Quantum information or entanglement transfer between subsystems

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The size of quantum information – or entanglement – transfer rates between subsystems is a generic question in problems ranging from decoherence in quantum computation and sensing, to quantum underpinnings of thermodynamics, to the behavior of quantum black holes. We investigate such rates for given couplings between subsystems, for sufficiently random subsystem evolution, and find evidence for a conjectured relation of these rates to the size of the couplings. This provides a direct connection between entanglement transfer and the microphysical couplings in the Hamiltonian that are responsible for it.

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I. INTRODUCTION AND MOTIVATION

Consider two quantum subsystems A and B of a quantum system, that are coupled by a Hamiltonian including interactions between A and B. In general, the couplings will transfer quantum information between the subsystems. This can be characterized in terms of entanglement: one can take a copy of A, maximally entangle it with A, and ask how fast the evolution (with trivial evolution of the copy) transfers that entanglement to B. In general, this rate will depend on the interactions, and on the nature of the terms of the Hamiltonian that act solely on A or on B.

This is a very general problem, encountered in various contexts. One example is that of a quantum computer or sensor interacting with its environment. In that case, if the interactions transfer entanglement from a subsystem of the quantum system to its environment, this leads to decoherence and degradation of the computing or sensing ability, and indicates how fast the quantum system evolves to classical behavior (see *e.g.* [1, 2] and references therein). This problem also makes contact with the question of deriving thermodynamics from quantum dynamics[3, 4]. Specifically, if A and B are large quantum systems, that can for example be thought of as near equilibrium, then the information transfer between them is related to the entropy transfer between them, which may for example be driven by a thermal gradient. A final example is that of a black hole. In a quantum field theory description, a black hole will build up entanglement with its environment, either through absorption, or by emitting Hawking particles entangled with its internal state. But, if a black hole decays and disappears, in order to maintain a quantum-mechanical (unitary) description, this entanglement must be transferred back to its environment during this decay. An important question is what couplings to the environment could suffice[5–7] to transfer sufficient information.

There are different measures of information transfer and decoherence; this paper will focus on mutual information, which will be used to characterize entanglement transfer. For large subsystems, coupled by products of operators in the two subsystems, and undergoing sufficiently random evolution, a conjecture about the relation between the information transfer rate and the couplings was made in [7]. This paper will investigate the question of the rate, and provide evidence for this conjecture.[1] One aspect of our approach is to connect the information transfer to decay probabilities. Specifically, for subsystem dimensions $|B| \gg |A|$, and if the initial energy of B is comparable or smaller than that of A, one typically expects A to decay to its lower-energy states, creating excitations in B. This will be directly related to transfer of entanglement from A to B. Decay probabilities, in turn, can be approximately calculated in perturbation theory, giving a means to estimate information transfer rates. Alternatively, one can study non-perturbative numerical evolution for certain simple models. These provide complementary calculations of information transfer rates, agreeing with and extending perturbative results.

II. SET UP AND CONJECTURE

Our goal is to describe how information transfers between two subsystems A and B of a bigger system, whose Hilbert space we assume to be of the form $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We take the Hamiltonian to be of the form

$$H = H_A + H_B + H_I , \tag{1}$$

where H_A and H_B act only on their respective subsystems, and H_I is an interaction term which can be expanded in terms of products of operators acting on A and B respectively,

$$H_I = \sum_{\gamma=1}^{\chi} C_\gamma \mathcal{O}_A^\gamma \mathcal{O}_B^\gamma , \tag{2}$$

with coefficients C_γ. Transfer of quantum information from A to B can be understood by introducing a copy $\bar{A}$ of A, and beginning with a maximally entangled state of A and $\bar{A}$, times some initial state for B. Then, the mutual information $I(A, \bar{A})$, defined by

$$I(A, \bar{A}) = S(A) + S(\bar{A}) - S(A\bar{A}) , \tag{3}$$

with $S(X) = -\mathrm{Tr}\left(\rho_X \log \rho_X\right)$, will initially be $I(A, \bar{A}) = 2 \log |A|$, and $I(B, \bar{A})$, similarly defined, will initially be zero. Evolution via (1) will in general transfer information, alternately understood as entanglement with $\bar{A}$, from A

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[1] The related question of bounds on such transfer rates has been studied in [8–10].

to B, and so we expect that initially $I(A, \bar{A})$ decreases and $I(B, \bar{A})$ grows[2]. We wish to understand how these changes depend on features of the Hamiltonian (1).

While this is an interesting general problem, we will impose some additional structure. We assume that $|B| \gg |A|$, so that B has plenty of "capacity" to absorb information from A, and also that the eigenvalues of H_A have a typical scale $\mathcal{E}$. In that case, by energy conservation,[3] evolution by H will typically excite states of B with energy $\mathcal{E}$ above its initial state – assuming H_I permits such transitions – so $\mathcal{E}$ functions as a common scale and, *e.g.*, states of B of much higher energy need not be considered. In the limit $|B| \gg |A|$, as A decays its number of available states decreases, reducing its capacity for entanglement. We also define $C_\gamma = c_\gamma \mathcal{E}$, and so describe H_I in terms of dimensionless coefficients and operators.

A conjecture for the information transfer rate was made in [7] in the case where also $|A| \gg 1$, where H_A and H_B are sufficiently random, and where O_A^γ and O_B^γ are independent (commuting) operators with a standard normalization to unit size. This conjecture states that for $c_\gamma \lesssim 1$,

$$\frac{dI(B, \bar{A})}{dt} = -\frac{dI(A, \bar{A})}{dt} = \mathcal{K}\mathcal{E}\sum_{\gamma=1}^{\chi} c_\gamma^2 \tag{4}$$

with $\mathcal{K}$ constant. Here we use the relations $I(A, \bar{A}) - S(\bar{A}) = S(A) - S(B) = -[I(B, \bar{A}) - S(\bar{A})]$, which follow from the fact that $S(\bar{A})$ is constant (since $\bar{A}$ is not interacting). For random matrices, [7] used the operator norm[4] $\|O\| = 1$, but as is seen below, more generally one should instead use the norm $|O|^2 = \mathrm{Tr}(O^2)/N$ where N is the subsystem dimension. This norm gives the typical size $\langle O^2 \rangle$ of diagonal matrix elements of O^2. The transfer of information takes place through the χ different operator "pathways,"[5] corresponding to the independent operators indexed by γ. Below we will also find evidence that the formula (4) holds in a time-averaged sense for small $|A|$.

III.   PERTURBATIVE ANALYSIS AND GENERAL TREATMENT

An intuition behind the preceding conjectures is that, given the assumption $|B| \gg |A|$, and for example the comparability of the sizes of the total energies, system A will decay through H_I, in the process transferring entanglement to B, and so the rate of transfer of entanglement is related to the decay rate. We begin by investigating the latter.

Let $|K\rangle$ be the energy eigenstates of H_A, and $|i\rangle$ of H_B; the latter are generically much more closely spaced than the former. Alternatively, we could consider the situation where H_I causes a transition from a more limited collection of initial eigenstates of A, say in a finite energy band, and study the information encoded in this subset of states; one can likewise consider a limited band for B. In this case, it also may be true that the final states $|K'\rangle$ of A to which H_I induces transitions is different from the initial subset. Denote the initial state of B as $|\psi_0\rangle$, and for simplicity assume it is an H_B eigenstate with energy E_0. We assume that the couplings are sufficiently small that first order perturbation theory in H_I gives a reasonable approximation, and transition rates are given by Fermi's Golden Rule. If $U(t)$ denotes the interaction-picture evolution operator, we find transition amplitudes

$$\langle i|\langle K'|U(t)|K\rangle|\psi_0\rangle = A_{K'i,K0}(t) \simeq -2i\langle i|\langle K'|H_I|K\rangle|\psi_0\rangle \frac{\sin(t\Delta E/2)}{\Delta E}e^{it\Delta E/2} \ , \tag{5}$$

where

$$\Delta E = E_{K'} + E_i - E_K - E_0 \ . \tag{6}$$

The total transition probability from $|K\rangle$ to the collection of final states $|K'\rangle$ is then given by

$$P_K(t) \simeq 4\sum_{K',i} |\langle i|\langle K'|H_I|K\rangle|\psi_0\rangle|^2 \frac{\sin^2(t\Delta E/2)}{(\Delta E)^2} = 2\pi t^2 \langle \psi_0|\langle K|H_I \mathcal{P}_{1/t} H_I|K\rangle|\psi_0\rangle \tag{7}$$

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[2] This does not have to be the case in general, since it is possible that the entropy of A remains maximal. In fact, this is what would happen if we considered evolution by a random unitary acting on AB.

[3] Here we use time independence of the hamiltonian (1), and the assumption that the interaction term H_I is sufficiently small.

[4] Since we are working in finite dimensions, the norm is just the size of the largest eigenvalue.

[5] A more natural nomenclature might be *channels*, but the terminology *quantum channel* is already used with a different definition.

where

$$\mathcal{P}_{1/t} = \sum_{K',i} |K'\rangle|i\rangle\langle i|\langle K'| \frac{1}{2\pi} \frac{\sin^2(t\Delta E/2)}{(t\Delta E/2)^2} \tag{8}$$

acts like a projector onto states in a band of energies of width $1/t$ about $\Delta E = 0$. For sufficiently smoothly-varying matrix elements of H_I and density of states $\rho(E)$, this yields Fermi's Golden Rule,

$$P_K(t) \simeq 2\pi t |H_I|^2 \rho(E_K + E_0) ; \tag{9}$$

where $|H_I|^2$ is a typical squared matrix element of H_I between initial and final states. The transition rate is approximately constant in t, and scales like the expectation value of H_I^2, restricted to the relevant range of states selected by $\mathcal{P}_{1/t}$.

If the information transfer rate matched the decay rate, that would demonstrate the essential features of the conjecture (4) (the factor of $\mathcal{E}$ in (4) arises from the combination of the density of states and the matrix element squared, which both contain the same scale). However, the former requires closer investigation. Specifically, consider the maximally entangled initial state

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{K=1}^{N} |\bar{K}\rangle|K\rangle , \tag{10}$$

where K ranges over the N states of either A, or of our initial state band of A. The evolution of a given state will be given in terms of amplitudes by

$$|K\rangle|\psi_0\rangle \rightarrow B_K(t)|K\rangle|\psi_0\rangle + A_{K'i,K0}(t)|K'\rangle|i\rangle , \tag{11}$$

with summation over i, K' implicit, and with

$$|B_K(t)|^2 = 1 - \sum_{K',i} |A_{K'i,K0}(t)|^2 = 1 - P_K(t) . \tag{12}$$

In (11) we have neglect terms where only one of $|\psi_0\rangle$, $|K\rangle$ changes, as suggested by energy conservation. The initial state (10) then evolves via (1), (11) to

$$|\Psi(t)\rangle = \frac{1}{\sqrt{N}} \sum_{K=1}^{N} |\bar{K}\rangle \left[B_K(t)|K\rangle|\psi_0\rangle + A_{K'i,K0}(t)|K'\rangle|i\rangle \right] , \tag{13}$$

and the entanglement transfer will be determined in terms of the entropies of the reduced density matrices ρ_A and ρ_B. These take the form

$$\rho_A(t) = \frac{1}{N} \sum_K \left(|B_K(t)|^2 |K\rangle\langle K| + \sum_i |\psi_{iK}(t)\rangle\langle\psi_{iK}(t)| \right) , \tag{14}$$

with

$$|\psi_{iK}(t)\rangle = \sum_{K'} A_{K'i,K0}(t)|K'\rangle , \tag{15}$$

and

$$\rho_B(t) = \frac{1}{N} \sum_K \left(|B_K(t)|^2 |\psi_0\rangle\langle\psi_0| + \sum_{K'} |\psi_{K'K}(t)\rangle\langle\psi_{K'K}(t)| \right) , \tag{16}$$

with

$$|\psi_{K'K}(t)\rangle = \sum_i A_{K'i,K0}(t)|i\rangle . \tag{17}$$

The problem is then to relate the resulting difference $S(B) - S(A)$, giving the mutual information, to the decay probabilities P_K.

IV. QUBIT DECAY

As an initial approach to the preceding problem, we consider a simple case, where the system A is a two-state system, *i.e.* a qubit. This will illustrate aspects of the more general problem.

Specifically, suppose A has two states, $|0\rangle$ and $|1\rangle$, which are eigenstates of H_A with eigenvalues zero and $\mathcal{E}$ respectively. If H_I does not commute with H_A and H_B, such a qubit will generically decay, producing excitation in B. We are interested in evolution of an initial state (see (10), (11))

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|\bar{0}\rangle|0\rangle + |\bar{1}\rangle|1\rangle \right) \otimes |\psi_0\rangle \, , \tag{18}$$

and transfer of the entanglement with $\bar{A}$ from A to B.

A. Analytic treatment - simplified model

For a simplified model of the evolution (11), take

$$
\begin{aligned}
|1\rangle|\psi_0\rangle &\rightarrow B(t)|1\rangle|\psi_0\rangle + \sum_i A_i(t)|0\rangle|i\rangle \\
|0\rangle|\psi_0\rangle &\rightarrow |0\rangle|\psi_0\rangle \, .
\end{aligned}
\tag{19}
$$

The decay amplitudes can be perturbatively calculated from (5), and satisfy, with $P(t)$ the total decay probability,

$$|B(t)|^2 = 1 - \sum_i |A_i(t)|^2 = 1 - P(t) \, . \tag{20}$$

The appropriate traces of the density matrix arising from the evolution of (19) then give

$$\rho_A = \frac{1}{2} \left[1 + P(t) \right] |0\rangle\langle 0| + \frac{1}{2} \left[1 - P(t) \right] |1\rangle\langle 1| \, , \tag{21}$$

$$\rho_B = \left[1 - \frac{P(t)}{2} \right] |\psi_0\rangle\langle\psi_0| + \frac{1}{2} |\psi(t)\rangle\langle\psi(t)| \, , \tag{22}$$

with

$$|\psi(t)\rangle = \sum_i A_i(t)|i\rangle \, . \tag{23}$$

These density matrices have von Neumann entropies

$$S(A) = -\frac{1}{2} \left\{ [1 + P(t)] \log \left[\frac{1 + P(t)}{2} \right] + [1 - P(t)] \log \left[\frac{1 - P(t)}{2} \right] \right\} \tag{24}$$

and

$$S(B) = -\left[1 - \frac{P(t)}{2} \right] \log \left[1 - \frac{P(t)}{2} \right] - \frac{1}{2} P(t) \log \left[\frac{P(t)}{2} \right] \, . \tag{25}$$

At $t = 0$, $S(A) = \log 2$ and $S(B) = 0$; at large time, if $P(t) \rightarrow 1$, $S(A) = 0$ and $S(B) = \log 2$. The initial mutual information $I(A, \bar{A}) = 2\log 2$ (see (3)) thus transfers to B, giving final $I(B, \bar{A}) = 2\log 2$.

The perturbative decay probability can be calculated using (7), and for a sufficiently dense collection of allowed final states of B, we expect that if $H_I = c\mathcal{E}O_A O_B$, then $P(t) \propto c^2\mathcal{E}t$ from (9). Perturbation theory fails when $P(t)$ approaches unity, but the perturbative calculation gives the characteristic decay time $T \sim 1/(c^2\mathcal{E})$ for it to do so. Given (24) and (25), $I(B, \bar{A}) = S(B) - S(A) + S(\bar{A})$ does not vary exactly linearly in time, but does transition in the decay time T from zero to $2\log 2$. Thus, the conjectured rate (4) is found in an average over this time.
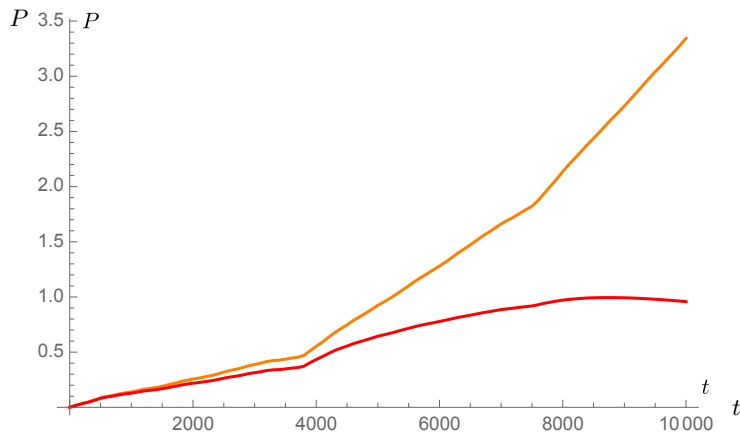
FIG. 1: The probability of decay as a function of time, computed numerically (red, bottom) and from perturbation theory (orange, top). The value of the coupling is set to $c = 1/400$.
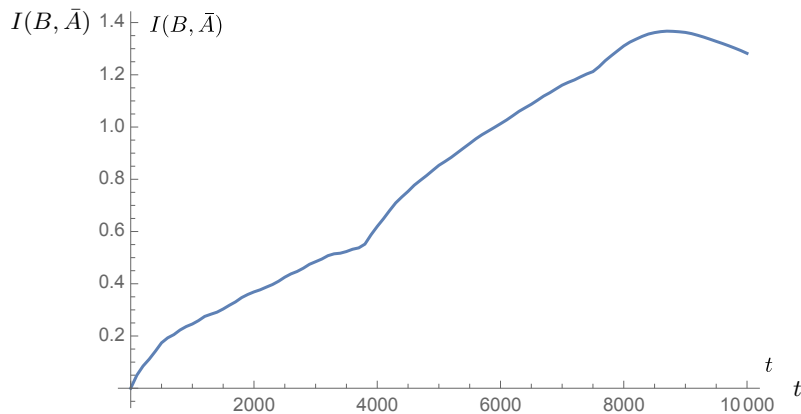

FIG. 2: The mutual information as a function of time, computed numerically and using the model of the previous section ($c = 1/400$). The plot shows perfect agreement, since both curves are plotted but are indistinguishable.

B. Numerical example

These statements can be checked beyond a perturbative analysis, through numerical evolution. Specifically, let the spectrum of H_A be $\{0, 1\}$ and consider a simple model where B contains 7 qubits. For H_B we choose the spectrum $\{0, 1 + i/600\}$, where i ranges between $(-63, 63)$. After fixing the spectra, the eigenvectors of H_A and H_B are drawn randomly according to the Haar measure. The interaction Hamiltonian is $H_I = c\sigma_A^x \sigma_{B_1}^x$, where $\sigma_{B_1}^x$ acts on one of the qubits in B. The non-perturbative numerical evolution can be studied for this model, to check the preceding statements. For the initial state of B we choose the ground state of H_B, $|\psi_0\rangle = |0\rangle$

Fig. 1 shows the comparison between the perturbative evaluation of the decay probability $P(t)$ and the numerical evolution. One can see that there is good agreement until $P \sim 1/2$, beyond which the perturbative result gives an increasingly worse approximation.

The mutual information $I(B, \bar{A}; t)$ can be evaluated in two different ways, either directly from the numerically-generated density matrices ρ_A and ρ_B, or in the model evolution of the preceding section, with $S(A)$ and $S(B)$ of (24), (25) evaluated in terms of the numerically-calculated $P(t)$. This model neglects transitions with large energy non-conservation, which should be a good approximation. Indeed fig. 2 shows that this gives excellent results for $I(B, \bar{A}; t)$, since the curves from the two approaches are plotted in that figure and are indistinguishable. Closer examination shows that they differ by very small amounts, $\sim 10^{-6}$.

Finally, the decay time, which Fig 2 shows is also the time for $I(B, \bar{A})$ to transition from zero to $2 \log 2$, can also be numerically determined. For example, define $T_{0.8}$ by $P(T_{0.8}) = 0.8$ as a characteristic decay time. Fig. 3 shows a plot of $T_{0.8}$ as a function of $1/c^2$, exhibiting the characteristic dependence of the decay time on c that was described above. So, numerical evolution supports the conjecture, with the proviso that for small A the rate should be viewed
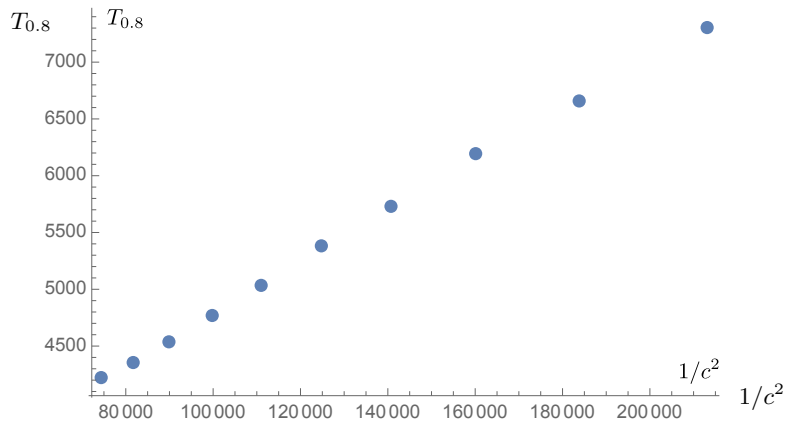
FIG. 3: The dependence of $T_{0.8}$ on the value of the coupling. $c = 1/500 + i/6000$ where i ranges between $(1, 10)$.

as a time-averaged information transfer rate.

V. NEXT DIRECTIONS

A. More general analysis

The preceding provides evidence for the conjecture (4) in the case where subsystem A is small; we would like to extend that to larger A. Specifically, if A has N states in the band of initial states that we consider, and if it transitions to a band of final states with $N' < N$ states, $S(A)$ is expected to decrease by $\log(N/N')$, and $S(B)$ to correspondingly increase. This will take place over the average transition time, $\bar{T}$, for the initial state band, suggesting an information transfer rate

$$\frac{dI(B, \bar{A})}{dt} = -\frac{dI(A, \bar{A})}{dt} \approx \frac{2\log(N/N')}{\bar{T}} \ . \tag{26}$$

It is plausible that the averaging over the different initial states, and the different transition times, improves the linear behavior in t, and this is in fact seen in simple multistate models[11]. An important future goal is to investigate such behavior more closely, for example from the structure of the general formulae (14), (16). Then, from (7), we estimate

$$\frac{1}{\bar{T}} \approx 2\pi t \langle H_I \mathcal{P}_{1/t} H_I \rangle \approx k\mathcal{E} \sum_{\gamma=1}^{\chi} c_\gamma^2 \tag{27}$$

with k a constant. Here we average over the initial state band; in the second equality, we assume that the operators O_A^γ, O_B^γ are normalized so that $\langle O^2 \rangle = 1$, again for typical allowed (energy-conserving) transitions. If the O are random matrices, this is the same as saying that the operator norm $\|O\| \approx 1$, as in [7], though the conjecture for other operators uses the former norm. Also in the more general case $\log(N/N') \gg 1$, we find $\mathcal{K} = 2k \log(N/N')$ in (4).

B. Applications

The preceding discussion is closely related to analogous problems in quantum computing and sensing. Better understanding the relation of decoherence rates to microphysics may help identify sources of the former. In fault-tolerant quantum computing (see [12] and references therein), one instead uses a failure measure based on an error rate, though the decoherence rate is in principle related. Generalization of the present discussion is also needed in that in quantum computing contexts, typically one considers a time-dependent Hamiltonian $H_A(t)$.

One can also anticipate a direct connection to thermodynamics, due to the close connection between thermodynamic entropy and von Neumann entropy. This paper has investigated transfer of the latter between subsystems, but when these subsystems are behaving thermodynamically, one expects a direct connection to the thermodynamic description of their behavior and entropy transfer.

Finally, the conjecture (4) was originally motivated by the problem of describing a black hole (BH) interacting with its environment. This led to an interesting result[7], which can be understood from the conjecture. In order to transfer information from BH to environment, there must be couplings mediating the transfer, which would be forbidden in a standard local field theory description. However, to have a close correspondence with field theory, one does not want these couplings to produce large deviations from field theory. Couplings to degrees of freedom near a BH can be described as interactions of the form (2) between operators acting on the BH and on these nearby field modes. The couplings can be interpreted in terms of extra fluctuations of the fields near the BH, *e.g.* the metric, that depend on the state of the BH. One finds[7] from (4) that operators of size $|O_{BH}| = 1$ transfer sufficient information into modes with BH-size wavelengths. The BH state space is expected to have very high dimension N, *e.g.* given by the Bekenstein-Hawking entropy. The characteristic size of the corresponding fluctuations is given in terms of the expectation value of O_{BH} in a typical state; if the O_{BH}'s behave like random matrices, this gives $\langle\psi|O_{BH}|\psi\rangle \sim 1/\sqrt{N}$ and so this size is very tiny. One may alternately describe this result in terms of correspondingly tiny couplings between BH energy eigenstates. In effect, tiny couplings to the huge number of BH states are sufficient to transfer enough information. This provides a possible way to save quantum mechanics (unitarity) for black holes, with "minimal" deviation from the quantum field theory approximation, and impact on infalling observers[7].

VI. ACKNOWLEDGEMENTS

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