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Conformality Lost in Efimov Physics

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A general mechanism for the loss of conformal invariance is the merger and disappearance of an infrared fixed point and an ultraviolet fixed point of a renormalization group flow. We show explicitly how this mechanism works in the case of identical bosons at unitarity as the spatial dimension d is varied. For d between the critical dimensions $d_1 = 2.30$ and $d_2 = 3.76$, there is loss of conformality as evidenced by the Efimov effect in the three-body sector. The beta function for an appropriate three-body coupling is a quadratic polynomial in that coupling. For $d < d_1$ and for $d > d_2$, the beta function has two real roots that correspond to infrared and ultraviolet fixed points. As d approaches d_1 from below and as d approaches d_2 from above, the fixed points merge and disappear into the complex plane. For $d_1 < d < d_2$, the beta function has complex roots and the renormalization group flow for the three-body coupling is a limit cycle.

Keywords: Bose gases, effective field theory, Efimov states, renormalization group.

I. INTRODUCTION

In a classic paper in 1970 [1], Vitaly Efimov discovered a remarkable phenomenon that can occur in a three-body system of nonrelativistic particles with short-range interactions if at least two of the three pairs of particles have two-body scattering length a much larger than their range. In the unitary limit $a \to \pm\infty$, there is a shallow two-body bound state exactly at the two-body threshold. Efimov showed that the three-body system in the unitary limit can have infinitely many shallow three-body bound states with an accumulation point at the three-body threshold. This phenomenon is called the *Efimov effect*, and the three-body bound states are called *Efimov states*. As the threshold is approached, the ratio of the binding energies of successive Efimov states approaches a universal constant that depends on the statistics and the mass ratios of the particles. This spectrum is evidence for discrete scale invariance in the three-body system.

The renormalization group (RG) has provided deep insights into quantum field theories in condensed matter, particle, and nuclear physics [2]. An RG flow describes how parameters that describe short-distance scales depend on the ultraviolet (UV) cutoff Λ. The simplest attractor for an RG flow is a fixed point. At a fixed point, the short-distance parameters are independent of the UV cutoff Λ, and this requires the system to be scale invariant. Another possible attractor for an RG flow is an RG *limit cycle*, a possibility first proposed by Wilson [3]. The RG flow at a limit cycle is around a closed loop. The RG flow completes one cycle every time the UV cutoff Λ is changed by a specific multiplicative factor λ_0. This requires the system to have discrete scale invariance with discrete scaling factor λ_0. The discrete scale invariance associated with the Efimov effect can be interpreted in terms of the RG flow of a three-body coupling with a limit cycle [4].

Scale invariance of a system may be compatible with a larger symmetry group of conformal transformations. For Lorentz-invariant quantum field theories in two space-time dimensions, Zamolodchikov has proved under general assumptions that scale invariance implies conformal invariance [5]. For relativistic quantum field theories with space-time dimension greater than two, there are no known counterexamples to the conjecture that scale invariance implies conformal invariance [6]. In nonrelativistic quantum field theories with Galilean invariance [7, 8], the conjecture that scale invariance implies conformal invariance has not been proven even in two space dimensions, but again there are no known counterexamples [6]. In this article, we will use the phrases "scale invariance" and "conformal invariance" interchangeably.

In a paper entitled "Conformality Lost" [9], Kaplan, Lee, Son, and Stephanov revealed a general mechanism for the loss of conformal invariance in a system. As a parameter α of a scale-invariant system approaches a critical value α_* where conformality is lost, an ultraviolet (UV) fixed point and an infrared (IR) fixed point of an appropriate RG flow merge together and disappear. The RG flow near the point where conformality is lost can be described by a simple model with a single dimensionless coupling g whose

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beta function has the form

$$\Lambda \frac{d}{d\Lambda} g = (\alpha - \alpha_*) - (g - g_*)^2 , \tag{1}$$

where Λ is the renormalization scale and g_* is a constant independent of g and α. For $\alpha > \alpha_*$, the coupling g has a UV fixed point g_+ and an IR fixed point g_-, which are given by

$$g_\pm = g_* \pm (\alpha - \alpha_*)^{1/2} . \tag{2}$$

As α decreases toward α_*, the two fixed points both approach g_*, and they merge together at $\alpha = \alpha_*$. For $\alpha < \alpha_*$, the zeros of the beta function have disappeared into the complex plane and conformality is lost.

In Ref. [9], the authors suggested that the loss of conformality associated with Efimov physics can be understood in terms of the merging of a UV fixed point and an IR fixed point for a three-body coupling, but they didn't show this explicitly. In the case of identical bosons in three spatial dimensions at unitarity ($a \to \pm\infty$), the authors pointed out that there is no tunable parameter analogous to α in Eq. (1). In this paper, we show explicitly how conformality is lost in the case of identical bosons at unitarity when the spatial dimension d is the external parameter. The Efimov effect disappears at critical dimensions $d_1 = 2.30$ and $d_2 = 3.76$ that were first derived by Nielsen, Fedorov, Jensen, and Garrido [10]. We show that the beta function for an appropriate three-body coupling is a quadratic polynomial analogous to Eq. (1) with coefficients that depend on d. For $d < d_1$ and $d > d_2$, the beta function has real zeros that correspond to a UV fixed point and an IR fixed point. As d enters the interval $d_1 < d < d_2$ where conformality is lost, the zeros of the beta function disappear into the complex plane, and the RG flow for the three-body coupling becomes a limit cycle. The continuous scaling symmetry is broken down to a discrete scaling symmetry that manifests itself in the Efimov effect.

This article has two additional sections followed by a summary. In Section II, we consider briefly the two-body sector for identical bosons in d dimensions. We determine the beta function for the two-body coupling. It has a zero that corresponds to a UV fixed point associated with conformal invariance in the two-body sector. In Section III, we consider in detail the three-body sector for identical bosons in d dimensions. We show that the beta function for a three-body coupling is a quadratic polynomial in the coupling analogous to Eq. (1). The behavior of its zeros as functions of d is compatible with the mechanism for the loss of conformality in Ref. [9].

II. TWO-BODY SECTOR

Nonrelativistic particles with short-range interactions in three spatial dimensions have a nontrivial *zero-range limit* in which the range of the interaction is taken to zero with the s-wave scattering length a held fixed. In the zero-range limit, a is the only interaction parameter, at least in the two-body sector. The *unitary limit* is defined by $a \to \pm\infty$. In the unitary limit, the two-body system is scale invariant, because the interactions no longer provide any length scale. The two-body scattering cross section has power-law dependence on the collision energy E: it saturates the s-wave unitarity bound $8\pi/E$.

We now consider identical bosons with zero-range interactions in d dimensions. The T-matrix element $\mathcal{T}(k)$ for the scattering of two particles with total energy $E = k^2$ in the center-of-momentum frame can be derived by solving a Lippmann-Schwinger equation [11]:

$$\mathcal{T}(k) = \frac{2(4\pi)^{d/2}/\Gamma\left(\frac{2-d}{2}\right)}{1/\lambda - e^{-i\pi(d-2)/2} k^{d-2}} . \tag{3}$$

For simplicity, we set $\hbar = 1$ and $m = 1$ here and in the remainder of this article. The single interaction parameter λ that characterizes the zero-range limit is proportional to the T-matrix element at $k = 0$. In $d = 3$, the interaction parameter is the scattering length: $\lambda = a$. The unitary limit is defined by $\lambda \to \pm\infty$. In this limit, the T-matrix element in Eq. (3) has power-law dependence on the momentum: $\mathcal{T}(k) \sim 1/k^{d-2}$. Since the interactions no longer provide any length scale, the two-body system is scale invariant. The T-matrix element in Eq. (3) goes to 0 as $d \to 2$ and as $d \to 4$, indicating that 2 and 4 are critical dimensions where the theory becomes noninteracting. The upper critical dimension $d = 4$ was first pointed out by Nussinov and Nussinov [12].

Identical bosons in the zero-range limit can be described by a local quantum field theory with an atom field ψ. The Lagrangian density is

$$\mathcal{L} = \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 \right) \psi - \frac{g_2}{4} \left(\psi^\dagger \psi\right)^2 - \frac{g_3}{36} \left(\psi^\dagger \psi\right)^3 , \tag{4}$$

where g_2 and g_3 are bare couplings for the two-body and three-body contact interaction. The two-body coupling g_2 can be tuned as a function of the UV cutoff to get the desired interaction parameter λ. In three dimensions, the three-body coupling g_3 is required to reproduce Efimov physics [13, 14]. No higher-body couplings are required. We will determine the RG flow of g_2 in d dimensions and use it to understand scale invariance in the two-body sector in the unitary limit.

The perturbative expansion of the off-shell two-body scattering amplitude in powers of g_2 is UV divergent. The amplitude can be regularized by imposing a sharp UV cutoff Λ on loop momenta in the center-of-mass frame.[1] The bare coupling g_2 must depend on Λ to compensate for the explicit dependence of the two-body scattering amplitude on Λ. The sum of the geometric series in g_2 has the same form as Eq. (3), with $1/\lambda$ replaced by a linear combination of $1/g_2$ and Λ^{d-2} with coefficients that are functions of d. In order to reproduce the T-matrix element, g_2 must be related to the interaction parameter λ via

$$g_2(\Lambda) = -\frac{2(4\pi)^{d/2}}{\Gamma\left(\frac{2-d}{2}\right)} \left[\frac{1}{\lambda} + \frac{2\sin\left(\frac{d}{2}\pi\right)}{(d-2)\pi}\Lambda^{d-2}\right]^{-1}. \tag{5}$$

We define a dimensionless two-body coupling $\hat{g}_2$ by multiplying g_2 by the power of the UV cutoff Λ required by dimensional analysis:

$$\hat{g}_2(\Lambda) = \frac{1}{(d-2)(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\Lambda^{d-2} g_2(\Lambda). \tag{6}$$

The d-dependent prefactor has been introduced in order to simplify the RG equation for $\hat{g}_2(\Lambda)$ below. The RG equation for $\hat{g}_2$ can be obtained by differentiating both sides of Eq. (6) with respect to $\log\Lambda$ and then eliminating Λ in favor of $\hat{g}_2(\Lambda)$:

$$\Lambda\frac{d}{d\Lambda}\hat{g}_2 = (d-2)\,\hat{g}_2\,(\hat{g}_2+1)\,. \tag{7}$$

Scale invariance arises at a fixed point of the RG flow for $\hat{g}_2$ or, equivalently, at a zero of the beta function defined by the right side of Eq. (7). The RG flow for $\hat{g}_2$ has two fixed points: a UV fixed point $\hat{g}_+ = -1$ and an IR fixed point $\hat{g}_- = 0$. Using Eqs. (5) and (6), we find that the UV fixed point corresponds to the unitary limit $\lambda \to \pm\infty$ and the IR fixed point corresponds to the non-interacting limit $\lambda \to 0$. At the UV fixed point, there are scale-invariant interactions in the two-body sector.

III. THREE-BODY SECTOR

In 1970, Vitaly Efimov discovered that a system of three identical bosons in three spatial dimensions in the unitary limit $a \to \pm\infty$ has a sequence of infinitely many shallow three-body bound states (Efimov states) whose binding energies have an accumulation point at the energy of the three-body scattering threshold [1]. As they approach the threshold, the ratio of the binding energies of successive Efimov states approaches a universal factor that is approximately $515 \approx 22.7^2$. The spectrum of Efimov states reflects a discrete scaling symmetry with discrete scaling factor $e^{\pi/s_0} \approx 22.7$, where $s_0 = 1.00624$. In subsequent papers, Efimov showed that the system of three identical bosons has universal properties characterized by discrete scaling behavior with the discrete scaling factor e^{π/s_0} not only in the unitary limit but whenever the scattering length a is much larger than the range of the interactions [16, 17].

From an RG perspective, discrete scaling behavior should be associated with an RG flow whose ultraviolet attractor is a limit cycle. Bedaque, Hammer, and van Kolck showed that in $d = 3$, the renormalization of the three-body coupling g_3 in Eq. (4) is indeed governed by a limit cycle [13, 14]. Braaten and Hammer derived the RG equation for the dimensionless coupling $\hat{g}_3 = \Lambda^4 g_3/144\pi^4$ [4]:

$$\Lambda\frac{d}{d\Lambda}\hat{g}_3 = \frac{1+s_0^2}{2}\left(\hat{g}_2^2 + \frac{\hat{g}_3^2}{\hat{g}_2^2}\right) + \left(3 - s_0^2 + 2\hat{g}_2\right)\hat{g}_3. \tag{8}$$

The beta function defined by the right side is a quadratic polynomial in $\hat{g}_3$ with coefficients that depend on $\hat{g}_2$. This structure is reminiscent of the beta function in Eq. (1) that describes the loss of conformality. Near the UV fixed point $\hat{g}_2 = -1$ of the two-body coupling, the zeros of the beta function in Eq. (8) are

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[1] The same expression for the two-body amplitude can be obtained by using dimensional regularization with power divergence subtraction [15] and with renormalization scale $\Lambda/2$, but no one has succeeded in deriving Efimov physics in the 3-body sector using dimensional regularization.

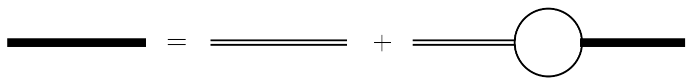

FIG. 1: Diagrammatic equation for the complete diatom propagator iD.

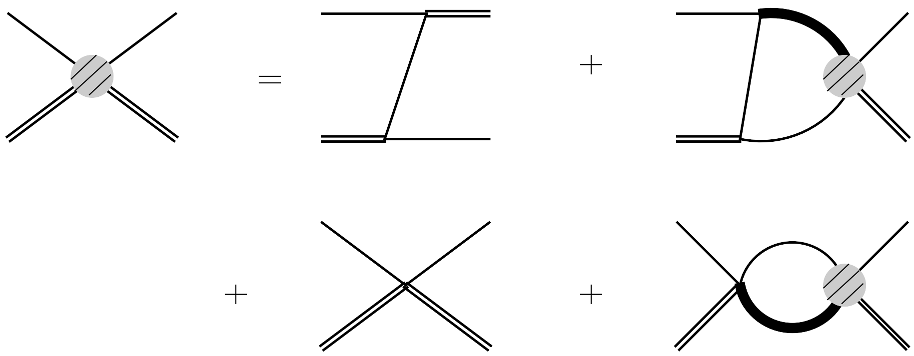

FIG. 2: Integral equation for the atom-diatom amplitude $\mathcal{A}$.

complex. Thus the RG flow for $\hat{g}_2$ and $\hat{g}_3$ does not have a UV fixed point. Instead the coupling constants flow in the ultraviolet to a limit cycle given by

$$\hat{g}_2(\Lambda) = -1, \quad \hat{g}_3(\Lambda) = -\frac{\cos\left[s_0 \log\left(\Lambda/\Lambda_*\right) + \arctan s_0\right]}{\cos\left[s_0 \log\left(\Lambda/\Lambda_*\right) - \arctan s_0\right]}, \tag{9}$$

where Λ_* is a constant momentum scale.

A. Atom-diatom amplitude

In Ref. [13, 14], Bedaque, Hammer, and van Kolck [BHvK] developed an effective field theory (EFT) to calculate universal results for three-body observables for three identical bosons in the zero-range limit in three spatial dimensions. We will use this EFT to describe the system in a variable spatial dimension d. The fields in this EFT are the dynamical atom field ψ and an auxiliary diatom field Δ. The BHvK Lagrangian density is

$$\mathcal{L}_{\text{BHvK}} = \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2\right)\psi + \frac{g_2}{4}\Delta^\dagger\Delta - \frac{g_2}{4}\left(\Delta^\dagger\psi^2 + \psi^{\dagger 2}\Delta\right) - \frac{g_3}{36}\left(\Delta\psi\right)^\dagger\left(\Delta\psi\right), \tag{10}$$

where g_2 and g_3 are bare two-body and three-body couplings. Using the equation of motion for Δ, we can eliminate the diatom field Δ to obtain the Lagrangian density

$$\mathcal{L} = \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2\right)\psi - \frac{g_2}{4}\frac{\left(\psi^\dagger\psi\right)^2}{1-\left(g_3/9g_2\right)\psi^\dagger\psi}. \tag{11}$$

Expanding the interaction term in powers of $\psi^\dagger\psi$ and truncating after the $\left(\psi^\dagger\psi\right)^3$ term, we recover the Lagrangian for identical bosons in Eq. (4). The additional operators $\left(\psi^\dagger\psi\right)^n$ with $n \geq 4$ from the expansion of the interaction potential in Eq. (11) are irrelevant operators at the UV fixed point $\hat{g}_2 = -1$ for the two-body coupling, so they can be ignored. Therefore, the quantum field theory described by the BHvK Lagrangian density in Eq. (10) is equivalent to the quantum field theory described by the Lagrangian density in Eq. (4).

The propagator of the auxiliary diatom field Δ is $4i/g_2$. It receives quantum corrections from a geometric series of bubble diagrams that can be obtained by iterating the diagrammatic equation in Fig. 1. The dependence of the three-body coupling g_3 on the UV cutoff Λ can be determined from the off-shell atom-diatom amplitude $\mathcal{A}$. The amplitude $\mathcal{A}$ satisfies the integral equation shown diagrammatically in Fig. 2. If $g_3 = 0$, the diagrams in the second line of Fig. 2 are zero, and the diagrammatic equation reduces to the Skorniakov-Ter-Martirosian (STM) integral equation [18]. In the limit $\Lambda \to \infty$, the solutions of the STM equation are not ultraviolet divergent, but they depend log-periodically on Λ. Requiring the log-periodic dependence to be cancelled by the diagrams in the second line of Fig. 2 determines the dependence of g_3 on Λ.

To obtain the RG equation for the three-body coupling g_3, we follow closely the path used in Ref. [4] to derive the beta function in Eq. (8), except that we carry out all the steps in dimension d instead of $d = 3$. The complete diatom propagator in d spatial dimensions including the quantum corrections is

$$iD\left(p_0, \boldsymbol{p}\right) = -i \frac{8(4\pi)^{d/2}}{\Gamma\left(\frac{2-d}{2}\right) g_2(\Lambda)^2} \left[1/\lambda - \left(-p_0 + p^2/4 - i\epsilon\right)^{(d-2)/2}\right]^{-1}. \tag{12}$$

All the dependence on the UV cutoff Λ is in the multiplicative factor $1/g_2^2$. In the unitary limit $\lambda \to \pm\infty$, the propagator in Eq. (12) has a pole in p_0 at the upper critical dimension $d = 4$. In the center-of-mass frame, the atom-diatom amplitude is a function of the incoming relative momentum $\boldsymbol{p}$, the outgoing relative momentum $\boldsymbol{k}$, and the total energy E. The s-wave atom-diatom amplitude $\mathcal{A}_s(p, k; E)$ can be obtained by averaging $\mathcal{A}(\boldsymbol{p}, \boldsymbol{k}; E)$ over the angle θ between $\boldsymbol{p}$ and $\boldsymbol{k}$ in d dimensions. To simplify the integral equation satisfied by $\mathcal{A}_s$, we also choose to multiply it by $4/g_2^2$:

$$\mathcal{A}_s(p, k; E) = \frac{4}{g_2^2} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{+1} d\cos\theta \, \left(1 - \cos^2\theta\right)^{(d-3)/2} \mathcal{A}(\boldsymbol{p}, \boldsymbol{k}; E). \tag{13}$$

The integral equation for $\mathcal{A}_s$ with a sharp UV cutoff Λ on the loop momentum in the center-of-mass frame is

$$\begin{aligned} \mathcal{A}_s\left(p, k; E\right) \; = \; & -\left[\frac{1}{E - p^2 - k^2 + i\epsilon} \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{d}{2} \end{matrix} \middle| \frac{p^2 k^2}{\left(E - p^2 - k^2 + i\epsilon\right)^2}\right) + \frac{\hat{G}(\Lambda)}{\Lambda^2}\right] \\ & - \frac{4\sin\left(\frac{d}{2}\pi\right)}{\pi} \int_0^\Lambda dq \, q^{d-1} \left[\frac{1}{E - p^2 - q^2 + i\epsilon} \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{d}{2} \end{matrix} \middle| \frac{p^2 q^2}{\left(E - p^2 - q^2 + i\epsilon\right)^2}\right) \right. \\ & \left. + \frac{\hat{G}(\Lambda)}{\Lambda^2}\right] \frac{\mathcal{A}_s\left(q, k; E\right)}{1/\lambda - \left(-E + 3q^2/4 - i\epsilon\right)^{(d-2)/2}}, \end{aligned} \tag{14}$$

where $\hat{G}(\Lambda)$ is a dimensionless three-body coupling defined by

$$\hat{G}(\Lambda) = \frac{\Lambda^2 g_3(\Lambda)}{9 g_2(\Lambda)^2}. \tag{15}$$

The dependence of the amplitude $\mathcal{A}_s$ on Λ has been suppressed in Eq. (14). The dependence of $\hat{G}$ on Λ can be determined by requiring the solutions to Eq. (14) to have well-behaved limits as $\Lambda \to \infty$ [13, 14]. Since $\hat{G}$ is the only coupling in the integral equation, the beta function for $\hat{G}$ can depend only on $\hat{G}$.

B. Asymptotic solutions

In the limit $\Lambda \to \infty$, the integral equation in Eq. (14) for the s-wave atom-diatom amplitude $\mathcal{A}_s(p, k, E)$ has asymptotic solutions for large p that have power-law dependence p^{s-1}. The possible exponents of p can be determined by neglecting the inhomogeneous terms in Eq. (14), replacing $\mathcal{A}_s(p, k, E)$ by Ap^{s-1}, setting E, $\widehat{G/\Lambda^2}$, and $1/\lambda$ to 0 inside the integral over q, and taking the upper endpoint Λ of the integral to ∞ [4]. After making the change of variable $q = xp$, the dependence on p drops out and the integral equation in Eq. (14) reduces to

$$1 = -\left(\frac{4}{3}\right)^{\frac{d-2}{2}} \frac{4\sin\left(\frac{d}{2}\pi\right)}{\pi} \int_0^\infty dx \, \frac{x^s}{1 + x^2} \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{d}{2} \end{matrix} \middle| \frac{x^2}{\left(1 + x^2\right)^2}\right). \tag{16}$$

The integral can be evaluated analytically by inserting the power series definition of the hypergeometric function ${}_2F_1$. After integrating the individual terms of the power series over x and then resumming the series, the resulting equation for s is

$$2\sin\left(\frac{d}{2}\pi\right) {}_2F_1\left(\begin{matrix} \frac{d-1+s}{2}, \frac{d-1-s}{2} \\ \frac{d}{2} \end{matrix} \middle| \frac{1}{4}\right) + \cos\left(\frac{s}{2}\pi\right) = 0. \tag{17}$$

The equation is invariant under $s \to -s$, so it is an equation for s^2.

The equation in Eq. (17) has infinitely many branches of solutions for s^2 as a function of the dimension d. The lowest branch of solutions is shown in Fig. 3. The value of s^2 decreases from 1 at the endpoints $d = 2, 4$

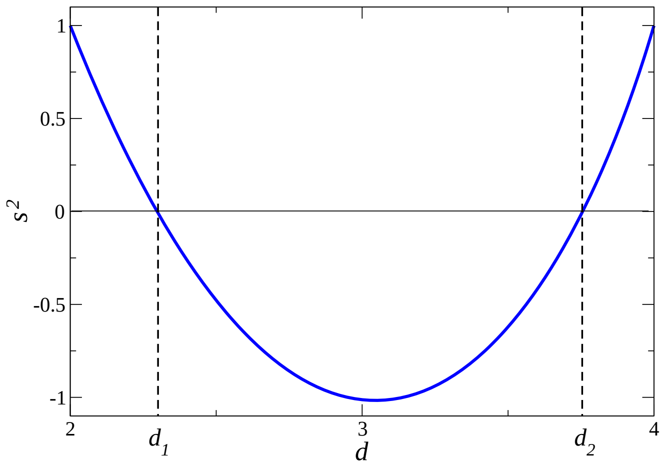

FIG. 3: The lowest branch of solutions of Eq. (17) for s^2 as a function of the dimension d. The solution for s^2 is negative in the interval $d_1 < d < d_2$ between the two vertical dotted lines.

to a minimum of -1.016 at $d = 3.04$. The higher branches of solutions have $s^2 > 9$. The only branch that is physically relevant is the lowest branch. The higher branches are irrelevant, because the integral over x in Eq. (16) is convergent at both the endpoints 0 and ∞ only if the exponent s satisfies the condition $-1 < \mathrm{Re}\,(s) < 1$. There are two critical dimensions at which Eq. (17) is satisfied by $s^2 = 0$:

$$d_1 = 2.30, \quad d_2 = 3.76. \tag{18}$$

These lower and upper critical dimensions for the Efimov effect were first derived in Ref. [10]. We will focus first on the regions $2 < d < d_1$ and $d_2 < d < 4$ and then on the region $d_1 < d < d_2$.

C. Conformality

In the regions $2 < d < d_1$ and $d_2 < d < 4$, the lowest branch of solutions of Eq. (17) for the exponent of p are $\pm s - 1$, where s is real and positive. The value of s depends on d, decreasing from 1 to 0 as d goes from 2 to d_1 and as d goes from 4 to d_2. The most general asymptotic solution of the integral equation in Eq. (14) as $p \to \infty$ is given by

$$\mathcal{A}_s\,(p, k; E) \longrightarrow A_+ p^{s-1} + A_- p^{-s-1}, \tag{19}$$

where $A_\pm$ are two constants that may depend on k, E, and λ. The inhomogeneous term in the integral equation determines one of the constants and the three-body coupling determines the other constant.

The Λ dependence of the dimensionless coupling $\hat{G}(\Lambda)$ can be obtained by inserting the asymptotic solution in Eq. (19) into the integral equation in Eq. (14) and requiring the integral over q to be independent of Λ [4]. The Λ dependence of the integral is proportional to

$$\begin{aligned}\int^\Lambda dq \left(\frac{1}{q^2} - \frac{\hat{G}(\Lambda)}{\Lambda^2} \right) \left(A_+ q^s + A_- q^{-s} \right) \\ = \left(\frac{A_+ \Lambda^{s-1}}{s-1} + \frac{A_- \Lambda^{-s-1}}{-s-1} \right) - \hat{G}(\Lambda) \left(\frac{A_+ \Lambda^{s-1}}{s+1} + \frac{A_- \Lambda^{-s-1}}{-s+1} \right).\end{aligned} \tag{20}$$

Setting this equal to 0 gives an expression for $\hat{G}(\Lambda)$ that depends on the ratio A_+/A_-. It can be simplified by setting $A_+/A_- = \Lambda_*^{-2s}$, where Λ_* is a constant momentum scale. The resulting expression for $\hat{G}(\Lambda)$ is

$$\hat{G}(\Lambda) = -\frac{\cosh\left[s \log\left(\Lambda/\Lambda_*\right) + \mathrm{arctanh}\, s\right]}{\cosh\left[s \log\left(\Lambda/\Lambda_*\right) - \mathrm{arctanh}\, s\right]}. \tag{21}$$

The RG equation for $\hat{G}(\Lambda)$ can be obtained by differentiating both sides of Eq. (21) with respect to $\log \Lambda$ and then eliminating Λ in favor of $\hat{G}(\Lambda)$:

$$\Lambda \frac{d}{d\Lambda} \hat{G} = \frac{1-s^2}{2} + \left(1+s^2\right) \hat{G} + \frac{1-s^2}{2} \hat{G}^2, \tag{22}$$

where s^2 is the function of d defined by Eq. (17). The beta function defined by the right side of Eq. (22) is a quadratic polynomial in $\hat{G}$. The RG flow for $\hat{G}$ has two fixed points where the beta function is zero: a UV fixed point $\hat{G}_+$ and an IR fixed point $\hat{G}_-$ given by

$$\hat{G}_\pm = -\frac{1 \pm s}{1 \mp s}. \tag{23}$$

As d increases towards d_1 or as d decreases towards d_2, s approaches 0 so the two fixed points $\hat{G}_-$ and $\hat{G}_+$ both approach -1. At unitarity, the two-body coupling is at its UV fixed point $\hat{g}_2 = -1$. If $\hat{G}$ is also at its UV fixed point G_+, the three-body sector is scale invariant and therefore presumably also conformally invariant.

D. Conformality lost

In the region $d_1 < d < d_2$, the lowest branch of solutions of Eq. (17) for the exponent of p are $\pm i s_0 - 1$, where s_0 is real and positive. The value of s_0 depends on d, increasing from 0 to 1.00812 and then decreasing to 0 as d increases from d_1 to 3.04 and then to d_2. At $d = 3$, its value is $s_0 = 1.00624$. The most general asymptotic solution of the integral equation in Eq. (14) as $p \to \infty$ is given by

$$\mathcal{A}_s\left(p, k; E\right) \longrightarrow A_+ p^{i s_0 - 1} + A_- p^{-i s_0 - 1}, \tag{24}$$

where $A_\pm$ are two constants that may depend on k, E, and λ.

Using the arguments in Section III C, one can again determine the Λ dependence of the dimensionless parameter $\hat{G}(\Lambda)$. Upon inserting the asymptotic solution in Eq. (24) into the integral equation in Eq. (14), the dependence of the integral on Λ has the form in Eq. (20) with s replaced by $i s_0$. The dependence on Λ cancels if $\hat{G}(\Lambda)$ has the form

$$\hat{G}(\Lambda) = -\frac{\cos\left[s_0 \log\left(\Lambda/\Lambda_*\right) + \arctan s_0\right]}{\cos\left[s_0 \log\left(\Lambda/\Lambda_*\right) - \arctan s_0\right]}, \tag{25}$$

where Λ_* is a constant momentum scale. This expression is a log-periodic function of Λ with period e^{π/s_0}, so the RG flow of $\hat{G}$ is a limit cycle. The RG equation for $\hat{G}$ can be obtained by differentiating both sides of Eq. (25) with respect to $\log \Lambda$ and then eliminating Λ in favor of $\hat{G}(\Lambda)$. The result is just the RG equation for $\hat{G}$ in Eq. (22) with $s^2 = -s_0^2$. Since $s^2 < 0$, the zeros of this beta function are complex valued. At unitarity, the two-body coupling is at its UV fixed point $\hat{g}_2 = -1$, so the two-body sector is scale invariant. However the three-body sector is not scale invariant. Instead it has discrete scale invariance with discrete scaling factor e^{π/s_0}.

We can define a dimensionless three-body coupling $\hat{g}_3$ by multiplying g_3 with the power of UV cutoff Λ required by dimensional analysis:

$$\hat{g}_3\left(\Lambda\right) = \frac{1}{\left[3(d-2)(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)\right]^2} \Lambda^{2d-2} g_3\left(\Lambda\right). \tag{26}$$

The d-dependent prefactor has been introduced in order to simplify the RG equation for $\hat{g}_3$ below. The RG equation for $\hat{g}_3 = \hat{g}_2^2 \, \hat{G}$ can be derived by using the RG equations for $\hat{G}(\Lambda)$ in Eq. (22) and for $\hat{g}_2$ in Eq. (7):

$$\Lambda \frac{d}{d\Lambda} \hat{g}_3 = \frac{1-s^2}{2} \left(\hat{g}_2^2 + \frac{\hat{g}_3^2}{\hat{g}_2^2} \right) + \left(2d - 3 + s^2 + 2(d-2)\hat{g}_2\right) \hat{g}_3, \tag{27}$$

where s^2 is the function of d defined by Eq. (17). Upon setting $d = 3$ and $s^2 = -s_0^2$, where $s_0 = 1.00624$, this reduces to the RG equation in Eq. (8) derived in Ref. [4].

IV. SUMMARY

In Ref. [9], Kaplan, Lee, Son, and Stephanov revealed a general mechanism for the loss of conformal invariance in a system. Near the point where conformality is lost, the beta function of an appropriate coupling is a quadratic function of the coupling as in Eq. (1). As an external parameter is tuned to the critical value where conformality is lost, an IR fixed point and a UV fixed point of the coupling merge together and then disappear.

In this work, we have shown explicitly the mechanism for loss of conformal invariance associated with Efimov physics in the case of identical bosons at unitarity with variable spatial dimension d. The beta function for the dimensionless three-body coupling $\hat{G}$ defined in Eq. (15) is the quadratic polynomial in $\hat{G}$ given in Eq. (22). Its coefficients depend on d through s^2, which is a solution to Eq. (17). The critical dimensions at which conformality is lost are $d_1 = 2.30$ and $d_2 = 3.76$ [10]. In the regions $2 < d < d_1$ and $d_2 < d < 4$, the beta function for $\hat{G}$ has real zeros and its RG flow has IR and UV fixed points. As d increases through d_1 or decreases through d_2, the IR and UV fixed points approach each other, merge together, and then disappear into the complex plane. In the region $d_1 < d < d_2$ where conformality is lost, the beta function for $\hat{G}$ has complex zeros and its RG flow is a limit cycle.

In the case of identical bosons A and an additional species B of fermion or boson, the critical dimensions for the Efimov effect in the AAB system depend on the mass ratio m_A/m_B [19]. The dependence of the critical dimensions on m_A/m_B is surprisingly weak, and they reduce to d_1 and d_2 for identical bosons at $m_A/m_B = 1$. The beta function for the 3-body coupling that is associated with the loss of conformality could presumably be derived in a similar way to the method we used for identical bosons.

In the case of two species A and B of fermions in 3 spatial dimensions, the ratio m_A/m_B of the masses plays the role of an external parameter that controls the loss of conformality. The Efimov effect occurs in the p-wave channel at unitarity for two heavy fermions and one light fermion when the mass ratio m_A/m_B exceeds 13.6 [20]. It would be interesting to provide an RG perspective on the loss of conformality associated with Efimov physics in this system by determining the beta function for the appropriate 3-body coupling.

The unitary Fermi gas, which consists of fermions with two spin states and infinitely large scattering length in 3 spatial dimensions, is a challenging many-body physics problem. The most successful analytic approach to this problem has been through interpolation in the number of spatial dimensions d using epsilon expansions around the critical dimensions $d = 2$ and $d = 4$ [11]. The ground state energy for the unitary Fermi gas has been calculated using next-to-next-to-leading-order epsilon expansions around both 2 and 4 [11, 21, 22]. Interpolation in d is essential because of the poor convergence properties of the epsilon expansions around 2 and 4. The unitary Bose gas, which consists of identical bosons with infinitely large scattering length in 3 spatial dimensions, is an even more challenging many-body physics problem because of the Efimov effect. The epsilon expansion around the upper critical dimension 4 has been applied to the unitary Bose gas [23]. However interpolation in d using epsilon expansions around both 2 and 4 seems to be more promising. The understanding of the RG flow near the critical dimensions 2.30 and 3.76 for the loss of conformality that we have presented in this paper could be essential for applying interpolation in d to the unitary Bose gas in 3 dimensions.

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[1] V. Efimov, Energy levels arising from resonant two-body forces in a three-body system, Phys. Lett. B **33**, 563 (1970).
[2] K.G. Wilson, The renormalization group and critical phenomena, Rev. Mod. Phys. **55**, 583 (1983).
[3] K.G. Wilson, The renormalization group and strong interactions, Phys. Rev. D. **3**, 1818 (1971).
[4] E. Braaten and H.-W. Hammer, Universality in few-body systems with large scattering length, Phys. Rep. **428**, 259 (2006) [cond-mat/0410417].
[5] A.B. Zamolodchikov, "Irreversibility" of the flux of the renormalization group in a 2D field theory, JETP Lett. **43**, 730 (1986).
[6] Y. Nakayama, Scale invariance vs conformal invariance, Phys. Rept. **569**, 1 (2015) [arXiv:1302.0884].
[7] C.R. Hagen, Scale and conformal transformations in Galilean-covariant field theory, Phys. Rev. D. **5**, 377 (1972).
[8] U. Niederer, The maximal kinematical invariance group of the free Schrödinger equation, Helv. Phys. Acta. **45**, 802 (1972).
[9] D.B. Kaplan, J.W. Lee, D.T. Son, and M.A. Stephanov, Conformality lost, Phys. Rev. D. **80**, 125005 (2009) [arXiv:0905.4752].
[10] E. Nielsen, D.V. Fedorov, A.S. Jensen, and E. Garrido, The three-body problem with short-range interactions, Phys. Rep. **347**, 373 (2001).
[11] Y. Nishida and D.T. Son, Fermi gas near unitarity around four and two spatial dimensions, Phys. Rev. A **75**, 063617 (2007) [cond-mat/0607835].

[12] Z. Nussinov and S. Nussinov, Triviality of the BCS-BEC crossover in extended dimensions: Implications for the ground state energy, Phys. Rev. A. **74**, 053622 (2006) [cond-mat/0410597].
[13] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, Renormalization of the three-body system with short-range interactions, Phys. Rev. Lett. **82**, 463 (1999) [nucl-th/9809025].
[14] P.F. Bedaque, H.-W. Hammer, and U. van Kolck, The three-boson system with short range interactions, Nucl. Phys. A **646**, 444 (1999) [nucl-th/9811046].
[15] D.B. Kaplan, M.J. Savage, and M.B. Wise, A New expansion for nucleon-nucleon interactions, Phys. Lett. B **424**, 390 (1998) [nucl-th/9801034].
[16] V. Efimov, Weakly bound states of three resonantly interacting particles, Sov. J. Nucl. Phys. **12**, 589 (1971).
[17] V. Efimov, Low-energy properties of three resonantly interacting particles, Sov. J. Nucl. Phys. **29**, 546 (1979).
[18] G.V. Skorniakov and K.A. Ter-Martirosian, Three body problem for short range forces 1. Scattering of low energy neutrons by deuterons, Sov. Phys. JETP **4**, 648 (1957).
[19] D.S. Rosa, T. Frederico, G. Krein, and M. T. Yamashita, Efimov effect in D spatial dimensions in AAB systems, Phys. Rev. A **97**, 050701 (2018) [arXiv:1707.06616].
[20] V. Efimov, Energy levels of three resonantly interacting particles, Nucl. Phys. A **201**, 157 (1973).
[21] P.B. Arnold, J.E. Drut, and D.T. Son, Next-to-next-to-leading-order epsilon expansion for a Fermi gas at infinite scattering length, Phys. Rev. A **75**, 043605 (2007) [cond-mat/0608477].
[22] Y. Nishida, Ground-state energy of the unitary Fermi gas from the epsilon expansion, Phys. Rev. A **79**, 013627 (2009) [arXiv:0808.3826].
[23] S.-J. Jiang, W.-M. Liu, G.W. Semenoff, and F. Zhou, Universal Bose gases near resonance: a rigorous solution, Phys. Rev. A **89**, 033614 (2014) [arXiv:1307.4263].