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Batch Optimization of Frequency-Modulated Pulse for Robust Two-qubit Gates in Ion Chains

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Two-qubit gates in trapped ion quantum computers are generated by applying spin-dependent forces that temporarily entangle the internal state of the ion with its motion. Laser pulses are carefully designed to generate a maximally entangling gate between the ions while minimizing any residual entanglement between the motion and the ion. The quality of the gates suffers when actual experimental parameters differ from the ideal case. Here we improve the robustness of frequency-modulated Mølmer-Sørensen gates to motional mode frequency offsets by optimizing average performance over a range of systematic errors using batch optimization. We then compare this method to frequency modulated gates optimized for ideal parameters that include an analytic robustness condition. Numerical simulations show good performance up to 12 ions and the method is experimentally demonstrated on a two-ion chain.

I. INTRODUCTION

Trapped ion systems are one of the leading candidates for scalable quantum computing platform [1, 2]. In addition to the near-perfect coherence properties [3, 4] and single-qubit gates with error rates below 10^{-4} [5–8], trapped ion qubits have significant advantages in entangling gate fidelities. For systems with exactly two ions, state-of-art two-qubit gates reached a fidelity higher than 99.9% by applying state-dependent force with lasers [9, 10] or magnetic field gradients [11]. For larger systems, two-qubit gate fidelities $> 99\%$ for a four-ion chain [12] and $> 97\%$ for 13-ion and 17-ion chains [13, 14] have been reported. Trapped ion systems with many qubits are particularly promising as long-range Coulomb interactions between ions lead to all-to-all connectivity between qubits [13, 15].

The central challenge in achieving scalability is to perform high-fidelity entangling gates with a large number of qubits. Entangling gates are performed by briefly exciting the ions' normal modes of motion, which serve as carrier of quantum information [16, 17]. The driving field should be carefully controlled such that all motional modes are completely disentangled from the internal qubit states at the end of the gate, while the qubit states undergo a maximally entangling operation.

In the presence of noise and parameter drifts, pulse design is necessary to achieve fast and robust high-fidelity gates. One approach is to design the amplitudes of multi-chromatic beams that suppress the effect of noise [18–23].

Another way is to control the amplitude [24–29], phase [30–33], and/or frequency [12, 14, 34] modulation over many time segments, which has recently been applied to experiments with many ions [12, 14, 27–29, 31].

While the methods above lead to analytic robustness by guaranteeing high fidelity up to a certain order [22] for uncertainty in a control parameter, a promising approach is to find the robust pulse numerically using machine learning (ML)-inspired optimization algorithms. In particular, Ref.[35] shows that training with a large sample set and mini-batches of parameter offsets significantly improves the robustness of the optimized pulse on a generic Hamiltonian with control fields. For trapped ion systems, Ref.[36] demonstrates application of deep reinforcement learning to robust single-qubit gate.

In this work we improve on previous discrete and continuous frequency modulation (FM) schemes [12, 34]. We propose two algorithms for FM pulse optimization by training with large sample set and mini-batches, namely, s(ample)-robust and b(atch)-robust, following the notation of Ref.[35]. The rest of the paper is organized as follows. In Section II, we briefly review the theory of robust frequency-modulated Mølmer-Sørensen (MS) gate [34] and introduce the optimization schemes for s-robust and b-robust. In Section III, we show that s-robust and b-robust FM are significantly more robust to motional mode frequency drifts than previous robust FM. We also discuss the scalability of b-robust FM. In Section IV, we show the experimental results with a two-ion chain which demonstrate that b-robust FM is more robust to detuning errors than robust FM. Finally, we summarize our results and discuss future directions in Section V.

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II. ROBUST OPTIMIZATION METHODS FOR FREQUENCY-MODULATED MS GATES

The frequency-modulated MS gate uses a state-dependent force induced by lasers at the drive frequency modulated near the sideband frequencies [16, 17]. When addressed by lasers with correct optical phases, the ions j_1 and j_2 undergo the unitary evolution described by the following [34, 37]:

$$U(\tau) = \exp\left\{ \sum_{j=j_1, j_2} \sum_k \left((\alpha_k^j(\tau) \hat{a}_k^\dagger - \alpha_k^{j*}(\tau) \hat{a}_k) \hat{\sigma}_x^j \right) + i\Theta(\tau) \hat{\sigma}_x^{j_1} \hat{\sigma}_x^{j_2} \right\} \quad (1)$$

where

$$\alpha_k^j(\tau) = \frac{\Omega}{2} \eta_k^j \int_0^\tau e^{-i\theta_k(t)} dt \quad (2)$$

$$\Theta(\tau) = -\frac{\Omega^2}{2} \sum_k \eta_k^{j_1} \eta_k^{j_2} \int_0^\tau dt_1 \int_0^{t_1} dt_2 \sin(\theta_k(t_1) - \theta_k(t_2)) \quad (3)$$

Here, τ is the pulse length, Ω is the carrier Rabi frequency, η_k^j is the Lamb-Dicke parameter of ion j with respect to motional mode k , and $\hat{\sigma}_x^j$ is the bit-flip Pauli operator of ion j . Also,

$$\theta_k(t) = \int_0^t (\mu(t') - \omega_k) dt' \quad (4)$$

is the phase of motional mode k , which is the integral of detuning between the drive frequency $\mu(t)$ and mode frequency ω_k . The first term in Eq.1 describes state-dependent displacement of the motional modes, while the second term represents rotation with respect to the two-qubit axis $\hat{\sigma}_x^{j_1} \hat{\sigma}_x^{j_2}$.

For an ideal MS gate, the qubits should be completely disentangled from the motional modes ($\alpha_k^j(\tau) = 0 \forall j, k$) and the rotation angle $\Theta(\tau)$ should reach exactly $\pi/4$ at the gate's conclusion [29, 38]. Hence, the goal of robust FM is to modulate the drive frequency profile $\mu(t)$ such that $\alpha_k^j(\tau)$ and $|\Theta(\tau) - \pi/4|$ are sufficiently minimized, in the presence of mode frequency offsets ϵ_k , i.e. $\omega_k \rightarrow \omega_k + \epsilon_k$.

Minimizing $|\alpha_k^j(\tau)| \propto |\int_0^\tau e^{-i\theta_k(t)} dt|$ is the intuitive criterion for an optimized gate. However, such gate is sensitive to small changes $\epsilon_k \ll 1/\tau$. Instead, Ref.[34] induces robustness by minimizing the time-averaged displacement $|\alpha_{k,\text{avg}}^j| \propto \frac{1}{\tau} |\int_0^\tau \int_0^t e^{-i\theta_k(t')} dt' dt|$, which is proportional to the first-order correction of $|\alpha_k^j(\tau)|$ when $\omega_k \rightarrow \omega_k + \epsilon_k$. Note that a time-symmetric pulse can be used to guarantee that minimizing $|\alpha_{k,\text{avg}}^j|$ also minimizes $|\alpha_k^j(\tau)|$. This optimization scheme, which we call ‘‘robust FM’’, has been used in recent experiments with 4-ion [12] and 17-ion [14] chains. Similar approach

with amplitude and phase modulation [32, 33] have also been studied.

Although robust FM is shown to be robust to mode frequency offsets that are an order of magnitude smaller than $1/\tau$, it does not guarantee robustness to $\epsilon_k \lesssim 1/\tau$. Moreover, robustness of the angle $\Theta(\tau) \approx \pi/4$ to detuning errors is not enforced by this method.

Inspired by the recent work in applying machine learning with large sample set and mini-batches to quantum control [35], we present ‘‘s(ample)-robust’’ and ‘‘b(atch)-robust’’ FM, which further enhances the robustness of the two-qubit gate. Instead of minimizing the analytic first-order correction, we minimize the average of $|\alpha_k^j(\tau)|^2$ over an ensemble of offsets, thereby directly incorporating the robustness condition into the cost function. Similarly, we also include the robustness condition of the angle $\Theta(\tau)$ in our cost function. Note that optimizing robustness of displacement has been achieved to some extent by various methods [20, 22, 32], but not with the additional goal of optimizing robustness of angle [39], although this is crucial to reach high fidelity in the presence of motional frequency drifts. We find the optimal FM pulse $\mu(t)$ that minimizes the following cost function $C_{\mathcal{E}}$:

$$C_{\mathcal{E}} = \frac{1}{|\mathcal{S}_{\mathcal{E}}|} \sum_{\epsilon \in \mathcal{S}_{\mathcal{E}}} C(\epsilon)$$

$$C(\epsilon) = \sum_k (\alpha_k^{j_1}(\tau, \epsilon)^2 + \alpha_k^{j_2}(\tau, \epsilon)^2) + \frac{1}{2} (\Theta(\tau, \epsilon) - \frac{\pi}{4})^2 \quad (5)$$

Here, \mathcal{E} is the motional frequency uncertainty, and $\mathcal{S}_{\mathcal{E}}$ consists of offset vectors ϵ whose components ϵ_k are independently and randomly drawn from the normal distribution $\mathcal{N}(0, \mathcal{E})$. $\alpha_k^j(\tau, \epsilon)$ and $\Theta(\tau, \epsilon)$ are displacement and angle when $\omega_k \rightarrow \omega_k + \epsilon_k$. The two terms of $C(\epsilon)$ are simply the displacement error representing residual entanglement with the phonons and the angle error.

The carrier Rabi frequency Ω is updated at each iteration such that $\Theta(\tau, \mathbf{0}) = \pi/4$. Since the displacement error is $\propto \Omega^2$ and angle error is $\propto \Omega^4$, this cost function naturally finds the low- Ω solution. This differs from the robust FM approach, which sets Ω after the entire optimization [34], requiring explicit regularization to fit the experimental constraints.

For s-robust FM, we set $\mathcal{S}_{\mathcal{E}}$ as a fixed training set throughout the optimization. For b-robust FM, we set $\mathcal{S}_{\mathcal{E}}$ as a batch, which gets randomly updated at each iteration of the optimization. Therefore, while s-robust FM calculates the cost function repeatedly with a certain set of samples, b-robust FM computes the cost function with a different batch generated from the error distribution throughout the entire optimization. In this work we set the training set size as 100 for s-robust and batch size as 10 for b-robust. For the batch method, we choose the adaptive moment estimation (ADAM) [40] optimizer to stabilize the gradient during training. We obtain sufficiently good results without hyperparameter tuning.

III. COMPARISON OF OPTIMIZATION METHODS

Fig.1a shows examples of continuous and discrete pulses from robust and b-robust FM optimization over a four-ion chain. Note that b-robust pulses do not have the even-pulse constraint, thus having twice as many degrees of freedom as robust pulses. This allows b-robust FM to explore a wider range of pulse shapes.

Continuous and discrete pulses have different time complexities of evaluating the gradient of angle $\Theta(\tau)$ over the pulse $\mu(t)$, which is the most time-consuming routine of the optimization. For continuous pulses, each neighboring steps are connected by sub-steps that follow a cosine envelope, and the evaluation time is quadratic to the number of sub-steps. However, for discrete pulses, the stepwise-constant form allows efficient evaluation of the gradient of $\Theta(\tau)$, requiring time linear to the number of steps.

Fig.1b shows the learning curves of robust and b-robust optimization. For robust FM, the cost function quickly and smoothly drops to lower than 10^{-6} . However, this only guarantees very accurate gate solution at close to zero offsets. Meanwhile, for b-robust FM, we set the motional frequency uncertainty \mathcal{E} as $2\pi \cdot 1\text{kHz}$. The cost function experiences larger fluctuations as a new batch of parameter offsets are used for optimization at each iteration. Although the cost function only reaches $\approx 10^{-3}$, we expect our gate fidelity to be robust, $\mathcal{F} \geq 1 - 10^{-3}$, against all mode frequency offsets within the optimized range.

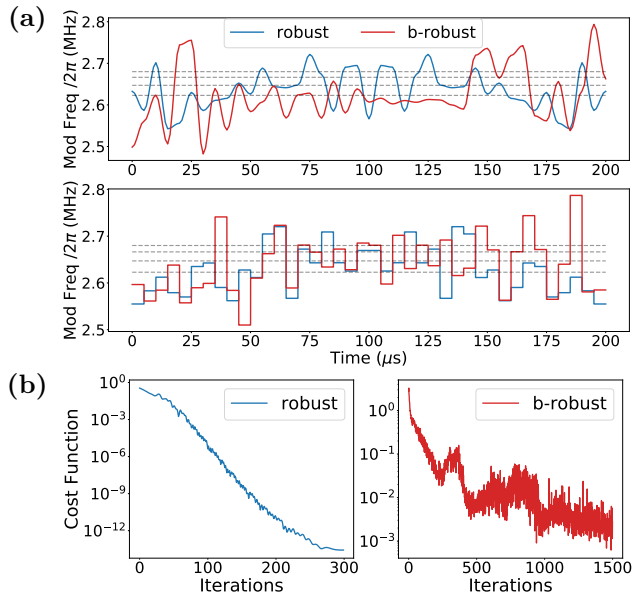


FIG. 1. (a) Continuous (above) and discrete (below) pulses optimized by robust and b-robust FM. Grey lines are the side-band frequencies of a 4-ion chain. Robust FM pulses are time-symmetric, while b-robust FM pulses are not. (b) Learning curves of robust and b-robust FM optimization.

To evaluate the robustness performance, we calculate the average unitary gate fidelity over the test set $T_{\mathcal{E}}$ of mode frequency offsets. The unitary fidelity can be expressed as $\mathcal{F} = \frac{1}{D} \text{Tr}(\mathcal{U}^\dagger U(\tau))$, where $U(\tau)$ is the unitary evolution in Eq.1, \mathcal{U} is the target unitary, and D is the Hilbert space dimension [41]. In terms of displacement and angle, the average fidelity can be expressed to second order as the following (see supplementary information of Ref.[33] for the derivation):

$$\mathcal{F}_{\mathcal{E}} = \frac{1}{|T_{\mathcal{E}}|} \sum_{\epsilon \in T_{\mathcal{E}}} \mathcal{F}(\epsilon)$$

$$\mathcal{F}(\epsilon) = \cos\left(\Theta(\tau, \epsilon) - \frac{\pi}{4}\right) \cdot \left(1 - \sum_k (\alpha_k^{j_1}(\tau, \epsilon)^2 + \alpha_k^{j_2}(\tau, \epsilon)^2) \left(\bar{n}_k + \frac{1}{2}\right)\right) \quad (6)$$

where \bar{n}_k is the mean phonon number of mode k and $T_{\mathcal{E}}$ is the test set of motional frequency uncertainty \mathcal{E} , constructed similarly as $S_{\mathcal{E}}$. In order to evaluate robustness, the test set is completely random and independent of the training set or mini-batches for optimization. We choose the test set size to be 1000. For an initial state with average 0.5 phonons, fidelity is simply equal to one minus the cost function to leading order in the errors: $\mathcal{F}(\epsilon) = 1 - C(\epsilon)$.

Fig.2a shows the simulated average error $1 - \mathcal{F}_{\mathcal{E}}$ of pulses optimized by non-robust, robust, s-robust, and b-robust FM, for various values of mode frequency uncertainty \mathcal{E} . We use a pulse of length $200\mu\text{s}$ to perform an MS gate on the first two ions in a four-ion chain. Note that each point of s-robust and b-robust data is optimized with the respective range \mathcal{E} . We perform 300 iterations for non-robust and robust FM and 1500 iterations for s- and b-robust FM. Since optimization performance slightly depends on the choice of the random initial guess pulse, we perform ten trials and choose the optimized pulse with the best average fidelity over a cross-validation set, constructed randomly and independently from the test set.

We find that s-robust and b-robust pulses have significantly smaller average error than robust pulses, for error range $\mathcal{E}/2\pi \geq 0.5\text{kHz}$ using continuous pulses, and for $\mathcal{E}/2\pi \geq 1\text{kHz}$ using discrete pulses. Notably, continuous s- and b-robust pulses have ~ 0.99 average fidelity over an offset range of $\mathcal{E}/2\pi = 5\text{kHz}$. This shows that s- and b-robust FM can be robust to offsets as large as the inverse of pulse length $1/\tau = 5\text{kHz}$. In general, b-robust performs slightly better than s-robust, despite having ten times fewer cost function and gradient evaluations than s-robust. This can be understood as the advantage of exploring various values of offsets ϵ , thus reducing the gap between the training curve and the testing curve [35].

Fig.2b and 2c visualize the displacement and angle errors of robust and b-robust pulses where the motional mode frequencies have drifted by $\epsilon_k/2\pi = 1\text{kHz}$ for all four modes. As expected, the b-robust pulse has smaller

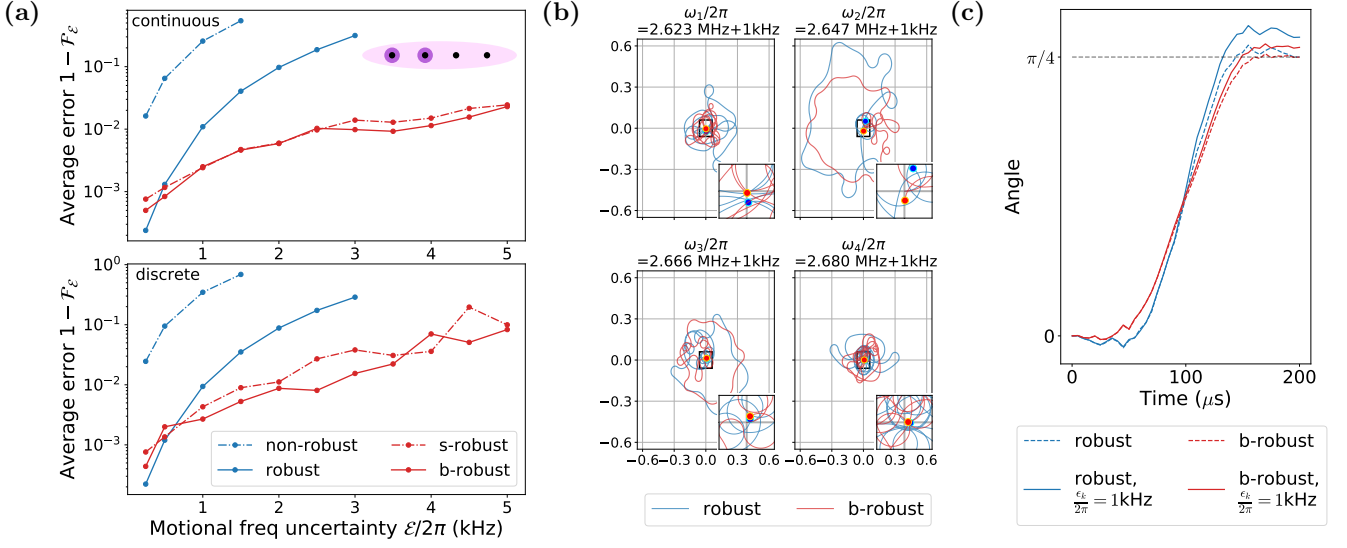


FIG. 2. (a) Simulated unitary gate errors averaged over test set of offsets drawn from distributions of various uncertainties \mathcal{E} . $200\mu\text{s}$ pulse on the first two ions of a four-ion chain is used. Each s- and b-robust pulse is optimized over the corresponding uncertainty \mathcal{E} . Except when \mathcal{E} is too small, s- and b-robust FM are significantly more robust than robust FM. (b) Displacements (arbitrary units) of motional modes during when discrete robust and b-robust pulses are applied, where mode frequencies drifted by 1kHz. Displacements at the end (circles) are overall closer to the origins when b-robust pulse is applied. (c) Angle $\Theta(t)$ during when discrete robust and b-robust pulses are applied. When no drifts occurred (dashed lines), angle reaches exactly $\pi/4$ at the end of both robust and b-robust pulses. When uniform drift of 1kHz occurs (solid lines), angle is closer to $\pi/4$ when b-robust pulse is applied. For both (b) and (c), b-robust pulse is optimized over $\mathcal{E} = 2\pi \cdot 1\text{kHz}$.

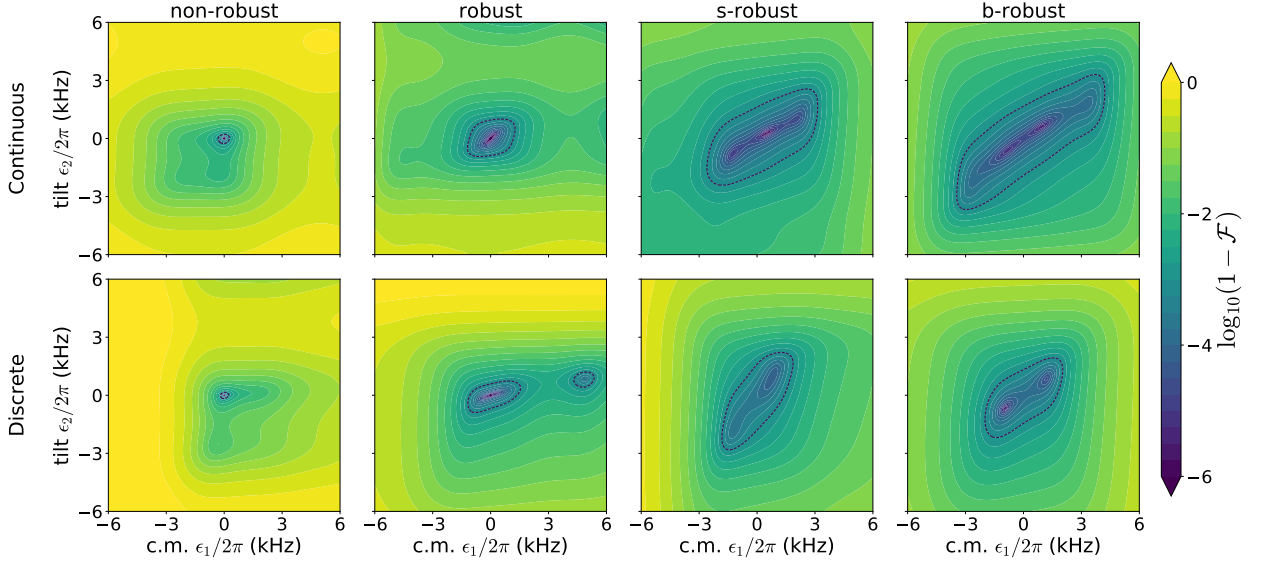


FIG. 3. Error landscapes over frequency offsets of motional center-of-mass and tilt modes, simulated for various FM pulses. $200\mu\text{s}$ pulse on a two-ion chain is used. The s- and b-robust pulses are optimized over mode frequency uncertainty $\mathcal{E} = 2\pi \cdot 1\text{kHz}$. The regions where error is lower than 10^{-3} are marked with dashed contour lines. The s- and b-robust pulses are clearly more robust over a wider region of offsets.

errors in both displacement and angle.

To visualize the robustness of various FM methods, Fig.3 plots the error landscapes over motional frequency offsets. We use a pulse of length $200\mu\text{s}$ for an MS gate on a two-ion chain, with offsets of center of mass mode

(ϵ_1) and tilt mode (ϵ_2). Both continuous and discrete pulses are used. The s-robust and b-robust pulses are optimized over mode frequency uncertainty $\mathcal{E}/2\pi = 1\text{kHz}$. The “high-fidelity regions” where error is less than 10^{-3} are marked with dashed contour lines. For continuous

pulses, the high-fidelity region is 4.5 and 6.4 times larger with s-robust and b-robust pulse, respectively, than with the robust pulse. For discrete pulses, the high-fidelity region is 2.8 times larger with both s-robust and b-robust pulses than with robust pulse. This shows that we can achieve significantly enhanced robustness with s-robust and b-robust FM. Also, note that continuous b-robust pulse has 2.5 times larger high-fidelity region than discrete b-robust pulse.

The error landscapes of s-robust and b-robust pulses have two or three peaks of high fidelity that are clearly separated from the origin. The average position of the peaks is near the origin, thus guaranteeing high fidelity at zero-offset as well. A large high-fidelity region that encompasses all peaks is formed. This provides an understanding of how s-robust and b-robust FM are able to achieve significantly better robustness than robust FM, whose landscape has a single sharp peak at the origin. Note that a double-peak landscape is also observed in Ref.[35], which performs mini-batch optimization over errors in coupling strengths.

Now we discuss the scalability of robust and b-robust FM. Unlike various generic pulse optimization algorithms whose computational cost increases exponentially with the number of qubits [42–44], robust FM for trapped ion system has linear computational cost, which makes the algorithm applicable for large-scale systems [45]. Our b-robust FM method also inherits this advantage.

Fig.4 shows the performance of robust and b-robust FM optimized in ion chains in a harmonic trap potential, with number of ions ranging from 2 to 12. For 2- and 4-ion chains, MS gates on all pairs of ions are simulated. For ion chains of length $N \geq 6$, MS gates on all pairs in the sub-chain of length $N - 2$, excluding the ions at the edges that are too weakly coupled to the motional modes, are simulated. The error bars indicate the standard deviation over the ion pairs. We use $400\mu\text{s}$ pulses, both continuous and discrete. The b-robust pulses are optimized over motional frequency uncertainty $\mathcal{E}/2\pi = 0.5\text{kHz}$. Only for continuous b-robust optimization, we minimize the displacement error $C(\epsilon) = \sum_k (\alpha_k^{j_1}(\tau, \epsilon)^2 + \alpha_k^{j_2}(\tau, \epsilon)^2)$ instead of the entire error as in Eq.5, due to computational time issues. 1500 iterations are performed for each optimization.

Fig.4a plots the average fidelity $\mathcal{F}_{\mathcal{E}}$, where $\mathcal{E}/2\pi = 0.5\text{kHz}$, and Fig.4b plots the Rabi frequency Ω . For both continuous and discrete pulses, b-robust FM finds the pulse solution with higher average fidelity and lower Ω , and the advantages are more significant as the number of ions increases. Note that explicit regularization of Ω is possible for both methods, but at the cost of lower average fidelity. Nonetheless, the cost function of b-robust FM for discrete (continuous) pulses scales as Ω^4 (Ω^2) with the frequency offset, which naturally leads to convergence to low- Ω solution. We expect further reduction in Ω by carefully choosing the initial guess pulse for each pair of ions, as well as designing the shape of trap potential for even spacing

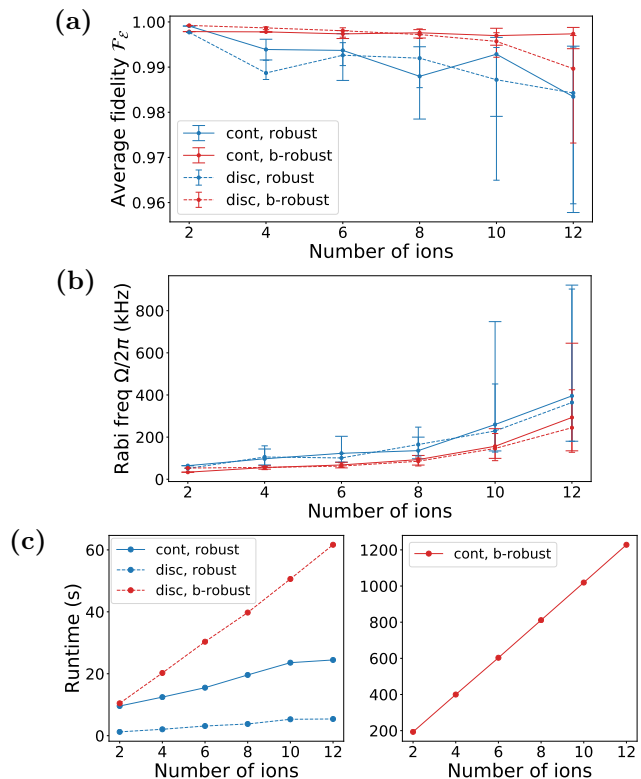


FIG. 4. Scalability of robust and b-robust FM, simulated up to 12-ion chain. Error bars represent the standard deviation over all ion pairs on which gates are applied. For number of ions ≥ 6 , ions at the edges are not used for gates. Both continuous and discrete pulses of length $400\mu\text{s}$ are used. (a) Average fidelity over $\mathcal{E} = 2\pi \cdot 0.5\text{kHz}$. (b) Rabi frequency. (c) Runtime of single pulse optimization. Calculations are performed on a consumer laptop with 1.60GHz Intel Core i5 CPU and 16.0 GB RAM.

between ions [45].

Fig.4c plots the runtime of single pulse optimization of each FM method, executed in a standard consumer laptop. As expected, runtimes scale linearly with the number of ions. Runtime of discrete b-robust FM is more than ten times longer than that of discrete robust FM, due to batch size of 10 and additional computation of $\Theta(\tau)$. Nonetheless, even in a 12-ion chain, discrete b-robust FM optimizes within $\sim 1\text{min}$, making it a practical candidate for actual experiments.

For continuous b-robust FM, the runtime is ~ 20 times longer than discrete b-robust FM, even though we minimize only the displacement error. The most time-consuming routine is evaluating $\Omega \propto \Theta(\tau, \mathbf{0})^{1/2}$ and its gradient at each iteration, which is quadratic to the number of sub-steps in the continuous case. However, we still find continuous b-robust FM a promising scheme for larger-scale systems, as for a 12-ion chain, $\mathcal{F}_{\mathcal{E}}$ is significantly higher (average 99.7% over ion pairs) than the other FM methods. We note that runtimes could be improved by parallelization using GPUs and development

of faster algorithms for continuous pulses.

IV. EXPERIMENT

We compare the experimental results of implementing discrete robust and b-robust FM pulses of length $120 \mu\text{s}$ on a 2-ion chain of $^{171}\text{Yb}^+$. The detailed experimental setup is described in Ref.[12]. The RF source for modulating the control lasers has been upgraded from direct digital synthesizers (AD9912) to an RF system on chip (ZCU111) driven by firmware from Sandia National Laboratories [46].

FM and AM pulses require careful tracking of AC Stark shifts during the modulation sequence. This detail is suppressed in most derivations since from a theoretical viewpoint it is simply bookkeeping. For our system, the dominant AC Stark shift is fourth-order by design and FM leads to negligible changes in the Stark shift. This may not be the case for other ion qubits where tracking the AC Stark shift will be critical.

After initializing the qubits to the $|00\rangle$ state, we apply a sequence of five MS gates, which ideally generates the maximally entangled state $(|00\rangle + i|11\rangle)/\sqrt{2}$. To evaluate the effect of motional frequency drifts, we apply pulses with various detuning offsets. Fig.5a, 5b show that with b-robust FM, $|00\rangle$ and $|11\rangle$ populations deviate from 0.5 more slowly as detuning offset increases, compared to robust FM. Also, population of unwanted odd-parity states is more suppressed with b-robust pulses. This indicates that b-robust FM is more robust to detuning error than robust FM.

Fig.5c plots the simulated MS gate errors of discrete robust and b-robust FM, both with and without dissipative noise. Each error is averaged over a sequence of five gates. We use a master equation [47] to simulate MS gate under dissipative noise, which consists of motional dephasing, laser dephasing, and motional heating (see the Supplemental Material of Ref.[12] for details). The noise parameters that describe the current experiment are the following: motional coherence time 8 ms, laser coherence time 333 ms, and motional heating rate 400 quanta/s and 40 quanta/s for center-of-mass mode and tilt mode, respectively. Although the peak gate fidelity of b-robust FM is lower than robust FM without dissipative noise, it is slightly higher (99.81%) than robust FM (99.77%) with the noise. Appendix A shows that b-robust FM is more robust to slow dephasing noise than robust FM, in the presence of motional frequency drifts.

The peak fidelity of b-robust FM occurs at a detuning offset -0.4kHz . This is because optimizing over mini-batches does not necessarily set the peak fidelity to be exactly at zero detuning. Simulations with dissipative noise predict that b-robust FM has slightly lower fidelity at zero detuning (99.74%) than robust FM.

Appendix B shows that the b-robust FM pulse achieves MS gate fidelity 99.08(7)% in the experiment. Note that this is lower than the MS gate fidelity 99.49(7)% reported

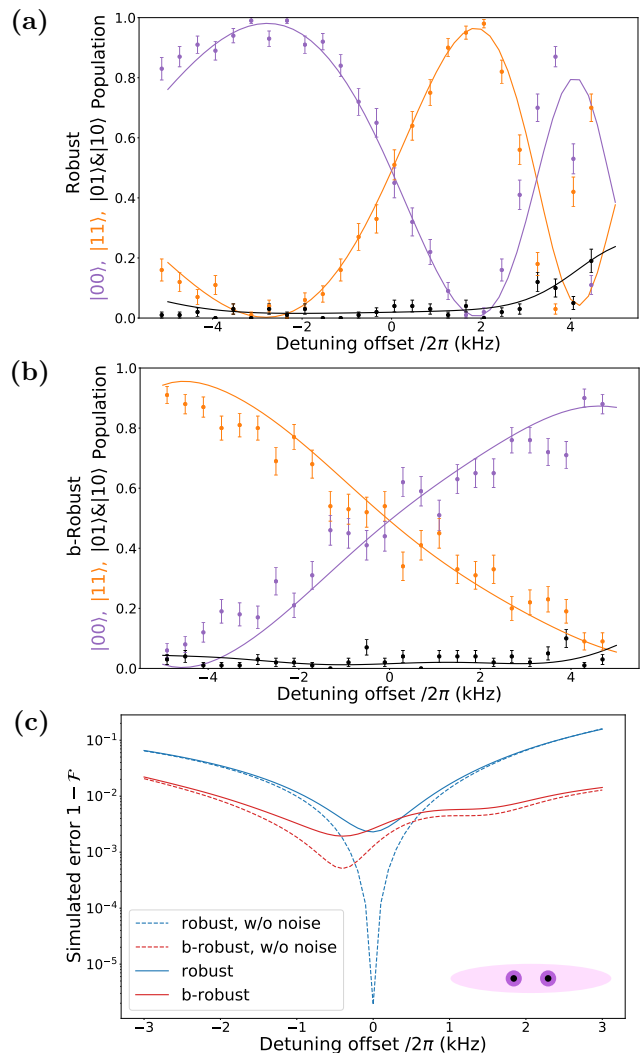


FIG. 5. (a,b) Experimental (points) and simulated (lines) state populations, over a range of detuning offsets, after sequences of five discrete (a) robust and (b) b-robust FM pulses are applied. Error bars represent the shot noise. Smaller slope of even-parity curves and flatter odd-parity curve indicate that b-robust FM is more robust to detuning errors than robust FM. (c) Gate errors averaged over sequences of five gates, simulated with (solid) and without (dashed) dissipative noise. Peak error is lower for b-robust in the presence of noise.

in Ref.[12], which uses robust FM pulse on the same system. The gate operates at zero-detuning, calibrated to the point where crossover between the populations of $|00\rangle$ and $|11\rangle$ states occurs in the experiment described in Fig.5b. In future experiments with b-robust FM, the detuning offset should be calibrated to the expected gate fidelity peak. Also, the gate suffers from the high heating rate of the transverse center-of-mass mode and the off-resonant coupling to the motional modes of other directions, which is ignored in the gate pulse design. We expect the gate fidelity to be improved when the trap

is operated at a higher RF voltage, which corresponds to higher transverse mode frequency, lower heating rate, and smaller off-resonant coupling. However, the RF voltage in our experiments is currently limited by several malfunctioning electrodes on the surface trap. Although we disable those electrodes, the impedance of the trap changes, and the ions are observed to be unstable when the center-of-mass transverse mode frequency is higher than $2\pi \cdot 2.1$ MHz.

V. CONCLUSION

In this work, we presented the s-robust and b-robust FM pulse optimization schemes for two-qubit entangling gates in trapped ion systems. We improved on the robust FM scheme [34] by application of ML-inspired techniques, using large sample set and mini-batches respectively. In our schemes, robustness is directly enforced by defining the cost function as displacement and angle errors averaged over various values of motional frequency offsets. Our results have shown that s- and b-robust FM achieve robustness significantly improved from that of robust FM, finding pulse solutions with multiple peaks in the fidelity landscape. Scalability of b-robust FM, in terms of high average fidelity, low laser power requirement, and reasonable optimization runtime, is demonstrated for up to 12 ions. Finally, we provide proof-of-concept experimental results that demonstrate improved robustness when using b-robust FM. We expect that b-robust FM has more significant advantages over robust FM in more complicated experiments with larger number of ions and uncertainty in motional frequencies, as shown in Fig. 4a and 2a.

Immediate directions include analyzing the trade-off of using the approximated error model as in Eq. 1-5 versus using a more realistic model such as a master equation for the cost function of b-robust optimization. Another approach is collecting the samples of gate errors at various parameter offsets directly with the experimental apparatus.

The idea of b-robust can be extended to other types of pulse modulation and noise. One future direction is to extend the mini-batch optimization scheme to find pulse solutions that are robust to fast time-varying noise, when combined with quantum oscillator noise spectroscopy [48]. In general, we expect that ML-inspired pulse optimization tools for robust quantum control will make significant contribution to high-fidelity operations, not only in trapped-ion systems but also in various other quantum computing platforms [35].

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APPENDIX A: ROBUSTNESS TO DEPHASING NOISE

Minimizing the time-averaged displacement $|\alpha_{k,\text{avg}}^j| \propto \frac{1}{\tau} |\int_0^\tau \int_0^t e^{-i\theta_k(t')} dt' dt|$ as in robust FM achieves robustness not only to systematic frequency offsets but also to time-dependent fluctuations in motional mode frequencies and laser amplitude, as demonstrated in both simulations and experiments with phase modulation (PM) [32]. Here we prove that minimizing $|\alpha_{k,\text{avg}}^j|$ achieves robustness to slow dephasing noise. Then we provide simulation results that show b-robust FM is more robust to dephasing noise than robust FM in the presence of motional frequency drifts, despite only minimizing the final displacements $|\alpha_k^j(\tau, \epsilon)|$.

Consider a time-dependent phase fluctuation $\varphi(t)$ caused by motional and/or optical dephasing noise. We assume the fluctuation is small: $|\varphi(t)| \ll 1$ ($0 \leq t \leq \tau$). We also assume the fluctuation is slow compared to the inverse gate time:

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}(\omega) e^{i\omega t} d\omega \approx \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} \tilde{\varphi}(\omega) e^{i\omega t} d\omega \quad (\text{A1})$$

where $\tilde{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$ and $\omega_c \ll 1/\tau$ is the cutoff frequency. We consider the case where the final displacement is set to zero when there is no dephasing noise. Replacing the phase $\theta(t)$ with $\theta(t) + \varphi(t)$, we evaluate the displacement by the following:

$$\begin{aligned} \alpha_k^j(\tau) &\propto \int_0^\tau e^{-i(\theta_k(t) + \varphi(t))} dt \approx \int_0^\tau e^{-i\theta_k(t)} (1 - i\varphi(t)) dt \\ &= -i \int_0^\tau e^{-i\theta_k(t)} \varphi(t) dt \\ &\approx \frac{-i}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega \tilde{\varphi}(\omega) e^{i\omega t} \int_0^\tau dt e^{-i\theta_k(t)} \\ &\approx \frac{-i}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega \tilde{\varphi}(\omega) \int_0^\tau dt e^{-i\theta_k(t)} (1 + i\omega t) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega \omega \tilde{\varphi}(\omega) \int_0^\tau dt t e^{-i\theta_k(t)} \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} d\omega \omega \tilde{\varphi}(\omega) \int_0^\tau dt \int_0^t dt' e^{-i\theta_k(t')} \\ &\propto \alpha_{k,\text{avg}}^j \end{aligned} \quad (\text{A2})$$

where in the second to last step we perform integration by parts. Therefore we conclude that $\alpha_{k,\text{avg}}^j \approx 0$ achieves first-order robustness to slow dephasing noise.

To evaluate the robustness to slow dephasing noise in the presence of motional frequency drifts, we compute

the time-averaged displacements averaged over a test set of motional frequency uncertainty \mathcal{E} :

$$C_{\mathcal{E}}^{\text{avg}} = \frac{1}{|T_{\mathcal{E}}|} \sum_{\epsilon \in T_{\mathcal{E}}} C^{\text{avg}}(\epsilon) \quad (\text{A3})$$

$$C^{\text{avg}}(\epsilon) = \sum_k (\alpha_{k,\text{avg}}^{j_1}(\epsilon)^2 + \alpha_{k,\text{avg}}^{j_2}(\epsilon)^2)$$

Fig 6 plots $C_{\mathcal{E}}^{\text{avg}}$ of pulses optimized by non-robust, robust, s-robust, and b-robust FM, for various uncertainties \mathcal{E} . As in Fig.2a, we use $200\mu\text{s}$ pulse on the first two ions in a four-ion chain. We find that s-robust and b-robust pulses have significantly smaller $C_{\mathcal{E}}^{\text{avg}}$ than robust FM when $\mathcal{E}/2\pi \geq 1\text{kHz}$. While s- and b-robust FM minimize the final displacements over the uncertainty range, they naturally minimize the time-averaged displacements to satisfy the condition for robustness to motional frequency offsets. This also leads to robustness to dephasing noise, which shares the same condition.

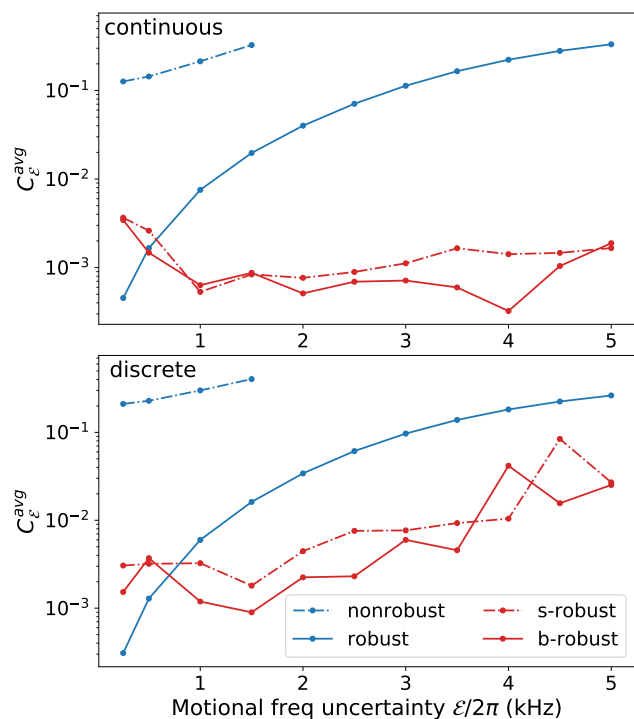


FIG. 6. Time-averaged displacements averaged over test set of offsets drawn from distributions of various uncertainties \mathcal{E} . $200\mu\text{s}$ pulse on the first two ions of a four-ion chain is used. Each s- and b-robust pulse is optimized over the corresponding uncertainty \mathcal{E} . Except when \mathcal{E} is too small, s- and b-robust FM have significantly smaller time-averaged displacements than robust FM. This implies that s- and b-robust FM are more robust to dephasing noise in the presence of motional frequency drifts.

Notably, for continuous b-robust FM, $C_{\mathcal{E}}^{\text{avg}} < 10^{-3}$ when $1\text{kHz} \leq \mathcal{E}/2\pi \leq 4\text{kHz}$. This leads to the displacement errors reduced by one to two orders of magnitude

compared to robust FM, because the displacement errors are proportional to the time-averaged displacements. In this range, the rotation angle errors of the b-robust pulse dominate the displacement errors.

APPENDIX B: EXPERIMENTAL GATE FIDELITY MEASUREMENT

We experimentally measure the MS gate fidelity of discrete b-robust FM on a two-ion chain, using the method in Ref.[12]. We initialize the qubits to $|00\rangle$ and then apply a sequence of 1, 5, and 13 MS gates to generate the maximally entangled state $(|00\rangle + i|11\rangle)/\sqrt{2}$. Population of $|01\rangle$ and $|10\rangle$ states and parity contrast are used to measure the state fidelity [49]. Using that the stochastic error accumulates linearly, coherent error accumulates quadratically, and the state preparation and measurement (SPAM) error remains constant, we extract the gate fidelity without the SPAM error from the linear fit. From Fig.7, we measure the gate fidelity 99.08(7)%. The data agree with the linear fit, indicating negligible coherent error.

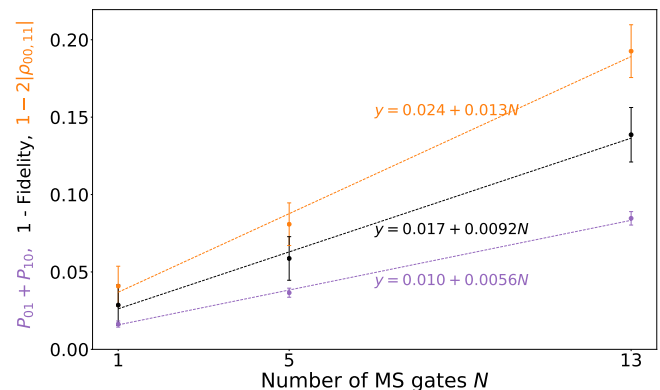


FIG. 7. Experimental errors of the maximally entangled state generated by sequences of repeated MS gates. The purple, orange, and black points represent the population leakage to $|01\rangle$ and $|10\rangle$ states, the loss of parity contrast, and the final state error, respectively. Gate error is given by the slope of the black points' linear fit.

APPENDIX C: CONNECTIVITY OF A 10-ION CHAIN

To understand the performance of b-robust FM in a larger system, Fig.8 plots the connectivity of a 10-ion chain. MS gate for each ion pair is optimized with continuous b-robust FM of pulse length $400\mu\text{s}$. The ions at the edges (1 and 10) are not used.

We expect to have a fully-connected 8-qubit device with fidelities ranging from 0.993 to 0.9998, even with uncertainty $\mathcal{E} = 2\pi \cdot 0.5\text{kHz}$ in the motional mode frequencies. Larger Rabi frequency is required for pairs

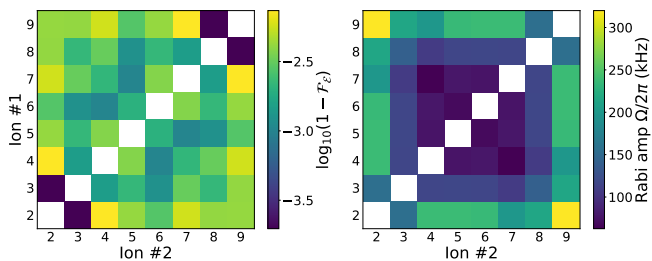


FIG. 8. Connectivity of a 10-ion chain, simulated with continuous b-robust FM. Pulse length is $400\mu\text{s}$. Left: average error over motional frequency uncertainty $\mathcal{E} = 2\pi \cdot 0.5\text{kHz}$. Right: Rabi frequency.

that include the ion(s) close to the edges (2 or 9), due to smaller participation in the excited modes. This can be improved by carefully choosing the frequency offset of the initial guess pulse and shaping the trap potential to have evenly spaced ions [45].

APPENDIX D: BATCH SIZE AND OPTIMIZATION PERFORMANCE

To choose the optimal batch size for b-robust optimization, Fig.9 plots the learning curves for various batch sizes as in Ref.[35], as well as the Rabi frequencies of the optimized pulses. Discrete pulse of length $200\mu\text{s}$ on the first two ions of a four-ion chain is used to optimize over

the motional frequency uncertainty $\mathcal{E} = 2\pi \cdot 1\text{kHz}$. Since the runtime is proportional to the number of evaluations of the cost function $C(\epsilon)$ at a certain frequency offset, we fix batch size times number of iterations as 15,000.

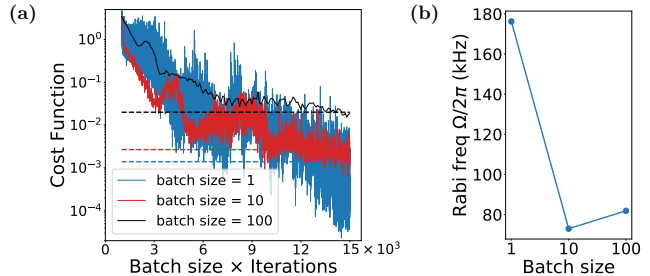


FIG. 9. (a) Learning curves of discrete b-robust FM optimization on a 4-ion chain with various batch sizes. Dashed lines represent the final average error $1 - \mathcal{F}_{\mathcal{E}}$, where $\mathcal{E} = 2\pi \cdot 1\text{kHz}$. (b) Rabi frequency of the optimized pulse for each batch size.

Despite evaluating the cost function on same number of samples, b-robust optimization with smaller batch size leads to higher fidelity $\mathcal{F}_{\mathcal{E}}$. This can be interpreted as the batch-induced noise, which is shown in the fluctuations of the learning curve, leading to enhanced robustness [35]. However, when the batch size is 1, the Rabi frequency Ω of the optimized pulse is significantly higher than when the batch size is 10 or 100. This justifies our choice of batch size as 10. We note that after parallelization, increasing the batch size while fixing the number of iterations does not necessarily increase the runtime.

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