum circuits. Moreover, with modulated DC spin current, the STO maser can generate high quality microwave pulses with controlled envelopes, which holds potential applications in scalable quantum information processing.

In summary, we studied the dynamics of a multi-domain STO located within a microwave resonator. We found that different from free STOs, a STO coupled with a microwave resonator can exhibit coherent oscillations even in large, extended ferromagnetic thin films. In the coherent region, the microwave emission power increases with the dimension of the magnetic film while the generation linewidth reduces. The requirement for reaching the coherent region can be satisfied by increasing the volume of magnetic films and using two dimensional resonator design that can be achieved with existing technologies. The operation mechanism of this device is reminiscent of a maser, with a large area STO serving as the nonlinear gain medium. Although the system is described by classical dynamical equations without referring to quantized energy levels, we note that essential laser physics can be understood in classical physics [44], with examples of free-electron lasers [57, 58] and Josephson junction lasers [59, 60]. By harnessing spin-photon coupling and spin orbit torque, we expect the demonstrated results can enable compact, highly coherent, on-chip microwave sources that are beneficial for applications in both classical and quantum electronics domains.

### ACKNOWLEDGMENTS

The work is supported by AFOSR under Grant No. FA9550-19-1-0048, and National Science foundation under Grant No. ECCS-1653553. J.T.H. acknowledge support from Mathworks fellowship. Simulations were performed on the MIT/IBM Satori GPU supercomputers supported by the Massachusetts Green High Performance Computing Center (MGHPCC).

### APPENDIX A: THERMAL NOISES FOR MICROMAGNETIC SIMULATIONS WITH A MICROWAVE RESONATOR

In this section, we provide derivations of thermal noise terms in Eq. (1-3). In the main text, we have shown that the dynamical equations for each magnetic cell  $i$  within a magnetic film and the serial LCR circuit with the mutual coupling terms can be written as:

$$\frac{d\hat{\mathbf{m}}_i}{dt} = \boldsymbol{\tau}_i - \gamma \hat{\mathbf{m}}_i \times \mathbf{b}_{\text{rf}} I \quad (\text{A1})$$

$$\frac{dI}{dt} = \frac{V}{L} - \frac{R}{L} I - \frac{b_{\text{rf}} M_s V_c}{L} \sum_i \frac{d\hat{m}_{xi}}{dt} \quad (\text{A2})$$

$$\frac{dV}{dt} = -\frac{I}{C}. \quad (\text{A3})$$

To consider finite temperature effect, the agitation from thermal noise onto the dynamics of magnet and the LCR resonator should be included. For magnetic dynamics, we used the torque from the thermal effect that is included in the standard Mumax3 package [46]. For the LCR resonator, we define a complex amplitude  $c_r = \sqrt{p_r} e^{i\phi_r} = I + iV/Z$ , where  $Z = \sqrt{L/C}$  is characteristic impedance. Eq. (A2) and (A3) without the coupling term can then be combined into a single equation  $\frac{dc_r}{dt} = -i\omega_r c_r - \frac{R}{2L} c_r + f_r(t)$ , with a complex thermal noise term  $f_r(t)$ , where  $\omega_r = 1/\sqrt{LC}$  is the resonance frequency. (Nonresonant terms proportional to  $c_r^*$  are dropped.) Following Slavin *et al.* [1], we consider Gaussian white noise such that  $\langle f_r(t) f_r(t') \rangle = 0$  and  $\langle f_r(t) f_r^*(t') \rangle = 2D_r \delta(t - t')$ , where  $D_r$  is the diffusion coefficient. Under Fokker-Planck equation, the stationary probability density function  $\mathcal{P}_r(p_r, \phi_r, t)$  satisfies:  $\frac{d}{dp_r} \left[ 2p_r \frac{R}{2L} \mathcal{P}_r + 2p_r D_r \frac{\partial \mathcal{P}_r}{\partial p_r} \right] = 0$  [1]. In the meantime,  $\mathcal{P}_r$  needs to satisfy the Boltzmann distribution under thermal equilibrium at temperature  $T$ ,  $\mathcal{P}_r \propto e^{-E_r/k_B T}$ , where  $E_r = CV^2/2 + LI^2/2 = Lp_r/2$  is the energy stored in the resonator. This sets  $D_r = Rk_B T/L^2$ . Rewrite the thermal noise terms for  $c_r$  into equations of  $I$  and  $V$ , we

obtain:

$$\frac{d\hat{\mathbf{m}}_i}{dt} = \boldsymbol{\tau}_i - \gamma \hat{\mathbf{m}}_i \times \mathbf{b}_{\text{rf}} I \quad (\text{A4})$$

$$\frac{dI}{dt} = \frac{V}{L} - \frac{R}{L} I - \frac{b_{\text{rf}} M_s V_c}{L} \sum_i \frac{d\hat{m}_{xi}}{dt} + f_I(t) \quad (\text{A5})$$

$$\frac{dV}{dt} = -\frac{I}{C} + f_V(t) \quad (\text{A6})$$

where  $\langle f_I(t)f_I(t') \rangle = Rk_B T/L^2 \delta(t-t')$  and  $\langle f_V(t)f_V(t') \rangle = Rk_B T/LC \delta(t-t')$ , respectively.

## APPENDIX B: THEORY OF NONLINEAR AUTO-OSCILLATOR COUPLED WITH LINEAR RESONATOR

For this section, we consider a spin-torque oscillator (STO) modeled as a macrospin which is coupled to an LCR resonator at  $T = 0$  K as described by Eq. (A1-A3). The results from this section will be used as a basis for the derivation of the synchronization condition for coupling multi-domain oscillators with a resonator. Here we first recap some of the key results for a STO without spin-photon coupling given by Slavin *et al.* [50] and consider the effects of coupling terms later. For a macrospin with  $\mathbf{M}$  magnetic moment per unit volume (magnetization), the Landau-Lifshitz-Gilbert (LLG) equation with spin-transfer term can be written as:

$$\frac{d\mathbf{M}}{dt} = \gamma \left( \mathbf{M} \times \frac{\delta W_0}{\delta \mathbf{M}} \right) + \mathbf{T}_\epsilon. \quad (\text{B1})$$

Here  $W_0 = \int (-\mathbf{H}_0 \cdot \mathbf{M} + 2\pi(\mathbf{M} \cdot \hat{\mathbf{z}})^2 - \frac{H_a}{2M_s}(\mathbf{M} \cdot \hat{\mathbf{z}})^2) d\mathbf{r}$  is the free-energy with  $H_a$  being anisotropy field, and  $\mathbf{T}_\epsilon = \mathbf{T}_h + \mathbf{T}_d + \mathbf{T}_s$  includes contributions from external microwave field  $\mathbf{T}_h = -\gamma(\mathbf{M} \times \mathbf{h})$ , Gilbert damping  $\mathbf{T}_d = \frac{\alpha}{M_s}(\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t})$ , and spin torque  $\mathbf{T}_s = \frac{\beta J_s}{M_s}(\mathbf{M} \times (\mathbf{M} \times \hat{\mathbf{p}}))$  where  $\beta = \frac{g\mu_B}{2eM_s t}$  with  $t$  being thickness of the magnetic film and  $J_s$  being the spin current density. Here we consider  $\mathbf{H}_0 = H_0 \hat{\mathbf{y}}$ ,  $\mathbf{h} = h_x \hat{\mathbf{x}}$ , and that the injected spin moments are oriented along  $-y$  axis. Due to the fixed length constraint  $|\mathbf{M}| = M_s$ , there are two independent degrees of freedom, which can be conveniently described by a complex circular precession amplitude  $a_m$  given by Holstein-Primakoff transformation  $a_m = (M_z - iM_x)/\sqrt{2M_s(M_s + M_y)}$ . The inverse transformation is  $\mathbf{M} = M_s(1 - 2|a_m|^2)\hat{\mathbf{y}} + M_s\sqrt{1 - |a_m|^2}((\hat{\mathbf{z}} + i\hat{\mathbf{x}})a_m + (\hat{\mathbf{z}} - i\hat{\mathbf{x}})a_m^*)$ . Clearly, only  $a_m$ 's satisfying  $|a_m| \leq 1$  have physical meaning. The equation of motion (EOM) can be written as:

$$\frac{da_m}{dt} = -i\frac{\delta \mathcal{H}_0}{\delta a_m^*} + F_a \quad (\text{B2})$$

where  $\mathcal{H}_0 = \gamma W_0/2M_s$  is macrospin Hamiltonian and  $F_a = \frac{\partial a_m}{\partial \mathbf{M}} \cdot \mathbf{T}_\epsilon$ . Because of the elliptical orbit of magnetization precession, the Hamiltonian will take a simpler form in elliptical precession amplitudes, called

$b_m$ . The relation between  $a_m$  and  $b_m$  amplitudes is  $a_m = ub_m - vb_m^*$  with  $u = \sqrt{(\omega_H + \omega_0)/2\omega_0}$  and  $v = \sqrt{(\omega_H - \omega_0)/2\omega_0}$ , where  $\omega_H = \gamma(H_0 + 2\pi M_{\text{eff}})$  and  $\omega_0 = \gamma\sqrt{H_0(H_0 + 4\pi M_{\text{eff}})}$  with  $4\pi M_{\text{eff}} = 4\pi M_s - H_a$ . This is a canonical transformation analogous to Bogoliubov transformation. In terms of  $b_m$ , the EOM is:

$$\frac{db_m}{dt} = -i\frac{\delta \mathcal{H}_0}{\delta b_m^*} + F_b \quad (\text{B3})$$

where  $F_b = uF_a + vF_a^*$ . Following the convention utilized in Slavin *et al.* [50], a new complex amplitude  $c_m = \sqrt{\omega_H/\omega_0}b_m$  is defined, where subscript  $m$  stands for magnetic dynamics to distinguish from resonator dynamics introduced later. By neglecting nonresonant terms and keeping leading order terms, we finally write the EOM in terms of  $c_m$ :

$$\frac{dc_m}{dt} = -i\frac{\delta \mathcal{H}_c}{\delta c_m^*} - \Gamma_+(|c_m|^2)c_m + \Gamma_- (|c_m|^2)c_m + F_{ch}. \quad (\text{B4})$$

Here  $\mathcal{H}_c = \omega_H \mathcal{H}_0/\omega_0 = \omega_0|c_m|^2 + \frac{K}{2}|c_m|^4$ , with  $\omega_0|c_m|^2$  being the linear term and  $\frac{K}{2}|c_m|^4$  corresponding to the non-linear frequency shift,  $K = \gamma\frac{2\omega_0}{\omega_H}(-(3(u^2+v^2)^2-1)+6uv(u^2+v^2))\pi M_{\text{eff}}$ . The second and third terms on the right hand side of Eq. (B4) represent the positive non-linear damping  $\Gamma_+(|c_m|^2) = \Gamma_0(1 + Q_1|c_m|^2 + Q_2|c_m|^4)$  due to Gilbert damping, and negative nonlinear damping  $\Gamma_- (|c_m|^2) = \Gamma_J(1 - |c_m|^2)$  due to spin torque effect, with  $\Gamma_0 = \alpha\omega_H$ ,  $\Gamma_J = \beta J_s$ ,  $Q_1 = 2 - 3(\omega_0/\omega_H)^2 - 2\omega_H'/\omega_H$ , and  $Q_2 = (2 - 3(\omega_H'/\omega_H) + (\omega_H'/\omega_H)^2)(\omega_H'/\omega_H)$  with  $\omega_H' = \gamma 2\pi M_{\text{eff}}$ . In the following discussions, we consider the case with  $\Gamma_J > \Gamma_0$ , corresponding to the operation regime of STOs. The external microwave field effect contributes to an additional driving term  $F_{ch} = \gamma\sqrt{\omega_H/\omega_0}(u+v)(c_r + c_r^*)b_{\text{rf}}/4$ , up to leading order. These results are adapted from Slavin *et al.* [50] after considering our proposed device geometry.

In the following, we consider the coupling between the macrospin STO and the LCR circuit, which is a coupling scheme raised in the current work and has not been visited in previous publications. We first note that the effect of LCR circuit on macrospin is to produce an effective magnetic field  $\mathbf{h} = h_x \hat{\mathbf{x}} = b_{\text{rf}} I \hat{\mathbf{x}}$ , resulting in  $F_{ch} = \gamma\sqrt{\omega_H/\omega_0}(u+v)(c_r + c_r^*)b_{\text{rf}}/4$ , where  $c_r = I + iV/Z$  is the complex resonator amplitude defined in the previous section. To consider the effect of STO dynamics on the LCR resonator, we rewrite Eq. (A2) and (A3) using  $c_r$ :

$$\frac{dc_r}{dt} = -i\omega_r c_r - \frac{R}{2L}c_r - \frac{b_{\text{rf}}}{L}\frac{dm_x}{dt} \quad (\text{B5})$$

where  $\omega_r = 1/\sqrt{LC}$  (Terms proportional to  $c_r^*$  are dropped). Since  $m_x$  is correlated with  $c_m$  through the transformations introduced above, we can rewrite  $\frac{dm_x}{dt} = iM_s V_c(u+v)\sqrt{\frac{\omega_0}{\omega_H}}(\frac{dc_m}{dt} - \frac{dc_m^*}{dt})$  up to leading order, where



$V_c$  is volume of the macrospin cell. Substituting the expressions of  $F_{ch}$  and  $\frac{dm_x}{dt}$  into Eq. (B4) and (B5) and only keeping leading order terms, we get equations which describe dynamics of the coupled system regarding the two complex amplitudes  $c_m$  and  $c_r$ :

$$\frac{dc_m}{dt} + i\omega(|c_m|^2)c_m + \Gamma(|c_m|^2)c_m = F_m(c_r) \quad (\text{B6})$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q}c_r = F_r(c_m) \quad (\text{B7})$$

where  $\Gamma(|c_m|^2) = \Gamma_+(|c_m|^2) - \Gamma_-(|c_m|^2)$ ,  $\omega(|c_m|^2) = \omega_0 + K|c_m|^2$ ,  $F_m(c_r) = g_m c_r$ ,  $F_r(c_m) = -g_r c_m$ , and  $Q = \sqrt{L/C}/R$  is the quality factor of LCR resonator. The parameters for the coupling terms are  $g_m = \gamma b_{rf}/4$  and  $g_r = \omega_0 b_{rf} M_s V_c / L$ . In Eq. (B6) and (B7), we dropped terms proportional to  $c_m^*$  and  $c_r^*$  under rotating wave approximation [50]. In the main text, we consider the case where the demagnetization field is nearly compensated with the perpendicular anisotropy, therefore, we have set  $\omega_H \approx \omega_0$  and  $u + v = 1$  in the equations above. We note that  $g_m \propto b_{rf}$  and  $g_r \propto b_{rf} N_s$ , where  $N_s$  is the number of spins. Eq. (B6) and (B7) represent coupling between a nonlinear auto-oscillator and a linear passive resonator, different from mutual synchronization between two STOs (nonlinear auto-oscillators) in previous literatures [1]. To proceed, by expressing  $c_m = \sqrt{p_m} e^{i\phi_m}$  and  $c_r = \sqrt{p_r} e^{i\phi_r}$ , we separate Eq. (B6) and (B7) into equations about the magnitude and about the phase:

$$\frac{dp_m}{dt} + 2\Gamma(p_m)p_m = 2g_m \sqrt{p_r p_m} \cos(\phi_r - \phi_m) \quad (\text{B8})$$

$$\frac{d\phi_m}{dt} + \omega(p_m) = g_m \sqrt{\frac{p_r}{p_m}} \sin(\phi_r - \phi_m) \quad (\text{B9})$$

$$\frac{dp_r}{dt} + \frac{\omega_r}{Q} p_r = -2g_r \sqrt{p_r p_m} \cos(\phi_r - \phi_m) \quad (\text{B10})$$

$$\frac{d\phi_r}{dt} + \omega_r = g_r \sqrt{\frac{p_m}{p_r}} \sin(\phi_r - \phi_m). \quad (\text{B11})$$

Without the coupling term which is proportional to  $g_m$ , the relevant solution to Eq. (B8) and (B9) is the persistent macrospin precession with stationary power  $p_0$  satisfying  $\Gamma_+(p_0) = \Gamma_-(p_0)$  and generation frequency  $\omega_g = \omega(p_0)$ . In the limit of weak coupling, we can expand Eq. (B8) and (B9) around the stationary point by treating power deviation  $\delta p_m = p_m - p_0$  as a small signal [1]:

$$\frac{d\delta p_m}{dt} + 2\Gamma_p \delta p_m = 2g_m \sqrt{p_r p_0} \cos(\phi_r - \phi_m) \quad (\text{B12})$$

$$\frac{d\phi_m}{dt} + \omega_g + K\delta p_m = g_m \sqrt{\frac{p_r}{p_0}} \sin(\phi_r - \phi_m) \quad (\text{B13})$$

where  $\Gamma_p = (G_+ - G_-)p_0$  with  $G_+ = \frac{d\Gamma_+}{dp}|_{p_0}$  and  $G_- = \frac{d\Gamma_-}{dp}|_{p_0}$ . Defining  $\nu = K/(G_+ - G_-)$  and keeping leading order terms, the dynamics of Eq. (B8-B11)

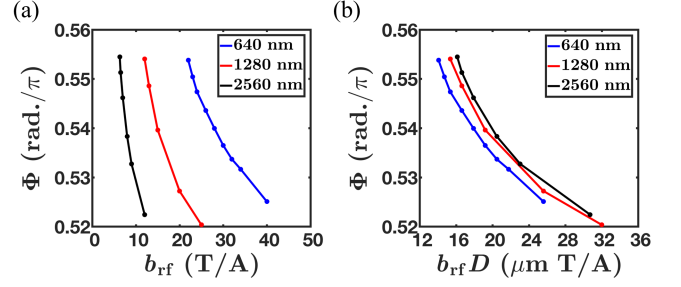


FIG. 5. Locked phase  $\Phi = \phi_r - \phi_m$  as a function of (a)  $b_{rf}$  and (b)  $b_{rf}D$  for different diameter  $D$ , with  $b_{rf} \geq b_{rf,c}$ . In this simulation,  $T = 0$  K and the lateral size of cells is chosen to be 2.5 nm.

can be captured by the following equations:

$$\frac{d\phi_m}{dt} + \omega_g = g_m \sqrt{1 + \nu^2} \sqrt{\frac{p_r}{p_0}} \sin(\phi_r - \phi_m - \beta_m) \quad (\text{B14})$$

$$\frac{d\phi_r}{dt} + \omega_r = g_r \sqrt{\frac{p_0}{p_r}} \sin(\phi_r - \phi_m) \quad (\text{B15})$$

$$\frac{dp_r}{dt} + \frac{\omega_r}{Q} p_r = -2g_r \sqrt{p_r p_0} \cos(\phi_r - \phi_m) \quad (\text{B16})$$

where  $\beta_m = \arctan(\nu)$  is a phase due to nonlinear frequency shift. Eq. (B14-B16) serve as a basis for deriving synchronization condition in a resonator-coupled multidomain STO in the next section. Moreover, this macrospin theory also describes the STO maser in the synchronized region with  $b_{rf} \geq b_{rf,c}$ , as discussed below.

We seek a stationary solution where  $dp_r/dt = 0$ , which suggests  $\cos(\phi_r - \phi_m) < 0$  and  $\Phi = \phi_r - \phi_m$  a constant in time indicating phase locking. Then we have:

$$p_r = 4p_0 \left[ g_r \frac{Q}{\omega_r} \cos \Phi \right]^2 \quad (\text{B17})$$

$$\omega_r - \omega_g = -\frac{\omega_r}{2Q} \tan \Phi + 2G \frac{Q}{\omega_r} \cos \Phi \sin(\Phi - \beta_m) \quad (\text{B18})$$

where  $G = g_m g_r \sqrt{1 + \nu^2}$ . Eq. (B18) determines the stationary locked phase  $\Phi$ , which then determines resonator power  $p_r$  in Eq. (B17). Note that  $g_m \propto b_{rf}$  and  $g_r \propto b_{rf} N_s$ . Therefore, the locked phase  $\Phi$  depends on  $b_{rf}$  and  $N_s$  through  $G \propto b_{rf}^2 N_s$ . In our simulations,  $a_m \approx c_m$  due to the nearly compensated magnetic anisotropy, and we approximate  $\phi_m = \arctan(-\langle \hat{m}_x \rangle / \langle \hat{m}_z \rangle)$ . The locked phases  $\Phi$  as a function of diameter  $D$  and  $b_{rf}$  are extracted from the simulation and plotted in Fig. 5. We note that  $\cos \Phi < 0$  and  $\Phi$  depends on  $b_{rf}$  and  $N_s$  through the combination  $b_{rf} D \propto b_{rf} N_s^2$  as illustrated in Fig. 5(b), consistent with the macrospin theory.

Besides, we can obtain the microwave emission frequency by inserting Eq. (B17) into Eq. (B15), obtaining  $\frac{d\phi_r}{dt} = -\omega_r(1 + \tan(\Phi)/2Q)$ . This indicates a relation

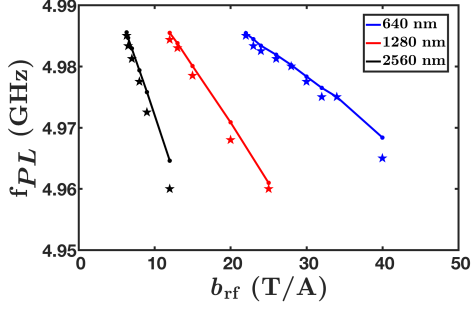


FIG. 6. Phase-locked frequency  $f_{PL}$  with  $b_{rf} \geq b_{rf,c}$ . Pentagram dots represent the peak frequencies extracted from simulation. Solid curves represent calculations using macrospin theory and the locked phases presented in Fig. 5.

between phase-locked frequency and phase:

$$\omega_{PL} = \omega_r \left( 1 + \frac{1}{2Q} \tan \Phi \right). \quad (B19)$$

In Fig. 6, the theoretical phase-locked frequencies calculated using Eq. (B19) and phases in Fig. 5 are shown in solid curves, which agree well with the peak frequencies obtained from simulation, represented by pentagram dots.

### APPENDIX C: SYNCHRONIZATION CONDITION FOR MULTIDOMAIN SPIN TORQUE OSCILLATOR WITH MICROWAVE RESONATOR

We model multidomain STO as consisting of  $N_c$  magnetic cells with volume  $V_c$ . As shown in earlier literatures, in the limit of large  $N_c$  short range interactions such as nearest-neighbor interaction from exchange and power-law interaction from dipolar field are insufficient to achieve synchronization of different cells. Moreover, it is mathematically challenging to derive analytical solution for synchronization in the presence of these short range coupling terms. Here, we only consider the interaction between magnetic cells and the LCR resonator. We show below that this model captures features of our simulation results semi-quantitatively. The dynamical equations of the system at  $T = 0$  K are the generalization of Eq. (B6) and (B7):

$$\frac{dc_i}{dt} + i\omega_i(|c_i|^2)c_i + \Gamma_{+i}(|c_i|^2)c_i - \Gamma_{-i}(|c_i|^2)c_i = F_{mi}(c_r) \quad (C1)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = \sum_i F_{ri}(c_i) \quad (C2)$$

where  $i$  is the cell index. In general, the functions  $\omega_i$ ,  $\Gamma_{+i}$ ,  $\Gamma_{-i}$ ,  $F_{mi}$ , and  $F_{ri}$  have  $i$ -dependent functional forms due on spatial variation of equilibrium effective field. For each

unit cell with complex amplitude  $c_i = \sqrt{p_i}e^{i\phi_i}$ , one can expand the equation around the stationary working point with corresponding stationary power  $p_{i0}$  and generation frequency  $\omega_{gi}$  as we did in Eq. (B14):

$$\frac{d\phi_i}{dt} + \omega_{gi} = g_m \sqrt{1 + \nu_i^2} \sqrt{\frac{p_r}{p_{i0}}} \sin(\phi_r - \phi_i - \beta_i) \quad (C3)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = \sum_i -g_{ri} \sqrt{p_{i0}} e^{i\phi_i} \quad (C4)$$

where the definition of  $g_{mi}$ ,  $\nu_i$  and  $\beta_i$  are in line with the corresponding definitions of  $g_m$ ,  $\nu$ , and  $\beta_m$  in previous section. These equations describe a complicated dynamical system with dimension  $N_c + 2$ . To obtain qualitative features of the system, we set all parameters except  $\omega_{gi}$  to their average value  $\nu_i = \nu$ ,  $p_{i0} = p_0$ ,  $\beta_i = \beta_m$ ,  $g_{ri} = g_r$ . In the limit of large  $N_c$ , we assume the frequency and phase distribution of magnetic cells satisfy a probability function  $f(\phi, \omega, t)$  where  $f(\phi, \omega, t)d\phi d\omega$  describes the fraction of cells in phase  $(\phi, \phi + d\phi)$  and frequency  $(\omega, \omega + d\omega)$  [53]. We define the phase order parameter:

$$\Psi = \frac{1}{N_c} \sum_i e^{i\phi_i} = \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} f(\phi, \omega, t) e^{i\phi} d\phi \quad (C5)$$

and we can rewrite Eq. (C3) and (C4) as:

$$\frac{d\phi}{dt} + \omega = g_m \sqrt{1 + \nu^2} \sqrt{\frac{p_r}{p_0}} \sin(\phi_r - \phi - \beta_m) \quad (C6)$$

$$\frac{dc_r}{dt} + i\omega_r c_r + \frac{\omega_r}{2Q} c_r = -g_r \sqrt{p_0} N_c \Psi. \quad (C7)$$

The function  $f(\phi, \omega, t)$  satisfies continuity equation  $\frac{\partial f}{\partial t} + \partial_\phi(f\dot{\phi}) = 0$ , where  $\dot{\phi}$  is given by Eq. (C6). Following the Ott-Antonsen ansatz [53, 54], we have a tentative solution  $f(\phi, \omega, t)$  expressed in Fourier series:

$$f(\phi, \omega, t) = \frac{h(\omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} [\alpha(\omega, t)]^n e^{in\phi} + c.c. \right] \quad (C8)$$

where the Fourier coefficients take functional form of  $[\alpha(\omega, t)]^n$ , which has been used to describe systems with large number of oscillators. We note that  $h(\omega) = \int_0^{2\pi} f(\phi, \omega, t) d\phi$  is the distribution of generation frequency of magnetic cells. Substituting Eq. (C8) and Eq. (C6) into the continuity equation, we have:

$$\begin{aligned} \frac{\partial \alpha}{\partial t} - i\omega \alpha &= \frac{g_m \sqrt{(1 + \nu^2)p_r}}{2\sqrt{p_0}} \left[ e^{-i(\phi_r - \beta_m)} - \alpha^2 e^{i(\phi_r - \beta_m)} \right]. \end{aligned} \quad (C9)$$

Moreover, within Ott-Antonsen ansatz, we note that the phase order parameter can be written  $\Psi = \int_{-\infty}^{\infty} d\omega h(\omega) \alpha^*(\omega, t)$ . In order to obtain qualitative results, we assume that frequency distribution follows Lorentzian distribution  $h(\omega) = \frac{\Delta_m}{\pi} \frac{1}{(\omega - \bar{\omega})^2 + \Delta_m^2}$  which

would simplify the expression of phase order parameter to  $\Psi = [\alpha(\bar{\omega} + i\Delta_m, t)]^*$ , using contour integration in the upper half complex plane [53]. As a result, Eq. (C9) governs the time evolution of  $\Psi$ , which together with Eq. (C7) specifies the dynamics of the system:

$$\frac{\partial \Psi}{\partial t} + i(\bar{\omega} - i\Delta_m)\Psi = \frac{g_m\sqrt{1+\nu^2}}{2\sqrt{p_0}} [c_r e^{-i\beta_m} - \Psi^2 c_r^* e^{i\beta_m}]. \quad (\text{C10})$$

To proceed, we write  $\Psi = \Psi_0 e^{i\phi_0}$ , where the phase  $\phi_0$  denotes the phase of the order parameter for magnetic dynamics. Separate Eq. (C10) and (C7) into phase and amplitude parts:

$$\frac{d\Psi_0}{dt} = -\Delta_m \Psi_0 + G_m \left( \frac{1 - \Psi_0^2}{2} \right) \cos(\phi_r - \phi_0 - \beta_m) \quad (\text{C11})$$

$$\frac{d\phi_0}{dt} + \bar{\omega} = G_m \left( \frac{1 + \Psi_0^2}{2} \right) \sin(\phi_r - \phi_0 - \beta_m) \quad (\text{C12})$$

$$\frac{dp_r}{dt} + \frac{\omega_r}{Q} p_r = -2g_r \sqrt{p_r p_0} N_c \Psi_0 \cos(\phi_r - \phi_0) \quad (\text{C13})$$

$$\frac{d\phi_r}{dt} + \omega_r = g_r \sqrt{\frac{p_0}{p_r}} N_c \Psi_0 \sin(\phi_r - \phi_0) \quad (\text{C14})$$

where  $G_m = g_m \sqrt{1+\nu^2} \sqrt{\frac{p_r}{p_0}}$ . We note that Eq. (C12-C14) generalizes Eq. (B14-B16), and they become identical in the case of  $\Psi_0 = 1$ , consistent with macrospin picture. The parameter  $g_r$  in Eq. (C13-C14) represents coupling from a single magnetic cell, which is accompanied with the number of cells  $N_c$  accounting for proper size scaling. Eq. (C11) is a new equation which quantifies the coherence of magnetic dynamics. In the absence of coupling to the resonator ( $g_m = 0$ ), one can verify that  $\Psi_0 = 0$  is a stable solution, corresponding to complete incoherence. However, as described below, when the mutual coupling strength represented by  $g_m g_r N_c$  reaches a critical value, this incoherence solution will lose its stability and the alternative phase-locked solution will describe the long-term dynamics of the hybrid system. To find this phase boundary, we linearize Eq. (C10) near  $\Psi_0 = 0$  by discarding the  $\Psi^2$  term. Assuming  $\Psi = \Psi_0 e^{(\lambda - i\Omega)t}$  and  $c_r = c_{r0} e^{(\lambda - i\Omega)t}$  in linearized Eq. (C7) and (C10), we have:

$$\begin{aligned} & [\lambda + \Delta_r - i(\Omega - \omega_r)] [\lambda + \Delta_m - i(\Omega - \bar{\omega})] \\ &= \frac{1}{2} g_m g_r N_c \sqrt{1+\nu^2} e^{-i\beta_m} \end{aligned} \quad (\text{C15})$$

where  $\Delta_r = \omega_r/2Q$  corresponds to the linewidth of the resonator. The phase boundary for the growth of coherence from the  $\Psi_0 = 0$  state is given by the condition  $\lambda = 0^+$  in Eq. (C15), which can be reached by a certain critical value of  $g_m g_r N_c$ . We define the corresponding threshold  $b_{\text{rf}}$  value as  $b_{\text{rf},c}$ . Under  $\lambda = 0^+$ , by letting the real and imaginary parts on both sides of Eq. (C15)

equal to each other, we arrive at the following equation for  $b_{\text{rf},c}$ :

$$\begin{aligned} & \frac{[\delta\omega\Delta_r - \frac{1}{2}G(b_{\text{rf},c})\sin\beta_m][\delta\omega\Delta_m + \frac{1}{2}G(b_{\text{rf},c})\sin\beta_m]}{(\Delta_m + \Delta_r)^2} \\ &= -\frac{1}{2}G(b_{\text{rf},c})\cos\beta_m - \Delta_r\Delta_m \end{aligned} \quad (\text{C16})$$

where  $\delta\omega = \bar{\omega} - \omega_r$ ,  $G(b_{\text{rf},c}) = g_{m,c}g_{r,c}N_c\sqrt{1+\nu^2} \propto b_{\text{rf},c}^2 N_s$ , with  $N_s$  being the number of total spins. Note that the threshold equation Eq. (C16) depends on  $b_{\text{rf},c}$  and  $N_s$  only through  $G(b_{\text{rf},c}) \propto b_{\text{rf},c}^2 N_s$ , which leads to the conclusion that  $b_{\text{rf},c} \propto 1/\sqrt{N_s}$ , consistent with simulation results presented in the main text. Eq. (C16) is a generalized result of the famous Millennium bridge problem by extending it to nonlinear oscillators of STO with nonlinear frequency shift [1, 54, 61]. Above the threshold coupling  $b_{\text{rf}} \geq b_{\text{rf},c}$ , the stationary solution is described by the phase-locked solution with nonzero  $\Psi_0$  given in Eq. (C11-C14). Combining Eq. (C12) and (C14) by defining  $\Phi = \phi_r - \phi_0$ , the phase-locked stationary solution is described by:

$$\Delta_m = -G(b_{\text{rf}}) \frac{Q}{\omega_r} (1 - \Psi_0^2) \cos\Phi \cos(\Phi - \beta_m) \quad (\text{C17})$$

$$\omega_r - \bar{\omega} = -\frac{\omega_r}{2Q} \tan\Phi + Z_0 G(b_{\text{rf}}) \frac{Q}{\omega_r} \cos\Phi \sin(\Phi - \beta_m) \quad (\text{C18})$$

$$p_r = 4p_0 \left[ \Psi_0 g_r N_c \frac{Q}{\omega_r} \cos\Phi \right]^2 \quad (\text{C19})$$

where  $Z_0 = (1 + \Psi_0^2)\Psi_0$ . The unknowns in Eq. (C17-C19) are  $\Psi_0$ ,  $\Phi$  and  $p_r$ , with  $\Delta_m, \bar{\omega}$ ,  $g$ , and  $\beta_m$  being material and biasing dependent parameters. Eq. (C17-C18) depend on  $b_{\text{rf}}$  through function  $G(b_{\text{rf}}) \propto b_{\text{rf}}^2 N_s$ . As a consequence,  $\Psi_0$  and  $\Phi$  can be determined from those two equations as functions of  $b_{\text{rf}}^2 N_s$  when other parameters are fixed. Therefore, for samples with different dimensions, if we pick  $b_{\text{rf}}$  slightly above the corresponding threshold value of  $b_{\text{rf},c}$ , as we did in Fig. 3(d) of the main text, we will get the same  $\Psi_0$  and  $\Phi$  since  $b_{\text{rf}}^2 N_s$  remains a constant. From Eq. (C19), we then get  $p_r \propto (g_r N_c)^2 \propto (b_{\text{rf}} N_s)^2 \propto N_s$ , which describes simulation results presented in the main text.

#### APPENDIX D: PARAMETER SET IN ANALYTICAL MODEL

With Eq. (C16-C19), the theoretical model explains the simulation results of  $b_{\text{rf},c}$  and  $E_r = Lp_r/2$  in main text. With  $L = 1.56$  nH,  $M_s = 5.5 \times 10^5$  A/m, and  $\omega_0 \approx 2\pi \times 5.44$  GHz utilized in the simulations, the values of  $g_m = \gamma b_{\text{rf}}/4$  and  $g_r N_c = \omega_0 b_{\text{rf}} M_s V_c N_c / L$  for different  $b_{\text{rf}}$  and  $D$  can be determined. Moreover, we estimate the value of  $p_0 \approx 0.3167$  from the simulation of uncoupled case. Under the chosen optimal parameters of  $\Delta_m =$

$2\pi \times 0.326$  GHz,  $\omega_r - \bar{\omega} = 2\pi \times 0.28$  GHz, and  $\beta_m = 2.43$ , the analytical model gives red curves in Fig. 3(c)

and (d) in the main text. For a self-consistency check,  $\nu = \tan \beta_m < 0$  gives  $K < 0$ , which is consistent with the fact that  $\bar{\omega} < \omega_0$ .

- 
- [1] A. Slavin and V. Tiberkevich, Nonlinear Auto-Oscillator Theory of Microwave Generation by Spin-Polarized Current, *IEEE Trans. Magn.* **45**, 1875 (2009).
  - [2] T. Chen, R. K. Dumas, A. Eklund, P. K. Muduli, A. Houshang, A. A. Awad, P. Dürrenfeld, B. G. Malm, A. Rusu, and J. Åkerman, Spin-Torque and Spin-Hall Nano-Oscillators, *Proc. IEEE* **104**, 1919 (2016).
  - [3] S. I. Kiselev, J. C. Sankey, I. N. Krivorotov, N. C. Emley, R. J. Schoelkopf, R. A. Buhrman, and D. C. Ralph, Microwave oscillations of a nanomagnet driven by a spin-polarized current, *Nature* **425**, 380 (2003).
  - [4] W. H. Rippard, M. R. Pufall, S. Kaka, S. E. Russek, and T. J. Silva, Direct-Current Induced Dynamics in  $\text{Co}_{90}\text{Fe}_{10}/\text{Ni}_{80}\text{Fe}_{20}$  Point Contacts, *Phys. Rev. Lett.* **92**, 027201 (2004).
  - [5] I. N. Krivorotov, N. C. Emley, J. C. Sankey, S. I. Kiselev, D. C. Ralph, and R. A. Buhrman, Time-Domain Measurements of Nanomagnet Dynamics Driven by Spin-Transfer Torques, *Science* **307**, 228 (2005).
  - [6] A. A. Tulapurkar, Y. Suzuki, A. Fukushima, H. Kubota, H. Maehara, K. Tsunekawa, D. D. Djayaprawira, N. Watanabe, and S. Yuasa, Spin-torque diode effect in magnetic tunnel junctions, *Nature* **438**, 339 (2005).
  - [7] J. Torrejon, M. Riou, F. A. Araujo, S. Tsunegi, G. Khalsa, D. Querlioz, P. Bortolotti, V. Cros, K. Yakushiji, A. Fukushima, H. Kubota, S. Yuasa, M. D. Stiles, and J. Grollier, Neuromorphic computing with nanoscale spintronic oscillators, *Nature* **547**, 428 (2017).
  - [8] M. Romera, P. Talatchian, S. Tsunegi, F. Abreu Araujo, V. Cros, P. Bortolotti, J. Trastoy, K. Yakushiji, A. Fukushima, H. Kubota, S. Yuasa, M. Ernoult, D. Vodenicarevic, T. Hirtzlin, N. Locatelli, D. Querlioz, and J. Grollier, Vowel recognition with four coupled spin-torque nano-oscillators, *Nature* **563**, 230 (2018).
  - [9] A. M. Deac, A. Fukushima, H. Kubota, H. Maehara, Y. Suzuki, S. Yuasa, Y. Nagamine, K. Tsunekawa, D. D. Djayaprawira, and N. Watanabe, Bias-driven high-power microwave emission from MgO-based tunnel magnetoresistance devices, *Nat. Phys.* **4**, 803 (2008).
  - [10] D. Houssameddine, U. Ebels, B. Delaët, B. Rodmacq, I. Firastrau, F. Ponthenier, M. Brunet, C. Thirion, J.-P. Michel, L. Prejbeanu-Buda, M.-C. Cyrille, O. Redon, and B. Dieny, Spin-torque oscillator using a perpendicular polarizer and a planar free layer, *Nat. Mater.* **6**, 447 (2007).
  - [11] V. S. Pribiag, I. N. Krivorotov, G. D. Fuchs, P. M. Braganca, O. Ozatay, J. C. Sankey, D. C. Ralph, and R. A. Buhrman, Magnetic vortex oscillator driven by d.c. spin-polarized current, *Nat. Phys.* **3**, 498 (2007).
  - [12] I. M. Miron, K. Garello, G. Gaudin, P.-J. Zermatten, M. V. Costache, S. Auffret, S. Bandiera, B. Rodmacq, A. Schuhl, and P. Gambardella, Perpendicular switching of a single ferromagnetic layer induced by in-plane current injection, *Nature* **476**, 189 (2011).
  - [13] L. Liu, C.-F. Pai, Y. Li, H. W. Tseng, D. C. Ralph, and R. A. Buhrman, Spin-Torque Switching with the Giant Spin Hall Effect of Tantalum, *Science* **336**, 555 (2012).
  - [14] V. E. Demidov, S. Urazhdin, H. Ulrichs, V. Tiberkevich, A. Slavin, D. Baither, G. Schmitz, and S. O. Demokritov, Magnetic nano-oscillator driven by pure spin current, *Nat. Mater.* **11**, 1028 (2012).
  - [15] L. Liu, C.-F. Pai, D. C. Ralph, and R. A. Buhrman, Magnetic Oscillations Driven by the Spin Hall Effect in 3-Terminal Magnetic Tunnel Junction Devices, *Phys. Rev. Lett.* **109**, 186602 (2012).
  - [16] V. E. Demidov, S. Urazhdin, E. R. J. Edwards, M. D. Stiles, R. D. McMichael, and S. O. Demokritov, Control of Magnetic Fluctuations by Spin Current, *Phys. Rev. Lett.* **107**, 107204 (2011).
  - [17] A. Smith, K. Sobotkiewicz, A. Khan, E. A. Montoya, L. Yang, Z. Duan, T. Schneider, K. Lenz, J. Lindner, K. An, X. Li, and I. N. Krivorotov, Dimensional crossover in spin Hall oscillators, *Phys. Rev. B* **102**, 054422 (2020).
  - [18] Z. Duan, A. Smith, L. Yang, B. Youngblood, J. Lindner, V. E. Demidov, S. O. Demokritov, and I. N. Krivorotov, Nanowire spin torque oscillator driven by spin orbit torques, *Nat. Commun.* **5**, 5616 (2014).
  - [19] J. Grollier, V. Cros, and A. Fert, Synchronization of spin-transfer oscillators driven by stimulated microwave currents, *Phys. Rev. B* **73**, 060409(R) (2006).
  - [20] S. Tsunegi, T. Taniguchi, R. Lebrun, K. Yakushiji, V. Cros, J. Grollier, A. Fukushima, S. Yuasa, and H. Kubota, Scaling up electrically synchronized spin torque oscillator networks, *Sci. Rep.* **8**, 13475 (2018).
  - [21] S. Kaka, M. R. Pufall, W. H. Rippard, T. J. Silva, S. E. Russek, and J. A. Katine, Mutual phase-locking of microwave spin torque nano-oscillators, *Nature* **437**, 389 (2005).
  - [22] F. B. Mancoff, N. D. Rizzo, B. N. Engel, and S. Tehrani, Phase-locking in double-point-contact spin-transfer devices, *Nature* **437**, 393 (2005).
  - [23] S. M. Rezende, F. M. de Aguiar, R. L. Rodríguez-Suárez, and A. Azevedo, Mode Locking of Spin Waves Excited by Direct Currents in Microwave Nano-oscillators, *Phys. Rev. Lett.* **98**, 087202 (2007).
  - [24] A. Ruotolo, V. Cros, B. Georges, A. Dussaux, J. Grollier, C. Deranlot, R. Guillemet, K. Bouzehouane, S. Fusil, and A. Fert, Phase-locking of magnetic vortices mediated by antivortices, *Nat. Nanotechnol.* **4**, 528 (2009).
  - [25] N. Locatelli, A. Hamadeh, F. Abreu Araujo, A. D. Belanovsky, P. N. Skirdkov, R. Lebrun, V. V. Naletov, K. A. Zvezdin, M. Muñoz, J. Grollier, O. Klein, V. Cros, and G. de Loubens, Efficient Synchronization of Dipolarly Coupled Vortex-Based Spin Transfer Nano-Oscillators, *Sci. Rep.* **5**, 17039 (2015).
  - [26] M. Zahedinejad, A. A. Awad, S. Muralidhar, R. Khymyn, H. Fulara, H. Mazraati, M. Dvornik, and J. Åkerman, Two-dimensional mutually synchronized spin Hall nano-oscillator arrays for neuromorphic computing, *Nat. Nanotechnol.* **15**, 47 (2020).
  - [27] J.-V. Kim, V. Tiberkevich, and A. N. Slavin, Generation Linewidth of an Auto-Oscillator with a Nonlinear Fre-

- quency Shift: Spin-Torque Nano-Oscillator, *Phys. Rev. Lett.* **100**, 017207 (2008).
- [28] W. H. Rippard, M. R. Pufall, S. Kaka, T. J. Silva, S. E. Russek, and J. A. Katine, Injection Locking and Phase Control of Spin Transfer Nano-oscillators, *Phys. Rev. Lett.* **95**, 067203 (2005).
- [29] B. Georges, J. Grollier, M. Darques, V. Cros, C. Deranlot, B. Marciilhac, G. Faini, and A. Fert, Coupling Efficiency for Phase Locking of a Spin Transfer Nano-Oscillator to a Microwave Current, *Phys. Rev. Lett.* **101**, 017201 (2008).
- [30] V. E. Demidov, H. Ulrichs, S. V. Gurevich, S. O. Demokritov, V. S. Tiberkevich, A. N. Slavin, A. Zhodud, and S. Urazhdin, Synchronization of spin Hall nano-oscillators to external microwave signals, *Nat. Commun.* **5**, 3179 (2014).
- [31] S. Tsunegi, E. Grimaldi, R. Lebrun, H. Kubota, A. S. Jenkins, K. Yakushiji, A. Fukushima, P. Bortolotti, J. Grollier, S. Yuasa, and V. Cros, Self-Injection Locking of a Vortex Spin Torque Oscillator by Delayed Feedback, *Sci. Rep.* **6**, 26849 (2016).
- [32] S. Tamaru, H. Kubota, K. Yakushiji, S. Yuasa, and A. Fukushima, Extremely Coherent Microwave Emission from Spin Torque Oscillator Stabilized by Phase Locked Loop, *Sci. Rep.* **5**, 18134 (2015).
- [33] Ö. O. Soykal and M. E. Flatté, Strong Field Interactions between a Nanomagnet and a Photonic Cavity, *Phys. Rev. Lett.* **104**, 077202 (2010).
- [34] H. Huebl, C. W. Zollitsch, J. Lotze, F. Hocke, M. Greifenstein, A. Marx, R. Gross, and S. T. B. Goennenwein, High Cooperativity in Coupled Microwave Resonator Ferrimagnetic Insulator Hybrids, *Phys. Rev. Lett.* **111**, 127003 (2013).
- [35] X. Zhang, C.-L. Zou, L. Jiang, and H. X. Tang, Strongly Coupled Magnons and Cavity Microwave Photons, *Phys. Rev. Lett.* **113**, 156401 (2014).
- [36] Y. Tabuchi, S. Ishino, T. Ishikawa, R. Yamazaki, K. Usami, and Y. Nakamura, Hybridizing Ferromagnetic Magnons and Microwave Photons in the Quantum Limit, *Phys. Rev. Lett.* **113**, 083603 (2014).
- [37] L. Bai, M. Harder, Y. P. Chen, X. Fan, J. Q. Xiao, and C.-M. Hu, Spin Pumping in Electrodynamically Coupled Magnon-Photon Systems, *Phys. Rev. Lett.* **114**, 227201 (2015).
- [38] Y. Tabuchi, S. Ishino, A. Noguchi, T. Ishikawa, R. Yamazaki, K. Usami, and Y. Nakamura, Coherent coupling between a ferromagnetic magnon and a superconducting qubit, *Science* **349**, 405 (2015).
- [39] Y. Cao, P. Yan, H. Huebl, S. T. B. Goennenwein, and G. E. W. Bauer, Exchange magnon-polaritons in microwave cavities, *Phys. Rev. B* **91**, 094423 (2015).
- [40] W. Yu, J. Wang, H. Y. Yuan, and J. Xiao, Prediction of Attractive Level Crossing via a Dissipative Mode, *Phys. Rev. Lett.* **123**, 227201 (2019).
- [41] Y. Li, T. Polakovic, Y.-L. Wang, J. Xu, S. Lendinez, Z. Zhang, J. Ding, T. Khaire, H. Saglam, R. Divan, J. Pearson, W.-K. Kwok, Z. Xiao, V. Novosad, A. Hoffmann, and W. Zhang, Strong Coupling between Magnons and Microwave Photons in On-Chip Ferromagnet-Superconductor Thin-Film Devices, *Phys. Rev. Lett.* **123**, 107701 (2019).
- [42] J. T. Hou and L. Liu, Strong Coupling between Microwave Photons and Nanomagnet Magnons, *Phys. Rev. Lett.* **123**, 107702 (2019).
- [43] W. E. Lamb, Theory of an Optical Maser, *Phys. Rev.* **134**, A1429 (1964).
- [44] M. Borenstein and W. E. Lamb, Classical Laser, *Phys. Rev. A* **5**, 1298 (1972).
- [45] J. D. Breeze, E. Salvadori, J. Sathian, N. M. Alford, and C. W. M. Kay, Continuous-wave room-temperature diamond maser, *Nature* **555**, 493 (2018).
- [46] A. Vansteenkiste, J. Leliaert, M. Dvornik, M. Helsen, F. Garcia-Sanchez, and B. Van Waeyenberge, The design and verification of MuMax3, *AIP Adv.* **4**, 107133 (2014).
- [47] For modified Mumax3 codes with spin-resonator coupling, see: [https://github.com/jthou0129/Mumax3\\_spinresonator.git](https://github.com/jthou0129/Mumax3_spinresonator.git).
- [48] B. Divinskiy, S. Urazhdin, S. O. Demokritov, and V. E. Demidov, Controlled nonlinear magnetic damping in spin-Hall nano-devices, *Nat. Commun.* **10**, 5211 (2019).
- [49] M. Göppl, A. Fragner, M. Baur, R. Bianchetti, S. Filipp, J. M. Fink, P. J. Leek, G. Puebla, L. Steffen, and A. Wallraff, Coplanar waveguide resonators for circuit quantum electrodynamics, *J. Appl. Phys.* **104**, 113904 (2008).
- [50] A. Slavin and V. Tiberkevich, Excitation of Spin Waves by Spin-Polarized Current in Magnetic Nano-Structures, *IEEE Trans. Magn.* **44**, 1916 (2008).
- [51] H. Hong, H. Park, and M. Y. Choi, Collective synchronization in spatially extended systems of coupled oscillators with random frequencies, *Phys. Rev. E* **72**, 036217 (2005).
- [52] V. Flovik, F. Macià, and E. Wahlström, Describing synchronization and topological excitations in arrays of magnetic spin torque oscillators through the Kuramoto model, *Sci. Rep.* **6**, 32528 (2016).
- [53] E. Ott and T. M. Antonsen, Low dimensional behavior of large systems of globally coupled oscillators, *Chaos* **18**, 037113 (2008).
- [54] M. M. Abdulrehem and E. Ott, Low dimensional description of pedestrian-induced oscillation of the Millennium Bridge, *Chaos* **19**, 013129 (2009).
- [55] M. W. Keller, M. R. Pufall, W. H. Rippard, and T. J. Silva, Nonwhite frequency noise in spin torque oscillators and its effect on spectral linewidth, *Phys. Rev. B* **82**, 054416 (2010).
- [56] J. C. Bardin, D. H. Slichter, and D. J. Reilly, Microwaves in Quantum Computing, *IEEE J. Microw.* **1**, 403 (2021).
- [57] F. A. Hopf, P. Meystre, M. O. Scully, and W. H. Louisell, Classical theory of a free-electron laser, *Opt. Commun.* **18**, 413 (1976).
- [58] W. B. Colson, One-body electron dynamics in a free electron laser, *Phys. Lett. A* **64**, 190 (1977).
- [59] M. C. Cassidy, A. Bruno, S. Rubbert, M. Irfan, J. Kammerhuber, R. N. Schouten, A. R. Akhmerov, and L. P. Kouwenhoven, Demonstration of an ac Josephson junction laser, *Science* **355**, 939 (2017).
- [60] S. H. Simon and N. R. Cooper, Theory of the Josephson Junction Laser, *Phys. Rev. Lett.* **121**, 027004 (2018).
- [61] S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, Crowd synchrony on the Millennium Bridge, *Nature* **438**, 43 (2005).