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Leggett-Garg tests of macro-realism for dynamical cat-states evolving in a nonlinear medium

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We show violations of Leggett-Garg inequalities to be possible for single-mode cat-states evolving dynamically in the presence of a nonlinear quantum interaction arising from, for instance, a Kerr medium. In order to prove the results, we derive a generalised version of the Leggett-Garg inequality involving different cat-states at different times. The violations demonstrate failure of the premise of macro-realism as defined by Leggett and Garg, provided extra assumptions associated with experimental tests are valid. With the additional assumption of stationarity, violations of the Leggett-Garg inequality are predicted for the multi-component cat-states observed in the Bose-Einstein condensate and superconducting circuit experiments of Greiner et al. [Nature **419**, 51, (2002)] and Kirchmair et al. [Nature, **495**, 205 (2013)]. The violations demonstrate a mesoscopic quantum coherence, by negating that the system can be in a classical mixture of mesoscopically distinct coherent states. Higher orders of nonlinearity are also studied and shown to give strong violation of Leggett-Garg inequalities.

I. INTRODUCTION

In quantum mechanics, the Schrodinger cat-state is of interest because it is a superposition of two macroscopically distinct states [1]. This gives a paradox, because the Copenhagen interpretation of a system in such a superposition is that the system cannot be regarded as being in one state or the other, prior to a measurement. The assumption that a macroscopic system is in one or other of two macroscopically distinct states prior to measurement is referred to as “macroscopic realism”. Cat-states have been created in laboratories [2–6]. However, cat-states are usually signified by evidence that the system is in a superposition, rather than a classical mixture, of the two states. This evidence is presented within the framework of quantum mechanics.

In 1985, Leggett and Garg were motivated to test macroscopic realism directly, without assumptions based on the validity of quantum mechanics [7]. This gives the potential for a stronger demonstration of a cat-state. Leggett and Garg originally considered a dynamical system that is always found, by some measurement, to be in one of two macroscopically distinct states at any given time e.g. Schrodinger’s cat is always found to be dead or alive. They derived inequalities which if violated negated the validity of a form of macroscopic realism, commonly referred to as “*macro-realism*”. Macro-realism involves an additional premise, called macroscopic noninvasive measurability. The beauty of the Leggett-Garg inequality is that (similar to Bell inequalities [8]) a whole class of classical hidden variable theories can be potentially falsified by an experiment.

There has been experimental evidence for violations of Leggett-Garg inequalities [9–16]. Many of the tests and proposals to date however have not addressed macroscopic states, some exceptions being Refs. [15–17] which address systems such as a superconducting qubit, or a

single atom. There have also been proposals given for tests of macro-realism using macroscopic or mesoscopic atomic systems [18–21]. One of the most commonly considered cat-states is the superposition of two single-mode coherent states, given as

$$|\psi\rangle = c_1|\alpha_1\rangle + c_2|\alpha_2\rangle \quad (1)$$

(c_i are complex amplitudes) where $|\alpha\rangle$ is a coherent state [22, 23]. This cat-state has been successfully created in a single mode microwave field, with a 100 photon separation ($|\alpha_1 - \alpha_2|^2 \sim 100$) between the two distinct states, using a superconducting circuit [4, 6]. This is one of the largest cat-states ever created in a laboratory, with a record quantifiable macroscopic quantum coherence [2]. Similar cat-states have been created in a Bose-Einstein condensate [5], giving the potential to test macro-realism for many atoms. To the best of our knowledge however, a Leggett-Garg test for these cat-state systems has not yet been proposed.

In this paper, we give such a test. We show how to test for Leggett-Garg’s macro-realism for cat-states of the type (1). In fact, we will consider the multi-component cat-states $|\psi\rangle = \sum_i c_i |\alpha_i\rangle$, defined as a superposition of *multiple* coherent states. Here, we assume the phase-space separation of the coherent states $|\alpha_i\rangle$ and $|\alpha_j\rangle$ is large i.e. $|\alpha_i - \alpha_j| \rightarrow \infty$. In this way, we give an experimental proposal that relates directly to the cat-state experiments of Greiner et al. [5] and Kirchmair et al. [6], thus giving the possibility of strong tests of macro-realism involving large micro-wave cat-states, and cat-states with a large number of atoms.

We study the specific case considered by Yurke and Stoler [23], where a single oscillator or field mode prepared in a coherent state undergoes a nonlinear interaction described by the Hamiltonian

$$H_{NL} = \Omega \hat{n}^k \quad (2)$$

($k > 1$). It is well known that the interaction (2) leads to the formation of cat-states [5, 6, 22–24]. The details of the dynamical evolution depend on whether k is odd or even. For simplicity in this paper, we focus only on even k , and consider the cases $k = 2$ and $k > 2$ separately. Cat-states are also formed for odd k [23] however, and the techniques presented in this paper may well be useful for this case. Here Ω is the strength of the non-linearity and $\hat{n} = \hat{a}^\dagger \hat{a}$ is the number operator, where $\hat{a}$, $\hat{a}^\dagger$ are the bosonic destruction and annihilation operators. We show in this paper that, by considering three successive times of evolution, one can violate a Leggett-Garg inequality based on the Leggett-Garg assumptions of macro-realism. In order to demonstrate this result, we derive generalisations of the Leggett-Garg inequalities originally put forward by Jordan et al. [17].

In Section V of this paper, we consider the case where, $k = 2$, which corresponds to a Kerr nonlinearity. As explained above, this Hamiltonian has been realised experimentally for Bose-Einstein condensates [5, 24] and, more recently, using superconducting circuits [6]. In both experiments, the collapse and revival of a coherent state were observed, with intermediate states formed that are strongly suggestive of cat-states. In Section IV of this paper, we demonstrate violations of the Leggett-Garg inequality for even values of k greater than 2. This case is presented because of the simplicity of the predictions for violating a Leggett-Garg inequality, and because larger violations are predicted. Although more difficult with current technology, an experiment with higher quantum nonlinearities $k > 2$ may become feasible. For instance, such nonlinearities have been proposed for the generation of entangled triplets of photons [25].

In summary, our proposal for testing Leggett-Garg macro-realism with $k = 2$ corresponds to the highly non-linear regime of the experiments of Greiner et al. [5] and Kirchmair et al. [6]. The times we propose for the Leggett-Garg tests are within the timescale over which these experiments demonstrate the collapse and revival of the coherent state, suggesting a Leggett-Garg experiment to be highly feasible. As summarised in the second paragraph of the Introduction however, the macro-realism assumptions introduced by Leggett and Garg involve an additional assumption about measurements. This can create extra complexities for the experimental realisations of the Leggett-Garg inequality. In Section VI we give a specific discussion of how one may achieve the test of Leggett-Garg inequalities in the experiments of Greiner et al. [5] and Kirchmair et al. [6], based on the additional assumption of *stationarity* [12, 13]. This assumption has been applied to demonstrate quantum coherence and violations of Leggett-Garg inequalities for photons and neutrinos [12, 13]. We also outline alternative strategies based on weak measurements [11, 21, 26, 27]. Despite the additional assumptions, we argue that a successful demonstration of the violation of the Leggett-Garg inequality would give a rigorous confirmation of the formation of the superposition cat-state at the intermediate

times. This is because the violation could not be achieved for a mixture of coherent states. A discussion of the implications and potential loopholes of such experiments is given in the Sections VI and VII.

II. GENERALIZED LEGGETT-GARG INEQUALITIES

In this Section, we derive the Leggett-Garg inequalities to be used in the remaining sections of the paper. Leggett and Garg introduced two premises as part of their definition of macroscopic realism. The first premise is called “macroscopic realism per se”: a system must always be in one *or* other of the macroscopically distinguishable states, prior to any measurement being made. The second premise is called “macroscopic noninvasive measurability”: a measurement exists that can reveal which state the system is in, with a negligible effect on the subsequent macroscopic dynamics of the system. These two premises are used to derive a Leggett-Garg inequality.

Let us assume that at each time t_i , the system is in one of two macroscopically distinct states symbolised by $\varphi_1(t_i)$ and $\varphi_2(t_i)$. At each of three times t_1, t_2, t_3 , a measurement $\hat{S}(t_i)$ is performed to indicate which state the system is in. In the original Leggett-Garg treatment, the result of the measurement is denoted by $S(t_i) = 1$ if the system is found to be in $\varphi_1(t_i)$, and $S(t_i) = -1$ if the system is found to be in $\varphi_2(t_i)$. This choice was in analogy with the Pauli spin-1/2 outcomes chosen by Bell in his derivation of Bell inequalities [8].

In this paper, it will be useful to generalise the treatment, so that the results of the measurements are denoted by a value $S(t_i)$ where $|S(t_i)| \leq 1$. This will allow us, in particular, to define the value of an outcome to be 0. A similar approach was taken for Bell inequalities and Clauser-Horne inequalities, when generalised to account for outcomes of no detection of a particle [29]. We will also allow for the possibility that the macroscopically distinct states $\varphi_1(t_i)$ and $\varphi_2(t_i)$ defined at the different times t_i can be different, and also that the values $S(t_i)$ assigned at the different times can vary. These generalisations make possible the derivation of inequalities that can be violated for the dynamics under the Hamiltonian (2).

Let us therefore consider a general case, where the outcome of the measurement $\hat{S}(t_i)$ is denoted by $S(t_i) = x_1(t_i)$ if the system is found to be in $\varphi_1(t_i)$, and denoted by $S(t_i) = x_2(t_i)$ if the system is found to be in $\varphi_2(t_i)$, where $|x_{1,2}^{(i)}| \leq 1$. For the initial time t_1 , we will take $x_1(t_1) = +1$ and $x_2(t_1) = -1$, as in the original derivation of Leggett and Garg [7]. However, at the times t_2 and t_3 , the values can be less than 1. We will extend the derivation of the Leggett-Garg inequality derived by Jordan et al [17] to account for this case.

Following the original derivation given by Leggett and Garg [7], assuming macroscopic realism per se, we can assign to the system at the times t_i a hidden variable λ_i that

predetermines the value of $S(t_i)$ prior to the measurement $\hat{S}(t_i)$. According to the assumption of macroscopic realism per se, the system is always in one *or* other of the states. Hence, the value of the hidden variable is a predetermined property of the system, regardless of whether the measurement $\hat{S}(t_i)$ takes place. We comment that in this context where measurements distinguish between the two macroscopically distinguishable states, the predetermination follows from the assumption of “macroscopic realism per se”. In the more general context however, which includes small quantum systems, the assumption of predetermination is often referred to as counterfactual definiteness [30].

Considering the three different times t_i , we consider three hidden variables λ_1 , λ_2 and λ_3 . Assuming macroscopic realism, the value $S(t_i)$ of the measurement is determined by the value of the hidden variable λ_i . If the system at time t_i is in state $\varphi_1(t_i)$, then we assign the value $x_1(t_i)$ to the hidden variable λ_i . If the system at time t_i is in state $\varphi_2(t_i)$, then we assign the value $x_2(t_i)$ to the hidden variable λ_i . The hidden variables λ_i assume a value that coincides with the values of the possible results $S(t_i)$ of the measurement. In the original Leggett-Garg analysis, the hidden variables therefore assume a value of $+1$ or -1 . In our generalised case, the hidden variables assume values bounded by 1 i.e. $|\lambda_i| \leq 1$.

Always then, $|\lambda_1| = 1$ and $|\lambda_2|, |\lambda_3| \leq 1$. This allows us to carry out the proof. Simple algebra shows that [7, 17]

$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 - \lambda_1 \lambda_3 \leq 1 \quad (3)$$

because each λ_i is bounded by 1. This may be proved straightforwardly. The value of λ_1 is either 1 or -1 . Suppose $\lambda_1 = 1$. The maximum value of the function $F = \lambda_2 + \lambda_2 \lambda_3 - \lambda_3$ over the domain $|\lambda_2|, |\lambda_3| \leq 1$ is readily determined to be 1. This can be seen by graphical means. Alternatively, this can be seen by noting the stationary point is given by coordinates $(\lambda_2, \lambda_3) = (1, -1)$ and by considering the values of F at the boundaries: When $\lambda_2 = 1$, $F = 1$; when $\lambda_2 = -1$, $F = -1 - 2\lambda_3 \leq 1$; when $\lambda_3 = 1$, $F = 2\lambda_2 - 1 \leq 1$; when $\lambda_3 = -1$, $F = 1$. Thus, $F \leq 1$ for all λ_2, λ_3 in the domain $|\lambda_2|, |\lambda_3| \leq 1$. Next, one considers $\lambda_1 = -1$, to show in this case it is also true that $F \leq 1$.

Following the original derivation of the Leggett-Garg inequality, one now applies the second assumption of macro-realism. One assumes that a macroscopically non-invasive measurement is made on the system to determine the value $S(t_i)$ at each time t_i . Then one considers the two-time correlation functions defined by $\langle S(t_i)S(t_j) \rangle$. Using the premises, these moments are given $\langle S(t_i)S(t_j) \rangle = \langle \lambda_i \lambda_j \rangle$, this leads to the Leggett-Garg inequality

$$\langle S_1 S_2 \rangle + \langle S_2 S_3 \rangle - \langle S_1 S_3 \rangle \leq 1 \quad (4)$$

where for simplicity of notation we have introduced the abbreviation $S_i \equiv S(t_i)$ and $\langle S_i S_j \rangle \equiv \langle S(t_i)S(t_j) \rangle$. The inequality is derived based on the assumptions of macro-

realism. The violation of the inequality for an appropriate experiment therefore falsifies the macro-realism premises. This inequality was originally derived in Ref. [17] for the case where $\lambda_i = \pm 1$. Violations of the inequality are predicted for states evolving according to $\langle S_i S_j \rangle = \cos 2(t_j - t_i)$, as can be seen by putting $t_1 = 0$, $t_2 = \pi/6$, $t_3 = \pi/3$ (or $t_3 = 5\pi/12$).

The obvious difficulty with carrying out a Leggett-Garg experiment is the evaluation of the moments $\langle S_i S_j \rangle$ which are made under the assumption of a macroscopically noninvasive measurement. There are several ways the moment $\langle S_2 S_3 \rangle$ can be evaluated for an experimental test of the inequality (refer Refs. [9–14, 16–19, 21]). Experiments require justification that any measurement made at time t_2 does not interfere with the subsequent macroscopic evolution of the system. One method proposed in the original Leggett and Garg paper is an ideal negative-result measurement. Another approach is to make a weak measurement of the type proposed by Aharonov, Albert and Vaidman [11, 21, 26–28]. Such a weak measurement does not fully collapse the wave function at t_2 , but allows one to infer the average $\langle S_2 S_3 \rangle$ over a series of runs. Alternatively, one may argue along the lines of “measure and re-prepare” and “stationarity” [12, 13]. This allows a test of the inequality by measuring two-time ensemble averages only. The argument is as follows: If the system is indeed in one of the states $\psi_1(t_2)$ and $\psi_2(t_2)$ at time t_2 , the experimentalist can determine $\langle S_2 S_3 \rangle$ by first measuring which of the states the system is in at t_2 , and then re-preparing that state (either $\psi_1(t_2)$ and $\psi_2(t_2)$), to determine the value of S_3 at the later time t_3 . The details of this approach will be given in Sections IV and VI.

III. MODEL

In this paper, we show how the Leggett-Garg inequalities can be violated for dynamical cat-states formed under the evolution of a nonlinear interaction. In this Section, we explain the theoretical predictions for the dynamical solutions. Following Yurke and Stoler [23], we consider the evolution of a single mode system prepared in a coherent state under the influence of a nonlinear Hamiltonian written in the Schrodinger picture as

$$H = \omega \hat{n} + \Omega \hat{n}^k \quad (5)$$

We will restrict to consider k even. The odd case requires a different analysis, because the evolution is different and gives rise to different types of cat-states. The anharmonic term is proportional to $\hat{n}^k$ where $\hat{n}$ is the mode number operator and the integer $k > 1$ represents the order of the nonlinearity. The ω is the frequency of the harmonic oscillator and we choose units such that $\hbar = 1$. In the interaction picture, the evolution of the state can be readily determined. The initial coherent state is of the form

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_n \alpha^n \frac{1}{\sqrt{n!}} |n\rangle \quad (6)$$

where $|n\rangle$ is the n -particle eigenstate. The state after a time t is

$$|\alpha, t\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_n \alpha^n \frac{\exp(-i\phi_n)}{\sqrt{n!}} |n\rangle \quad (7)$$

where $\phi_n = \Omega t n^k$.

It is known that at certain times the system evolves into a superposition of distinct coherent states [22–24]. For $k > 2$, after a time $t = \pi/2\Omega$ the system is in a cat-state with coherent amplitudes π out of phase. At $t = \pi/\Omega$, the system is again in a coherent state $|\alpha\rangle$. At double this time, there is a revival back to the original coherent state $|\alpha\rangle$. Thus we observe cyclic behaviour [5, 6, 23, 24] and the sign of Ω acts only to reverse the direction of evolution of the states. Plots of the Q functions representing the different states are illustrated in Figure 1 for even values of k greater than 2.

Figure 1. The evolution of multi-component cat-states for a system described by the nonlinear Hamiltonian eq. (5) where k is any even number greater than 2. The contour graphs show surface plots of $Q(\alpha)$ with $\alpha_0 = 5$. Plots are given for the scaled times $t = 0$, $t = \pi/6\Omega$, $t = \pi/4\Omega$, $t = \pi/2\Omega$, $t = 3\pi/4\Omega$ and $t = \pi/\Omega$ (in dimensionless units).

For the purpose of the Leggett-Garg tests, the times t_2 and t_3 will correspond to the system being in a cat-superposition state of some sort, where the amplitudes of the coherent states are well separated in phase space. For instance, Figure 1 shows such cat-superposition states at times $t = \pi/6\Omega$, $t = \pi/4\Omega$, $t = \pi/2\Omega$ and $t = 3\pi/4\Omega$. We allow in general that the cat-states may be different at the different times. This deviates from the traditional Leggett-Garg test, where the system is in a superposition of the same two states at all times. The generalised Leggett-Garg inequalities, described in Section II, allow more flexibility to analyst Leggett-Garg violations.

IV. LEGGETT-GARG VIOLATIONS FOR NONLINEARITY $k > 2$, k EVEN

In this Section, we consider $k > 2$ and k even. We take $t_1 = 0$, when the system is prepared in a coherent state $|\alpha\rangle$ (where α is real) and consider the subsequent times $t_2 = \pi/4\Omega$, $t_3 = 3\pi/4\Omega$. The analytic expression at time $t = \pi/4\Omega$ can be readily evaluated [23]. When $t = \pi/4\Omega$, $e^{-i\Phi_n} = \exp(-i\pi n^k/4)$. Therefore at $t = \pi/4\Omega$ one has $e^{-i\Phi_n} = (-1)^{n/2}$ when n is even and $k = 2$, and $e^{-i\Phi_n} = 1$ when n is even and $k > 2$. When n is odd and k even, one has $e^{-i\Phi_n} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.

For our case of interest, when $k > 2$ and k is even, the state generated at time $t = \pi/4\Omega$ is

$$|\alpha, \pi/4\Omega\rangle = \frac{1}{2} \{ (1 + e^{-i\pi/4})|\alpha\rangle + (1 - e^{-i\pi/4})|-\alpha\rangle \} \quad (8)$$

At $t = 3\pi/4\Omega$, the solution is

$$|\alpha, 3\pi/4\Omega\rangle = \frac{1}{2} \{ (1 - e^{-i\pi/4})|\alpha\rangle + (1 + e^{-i\pi/4})|-\alpha\rangle \} \quad (9)$$

This compares with

$$\frac{1}{\sqrt{2}} (e^{-i\frac{\pi}{4}}|\alpha\rangle + e^{+i\frac{\pi}{4}}|-\alpha\rangle) \quad (10)$$

at the time $t = \pi/2\Omega$. The Q functions for the states generated at the four times $t = 0$, $\pi/4\Omega$, $\pi/2\Omega$ and $3\pi/4\Omega$ are plotted in Figure 2. Also plotted in Figure 3 is the value of the probability density $P(x)$ for a measurement x on each of the states, where $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$. The calculations for $P(x)$ and the Q function are outlined in the Appendix.

We now evaluate the Leggett-Garg inequality (4) using $t_1 = 0$, $t_2 = \pi/4\Omega$ and $t_3 = 3\pi/4\Omega$. Since $S_1 = 1$, we evaluate $\langle S_1 S_2 \rangle$ as $\langle S_2 \rangle = P_+^{(2)} - P_-^{(2)}$, where $P_+^{(i)} = \int_0^\infty P(x) dx$ is the probability of result for x being greater than or equal to 0 at time t_i , and $P_-^{(i)} = \int_{-\infty}^0 P(x) dx$ is the probability of result for x being less than 0 at time t_i . By integration, we find $P_+^{(2)} = 0.8535$ and $P_-^{(2)} = 0.1465$ ($\alpha \geq 2$). Therefore $\langle S_1 S_2 \rangle = 0.7070$. Similarly, we see that $\langle S_1 S_3 \rangle = -\langle S_1 S_2 \rangle$ (refer Figure 2).

Figure 2. The evolution of two-component cat-states for a system described by the nonlinear Hamiltonian eq. (5) where k is any even number greater than 2. The contour graphs show surface plots of $Q(\alpha)$ with $\alpha_0 = 3$. The plots are given for the times $t = 0$, $t = \pi/4\Omega$, $t = \pi/2\Omega$ and $t = 3\pi/4\Omega$.

Figure 3. The evolution of two-component cat-states for a system described by the nonlinear Hamiltonian eq. (5) where k is any even number greater than 2. The graphs show plots of $P(x)$ with $\alpha = 5$. The figures (a)-(d) give $P(x)$ at $t = 0$, $t = \pi/4\Omega$, $t = \pi/2\Omega$, and $t = 3\pi/4\Omega$ respectively.

Determining $\langle S_2 S_3 \rangle$

We have evaluated the two-time moments $\langle S_1 S_2 \rangle$ and $\langle S_1 S_3 \rangle$ straightforwardly without consideration of any intermediate measurement that might be made at time t_2 . This is justified for two reasons. First, in the Leggett-Garg derivation, such a measurement is required to be macroscopically noninvasive (refer Section VI). Also, the moments $\langle S_1 S_2 \rangle$ and $\langle S_1 S_3 \rangle$ are predicted to be a func-

tion of the time differences $t_2 - t_1$ and $t_3 - t_1$ respectively, which justifies the assumption of stationarity, that the two-time moments are invariant under time translation.

These considerations justify an approach that can be used to determine $\langle S_2 S_3 \rangle$. To evaluate $\langle S_2 S_3 \rangle$, we use the “measure and re-prepare” approach, discussed at the end of the Section II. Specifically, we will use the expression

$$\langle S_2 S_3 \rangle = P_+^{(2)} \langle S_2 S_3 \rangle_+ + P_-^{(2)} \langle S_2 S_3 \rangle_- \quad (11)$$

where $P_\pm^{(2)}$ is the probability that the system at time t_2 has a positive or negative value for x . This probability can be measured experimentally. For sufficiently large α , this probability is equal to the probability the system can be found to be in the $|\pm\alpha\rangle$ state. Here we denote $\langle S_2 S_3 \rangle_+$ as the average of $\langle S_2 S_3 \rangle$ given the state is prepared in the state $|\alpha\rangle$ at time t_2 . Similarly, $\langle S_2 S_3 \rangle_-$ is the average of S after a time t given the state is prepared in the state $|- \alpha\rangle$ at time t_2 . Recall from Section II that we define the two-time correlation as $\langle S_2 S_3 \rangle \equiv \langle S(t_2) S(t_3) \rangle$.

The expression (11) is justified if we assume the Leggett-Garg premises. At time t_2 , the system is in a superposition of two states $|\psi_1\rangle$ and $|\psi_2\rangle$

$$|\psi\rangle = c_- |\psi_1\rangle + c_+ |\psi_2\rangle \quad (12)$$

where $|\psi_1\rangle = |-\alpha\rangle$ and $|\psi_2\rangle = |\alpha\rangle$. Here c_- and c_+ are probability amplitudes, where $P_\pm^{(2)} = |c_\pm|^2$ for large α . The assumption of Leggett-Garg macro-realism is that the system is in *one or the other* of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ at time t_2 (with probabilities $P_+^{(2)}$ and $P_-^{(2)}$ respectively). If we assume that at time t_2 the system *was* in state $|\alpha\rangle$, then this is the initial state for the calculation of $\langle S_2 S_3 \rangle$, which is then represented as $\langle S_2 S_3 \rangle_+$. We see that because $t_3 - t_2 = \pi/2\Omega$, if the system is indeed in the state $|\alpha\rangle$ at time t_2 , then the system at the later time t_3 is in the symmetric state with equal probability for $x > 0$ and $x < 0$, as evident from Figures 2 and 3. Therefore $\langle S_2 S_3 \rangle_+ = 0$. Similarly, if we take that the system at time t_2 *was* in state $|- \alpha\rangle$, then we can show that $\langle S_2 S_3 \rangle_- = 0$. Therefore $\langle S_2 S_3 \rangle = 0$. Thus, we evaluate the Leggett-Garg term as

$$\langle S_1 S_2 \rangle + \langle S_2 S_3 \rangle - \langle S_1 S_3 \rangle = 1.414 \quad (13)$$

This shows a violation of the Leggett-Garg inequality (4). The term “measure and re-prepare” is used to describe this technique because in principle, one can measure which state the system is in at time t_2 , and then re-prepare that state to determine $\langle S_2 S_3 \rangle$. The assumption of stationarity however means that the moments $\langle S_2 S_3 \rangle_\pm$ (which are predicted to be dependent only on the time difference $t_3 - t_2$) can be measured more conveniently in an independent experiment at any later or prior time.

V. LEGGETT-GARG VIOLATIONS FOR NONLINEARITY $k = 2$

For the case $k = 2$, the evolution of the cat-states is different to the case with $k \neq 2$, k even. We will consider other time intervals in order to obtain a violation of the Leggett-Garg inequality. The Appendix gives the analytical expressions for the states and the probabilities $P(x)$ at different times of evolution. The Q functions for the states generated at a selection of different times are plotted in Figures 4 and 5.

Figure 4. The evolution of multi-component cat-states for a system described by the nonlinear Hamiltonian eq. (5) where $k = 2$. The contour graphs show surface plots of $Q(\alpha)$ with $\alpha_0 = 5$ for the times $t = 0$, $t = \pi/6\Omega$, $t = \pi/4\Omega$, $t = \pi/3\Omega$, $t = \pi/2\Omega$, $t = 2\pi/3\Omega$ and $t = 3\pi/4\Omega$, $t = \pi/\Omega$ and $t = 2\pi/\Omega$.

Specifically, we will consider the times $t_1 = 0$, $t_2 = \pi/3\Omega$ and $t_3 = 2\pi/3\Omega$. This sequence for $\alpha = 3$ is plotted in Figure 6. To give violation of the LG inequality, we select the values of S_i differently at each of the times. This does not affect the derivation of the inequality, as shown in Section II. We define $S_1 = +1$ if $x \geq 0$ and $S_1 = -1$ otherwise. Similarly, we define $S_2 = +1$ if $x \geq 0$ and $S_2 = 0$ otherwise. Finally, we define $S_3 = -1$ if $x \leq 0$ and $S_3 = 0$ otherwise. Proceeding with the evaluation of the necessary moments, we find for $\alpha \geq 2$ on integrating $P(x)$ that $\langle S_1 S_2 \rangle = \frac{2}{3}$ and $\langle S_1 S_3 \rangle = -\frac{2}{3}$ (refer Table 1).

t_2	t_3
$x \geq 0, S_2 = +1, P_+ = \frac{2}{3}$	$x > 0, S_3 = 0, P_+ = \frac{1}{3}$
$x < 0, S_2 = 0, P_- = \frac{1}{3}$	$x \leq 0, S_3 = -1, P_- = \frac{1}{3}$

Table I. Table showing the probabilities for outcomes at times t_2 and t_3 as evaluated for $\alpha \geq 2$.

To evaluate $\langle S_2 S_3 \rangle$, we follow the “measure and re-prepare” approach explained in Sections II and IV. The expansion of the state at time t_2 is given as

$$|\psi\rangle = i\frac{1}{\sqrt{3}}|-\alpha\rangle + \frac{1}{\sqrt{3}}\exp(-i\pi/6)\left|e^{i\pi/3}\alpha\right\rangle + \frac{1}{\sqrt{3}}\exp(-i\pi/6)\left|e^{-i\pi/3}\alpha\right\rangle \quad (14)$$

This state is evident by the Q function given in the plot of Figure 6. At time t_2 , the system is thus in a superposition

$$|\psi\rangle = N_0(|\psi_1\rangle + |\psi_2\rangle) \quad (15)$$

where $|\psi_1\rangle = |-\alpha\rangle$ and

$$|\psi_2\rangle = N_2\left\{\left|e^{i\pi/3}\alpha\right\rangle + \left|e^{-i\pi/3}\alpha\right\rangle\right\} \quad (16)$$

The normalization constants are $N_0^{-2} = 3\{1 + 2\exp(-\frac{3}{2}\alpha^2)\cos(\frac{\sqrt{3}}{2}\alpha^2)\}$ and $N_2^{-2} = 2\{1 + \exp(-|\alpha|^2 - \frac{1}{2}\alpha^2)\cos(\frac{\sqrt{3}}{2}\alpha^2)\}$, noting the initial condition implies α real. The assumption of the macro-realism is that the system is *in one or the other* of the states $|\psi_1\rangle$ and $|\psi_2\rangle$ at time t_2 . Using equation (11), we thus evaluate $\langle S_2 S_3 \rangle = P_+^{(2)}\langle S_2 S_3 \rangle_+ + P_-^{(2)}\langle S_2 S_3 \rangle_-$ as given by Eq. (11), where $P_-^{(2)}$ is the probability the system is in state $|\psi_1\rangle$ at time t_2 , and $\langle S_2 S_3 \rangle_-$ is the two-time moment given the system is in the state $|\psi_1\rangle$ at time t_2 . Similarly, $P_+^{(2)}$ is the probability the system is in state $|\psi_2\rangle$ at time t_2 , and $\langle S_2 S_3 \rangle_+$ is the two-time moment given the system is in the state $|\psi_2\rangle$ at time t_2 . Following the “measure and re-prepare” procedure to evaluate $\langle S_2 S_3 \rangle_-$, we first assume the system *was* in $|\psi_1\rangle = |-\alpha\rangle$ at time $t_2 = \pi/3\Omega$. We then evaluate what would have been the state at the later time $2\pi/3\Omega$, after evolution for a time $t = \pi/3\Omega$, with the initial state being $|\psi_1\rangle$. The final state in this case is the tri-cat state depicted in Figures 4 and Figure 6 at $t = 2\pi/3\Omega$. From our definitions of S_2 and S_3 (Table 1), we see that $\langle S_2 S_3 \rangle_- = 0$.

Continuing with the evaluation of $\langle S_2 S_3 \rangle$ based on the “measure and re-prepare” strategy, we next assume the system *was* in $|\psi_2\rangle$ at the time t_2 . This state is depicted by its Q function in Figure 7a. We then take this state to be the initial state for a calculation where the system evolves for a time $\pi/3\Omega$, to obtain the state that would have been generated at the time $t_3 = 2\pi/3\Omega$ with initial state $|\psi_2\rangle$. The state generated is

Figure 5. The time sequence showing the evolution of multi-component cat-states corresponding to the experiment of Kirchmair et al. [6]. Here we take Ω to be negative, and $\alpha_0 = 2$ and $k = 2$ for comparison with their reported results. We note the correspondence $\Omega = -\Omega_K/2$ where Ω_K is the nonlinearity defined by the Hamiltonian used in Ref. [6]. The entire sequence was observed by them, except for the tri-cat state at $t = 2\pi/3$. Here we show the surface plots of the Q functions of the state at the given times.

Figure 6. Surface plot of $Q(\alpha)$ with $\alpha_0 = 3$ and $k = 2$ for the times $t = 0$, $t = \pi/3\Omega$ and $t = 2\pi/3\Omega$.

Figure 7. Surface plot of $Q(\alpha)$ with $\alpha_0 = 3$ and $k = 2$. Figure (a) represents the initial state given by equation (16). Figure (b) is the time evolved state formed after a time $\pi/3\Omega$, as given by equation (17).

$$|\psi, \pi/3\rangle_2 = N \left[\frac{2}{\sqrt{3}} e^{-i\pi/6} |\alpha\rangle + \frac{1}{3} (2 - e^{-i\pi/3}) \{ |-e^{i\pi/3}\alpha\rangle + |-e^{-i\pi/3}\alpha\rangle \} \right] \quad (17)$$

where $N^{-2} = 2[1 + \exp(-\frac{3}{2}\alpha^2) \cos(\frac{\sqrt{3}}{2}\alpha^2)]$. This state is depicted by its Q function in Figure 7b. One evaluates the $P(x)$ for (17), to find

Figure 8. Plot of $P(x)$ for the state eq. (17) depicted in Figure 7b. The fringe pattern on the left is due to the superposition of the coherent states $|-e^{i\pi/3}\alpha\rangle$ and $|-e^{-i\pi/3}\alpha\rangle$. The probability of obtaining positive and negative results for x are found on integration to be $P_+ = 0.6669$ and $P_- = 0.3331$.

$$P(x) = \frac{N^2}{\sqrt{\pi}} \exp(-x^2) \left\{ \frac{4}{3} \exp(2\sqrt{2}x\alpha - 2\alpha^2) + \frac{4}{3} \exp\left(\frac{x\alpha}{\sqrt{2}} - \frac{5\alpha^2}{4}\right) \cos\left(\frac{x\alpha\sqrt{6}}{2} + \frac{\alpha^2\sqrt{3}}{4}\right) + \frac{2}{3} \exp(-\sqrt{2}x\alpha - \frac{|\alpha|^2}{2}) [1 + \cos(\sqrt{6}x\alpha + \frac{\alpha^2\sqrt{3}}{2})] \right\} \quad (18)$$

as plotted in Figure 8. Evaluation of integrals gives probabilities of $P_+ = 2/3$ and $P_- = 1/3$ for obtaining a positive and negative result for measurement of x on this state (Figure 8). Thus

$$\langle S_2 S_3 \rangle_+ = P_+ \langle S_3 \rangle_+ + P_- \langle S_3 \rangle_- = -\frac{1}{3} \quad (19)$$

Using eq. (11), we find $\langle S_2 S_3 \rangle = -\frac{2}{9}$. This implies

$$\langle S_1 S_2 \rangle - \langle S_1 S_3 \rangle + \langle S_2 S_3 \rangle = 10/9 \quad (20)$$

A violation of the Leggett-Garg inequality is obtained for the $k = 2$ case.

VI. EXPERIMENTAL STRATEGY

Various experimental strategies can be used to demonstrate violation of the Leggett-Garg inequalities. These are documented in the literature [7, 9–14, 16–19, 21]. The inequality involves two-time correlation functions of three observables, S_1 , S_2 and S_3 , measured at three different times. A common strategy is to evaluate the two-time correlation moments by taking an ensemble average of an appropriate two-time moment with suitable initial states. As applied to this proposal, for two of the moments ($\langle S_1 S_2 \rangle$ and $\langle S_1 S_3 \rangle$) the initial state at the time t_1 is a coherent state, identical to the experiments [5, 6]. The state formed at the intermediate time t_2 is a superposition of two states ψ_1 and ψ_2 , which are well-separated and distinguishable by a measurement of a quadrature amplitude $\hat{x}$, as defined in Sections IV and V. The sign of the outcome for $\hat{x}$ determines the value of S_2 . The moments obtained experimentally if this measurement $\hat{x}$ were performed can be evaluated from the experimentally determined Q functions, given in Refs. [6]. The validity of the evaluation of the moments for a test of macro-realism is based on the validity of the macro-realism premises. It is assumed that the system prior to any measurement will be in one of several macroscopically distinct states available to it. The value S_1 at time t_1 is known by preparation: the coherent state at time t_1 is prepared as $|\alpha\rangle$, which has a positive value for outcome $\hat{x}$ so that $S_1 = 1$. The value S_3 or S_2 (for evaluation of $\langle S_1 S_2 \rangle$ and $\langle S_1 S_3 \rangle$) is measurable by a projective measurement of $\hat{x}$ and hence of S_3 (or S_2): It is assumed this measurement correctly measures which of the states the system was in (prior to the measurement) to a macroscopic level of precision – which is all that is necessary to determine S_i prior to the measurement. In evaluating $\langle S_1 S_3 \rangle$, the measurement at time t_2 does not need to be made.

The moment $\langle S_2 S_3 \rangle$ is measurable using different strategies, including weak and ideal negative-result measurements, as discussed in Refs. [7, 9–14, 16–19, 21]. Here, we suggest a simple approach, as in Refs. [12, 13, 16, 21]. It can be verified experimentally that at time t_2 , the system is in the superposition of two macroscopically distinguishable states ψ_1 and ψ_2 . The moment $\langle S_2 S_3 \rangle$ can be measured by first preparing the system in one state ψ_1 and then the other ψ_2 , and measuring the moment $\langle S_2 S_3 \rangle$ for each case. The final moment is evaluated from the weighted average, assuming a *stationarity*, that the system evolves similarly under time translations [12, 13]. If the system is indeed in one or other state at time t_2 (as the first macro-realism premise implies), then

the measured moment is justified to be the value $\langle S_2 S_3 \rangle$ that would be measured, if the ideal macroscopically non-invasive measurement could take place. A violation of the Leggett-Garg inequality observed with this approach then serves to invalidate the premise of macro-realism.

To carry out the evaluation of $\langle S_2 S_3 \rangle$ for $k = 2$, we note that the first state ψ_1 of the superposition formed at time t_2 is a coherent state, which can be prepared and evolved to the tri-cat as in Ref [6]. The second state ψ_2 of the superposition is itself a superposition, of two coherent states π out of phase. This state ψ_2 can be re-prepared, up to a rotation in phase space, by evolving the coherent state for a time $t = \pi/2\Omega$, as illustrated in Figure 4. Evidence for the generation of this state ψ_2 is given in the BEC and superconducting circuit experiments of Refs. [5, 6]. That this state is indeed generated can be established via tomography using the Q function. In the anticipated experiment, the re-prepared state ψ_2 is then evolved for a time corresponding to $\pi/3\Omega$. The predicted state after this evolution is given in Figure 6b, from which the moments $\langle S_2 S_3 \rangle$ can be evaluated as described in Section V. The re-prepared state used here is different to ψ_2 by a rotation in phase space. One can justify that the measured correlation $\langle S_2 S_3 \rangle$ is unchanged, assuming the invariance of moments under rotations in phase space. Alternatively, one can experimentally obtain the rotated state, by rotating the initial coherent state.

VII. CONCLUSION

In summary, we have derived generalizations of Leggett-Garg inequalities and demonstrated how the new inequalities can be used to test macro-realism for dynamical cat-states created by a nonlinearity. In particular, in Section V we demonstrate the feasibility of violating a Leggett-Garg inequality using a Kerr $\chi^{(3)}$ nonlinearity. This enables us to predict violation of Leggett-Garg inequalities for the experiments of Greiner et al. [5] and Kirchmair et al. [6]. These experiments observe the collapse and revival of a coherent state over the necessary timescales.

Finally, we comment on loopholes and on the significance of the proposed Leggett-Garg test. First, we need to assume that the system at time t_1 is reliably prepared in the coherent state, and that the measurement at time t_3 accurately records the value of S_3 of the state of the system prior to measurement. The violation of the inequality would then negate that the system is in one or other of the macroscopically distinguishable states ψ_1 and ψ_2 at the time t_2 (or similarly at time t_3). This is because a system in a classical mixture ρ_{mix} of these states at these times would satisfy the Leggett-Garg premises. (This is understood because the system is in one of the two macroscopically distinguishable states, and a measurement can be constructed that leaves these states unchanged). Hence if the system is in a classical mixture

ρ_{mix} of the two states, one could *not* generate a violation of the inequalities. The violation of the Leggett-Garg inequality thus gives a demonstration of a mesoscopic quantum coherence.

It could be argued that the failure of the mixture ρ_{mix} is also exemplified by the observation of the revival of the final coherent state as observed in the experiments of Refs. [5, 6], since a classical mixture ρ_{mix} would not give a such a revival. However, this latter argument does not rule out that the system might be describable as another mixture consistent with the macro-realism premises in a theory alternative to quantum mechanics. In this respect, the violation of the Leggett-Garg inequality, which does not rely on quantum mechanics, is designed to give a stronger conclusion.

Extra assumptions also exist for the Leggett-Garg test however, which create loopholes. The proposed strategy relies on the preparation of the states ψ_1 and ψ_2 at the time t_2 for the evaluation of the $\langle S_2 S_3 \rangle$, and we thus assume the state regenerated for the later measurement is the actual ψ_i . The objective of the Leggett-Garg inequality however is to falsify the premises of macro-realism for all theories, not only quantum mechanics. For this there is a potential loophole, since it could not be excluded that for an alternative theory, the system at time t_2 is in a state *microscopically* different to the state ψ_1 or ψ_2 . These states may be minimally disturbed by the measurement performed at t_2 , and it might be argued that this minimal disturbance may generate after evolution a macroscopic change to the state (and hence to S_3) at the later time t_3 . This would not affect the justification of the evaluation of $\langle S_2 S_3 \rangle$ or $\langle S_1 S_3 \rangle$. However, for the evaluation of $\langle S_2 S_3 \rangle$ one could then not exclude that a small difference to the state measured at t_2 results in a macroscopic difference to the outcome at the time t_3 . To eliminate such a loophole, one is left with the difficult task to regenerate all states that are microscopically different to ψ_1 and ψ_2 , and demonstrate that the evolution from time t_2 to t_3 does not change the value of S measured at t_3 . Alternatively, one can seek to perform the Leggett-Garg test using an ideal negative-result measurement or a weak measurement [7, 9].

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APPENDIX

Coherent state expansion of states

In the problems treated, we consider the initial state of the system to be a coherent state $|\psi, t=0\rangle = |\alpha\rangle$ where α is real. In some cases, it is well-known that the state created at a later time can be written as a superposition of a finite number of distinct coherent states. A summary of some such examples in given in the Table II. Important for this paper is that the state $|\psi, t=\pi/3\Omega\rangle$ (for $k=2$) is a superposition of three coherent states $|\alpha\rangle$, $|e^{i\pi/3}\alpha\rangle$ and $|e^{-i\pi/3}\alpha\rangle$. That this is so is suggested by the plot of the Q function, given in Figure 4. A simple analysis allows us to evaluate the probability amplitudes, given as $|\psi, t=\pi/3\Omega\rangle = A|\alpha\rangle + B|e^{i\pi/3}\alpha\rangle + C|e^{-i\pi/3}\alpha\rangle$. One can write the state at this time as

$$\begin{aligned} |\alpha, \pi/3\Omega\rangle &= e^{-|\alpha|^2/2} \sum_n A(-1)^n \alpha^n \frac{1}{\sqrt{n!}} |n\rangle \\ &\quad + e^{-|\alpha|^2/2} \sum_n B(e^{i\pi/3})^n \alpha^n \frac{1}{\sqrt{n!}} |n\rangle \\ &\quad + e^{-|\alpha|^2/2} \sum_n C(e^{-i\pi/3})^n \alpha^n \frac{1}{\sqrt{n!}} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_n \exp(-i\pi n^2/3) \alpha^n \frac{1}{\sqrt{n!}} |n\rangle \end{aligned}$$

Evaluation of the amplitudes A , B and C is done by simultaneously solving for the coefficients, where $n = 0, 1, 2, \dots$. By solving we can write the wavefunction as in eq. (14).

Evaluation of $P(x)$ and the Q function

We evaluate $P(x)$ using the expression $P(x_\theta)$ for the probability measurement: $P(x_\theta) = |\langle x_\theta | \psi \rangle|^2$ where $\hat{x}_\theta = (e^{i\theta}\hat{\alpha} + e^{-i\theta}\hat{\alpha}^\dagger)\sqrt{2}$. The generalized position representation $\psi_{\alpha,t}(x) = \langle x | \alpha, t \rangle$ can be written as [23]

$$\psi_\alpha(x_\theta) = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2} + \frac{2x|\alpha|e^{i(\theta+\phi)}}{\sqrt{2}} - \frac{|\alpha|^2 e^{2i(\theta+\phi)}}{2} - \frac{|\alpha|^2}{2}\right)$$

where $\alpha = |\alpha|e^{i\phi}$. By using the expression for $\theta = 0$, we see that $P(x) = \frac{1}{\pi^{1/2}} \exp(-x^2 + 2\sqrt{2}x\alpha - 2|\alpha|^2)$. A summary of the position probability distributions $P(x)$ evaluated for the various times is given in the Table III.

We also evaluate the Husimi Q [31] representation defined as $Q(\alpha) = \langle \alpha | \rho | \alpha \rangle / \pi = \langle \alpha | \psi \rangle \langle \psi | \alpha \rangle / \pi = |\langle \alpha | \psi \rangle|^2 / \pi$. At $t=0$, we take $\psi = |\alpha_0\rangle$. Then

$$Q(\alpha) = \frac{1}{\pi} \exp(-|\alpha|^2 - |\alpha_0|^2 + \alpha_0(\alpha^* + \alpha))$$

where we have considered α_0 is real. The remaining functions are calculated using that the inner product of two

coherent states is $\langle \alpha | \beta \rangle = \exp(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^* \beta)$. A summary of the functions $Q(\alpha)$ evaluated for various times is given in the table IV below.

	ψ_α	relevant k
$t = 0$	$ \alpha\rangle$	all k
$t = \pi/8\Omega$	$\frac{\sqrt{2}}{4} e^{-i\pi/8} \{ e^{i\pi/4}\alpha\rangle - -e^{i\pi/4}\alpha\rangle + e^{-i\pi/4}\alpha\rangle - -e^{-i\pi/4}\alpha\rangle \}$ $+ \frac{1}{4} \{ (1-i)(\alpha\rangle + -\alpha\rangle) + (1+i)(i\alpha\rangle + -i\alpha\rangle) \}$	$k = 2$
$t = \pi/4\Omega$	$\frac{1}{2} \{ -i\alpha\rangle + i\alpha\rangle + e^{-i\pi/4} \alpha\rangle - e^{-i\pi/4} -\alpha\rangle \}$	$k = 2$
	$\frac{1}{2} \{ (1 + e^{-i\pi/4}) \alpha\rangle + (1 - e^{-i\pi/4}) -\alpha\rangle \}$	$k > 2, k \text{ even}$
$t = \pi/3\Omega$	$i \frac{1}{\sqrt{3}} -\alpha\rangle + \frac{1}{\sqrt{3}} \exp(-i\pi/6) e^{i\pi/3}\alpha\rangle + \frac{1}{\sqrt{3}} \exp(-i\pi/6) e^{-i\pi/3}\alpha\rangle$	$k = 2$
$t = 3\pi/8\Omega$	$\frac{\sqrt{2}}{4} e^{-i3\pi/8} \{ e^{i\pi/4}\alpha\rangle - -e^{i\pi/4}\alpha\rangle + e^{-i\pi/4}\alpha\rangle - -e^{-i\pi/4}\alpha\rangle \}$ $+ \frac{1}{4} \{ (1+i)(\alpha\rangle + -\alpha\rangle) + (1-i)(i\alpha\rangle + -i\alpha\rangle) \}$	$k = 2$
$t = \pi/2\Omega$	$\frac{1}{\sqrt{2}} (e^{-i\pi/4} \alpha\rangle + e^{+i\pi/4} -\alpha\rangle)$	$k \text{ even}$
	$\frac{1}{2} (\alpha\rangle + -\alpha\rangle + -i\alpha\rangle - i\alpha\rangle)$	$k \text{ odd}$
$t = 5\pi/8\Omega$	$\frac{\sqrt{2}}{4} e^{-i5\pi/8} \{ e^{i\pi/4}\alpha\rangle - -e^{i\pi/4}\alpha\rangle + e^{-i\pi/4}\alpha\rangle - -e^{-i\pi/4}\alpha\rangle \}$ $+ \frac{1}{4} \{ (1-i)(\alpha\rangle + -\alpha\rangle) + (1+i)(i\alpha\rangle + -i\alpha\rangle) \}$	$k = 2$
$t = 2\pi/3\Omega$	$\frac{1-2e^{i\pi/3}}{3} \alpha\rangle + \frac{1}{\sqrt{3}} e^{i\pi/6} e^{i\pi/3}\alpha\rangle + \frac{1}{\sqrt{3}} e^{i\pi/6} e^{-i\pi/3}\alpha\rangle$	$k = 2$
$t = 3\pi/4\Omega$	$\frac{1}{2} \{ -i\alpha\rangle + i\alpha\rangle - e^{+i\pi/4} \alpha\rangle + e^{+i\pi/4} -\alpha\rangle \}$	$k = 2$
	$\frac{1}{2} \{ (1 - e^{+i\pi/4}) \alpha\rangle + (1 + e^{+i\pi/4}) -\alpha\rangle \}$	$k > 2, k \text{ even}$
$t = 7\pi/8\Omega$	$\frac{\sqrt{2}}{4} e^{-i7\pi/8} \{ e^{i\pi/4}\alpha\rangle - -e^{i\pi/4}\alpha\rangle + e^{-i\pi/4}\alpha\rangle - -e^{-i\pi/4}\alpha\rangle \}$ $+ \frac{1}{4} \{ (1+i)(\alpha\rangle + -\alpha\rangle) + (1-i)(i\alpha\rangle + -i\alpha\rangle) \}$	$k = 2$
$t = \pi/\Omega$	$ -\alpha\rangle$	all k

Table II. Summary of the multi-component cat-states formed at different times, for different k .

	$P(x)$	k
$t = 0$	$\frac{1}{\pi^{1/2}} \exp(-x^2 + 2\sqrt{2}x\alpha - 2 \alpha ^2)$	all k
$t = \pi/4\Omega$	$\frac{1}{2} \frac{1}{\pi^{1/2}} \exp(-x^2 - \alpha ^2) \{ 2 \sinh(\alpha ^2) + \exp(- \alpha ^2) \cosh(2\sqrt{2}x\alpha) + \exp(\alpha ^2) \cos(2\sqrt{2}x\alpha) + 2\sqrt{2} \cos(\sqrt{2}x\alpha) \sinh(\sqrt{2}x\alpha) \}$	$k = 2$
	$\frac{1}{\pi^{1/2}} \exp(-x^2 - 2 \alpha ^2) \{ \cosh(2\sqrt{2}x\alpha) + \frac{1}{\sqrt{2}} \sinh(2\sqrt{2}x\alpha) \}$	$k > 2, k \text{ even}$
$t = \pi/3\Omega$	$\frac{1}{3} \frac{1}{\pi^{1/2}} \exp(-x^2) \{ \exp(-2\sqrt{2}x\alpha - 2\alpha^2) + 2 \exp(\sqrt{2}x\alpha - \alpha^2/2) - 2 \exp(-\frac{\sqrt{2}}{2}x\alpha - 5\alpha^2/4) \cos(\frac{\sqrt{6}}{2}x\alpha - \frac{\sqrt{3}}{4}\alpha^2) + 2 \exp(\sqrt{2}x\alpha - \alpha^2/2) \cos(\sqrt{6}x\alpha - \frac{\sqrt{3}}{2}\alpha^2) \}$	$k = 2$
$t = \pi/2\Omega$	$\frac{1}{\pi^{1/2}} \exp(-x^2 - 2 \alpha ^2) \cosh(2\sqrt{2}x\alpha)$	$k \text{ even}$
$t = 3\pi/4\Omega$	$\frac{1}{2} \frac{1}{\pi^{1/2}} \exp(-x^2 - \alpha ^2) \{ 2 \sinh(\alpha ^2) + \exp(- \alpha ^2) \cosh(2\sqrt{2}x\alpha) + \exp(\alpha ^2) \cos(2\sqrt{2}x\alpha) - 2\sqrt{2} \cos(\sqrt{2}x\alpha) \sinh(\sqrt{2}x\alpha) \}$	$k = 2$
	$\frac{1}{\pi^{1/2}} \exp(-x^2 - 2 \alpha ^2) \{ \cosh(2\sqrt{2}x\alpha) - \frac{1}{\sqrt{2}} \sinh(2\sqrt{2}x\alpha) \}$	$k > 2, k \text{ even}$
$t = \pi/\Omega$	$\frac{1}{\pi^{1/2}} \exp(-x^2 - 2\sqrt{2}x\alpha - 2 \alpha ^2)$	all k

Table III. Summary of the probability distributions $P(x)$ for various t and k .

t	$Q(\alpha)$	k
$t = 0$	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2 + \alpha_0(\alpha^* + \alpha))$	all k
$t = \pi/4\Omega$	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2) \{ \cos(\alpha^* \alpha_0) + e^{-i\pi/4} \sinh(\alpha^* \alpha_0) \}$ $\times \{ \cos(\alpha_0 \alpha) + e^{+i\pi/4} \sinh(\alpha_0 \alpha) \}$	$k = 2$
	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2) \{ \cosh(\alpha \alpha_0 + \alpha^* \alpha_0)$ $+ \frac{1}{\sqrt{2}} \sinh(\alpha \alpha_0 + \alpha^* \alpha_0) + i \frac{1}{\sqrt{2}} \sinh(\alpha \alpha_0 - \alpha^* \alpha_0) \}$	$k > 2, k \text{ even}$
$t = \pi/3\Omega$	$\frac{1}{3} \frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2 + \frac{1}{2} \alpha_0 \alpha^* + \frac{1}{2} \alpha_0 \alpha) \{ \exp(-\frac{3}{2} \alpha \alpha_0) + 2ie^{i\pi/6} \cos(\frac{\sqrt{3}}{2} \alpha \alpha_0) \}$ $\times \{ \exp(-\frac{3}{2} \alpha^* \alpha_0) - 2ie^{-i\pi/6} \cos(\frac{\sqrt{3}}{2} \alpha^* \alpha_0) \}$	$k = 2$
$t = \pi/2\Omega$	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2) [\cosh(\alpha^* \alpha_0 + \alpha \alpha_0) - i \sinh(\alpha^* \alpha_0 - \alpha \alpha_0)]$	$k \text{ even}$
$t = 3\pi/4\Omega$	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2) \{ \cos(\alpha^* \alpha_0) - e^{+i\pi/4} \sinh(\alpha^* \alpha_0) \}$ $\times \{ \cos(\alpha_0 \alpha) - e^{-i\pi/4} \sinh(\alpha_0 \alpha) \}$	$k = 2$
	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2) \{ \cosh(\alpha^* \alpha_0 + \alpha \alpha_0)$ $- \frac{1}{\sqrt{2}} \sinh(\alpha^* \alpha_0 + \alpha \alpha_0) - i \frac{1}{\sqrt{2}} \sinh(\alpha^* \alpha_0 - \alpha \alpha_0) \}$	$k > 2, k \text{ even}$
$t = \pi/\Omega$	$\frac{1}{\pi} \exp(- \alpha ^2 - \alpha_0 ^2 - \alpha_0(\alpha^* + \alpha))$	all k

Table IV. $Q(\alpha)$ functions evaluated for various times and k .

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