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Analytic Generalized Description of a Perturbative Nonparaxial Elegant Laguerre-Gaussian Phasor for Ultrashort Pulses in the Time Domain

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An analytic expression for a polychromatic phasor representing an arbitrarily-short elegant Laguerre-Gauss (eLG) laser pulse of any spot size and LG mode is presented in the time domain as a non-recursive, closed-form perturbative expansion valid to any order of perturbative correction. This phasor enables the calculation of the complex electromagnetic fields for such beams without requiring the evaluation of any Fourier integrals. It is thus straightforward to implement in analytical or numerical applications involving eLG pulses.

I. INTRODUCTION

Perturbative models have long provided a straightforward means of calculating the electromagnetic (EM) fields of optical beams with various spatiotemporal structures [1–5]. To be generally applicable, such models must allow for the accurate description of beams which are focused to arbitrarily-small spot sizes, have arbitrarily-short temporal durations [5], and carry arbitrarily-many quanta of orbital angular momentum (OAM) [4–6], among other properties. The OAM carried by the beam manifests itself as an optical vortex [7–10], whereby the beam's phase exhibits a helical structure about the optical axis.

Perturbative models generally entail a power series expansion in a parameter that is small in the paraxial limit of loose focusing, such as $(k\mathrm{w}_0)^{-1}$ [1–5, 11–13] or $(k_\perp/k)$ [4], where k is the wave number and w_0 is the beam waist. The zeroth order term of such a series represents the optical beam in the paraxial limit, and higher order terms introduce nonparaxial corrections. Notably, the first-order correction introduces a longitudinal electric field that is characteristic of nonparaxial beams [1]. In practice, perturbative models retain terms only up to a predetermined order of perturbation, at which point the infinite series is truncated.

A perturbative model describing tightly-focused elegant Laguerre-Gauss (eLG) beams was presented by Bandres and Gutiérrez-Vega (BGV) [4], but this result was limited to a frequency-domain description for the case of monochromatic fields. Reference [5] extended this description in two ways: i) It modified the BGV model by introducing a frequency spectrum, thus allowing for the description of pulses with arbitrary temporal duration; and ii) It Fourier transformed this modified frequency-domain phasor into the time domain, from which one can obtain the EM fields by straightforward differentiation. The first two orders of perturbative correction to the time-domain phasor were also presented in Ref. [5], and a method for generating higher order corrections was described in detail.

A main benefit of using such perturbative models is the ability to calculate the EM fields using relatively simple expressions at each retained order of perturbative correction. While exact models, such as that of Ref. [14], accurately describe such beams in the frequency domain, it can be cumbersome to generate the corresponding time-domain descriptions, which are required for calculating the EM fields. In particular, the Fourier transformations necessary to bring the frequency domain models into the time domain are often difficult to carry out owing to the mathematically-complicated nature of the exact descriptions, particularly as the LG mode indices become large.

A major issue for perturbative descriptions, of course, is the convergence of the perturbation series describing the EM fields. For the model of Ref. [5], it was shown that the number of terms that must be retained in the perturbation series in order to achieve convergence depends not only on the spot size of the beam but also on the LG mode. For beams carrying large values of OAM (which can be created, e.g., in high-harmonic generation processes [10, 15, 16]), the perturbative order required to achieve convergence can become large. Thus, the ability to express a time-domain phasor to arbitrary perturbative order would be of great utility for general application of perturbative models to the calculation of EM fields in cases becoming increasing relevant in experiments involving tightly focused, highly structured pulses of light.

In this paper we generalize the second-order perturbative time-domain phasor results of Ref. [5] to arbitrarily-high perturbative order as a non-recursive, closed-form analytic expression. This generalized time domain phasor allows one to implement the perturbative model without requiring the explicit calculation of any Fourier integrals, which would be prohibitively difficult to calculate individually for each term of an arbitrarily-high order of perturbative correction. Instead, the EM fields can be calculated immediately from straightforward derivatives of the generalized time-domain phasor we present here.

This paper is organized as follows. In Sec. II the time-domain phasor of Ref. [5] including two orders of perturbative correction beyond the paraxial approximation is reviewed, and the third-order correction is explicitly derived in the time domain via Fourier integration. We then propose a generalization of this time-domain phasor that is valid to any perturbative order. In Sections III and IV, our proposed generalized time-domain phasor is derived analytically. In Sec. V we provide a numerical example showing the necessity for including high order terms in the perturbation expansion of the phasor in order to obtain good accuracy. We then summarize our results and present our conclusions in Sec. VI. In Appendix A, we derive the result of an integral involved in our analytical derivations. Finally, in Appendix B we present an alternative approach to the generalization of the time-domain phasor that may be of interest to mathematicians and mathematical physicists.

II. PERTURBATIVE EXPRESSIONS FOR THE TIME-DOMAIN PHASOR

A polychromatic time-domain phasor is an exact solution to the scalar Helmholtz equation. In Ref. [5] a second-order perturbative expression for this phasor was derived that is appropriate for describing the spatiotemporal profile of an arbitrarily-short laser pulse of any LG mode n, m focused to an arbitrarily-small spot size. The result in [5] is perturbative in the small parameter $\epsilon_c^2 \equiv c/(2z_R\omega_0)$, where z_R is the Rayleigh range, ω_0 is the central frequency of the pulse, and c is the speed of light. In Section II A, we extend this time-domain description up to the third order correction (i.e., up to order ϵ_c^6) via explicit Fourier transformation. Then in Section II B we compare the time-domain phasor to second-order with its third-order correction (in Eqs. (1) and (16) respectively) and suggest how the time-domain phasor can be almost completely predicted to any perturbative order. In Section IV of this paper, we then prove analytically (using some necessary results derived in Section III) the closed-form analytic expression of the time-domain phasor (proposed in Section II B) that is exact to any desired perturbative order.

A. Derivation of the Third-Order Correction

As derived in Ref. [5], the time-domain phasor $U(t)$ for any LG mode n, m, including all terms up to second-order in the perturbative parameter $\epsilon_c^2 \equiv c/(2z_R\omega_0)$ (i.e., up to the second-order correction to the paraxial solution), is

$$U^{(4)}(t) = \Lambda_{n,m}\Bigg[\sum_{j=0}^{n} c_{0,0}\xi^j T^{-\gamma-1)} + \frac{\epsilon_c^2}{\beta}\left(\sum_{j=0}^{n+1} c_{1,1}\xi^j T^{-\gamma} + \sum_{j=0}^{n+2} c_{1,2}\xi^j T^{-\gamma}\right) + \frac{\epsilon_c^4}{\beta^2}\left(\sum_{j=0}^{n+2} c_{2,2}\ \xi^j T^{-\gamma+1} + \sum_{j=0}^{n+3} c_{2,3}\ \xi^j T^{-\gamma+1} + \sum_{j=0}^{n+4} c_{2,4}\ \xi^j T^{-\gamma+1}\right)\Bigg]. \tag{1}$$

In Eq. (1), $\beta \equiv (1 + iz/z_R)$, $\gamma \equiv m/2 + s + j$, n is the radial LG index, m is the azimuthal LG index (i.e., the quantized OAM carried by the beam), and s is a spectral parameter related to the duration of the pulse [see Eq. (7)]. The spatiotemporal terms in Eq. (1) that occur in each perturbative order are defined as follows:

$$\xi \equiv \frac{\rho^2}{2c\beta z_R} \tag{2a}$$

$$T \equiv 1 + \frac{\omega_0}{s}\left(-\frac{iz}{c} + \xi + it\right) \tag{2b}$$

$$\Lambda_{n,m} \equiv (-1)^{n+m} 2^{2n+m}\sqrt{2\pi}n! \exp(i\phi_0) \times \xi^{m/2}\beta^{-(n+m/2+1)}\exp(im\phi), \tag{2c}$$

where cylindrical coordinates, $\mathbf{r} = (\rho, \phi, z)$, are used. The coordinate-independent coefficients in Eq. (1), $c_{N,p}(n, m, j)$, where N is the perturbative order of the term and $N \le p \le 2N$, are defined for $0 \le N \le 2$ as follows (in which their dependence on n, m, and j is suppressed):

$$c_{0,0} \equiv G_{n,m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s}\frac{\Gamma(\gamma+1)}{\Gamma(s+1)} \tag{3a}$$

$$c_{1,1} \equiv 2(n+1)G_{(n+1),m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s-1}\frac{\omega_0\Gamma(\gamma)}{\Gamma(s+1)} \tag{3b}$$

$$c_{1,2} \equiv -\frac{(n+2)!}{n!}G_{(n+2),m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s-1}\frac{\omega_0\Gamma(\gamma)}{\Gamma(s+1)} \tag{3c}$$

$$c_{2,2} \equiv 6\frac{(n+2)!}{n!}G_{(n+2),m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s-2}\frac{\omega_0^2\Gamma(\gamma-1)}{\Gamma(s+1)} \tag{3d}$$

$$c_{2,3} \equiv -4\frac{(n+3)!}{n!}G_{(n+3),m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s-2}\frac{\omega_0^2\Gamma(\gamma-1)}{\Gamma(s+1)} \tag{3e}$$

$$c_{2,4} \equiv \frac{1}{2}\frac{(n+4)!}{n!}G_{(n+4),m,j}\left(\frac{\omega_0}{s}\right)^{\gamma-s-2}\frac{\omega_0^2\Gamma(\gamma-1)}{\Gamma(s+1)}. \tag{3f}$$

The factors $G_{n,m,j}$ in Eq. (3) are defined as

$$G_{n,m,j} \equiv \frac{(-1)^j(n+m)!}{(n-j)!(m+j)!j!}; \tag{4}$$

they are coefficients of the expansion of the associated Laguerre polynomials [17]:

$$L_n^m(v) \equiv \sum_{j=0}^{n} G_{n,m,j}\ v^j. \tag{5}$$

We now derive the time-domain phasor, $U^{(6)}(t)$, which is correct to third order in the parameter ϵ_c^2. According to the procedure given in Ref. [5], we start from the perturbative, monochromatic frequency-domain phasor in Eq. (24) of Ref. [4], in which we retain terms $0 \le N \le N_o$:

$$
\begin{aligned}
U_{BGV}^{(2N_o)}(\mathbf{r},\omega) &= (-1)^{n+m}2^{2n+m}\exp(ikz+im\phi) \\
&\quad\times h^{2n+m+2}v^{m/2}\exp(-v) \\
&\quad\times \sum_{N=0}^{N_o}\left(\frac{h^2}{k^2\mathrm{w}_0^2}\right)^N f_{n,m}^{(2N)}(v) \\
&\equiv U_{0,BGV} + \frac{\epsilon^2}{\beta}\,U_{2,BGV} + \cdots + \frac{\epsilon^{2N_o}}{\beta^{N_o}}\,U_{2N_o,BGV}.
\end{aligned}
\tag{6}
$$

In Eq. (6), $h \equiv (1+iz/z_R)^{-1/2} = \beta^{-1/2}$ and $v \equiv h^2\rho^2/\mathrm{w}_0^2$ are dimensionless parameters, $\mathrm{w}_0 \equiv \sqrt{2z_R/k}$ is the beam waist, $\epsilon^2 \equiv (k\mathrm{w}_0)^{-2} = (c/2z_R\omega)$, and the first four factors $f_{n,m}^{(2N)}(v)$ $(0 \le N \le 3)$ in (6) are given in Eq. (18) below. In order to describe short-pulse fields, we multiply Eq. (6) by a Poisson-like frequency spectrum [18, 19],

$$
f(\omega) \equiv 2\pi e^{i\phi_0}\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\omega^s\exp(-s\omega/\omega_0)}{\Gamma(s+1)}\Theta(\omega), \tag{7}
$$

where s is the spectral parameter controlling the pulse duration, ϕ_0 is the initial phase of the pulse, and $\Theta(\omega)$ is the unit step function. Henceforth, we follow the prescription in Appendix B of Ref. [5] to derive here the third-order correction to the time-domain phasor.

Considering only the third-order term in Eq. (6), where $f_{n,m}^{(6)}(v)$ is given in Eq. (25) of Ref. [4] [see Eq. (18)(d) below], we make the replacements $\mathrm{w}_0 \to \sqrt{2z_R/k}$ and $k \to \omega/c$ to show explicitly the dependence on frequency. We also invoke here the condition of isodiffraction, which requires that z_R is independent of frequency [18–20]. The third-order frequency-domain phasor term is then

$$
\begin{aligned}
\frac{\epsilon^6}{\beta^3}U_{6,BGV} &= (-1)^{n+m}2^{2n+m}\exp(i\omega z/c+im\phi) \\
&\quad\times h^{2n+m+2}v^{m/2}\exp(-v)\left[\left(\frac{c}{2\omega\beta z_R}\right)^3\right. \\
&\quad\times\left\{20(n+3)!L_{n+3}^m(v) - 15(n+4)!L_{n+4}^m(v)\right. \\
&\quad\left.\left.+3(n+5)!L_{n+5}^m(v) - \frac{1}{6}(n+6)!L_{n+6}^m(v)\right\}\right].
\end{aligned}
\tag{8}
$$

Upon multiplying this result by the Poisson-like frequency spectrum in Eq. (7), the description becomes polychromatic. Therefore, the small parameter ϵ, which is appropriate for monochromatic fields, must be replaced by one that it is frequency independent,

$$
\epsilon^2 \equiv \frac{c}{2z_R\omega} = \frac{c}{2z_R\omega_0}\frac{\omega_0}{\omega} \equiv \epsilon_c^2\frac{\omega_0}{\omega}, \tag{9}
$$

where ϵ_c is now the requisite constant small parameter. Expressing the associated Laguerre polynomials in (8) as sums [see Eqs. (4) and (5)], substituting $v = \xi\omega$, and extracting powers of ω within the sums, we obtain finally,

$$
\begin{aligned}
U_6(\omega) &= \frac{\Lambda_{n,m}}{\Gamma(s+1)}\exp\left\{-\omega\left(-\frac{iz}{c}+\xi+\frac{s}{\omega_0}\right)\right\} \\
&\quad\times\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\Theta(\omega)\sqrt{2\pi}\epsilon_c^6}{\beta^3}\left[\sum_{j=0}^{n+3}\widetilde{c_{3,3}}\;\xi^j\omega^{\gamma-3}\right. \\
&\quad+\sum_{j=0}^{n+4}\widetilde{c_{3,4}}\;\xi^j\omega^{\gamma-3}+\sum_{j=0}^{n+5}\widetilde{c_{3,5}}\;\xi^j\omega^{\gamma-3} \\
&\quad\left.+\sum_{j=0}^{n+6}\widetilde{c_{3,6}}\;\xi^j\omega^{\gamma-3}\right],
\end{aligned}
\tag{10}
$$

where the variables defined in Eq. (2) and the text above it have been used, and the new constants, $\widetilde{c_{3,p}}(n,m,j)$, $3 \le p \le 6$, are defined as follows (in which indication of their dependence on n, m, j has been suppressed):

$$
\widetilde{c_{3,3}} \equiv 20\omega_0^3\frac{(n+3)!}{n!}G_{(n+3),m,j} \tag{11a}
$$

$$
\widetilde{c_{3,4}} \equiv -15\omega_0^3\frac{(n+4)!}{n!}G_{(n+4),m,j} \tag{11b}
$$

$$
\widetilde{c_{3,5}} \equiv 3\omega_0^3\frac{(n+5)!}{n!}G_{(n+5),m,j} \tag{11c}
$$

$$
\widetilde{c_{3,6}} \equiv -\frac{\omega_0^3}{6}\frac{(n+6)!}{n!}G_{(n+6),m,j}. \tag{11d}
$$

We now Fourier transform $U_6(\omega)$ to the time domain in order to obtain $U_6(t)$,

$$
\begin{aligned}
U_6(t) &= \frac{\Lambda_{n,m}}{\Gamma(s+1)}\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\epsilon_c^6}{\beta^3}\int_0^\infty\exp(-\omega\eta) \\
&\quad\times\left[\sum_{j=0}^{n+3}\widetilde{c_{3,3}}\;\xi^j\omega^{\gamma-3}+\sum_{j=0}^{n+4}\widetilde{c_{3,4}}\;\xi^j\omega^{\gamma-3}\right. \\
&\quad\left.+\sum_{j=0}^{n+5}\widetilde{c_{3,5}}\;\xi^j\omega^{\gamma-3}+\sum_{j=0}^{n+6}\widetilde{c_{3,6}}\;\xi^j\omega^{\gamma-3}\right]\mathrm{d}\omega\,,
\end{aligned}
\tag{12}
$$

where $\eta \equiv -iz/c + \xi + s/\omega_0 + it$ and the sign of the Fourier exponent has been chosen to describe a pulse traveling in the positive $\hat{\mathbf{z}}$ direction. Making use of the integral representation of the gamma function [c.f. Eq. (6.1.1) of Ref. [21]],

$$
\Gamma(\gamma+1) = \eta^{\gamma+1}\int_0^\infty \mathrm{d}\omega\,\omega^\gamma\exp(-\omega\eta), \quad \mathrm{Re}\,\eta > 0, \tag{13}
$$

the Fourier integrals in (12) can be evaluated to obtain,

$$U_6(t) = \Lambda_{n,m}\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\epsilon_c^6}{\beta^3}\left[\sum_{j=0}^{n+3}\overline{c_{3,3}}\;\xi^j\eta^{-(\gamma-2)} + \sum_{j=0}^{n+4}\overline{c_{3,4}}\;\xi^j\eta^{-(\gamma-2)} + \sum_{j=0}^{n+5}\overline{c_{3,5}}\;\xi^j\eta^{-(\gamma-2)} + \sum_{j=0}^{n+6}\overline{c_{3,6}}\;\xi^j\eta^{-(\gamma-2)}\right], \tag{14}$$

where $\overline{c_{3,p}} \equiv \widetilde{c_{3,p}}\,\Gamma(\gamma-2)/\Gamma(s+1)$ for $3 \le p \le 6$.

Taking now the overall prefactor $(s/\omega_0)^{s+1}$ in Eq. (14) inside each of the sums and using the definition of T in Eq. (2), we can write for any power q,

$$\left(\frac{s}{\omega_0}\right)^{s+1}\eta^{-q} = \left(\frac{s}{\omega_0}\right)^{s+1-q}T^{-q}. \tag{15}$$

Defining the coefficients $c_{3,p} \equiv \overline{c_{3,p}}\;(s/\omega_0)^{(s+3-\gamma)}$ for $3 \le p \le 6$, the final result for the third-order term $U_6(t)$ is:

$$U_6(t) = \Lambda_{n,m}\frac{\epsilon_c^6}{\beta^3}\left[\sum_{j=0}^{n+3}c_{3,3}\;\xi^j T^{-\gamma+2} + \sum_{j=0}^{n+4}c_{3,4}\;\xi^j T^{-\gamma+2} + \sum_{j=0}^{n+5}c_{3,5}\;\xi^j T^{-\gamma+2} + \sum_{j=0}^{n+6}c_{3,6}\;\xi^j T^{-\gamma+2}\right]. \tag{16}$$

B. Proposed Expression for the Phasor to Order N_o

Comparing the time-domain phasor to second-order in Eq. (1) with its third-order correction in Eq. (16), one surmises that its Nth order correction has the form:

$$U_{(2N)}(t) = \Lambda_{n,m}\left[\frac{\epsilon_c^{2N}}{\beta^N}\sum_{p=N}^{2N}\sum_{j=0}^{n+p}c_{N,p}\;\xi^j T^{-\gamma-1+N}\right]. \tag{17}$$

Before proving this result, one must first determine the general form of the coefficients $c_{N,p}(n,m,j)$. At least to order $N=3$, these coefficients are related to coefficients in the expressions for the factors $f^{2N}_{n,m}(v)$ that appear in the monochromatic frequency-domain phasor of BGV [4] presented in Eq. (6). The first four of these factors are given in Eq. (25) of Ref. [4], i.e., for $0 \le N \le 3$:

$$f^{(0)}_{n,m}(v) = n!L^m_n(v) \tag{18a}$$

$$f^{(2)}_{n,m}(v) = 2(n+1)!L^m_{n+1}(v) - (n+2)!L^m_{n+2}(v) \tag{18b}$$

$$f^{(4)}_{n,m}(v) = 6(n+2)!L^m_{n+2}(v) - 4(n+3)!L^m_{n+3}(v) + \frac{1}{2}(n+4)!L^m_{n+4}(v) \tag{18c}$$

$$f^{(6)}_{n,m}(v) = 20(n+3)!L^m_{n+3}(v) - 15(n+4)!L^m_{n+4}(v) + 3(n+5)!L^m_{n+5}(v) - \frac{1}{6}(n+6)!L^m_{n+6}(v). \tag{18d}$$

We illustrate the connection between the factors $f^{2N}_{n,m}(v)$ and the coefficients $c_{N,p}(n,m,j)$ for the second order case of $N=2$. Substituting Eq. (5) into Eq. (18)(c), we obtain for the factor $f^{(4)}_{n,m}(v)$:

$$f^{(4)}_{n,m}(v) = 6(n+2)!\sum_{j=0}^{n+2}G_{n+2,m,j}\;v^j - 4(n+3)!\sum_{j=0}^{n+3}G_{n+3,m,j}\;v^j + \frac{1}{2}(n+4)!\sum_{j=0}^{n+4}G_{n+4,m,j}\;v^j \tag{19}$$

Observe next that the coefficients $c_{N=2,p}(n,m,j)$ for $2 \le p \le 4$ in Eqs. (3)(d) - (3)(f) have the common factor X,

$$X \equiv \frac{1}{n!}\left(\frac{\omega_0}{s}\right)^{\gamma-s-2}\frac{\omega_0^2\Gamma(\gamma-1)}{\Gamma(s+1)}. \tag{20}$$

Comparing now the coefficients that multiply the common factor X in each of the Eqs. (3)(d) - (3)(f) respectively with the coefficents of v^j in each of the three summations in Eq. (19), one sees immediately that they are the same. However, we have derived these relations only for orders $0 \le N \le 3$ for which the factors $f^{2N}_{n,m}(v)$ are given in Ref. [4].

In order to obtain a closed-form analytic expression for the Nth-order correction to the time-domain phasor in Eq. (17), two tasks are therefore necessary. First, a general expression for the factors $f^{2N}_{n,m}(v)$ in the BGV frequency-domain phasor in Eq. (6) must be derived for any perturbation order N. This derivation is presented in Sec. III. Second, the Nth order term in the frequency-domain phasor expansion shown in Eq. (6) must be multiplied by a Poisson-like frequency spectrum in Eq. (7) and then it must be Fourier-transformed into the time-domain. This derivation is presented in Sec. IV.

For convenience, we present here the final result for the time-domain phasor correct to order N_o:

$$\begin{aligned} U^{(2N_o)}(t) &= \sum_{N=0}^{N_o} U_{(2N)}(t) \\ &= \Lambda_{n,m} \sum_{N=0}^{N_o} \left[\frac{\epsilon_c^{2N}}{\beta^N} \sum_{p=N}^{2N} \sum_{j=0}^{n+p} c_{N,p}\, \xi^j T^{-\gamma-1+N} \right], \end{aligned} \tag{21}$$

where the coefficients $c_{N,p}$ are given by

$$\begin{aligned} c_{N,p} \equiv \kappa_{N,p} G_{(n+p),m,j} \frac{(n+p)!}{n!} \omega_0^N \left(\frac{s}{\omega_0}\right)^{s-\gamma+N} \\ \times \frac{\Gamma(\gamma+1-N)}{\Gamma(s+1)}, \end{aligned} \tag{22}$$

where

$$\kappa_{N,p} \equiv \frac{(-1)^{p-N}}{(p-N)!} \binom{2N}{2N-p}. \tag{23}$$

Equations (21) - (23) are the main results of this work. They provide a closed-form, analytic expression for the time-domain phasor $U^{(2N_o)}(t)$ correct to an arbitrary perturbative order N_o in the parameter ϵ_c^2. This phasor can be utilized directly to calculate the fields for a general eLG mode without requiring the calculation of any Fourier integrals. It is easily confirmed that Eq. (21) is consistent with the result for $N_o = 2$ in Eq. (1) and that the $N = 3$ correction in Eq. (16) is consistent with Eq. (17) for $U_{(2N)}(t)$. A full derivation of the Fourier transformation necessary to obtain Eqs. (21) - (23) is presented in Sec. IV, after first deriving analytic expressions for the factors $f_{n,m}^{(2N)}(v)$ in the next section.

III. EXPLICIT DERIVATION OF $f_{n,m}^{(2N)}(v)$

In this section, we derive a general expression for the factors $f_{n,m}^{(2N)}(v)$ for any N. We begin by finding a generating function $\Psi(x,y)$ for the associated Laguerre polynomials with equal upper and lower indices, $L_n^n(y)$. We then connect this generating function to the results of BGV [4] in order to determine a general analytic expression for $f_{n,m}^{(2N)}(v)$.

A. Generating Function $\Psi(x,y)$ for $L_n^n(y)$

We seek a generating function for associated Laguerre polynomials having equal upper and lower indices,

$$\Psi(x,y) = \sum_{n=0}^{\infty} x^n L_n^n(y). \tag{24}$$

The associated Laguerre polynomial is expressible as an integral of a Bessel function of the first kind (see Eq. (22.10.14) of Ref. [21]):

$$L_n^n(y) = \frac{e^y y^{-n/2}}{n!} \int_0^{\infty} \mathrm{d}t\; e^{-t} t^{3n/2} J_n\left(2\sqrt{ty}\right). \tag{25}$$

By substituting Eq. (25) into Eq. (24), one obtains

$$\Psi(x,y) = e^y \int_0^{\infty} \mathrm{d}t\; e^{-t} \sum_{n=0}^{\infty} \left[\frac{a^n}{n!} J_n\left(2\sqrt{ty}\right)\right], \tag{26}$$

where $a \equiv x t^{3/2} y^{-1/2}$. This sum can be rewritten as a Bessel function using Eq. (19.9.1) of Ref. [22],

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\frac{a^n}{n!} J_n\left(2\sqrt{ty}\right)\right] &= J_0\left(\sqrt{4ty - 4a\sqrt{ty}}\right) \\ &= J_0\left(2i\sqrt{x}\sqrt{t^2 - \frac{ty}{x}}\right). \end{aligned} \tag{27}$$

Making this replacement in Eq. (26),

$$\Psi(x,y) = e^y \int_0^{\infty} \mathrm{d}t\; e^{-t} J_0\left(2i\sqrt{x}\sqrt{t^2 - \frac{ty}{x}}\right), \tag{28}$$

the integral can be carried out using Eq. (6.616.1) of Ref. [23],

$$\begin{aligned} \int_0^{\infty} \mathrm{d}x\; e^{-t} J_0\left(2i\sqrt{x}\sqrt{t^2 - \frac{ty}{x}}\right) \\ = \frac{1}{\sqrt{1-4x}} \exp\left[\frac{y}{2x}\left(\sqrt{1-4x} - 1\right)\right]. \end{aligned} \tag{29}$$

The result for the generating function in Eq. (24) is thus,

$$\Psi(x,y) = \frac{1}{\sqrt{1-4x}} \exp\left[y\left(1 + \frac{\sqrt{1-4x} - 1}{2x}\right)\right]. \tag{30}$$

B. Derivation of $f_{n,m}^{(2N)}(v)$ from $\Psi(x,y)$

In Ref. [4], the factors $f_{n,m}^{(2N)}(v)$ are generated from a sum over terms involving factors $G^{(2N)}$ that are not explicitly defined for $N > 3$. However, comparing Eqs. (16) and (22) of Ref. [4] (as shown explicitly in Ref. [5]), one sees that

$$\sum_{N=0}^{\infty} \epsilon^{(2N)} G^{(2N)} = \frac{1}{\sqrt{1-\epsilon^2\Omega}} \exp\left(\frac{\sqrt{1-\epsilon^2\Omega} - 1}{2\epsilon^2 h^2} + \frac{\Omega}{4h^2}\right), \tag{31}$$

where $\epsilon \equiv 1/(k\mathrm{w}_0)$ is the small parameter of the perturbation expansion and $\Omega \equiv \mathrm{w}_0^2 k_\perp^2$. By taking $x = \epsilon^2\Omega/4$

and $y = \Omega/(4h^2)$ in Eq. (30), we see immediately by comparison to Eq. (31) that

$$\Psi(x,y) = \sum_{n=0}^{\infty} x^n L_n^n(y) = \sum_{N=0}^{\infty} \epsilon^{(2N)} G^{(2N)}. \tag{32}$$

While not necessary, it is sufficient that the equality on the right-hand side of Eq. (32) is satisfied by setting the terms in each sum equal, i.e.,

$$G^{(2N)} = \left(\frac{\Omega}{4}\right)^N L_N^N\left(\frac{\Omega}{4h^2}\right). \tag{33}$$

Substituting $G^{(2N)}$ into the alternative expression for the monochromatic frequency-domain phasor given in Eq. (22) of Ref. [4], we obtain [cf. Eq. (6)],

$$\begin{aligned} U_{BGV}^{(2N_o)}(\mathbf{r},\omega) &= \frac{1}{2}(-1)^{n+m}\exp(ikz \pm im\phi)\mathrm{w}_0^{2n+m+2} \\ &\times \sum_{N=0}^{N_o}\left(\frac{1}{4k^2}\right)^N \int_0^\infty k_\perp^{2n+m+1} e^{-\mu^2 k_\perp^2} \\ &\times k_\perp^{2N} L_N^N\left(\mu^2 k_\perp^2\right) J_m(k_\perp \rho)\mathrm{d}k_\perp, \end{aligned} \tag{34}$$

in which our notation and that in Ref. [4] are related by $\mu^2 \equiv i(z - iz_R)/(2k) = [\mathrm{w}_0/(2h)]^2$, $k_\perp \equiv \alpha$, and $\rho \equiv r$. The integral in Eq. (34), defined as $I_{n,m}^{(2N)}(\rho,\mu)$ in Eq. (A2), is derived in Appendix A. Substituting the result in Eq. (A6) for $I_{n,m}^{(2N)}$, Eq. (34) becomes:

$$\begin{aligned} U_{BGV}^{(2N_o)}(\mathbf{r},\omega) &= (-1)^{n+m}2^{2n+m}\exp(ikz \pm im\phi) \\ &\times h^{2n+m+2} v^{m/2} e^{-v} \sum_{N=0}^{N_o}\left(\frac{h}{k\mathrm{w}_0}\right)^{2N} \\ &\times \left[\sum_{i=0}^{N} a_{N,i}(n+N+i)! L_{n+N+i}^m(v)\right], \end{aligned} \tag{35}$$

where the coefficients $a_{N,i}$ are defined in Eq. (A3). From Eq. (24) of Ref. [4], we have that

$$\begin{aligned} U_{BGV}^{(2N_o)}(\mathbf{r},\omega) &= (-1)^{n+m}2^{2n+m}\exp(ikz \pm im\phi) \\ &\times h^{2n+m+2} v^{m/2} e^{-v} \sum_{N=0}^{N_o}\left(\frac{h}{k\mathrm{w}_0}\right)^{2N} \\ &\times \left[f_{n,m}^{(2N)}\right]. \end{aligned} \tag{36}$$

Comparing Eqs. (35) & (36), and noting that the factors within the square brackets must be equal, we see that the general expression for the factors $f_{n,m}^{(2N)}$ of Ref. [4] is,

$$f_{n,m}^{(2N)}(v) = \sum_{i=0}^{N} a_{N,i}(n+N+i)! L_{n+N+i}^m(v). \tag{37}$$

Replacing the coefficients $a_{N,i}$ by their definition in Eq. (A3), and changing the summation index to $p \equiv N + i$, the factors $f_{n,m}^{(2N)}(v)$ are given explicitly by

$$f_{n,m}^{(2N)}(v) = \sum_{p=N}^{2N} \kappa_{N,p}(n+p)!\, L_{n+p}^m(v), \tag{38}$$

in which the coefficients $\kappa_{N,p}$ are defined in Eq. (23).

IV. EXPLICIT DERIVATION OF THE GENERALIZED TIME-DOMAIN PHASOR

In this section an explicit derivation of the generalized time-domain phasor up to arbitrary perturbative order N_o is provided, ultimately arriving at the expression given in Eq. (21). To this end, one starts with the monochromatic frequency-domain phasor of BGV, $U_{BGV}^{(2N_o)}(\mathbf{r},\omega)$, given in Eq. (6) [4, 5]. This phasor is then made polychromatic by multiplication with a Poisson-like frequency spectrum, $f(\omega)$, given in Eq. (7). Finally, a Fourier integral is performed to obtain the general time-domain phasor $U(\mathbf{r},t)$,

$$U^{(2N_o)}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t} f(\omega)\, U_{BGV}^{(2N_o)}(\mathbf{r},\omega)\;\mathrm{d}\omega, \tag{39}$$

where the negative exponential is chosen such that the resulting wave is traveling in the $+\hat{\mathbf{z}}$ direction. As described in Sec. II A, we assume the condition of isodiffraction.

A. Generalization in the Frequency Domain

We define the polychromatic frequency-domain phasor as $U^{(2N_o)}(\mathbf{r},\omega) \equiv f(\omega)U_{BGV}^{(2N_o)}(\mathbf{r},\omega)$, where $f(\omega)$ is given in Eq. (7) and $U_{BGV}^{(2N_o)}(\mathbf{r},\omega)$ is given in Eq. (6). This expression is correct to order N_o in the perturbative small parameter ϵ^2, which, however, depends on the frequency ω. Before carrying out the Fourier transform in Eq. (39), we therefore replace ϵ^2 in Eq. (6) by the frequency-independent small parameter ϵ_c^2 defined in Eq. (9). Then, all frequency dependent terms can be contained in new perturbative terms $\overline{U}_{2N}(\omega)$, namely,

$$\begin{aligned} &U^{(2N_o)}(\mathbf{r},\omega) \\ &= f(\omega)\left(U_{0,BGV} + \frac{\epsilon^2}{\beta}\,U_{2,BGV} + \cdots + \frac{\epsilon^{2N_o}}{\beta^{N_o}}\,U_{2N_o,BGV}\right) \\ &\equiv \overline{U}_0(\omega) + \frac{\epsilon_c^2}{\beta}\,\overline{U}_2(\omega) + \cdots + \frac{\epsilon_c^{2N_o}}{\beta^{N_o}}\,\overline{U}_{2N_o}(\omega), \end{aligned} \tag{40}$$

in which we have defined,

$$\overline{U}_{2N}(\omega) \equiv f(\omega)\frac{\omega_0^N}{\omega^N}\,U_{2N,BGV}. \tag{41}$$

Using Eqs. (6), (7), and (9), $\overline{U}_{2N}(\omega)$ in Eq. (41) may be written as,

$$\begin{aligned}\overline{U}_{2N}(\omega) &= (-1)^{n+m}2^{2n+m}\exp(ikz+im\phi+i\phi_0)\\ &\times\left(\frac{\omega_0}{\omega}\right)^N\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\omega^s\exp(-s\omega/\omega_0)}{\Gamma(s+1)}\Theta(\omega)\\ &\times(2\pi)h^{2n+m+2}v^{m/2}\exp(-v)f^{(2N)}_{n,m}(v).\end{aligned} \tag{42}$$

In order to make the frequency dependence of $\overline{U}_{2N}(\omega)$ in Eq. (42) explicit, we first substitute the expression for the associated Laguerre polynomial in Eq. (5) into the result in Eq (38) for $f^{(2N)}_{n,m}(v)$ to obtain,

$$f^{(2N)}_{n,m}(v) = \sum_{p=N}^{2N}\left[\kappa_{N,p}(n+p)!\sum_{j=0}^{n+p}G_{(n+p),m,j}\,v^j\right], \tag{43}$$

where the constants $G_{(n+p),m,j}$ are defined in Eq. (4). Finally, we extract the frequency dependence of v, using the vacuum dispersion relation, $k=\omega/c$, and the parameter definitions given in the text below Eq. (6), to obtain $v=\xi\omega$, where ξ is defined in Eq (2a). Substituting this latter expression for v and the result (43) for $f^{(2N)}_{n,m}(v)$ into Eq. (42), we obtain the Nth order term for the frequency-domain phasor in Eq. (40) as,

$$\begin{aligned}\overline{U}_{2N}(\omega) &= \sqrt{2\pi}\,\frac{\Lambda_{n,m}}{n!}\exp(i\omega z/c)\\ &\times\left(\frac{\omega_0}{\omega}\right)^N\left(\frac{s}{\omega_0}\right)^{s+1}\frac{\omega^s\exp(-s\omega/\omega_0)}{\Gamma(s+1)}\Theta(\omega)\\ &\times\omega^{m/2}\exp(-\xi\omega)\\ &\times\sum_{p=N}^{2N}\left[\kappa_{N,p}(n+p)!\sum_{j=0}^{n+p}G_{(n+p),m,j}\,\xi^j\,\omega^j\right],\end{aligned} \tag{44}$$

where $\Lambda_{n,m}$ is defined in Eq. (2c).

B. Generalization in the Time Domain

The time-domain representation of Eq. (44) is obtained through Fourier integration:

$$\begin{aligned}\overline{U}_{2N}(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega t}\,\overline{U}_{2N}(\omega)\,\mathrm{d}\omega\\ &= \frac{\omega_0^N\Lambda_{n,m}}{n!\Gamma(s+1)}\left(\frac{s}{\omega_0}\right)^{s+1}\sum_{p=N}^{2N}\left[\kappa_{N,p}(n+p)!\right.\\ &\left.\times\sum_{j=0}^{n+p}G_{(n+p),m,j}\xi^j\int_0^{\infty}\omega^{\gamma-N}\exp(-\eta\omega)\,\mathrm{d}\omega\right],\end{aligned} \tag{45}$$

where $\eta\equiv -iz/c+\xi+s/\omega_0+it$ and $\gamma\equiv s+m/2+j$. The integral is evaluated using Eq. (13), yielding

$$\begin{aligned}\overline{U}_{2N}(t) &= \frac{\omega_0^N\Lambda_{n,m}}{n!\,\Gamma(s+1)}\left(\frac{s}{\omega_0}\right)^{s+1}\sum_{p=N}^{2N}\left[\kappa_{N,p}(n+p)!\right.\\ &\left.\times\sum_{j=0}^{n+p}G_{(n+p),m,j}\xi^j\Gamma(\gamma+1-N)\eta^{-(\gamma+1-N)}\right].\end{aligned} \tag{46}$$

Moving all factors except $\Lambda_{n,m}$ into the inner sum, and making the substitutions indicated in Eqs. (15) and (22), the time-domain representation of the Nth order perturbative term $\overline{U}_{2N}$ takes the form,

$$\overline{U}_{2N}(t) = \Lambda_{n,m}\sum_{p=N}^{2N}\left[\sum_{j=0}^{n+p}c_{N,p}\xi^jT^{-\gamma-1+N}\right]. \tag{47}$$

Corresponding to the *frequency-domain* phasor to order N_o in Eq. (40), the generalized *time-domain* phasor including all terms up to perturbative order N_o is

$$\begin{aligned}U^{(2N_o)}(t) &= \sum_{N=0}^{N_o}\frac{\epsilon_c^{2N}}{\beta^N}\,\overline{U}_{2N}(t)\\ &=\Lambda_{n,m}\sum_{N=0}^{N_o}\left[\frac{\epsilon_c^{2N}}{\beta^N}\sum_{p=N}^{2N}\left\{\sum_{j=0}^{n+p}c_{N,p}\,\xi^jT^{-\gamma-1+N}\right\}\right],\end{aligned} \tag{48}$$

which agrees exactly with Eq. (21), as predicted.

V. TEST FOR ACCURACY OF THE PERTURBATIVE PHASOR

It is expected that the perturbative order in ϵ_c^2 necessary to obtain accurate values of the phasor will increase not only as the beam waist is reduced, but also as the radial and/or azimuthal LG indices are increased [5]. We illustrate this fact here using the simple numerical method suggested in Ref. [5] to check the convergence of our generalized perturbative phasor in the time domain. Specifically, a physically-correct description of the phasor requires that the wave equation is satisfied,

$$\nabla^2 U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}. \tag{49}$$

One may thus check convergence by comparing numerically both sides of this equation and requiring that

$$|\nabla^2 U| \approx \left|\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}\right|. \tag{50}$$

Such a comparison is shown in Fig. 1 for the case of an $LG_{0,7}$ mode and two different orders of perturbative correction. Also given for each comparison is the root mean

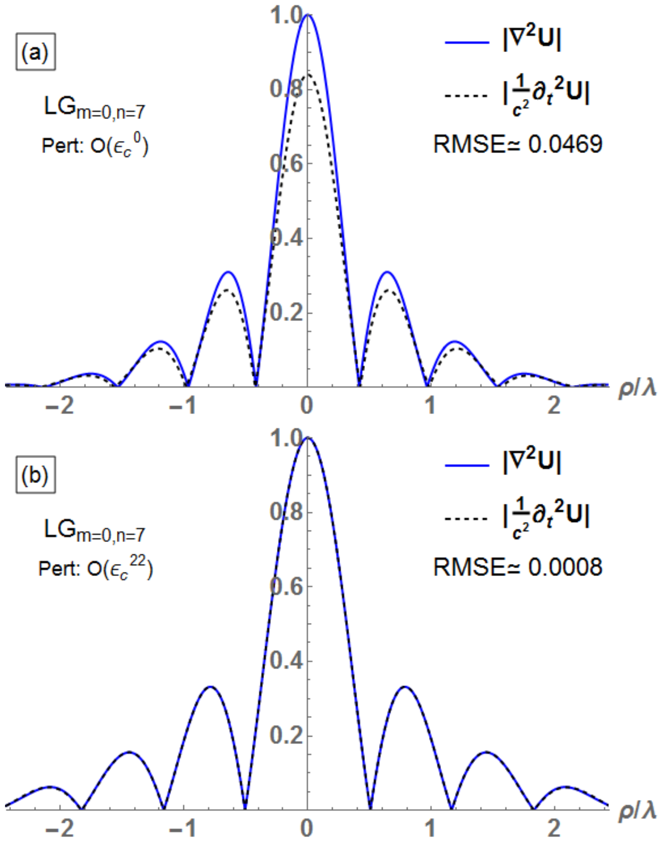

FIG. 1. Comparison of both sides of the wave equation [Eq. (50)] for the phasor, $|\nabla^2 U|$ and $|\partial_t^2 U/c^2|$, for the LG mode $m = 0, n = 7$, calculated for two different orders of perturbative correction. The phasor contains perturbation terms to order ϵ_c^0 in (a) and to order ϵ_c^{22} in (b). Inclusion of terms to $O(\epsilon_c^{22})$ is required for the RMSE to drop below 0.001, which we take here to indicate convergence. These plots were made near the beam waist using a spectral parameter $s = 70$, a beam waist $w_0 = 785$ nm, and a central wavelength $\lambda_0 = 800$ nm ($\epsilon_c^2 \approx 0.0253$).

squared error (RMSE), which is calculated near the beam waist on a finely-spaced grid of points extending over the range of ρ/λ shown in the plots in Fig. 1. Including only the lowest order perturbative term of order ϵ_c^0, one sees clearly in Fig. 1(a) that the two sides of Eq. (50) do not agree. Conversely, upon inclusion of corrective terms to $O(\epsilon_c^{22})$ in Fig. 1(b), the two sides of Eq. (50) agree to a very good approximation.

This example explicitly highlights the need for higher-order perturbative corrections in a generalized description. Both tight focusing and inclusion of high LG modes contribute to the complexity of the description. Thus, accuracy requires that higher order perturbative corrections are calculated in such cases.

VI. SUMMARY AND CONCLUSIONS

In summary, we have derived an analytic expression, postulated in Eq. (21) and derived explicitly in Eq. (48), for the time-domain phasor used to calculate the EM fields of an arbitrarily-tightly focused eLG beam of any LG mode and arbitrarily-short temporal duration. Our closed-form analytic result, obtained using the condition of isodiffraction, allows one to calculate the phasor to arbitrarily-high order N_o, in the perturbative small parameter ϵ_c^2 in Eq. (9), without having to evaluate any Fourier integrals. This model is thus straightforward to implement, either analytically or numerically.

The result in Eq. (48) generalizes the time-domain phasor that was presented up to order $N_o = 2$ in Ref. [5] (by a procedure requiring increasingly complicated Fourier integrals with increasing perturbative order N_o). Owing to increasing interest in laser-matter interactions involving structured light, accurate descriptions of high-OAM optical fields such as we have presented meet a current need. Reference [5] showed that higher-order perturbative corrections are required for the accurate description of high-OAM beams. Thus, having a closed-form analytical perturbative expression for the phasor to arbitrarily-high order N_o is a distinct advantage for applications involving eLG fields.

An alternative method for deriving the factors, $f^{(2N)}(v)$, is outlined in Appendix B, where the series expansion method of BGV [4] is followed explicitly. As discussed in Appendix B, there is a potential connection between that alternative method and the non-iterative derivation of integer partitions, which to our knowledge is an unsolved problem in the field of combinatorics in modern mathematics. Mathematicians or mathematical physicists may thus find this possible connection of significant interest.

VII. ACKNOWLEDGMENT

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Appendix A: Result for the Integral in Eq. (34)

In this appendix, we derive the result for the integral of a product of an associated Laguerre polynomial and a Bessel function that appears in Eq. (34) (in Section III above). We start from the integral in Eq. (8) of Ref. [4] (in which we have defined $\mu \equiv p$, $k_\perp \equiv \alpha$, and $\rho \equiv r$):

$$\int_0^\infty k_\perp^{2n+m+1} e^{-\mu^2 k_\perp^2} J_m(k_\perp \rho)\, \mathrm{d}k_\perp = \frac{n!}{2} \mu^{-(2n+m+2)} \left(\frac{\rho}{2\mu}\right)^m L_n^m\left(\frac{\rho^2}{4\mu^2}\right) \exp\left(-\frac{\rho^2}{4\mu^2}\right). \tag{A1}$$

We define now a similar integral,

$$I_{n,m}^{(2N)}(\rho,\mu) \equiv \int_0^\infty k_\perp^{2n+m+1} e^{-\mu^2 k_\perp^2} k_\perp^{2N} L_N^N\left(\mu^2 k_\perp^2\right) J_m(k_\perp \rho) \mathrm{d}k_\perp. \tag{A2}$$

The series representation of the associated Laguerre polynomials is given by Eq. (8.970.1) of Ref. [23]:

$$\begin{aligned} L_N^N(\mu^2 k_\perp^2) &= \sum_{i=0}^{N} \frac{(-1)^i}{i!} \binom{2N}{N-i} (\mu k_\perp)^{2i} \\ &\equiv \sum_{i=0}^{N} a_{N,i}\, (\mu k_\perp)^{2i}. \end{aligned} \tag{A3}$$

Substituting Eq. (A3) into Eq. (A2), we obtain

$$I_{n,m}^{(2N)}(\rho,\mu) = \sum_{i=0}^{N} a_{N,i} \mu^{2i} \int_0^\infty k_\perp^{2n+m+1+2N+2i} e^{-\mu^2 k_\perp^2} J_m(k_\perp \rho) \mathrm{d}k_\perp. \tag{A4}$$

This integral can be solved directly by application of Eq. (A1) with the replacement, $n \to (n+N+i)$:

$$\begin{aligned} I_{n,m}^{(2N)} &= \sum_{i=0}^{N} a_{N,i} \left[\frac{(n+N+i)!}{2} \mu^{-(2n+2N+m+2)} \right. \\ &\quad \left. \times \left(\frac{\rho}{2\mu}\right)^m L_{n+N+i}^m\left(\frac{\rho^2}{4\mu^2}\right) \exp\left(-\frac{\rho^2}{4\mu^2}\right)\right] \\ &= \frac{1}{2}\left(\frac{\rho}{2\mu}\right)^m \exp\left(-\frac{\rho^2}{4\mu^2}\right) \mu^{-(2n+2N+m+2)} \\ &\quad \times \sum_{i=0}^{N} a_{N,i} (n+N+i)! L_{n+N+i}^m\left(\frac{\rho^2}{4\mu^2}\right). \end{aligned} \tag{A5}$$

Using the definitions $\mu \equiv \mathrm{w}_0/(2h)$ [see text below Eq. (34)] and $v \equiv (h\rho/\mathrm{w}_0)^2$ [see text below Eq. (6)], we can write $v = \rho^2/(4\mu^2)$. Rewriting Eq. (A5) in terms of v and using $\mu \equiv \mathrm{w}_0/(2h)$, we obtain the following result for the integral defined in Eq. (A2):

$$\begin{aligned} I_{n,m}^{(2N)}(\rho,\mu) = &\ \frac{1}{2} v^{m/2} e^{-v} (2h/\mathrm{w}_0)^{2n+2N+m+2} \\ &\times \sum_{i=0}^{N} a_{N,i} (n+N+i)!\, L_{n+N+i}^m(v), \end{aligned} \tag{A6}$$

where the coefficients $a_{N,i}$ are defined in Eq. (A3).

Appendix B: An Alternative Method for Deriving the Factors $f^{(2N)}(v)$

In Ref. [4], the factors $f^{(2N)}(v)$ were originally calculated one at a time from each term in $G^{(2N)}$, which we introduced in Eq. (31). To calculate $G^{(2N)}$ for a particular N, one carries out a Taylor series expansion of the right-hand side of Eq. (31) about $\epsilon^2 = 0$, the first four terms of which are,

$$\begin{aligned} \sum_{j=0}^{\infty} \epsilon^{(2N)} G^{(2N)} = 1 &+ \epsilon^2 \left(\frac{\Omega}{2} - \frac{\Omega^2}{16h^2}\right) \\ &+ \epsilon^4 \left(\frac{3\Omega^2}{8} - \frac{\Omega^3}{16h^2} + \frac{\Omega^4}{512h^4}\right) \\ &+ \epsilon^6 \left(\frac{5\Omega^3}{16} - \frac{15\Omega^4}{256h^2} + \frac{3\Omega^5}{1024h^4} - \frac{\Omega^6}{24576h^6}\right) + O\left(\epsilon^8\right). \end{aligned} \tag{B1}$$

As one clearly sees, calculation of an arbitrarily high order term in this expansion in ϵ^2 is not simple. Referring to the right-hand side of Eq. (31) as $\mathfrak{F}$, by the product rule for differentiation, each ϵ^2 derivative acting on $\mathfrak{F}$ must act on both the prefactor and the exponential. The action of arbitrarily many such derivatives takes the form,

$$\begin{aligned} \frac{d^N \mathfrak{F}}{d(\epsilon^2)^N} = \sum_{i=0}^{N} \Bigg\{ &\left[\frac{\Omega^i}{(1-\epsilon^2\Omega)^{(1+2i)/2}} \prod_{i'=1}^{i} \left(\frac{2i'-1}{2}\right)\right] \\ &\times \left[\frac{d^{N-i}}{d(\epsilon^2)^{(N-i)}} e^{\mathfrak{E}}\right] \binom{N}{i} \Bigg\}, \end{aligned} \tag{B2}$$

where the first set of square brackets represents derivatives of the prefactor, the argument of the exponential in Eq. (31) is denoted by $\mathfrak{E} \equiv (\sqrt{1-\epsilon^2\Omega} - 1)/(2\epsilon^2 h^2) + \Omega/(4h^2)$, and the binomial coefficients occur owing to the product rule.

To evaluate the second set of square brackets in Eq. (B2), one requires an expression for arbitrarily many derivatives of an exponential function. This result can be found via Faà di Bruno's formula, which represents arbitrarily many derivatives of a composition of sufficiently differentiable functions [24, 25]. Faà di Bruno's formula, however, involves a sum over all possible integer partitions of the derivative order (cf. § 24.2.1 of Ref. [21]). Integer partitions are still an area of active research in combinatorics, and while there exist formulas for the *number* of partitions of an arbitrary integer there is, to our knowledge, presently no known analytical representation for the partitions themselves. As such, the partitions of a given integer are often generated through iterative algorithmic approaches [26, 27] (e.g., requiring knowledge of the partitions of q to calculate those of $q+1$). This prevents one from writing a non-recursive expression for the derivatives in Eq. (B2), and therefore from obtaining analytically a general solution for the coefficients of $f^{(2N)}(v)$ for arbitrary order N.

The method for deriving the factors $f^{(2N)}(v)$ in the main text gives them directly, whereas the alternative method described in this Appendix B involves integer partitions, which are typically derived iteratively. It remains an open question how this alternative method for deriving the factors $f^{(2N)}(v)$ is related to the method presented in the main text. Discovering this relationship may result in finding an analytical representation for the partitions of any integer. Researchers in combinatorics or mathematical physics may thus find this connection of interest.

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