Effective-field theory for one-dimensional nonrelativistic particles with contact interaction

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We consider a field theory describing interacting nonrelativistic particles of two types, which map to each other under time reversal, with point-like interaction. We identify a new type of interaction which depends on the relative velocity between the particles. We compute the renormalization group running of the coupling constants and find a fixed point and a fixed line. We show that the scattering amplitudes can be expressed in terms of three parameters. The result matches with a quantum mechanical analysis and represents the most general point-like interaction consistent with unitarity and time reversal invariance.

I.   INTRODUCTION

Effective field theory (EFT) [1, 2] has become a useful tool to investigate nonrelativistic systems. Starting fromn 1990s, EFT has been applied to few-nucleon systems in an attempt to achieve a simple description of low-energy nuclear physics [3]. It has been soon noticed that EFT is useful in the simple quantum-mechanical problem of a few nonrelativistic particles interacting through a pairwise potential [4–6]. Effective field theory provides a framework to parametrize low-energy properties of such systems in terms of a small number of free parameters appearing in the Lagrangian. The technique is very useful in the case of fermions at unitarity [7–9], where field-theory techniques have given additional insights to quantum mechanical problems. An example is the analysis of the Efimov effect using renormalization group [6, 10, 11].

In this paper, we apply effective field theory to the problem of nonrelativistic particles in one spatial dimension. The reason for us to revisit this seemingly trivial problem is the recent suggestion [12, 13] of a new Galilean-invariant interaction between nonrelativistic particles. In one spatial dimension, such an interaction is a pair-wise local interaction with strength proportional to the relative velocity of the participating particles.

We are also motivated by the possibility of finding new one-dimensional systems with nonrelativistic conformal symmetry [14], similar to fermions at unitarity in three dimensions and anyons in two dimensions. In one spatial dimension, the only theory found so far with this symmetry is the theory with four-particle contact interaction [15]. In field theory, conformal field theory emerges at RG fixed points, thus RG is the most convenient method to find such theories. Another motivation is to give an effective field theory interpretation of an extensive mathematical literature on self-adjoint extension of the 1D Hamiltonian (see Refs. [16, 17] and references therein).

The structure of the paper is as follows. In Sec. II, we consider a theory of two distinct species of particles which map to each other under parity. We construct a Lagrangian consistent with Galilean invariance containing only up to two derivatives. In Sec. III we evaluate the full scattering amplitudes (the transmission and reflection amplitudes). Then in Sec. IV we develop a renormalization group treatment of the two-body sector of the theory and find that the theory has a fixed point and a fixed line. We find the general form of scattering amplitudes, and make contact with quantum-mechanical calculations. In Sec. V we describe the physics at the fixed points and in some special cases of the RG flow. Section VI contains concluding remarks. The Appendix contains a purely quantum mechanical treatment of the problem.

II.   LAGRANGIAN

We consider two species of nonrelativistic particles, ψ_1 and ψ_2, living in one spatial dimension and interacting though a point-like interaction. The Lagrangian that we consider

is

$$\mathcal{L} = \sum_{a=1}^{2} \left(i\psi_a^\dagger \partial_t \psi_a - \frac{1}{2m_a} \partial_x \psi_a^\dagger \partial_x \psi_a \right) - \lambda \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2$$

$$+ \frac{i}{2}a(m_1 \psi_1^\dagger \psi_1 \psi_2^\dagger \overset{\leftrightarrow}{\partial}_x \psi_2 - m_2 \psi_2^\dagger \psi_2 \psi_1^\dagger \overset{\leftrightarrow}{\partial}_x \psi_1) - \frac{c}{4}(\psi_2^\dagger \overset{\leftrightarrow}{\partial}_x \psi_1^\dagger)(\psi_2 \overset{\leftrightarrow}{\partial}_x \psi_1). \quad (1)$$

where $\psi^\dagger \overset{\leftrightarrow}{\partial}_x \psi \equiv \psi^\dagger \partial_x \psi - \partial_x \psi^\dagger \psi$.

The theory does not generally have a parity symmetry $x \to -x$. However when $m_1 = m_2$, it is invariant under the combination of $x \to -x$ and $\psi_1 \leftrightarrow \psi_2$. The same is true for the combination of time reversal and $\psi_1 \leftrightarrow \psi_2$. It is easy to see also that the theory has Galilean symmetry. The most nontrivial check is for the interaction term proportional to a, which, in the first-quantized language, can be written as $m_1 m_2 a \{v - v', \, \delta(x - x')\}$, where v and v' are the velocities of the interacting particles. This interaction is obviously invariant under Galilean boosts.

It is useful to develop a power-counting scheme in our theory. In dealing with a nonrelativistic Lagrangian like (1), it is most natural to treat the particle mass m as dimensionless (in fact, soon we will set $m = 1$ in our calculation); hence if momentum has dimension one, then energy has dimension 2. From this naive power counting, λ has dimension 1 (i.e., that of momentum), a is dimensionless, and c has dimension -1. One may ask why one includes in the Lagrangian an irrelevant interaction. The reason, as we see in the next section, is that there exists a nonperturbative fixed point for c, in exactly the same manner as the naively irrelevant four-point interaction in 3D nonrelativistic theory has a UV fixed point. As we also see in the next section, the Feynman diagrams in theory (1) contains only one linear divergence, and this special property is not preserved if other interaction terms are added int the Lagrangian.
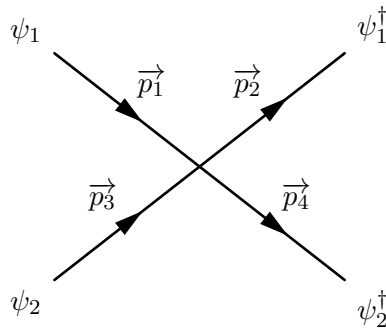
III. SCATTERING AMPLITUDES

FIG. 1. Four-particle interaction vertex

To compute the scattering amplitude, we first write down the vertex presented in Fig. 1,

$$V(p_1, p_3; p_2, p_4) = -i\lambda - \frac{i}{2}a[m_1(p_3 + p_4) - m_2(p_1 + p_2)] - \frac{i}{4}c(p_1 - p_3)(p_2 - p_4). \quad (2)$$

In further calculations we will set $m_1 = m_2 = 1$.

The total scattering amplitude is given by the sum of diagrams such as in Figure 2, for all possible numbers of loops. In the center-of-momentum frame, the incoming particles have momenta $\pm p$, and the total energy is E. If the particles are on-shell, then $E = p^2$. A diagram with n loops (and therefore $n + 1$ vertices, see Figure 2), contributes to the total amplitude a term of the form

$$A_{n+1} = \int \prod_{i=1}^{n} \frac{d\omega_i \, dq_i}{(2\pi)^2} \prod_{i=1}^{n} \frac{i}{\left(\frac{E}{2} + \omega_i - \frac{q_i^2}{2} + i\epsilon\right)} \frac{i}{\left(\frac{E}{2} - \omega_i - \frac{q_i^2}{2} + i\epsilon\right)} \prod_{i=1}^{n+1} V_i V_{n+1}, \quad (3)$$

where $V_i = V(q_{i-1}, q_i)$ is the vertex factor of the ith vertex, and where $q_0 \equiv p$ and $q_{n+1} = k$.
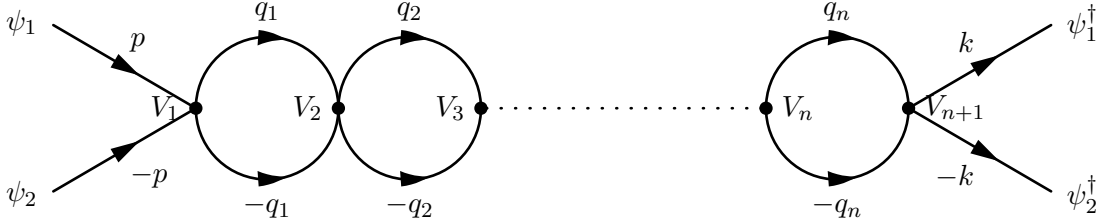

FIG. 2. Diagram with n loops and $n + 1$ vertices.

It is clear from Fig. 2 and the structure of the interaction vertex that in the product of V's, each q_i can only appear to the power of 0, 1, or 2. Thus, we only need to compute the integral

$$B_n = \int \frac{d\omega \, dq}{(2\pi)^2} \frac{i}{\frac{E}{2} + \omega - \frac{q^2}{2} + i\epsilon} \frac{i}{\frac{E}{2} - \omega - \frac{q^2}{2} + i\epsilon} q^n = \int \frac{dq}{2\pi} \frac{iq^n}{E - q^2 + i\epsilon}. \quad (4)$$

Obviously $B_1 = 0$. The integral for B_0 converges

$$B_0 = \frac{1}{2\sqrt{E}}, \quad (5)$$

and that for B_2 diverges linearly. Using a sharp momentum cutoff $|q| < \Lambda$, the latter is

$$B_2 = -\frac{i\Lambda}{\pi} + \frac{\sqrt{E}}{2}. \quad (6)$$

One may question whether one needs to use the same sharp cutoff in evaluating B_0, which would change Eq. (5) by $O(1/\Lambda)$. This is in fact not necessary: one can design a cutoff function so that Eqs. (5) and (6) are valid to any finite inverse powers of Λ.

Therefore, only combinations of vertex factors $V_1 V_2 ... V_{n+1}$ that have q_i to the power of 0 or 2 for all i will produce nonzero contributions to the total scattering amplitude. That

is, if one particular vertex involves a factor of q_i, then the next vertex has to have the same factor, and the combination of the two vertices results in a factor of B_2 in the scattering amplitude term corresponding to that particular diagram.

The full scattering amplitude satisfies the recursion relation

$$A(E; p, k) = V(p, k) + \int \frac{d\omega \, dq}{(2\pi)^2} V(p, q) \frac{i}{\frac{E}{2} + \omega - \frac{q^2}{2} + i\epsilon} \frac{i}{\frac{E}{2} - \omega - \frac{q^2}{2} + i\epsilon} A(E; p, k), \quad (7)$$

where

$$V(p, k) \equiv V(p, -p; k, -k) = -i\lambda + ia(p + k) - icpk. \quad (8)$$

To solve this equation, we note that the external momenta p and k can appear, through the left-most and right-most vertices, in powers 0 or 1 in the scattering amplitude. We thus can make the following ansatz

$$A(E; p, k) = A^0(E) + A^1(E)p + A^2(E)k + A^3(E)pk. \quad (9)$$

Here the A^i, $i = 0, 1, 2, 3$ are coefficients that depend on E, but not on p and k. Substituting the ansatz (9) into the recursion relation (7), one finds

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} -i\lambda \\ ia \\ ia \\ -ic \end{pmatrix} + \begin{pmatrix} -i\lambda B_0 & 0 & iaB_2 & 0 \\ 0 & -i\lambda B_0 & 0 & iaB_2 \\ iaB_0 & 0 & -icB_2 & 0 \\ 0 & iaB_0 & 0 & -icB_2 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}. \quad (10)$$

The solution to this linear system of equations is

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \frac{1}{(1 + i\lambda B_0)(1 + icB_2) + a^2 B_0 B_2} \begin{pmatrix} -i\lambda + (c\lambda - a^2)B_2 \\ ia \\ ia \\ -ic + (c\lambda - a^2)B_0 \end{pmatrix}. \quad (11)$$

Therefore,

$$A(p, k) = \frac{-i\lambda + (c\lambda - a^2)(B_2 + B_0 pk) + ia(p + k) - icpk}{(1 + i\lambda B_0)(1 + icB_2) + a^2 B_0 B_2}. \quad (12)$$

For forward scattering, $k = p$, and so the forward scattering amplitude is

$$A_F = \frac{-i\lambda + (c\lambda - a^2)(B_2 + B_0 p^2) + 2iap - icp^2}{(1 + i\lambda B_0)(1 + icB_2) + a^2 B_0 B_2}. \quad (13)$$

Backward scattering occurs when $k = -p$, and so the backward scattering amplitude is

$$A_B = \frac{-i\lambda + (c\lambda - a^2)(B_2 - B_0 p^2) + icp^2}{(1 + i\lambda B_0)(1 + icB_2) + a^2 B_0 B_2}. \quad (14)$$

In these equation B_0 and B_2 should be evaluated according to Eqs. (5) and (6) at the onshell value of the incoming energy $E = p^2$.

IV. RENORMALIZATION GROUP ANALYSIS

The amplitudes computed in the previous section depends on the coupling constant λ, a, c, and the momentum cutoff Λ. We now write down the renormalization group equation for the coupling constants, demanding that physical quantities (in this case, the scattering amplitudes) are invariant when one simultaneously changes the cutoff and the couplings. Introducing the dimensionless coupling constants $\tilde{\lambda} = \Lambda^{-1}\lambda$, $\tilde{a} = a$, $\tilde{c} = \Lambda c$, the renormalization group equations can be obtained from the condition

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_{\tilde{\lambda}} \frac{\partial}{\partial \tilde{\lambda}} + \beta_{\tilde{a}} \frac{\partial}{\partial \tilde{a}} + \beta_{\tilde{c}} \frac{\partial}{\partial \tilde{c}} \right) A_{F,B}(\Lambda, \tilde{\lambda}, \tilde{a}, \tilde{c}) = 0. \tag{15}$$

From this equation, which should be satisfies for all p, one obtains the beta-functions

$$\beta_{\tilde{\lambda}} = -\tilde{\lambda} + \frac{\tilde{a}^2}{\pi}, \tag{16a}$$

$$\beta_{\tilde{a}} = \frac{\tilde{a}\tilde{c}}{\pi}, \tag{16b}$$

$$\beta_{\tilde{c}} = \frac{\tilde{c}^2}{\pi} + \tilde{c}. \tag{16c}$$

Since the scattering amplitudes are exact, the beta functions are also exact. The fact that Eq. (15) can be satisfied for all p means that no additional interaction is generated by the RG apart from the ones that have been included in the Lagrangian.

Setting the beta functions to zero, we find that our theory has an isolated nontrivial fixed point at $(\tilde{\lambda}, \tilde{a}, \tilde{c}) = (0, 0, -\pi)$, as well as a line of fixed points at $(\tilde{\lambda}, \tilde{a}, \tilde{c}) = (\frac{1}{\pi}\tilde{a}^2, \tilde{a}, 0)$. The isolated fixed points have three relevant directions, while points on the fixed line have one relevant, one irrelevant and one marginal directions.

The most general solution to the RG equation $\Lambda(\partial g_i/\partial \Lambda) = \beta_{g_i}$ with $g_i = \tilde{c}, \tilde{a}, \tilde{\lambda}$ can be written as

$$\tilde{c}(\Lambda) = -\frac{2\pi\Lambda}{2\Lambda + \pi\mu}, \tag{17a}$$

$$\tilde{a}(\Lambda) = \frac{2\pi\alpha\mu}{2\Lambda + \pi\mu}, \tag{17b}$$

$$\tilde{\lambda}(\Lambda) = \frac{4\alpha^2\mu}{2\Lambda + \pi\mu} + \frac{2\gamma}{\Lambda}. \tag{17c}$$

where μ, α, and γ are integration constants. Note that μ and γ have dimension of momentum while α is dimensionless.

When $\gamma = 0$ the above flow interpolate between two fixed points: the isolated fixed point in the UV and a point on the fixed line in the IR. If $\gamma \neq 0$, in the IR λ blows up to $\pm\infty$ depending on the sign of γ. An interesting regime is $\mu \gg \gamma$. Then one first flows from the isolated fixed point ($\Lambda \gg \mu$) to a point on the fixed line with $\tilde{a} = 2\alpha$, lingers there for Λ in the interval $\mu \gg \Lambda \gg \gamma$, and then $\tilde{\lambda} \to \pm\infty$ when Λ drops below γ.

The invariance of the scattering amplitudes with respect to variation of the cutoff scale Λ implies that these amplitudes can be expressed completely in terms of the integration constants μ, α, and γ. In other words, the short-range interaction can be completely characterized by the parameters μ, α, and γ. In terms of these parameters, the forward and backward scattering amplitudes are

$$A_F = -2p\,\frac{p^2 + 2(\alpha\mu + i\alpha^2\mu + i\gamma)p - \gamma\mu}{p^2 + i(\mu + \alpha^2\mu + \gamma)p - \gamma\mu}\,, \tag{18a}$$

$$A_B = 2p\,\frac{p^2 + \gamma\mu}{p^2 + i(\mu + \alpha^2\mu + \gamma)p - \gamma\mu}\,, \tag{18b}$$

which corresponds to the transmission and reflection coefficients:

$$T = 1 + \frac{A_F}{2p} = \frac{i[(1 + i\alpha)^2\mu - \gamma]p}{p^2 + i(\mu + \alpha^2\mu + \gamma)p - \gamma\mu}\,, \tag{19a}$$

$$R = \frac{A_B}{2p} = \frac{p^2 + \gamma\mu}{p^2 + i(\mu + \alpha^2\mu + \gamma)p - \gamma\mu}\,. \tag{19b}$$

This coincides with the form of the scattering amplitudes derived in the Appendix, Eqs. (A7), by solving a quantum-mechanical problem of scattering from a point-like potential invariant under the combination of time reversal and particle exchange. As explained in the Appendix, such a potential is characterized by three complex numbers possessing the same phase. In our case, the three complex numbers are

$$\mathcal{A} = \frac{(1 + \alpha^2)\mu + \gamma}{(1 + i\alpha)^2\mu - \gamma}\,, \tag{20}$$

$$\mathcal{B} = \frac{2}{(1 + i\alpha)^2\mu - \gamma}\,, \tag{21}$$

$$\mathcal{C} = \frac{2\gamma\mu}{(1 + i\alpha)^2\mu - \gamma}\,. \tag{22}$$

They satisfy the condition $|\mathcal{A}|^2 - |\mathcal{B}||\mathcal{C}| = 1$, required by charge conservation and time reversal.

V. SPECIAL CASES

We now investigate the behavior of the scattering amplitudes in special cases.

A. Generic infrared behavior

Without any fine tuning, the behavior of the scattering amplitudes in the IR can be obtained by setting $p \to 0$ in Eqs. (19):

$$T = 0, \quad R = -1, \tag{23}$$

corresponding to total reflection. The minus sign in R can be understood by putting it into correspondence to the wave function of the 1D scattering problem

$$\psi(x) \sim \theta(-x) \sin kx. \tag{24}$$

where $\theta(x)$ is the step function. Such a wave function corresponds to scattering off a hard wall. It is also the behavior of the scattering amplitude in the IR for scattering on a delta-function potential (see below)

B. Delta-function potential

The familiar problem of scattering on a delta-function potential corresponds to $c = a = 0$. In this case the transmission and reflection amplitudes can be obtained from Eqs. (13) and (14):

$$T = \frac{2p}{2p + i\lambda}, \qquad R = -\frac{i\lambda}{2p + i\lambda}. \tag{25}$$

In the infrared $p \to 0$ we reproduce reflection on a hard wall, $T = 0$ and $R = -1$.

C. Fixed points

Now we consider the behavior of the scattering amplitudes when the coupling constants are at one of the fixed points. Scale invariance dictates, then, that the scattering amplitudes are momentum-independent.

Consider the isolated fixed point. We can approach the fixed point by setting $\gamma = 0$ and then taking the limit $p \gg \mu$. Then

$$R = 1, \qquad T = 0. \tag{26}$$

At the fixed point we have total reflection but, in contrast to the case of a hard-wall potential, the phase reflection amplitude is 1 instead of -1.

Now consider the system at a fixed point along the fixed line. Again we set $\gamma = 0$ but now $p \ll \mu$. One finds

$$T = \frac{1 + i\alpha}{1 - i\alpha}, \qquad R = 0, \tag{27}$$

which implies total transmission. The transmission amplitude, however, has a nonzero phase shift, $T = e^{i\delta}$, with

$$\delta = 2 \arctan \alpha. \tag{28}$$

D. Special RG flows

One can investigate the whole flow from the isolated fixed point to a point on the fixed line. Such a flow requires fine-tuning γ to 0. One finds then

$$T = \frac{i(1+i\alpha)^2\mu}{p+i(1+\alpha^2)\mu} = \frac{i(1+\alpha^2)\mu}{p+i(1+\alpha^2)\mu}e^{i\delta}, \qquad R = \frac{p}{p+i(1+\alpha^2)\mu}. \tag{29}$$

At $\alpha = 0$, this is the scattering off a "$\delta'(x)$ potential," defined as in Ref. [16]. For $\alpha \neq 0$, the transmission amplitude is modified by a constant phase.

We can investigate the RG flow along $c = 0$. For $\gamma \neq 0$, this is a flow between a point on the fixed line and $\gamma = \pm\infty$. For that we set $\mu \gg \gamma, p$. The amplitudes become

$$T = \frac{i(1+i\alpha)^2 p}{i(1+\alpha^2)p - \gamma}, \qquad R = \frac{\gamma}{i(1+\alpha^2)p - \gamma}, \tag{30}$$

which can be rewritten as

$$T = \frac{2p}{2p+i\tilde\lambda}e^{i\delta}, \qquad R = -\frac{i\tilde\lambda}{2p+i\tilde\lambda}, \tag{31}$$

where

$$\tilde\lambda = \frac{2\gamma}{1+\alpha^2}. \tag{32}$$

The scattering amplitudes thus have the same form as those of a delta-function potential, except that the transmission amplitude is multiplied by a momentum-independent phase.

VI. CONCLUSION

Using field-theory methods, we have analyzed a model involving nonrelativistic particles of two types interacting through a contact interaction. A novel feature of the model is the presence of a velocity-dependent interaction, which is still consistent with parity and time reversal if these symmetries also exchange the two types of particles.

Analyzing the RG group equations, we find that the theory has an isolated fixed point and a line of fixed points. At the fixed line, the scattering amplitudes are trivial, up to a constant phase in the transmission amplitude. In the general case, we find that the scattering amplitudes depend on three parameters, in perfect correspondence with the quantum mechanical analysis.

It would be interesting to explore the few-body and many-body aspects of the system. We defer this problem to future work.

ACKNOWLEDGMENTS

Appendix A: Quantum mechanical analysis

Denote the wavefunction of a system of two particles as $\psi(x, y)$, where x is the coordinate of the particle of species 1 and y the coordinate of the particle of species 2. Denote by $z = x - y$ the relative coordinate of the two particles. The most general interaction between particles at $x = y$ can be parametrized by a boundary condition across $z = 0$ [16]:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{z=+0} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}_{z=-0}, \tag{A1}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are complex number characterizing the short-range interaction. Particle number conservation requires that for two arbitrary wave functions ψ_1 and ψ_2, the matrix element of the particle number current between the two states, $\psi_1^* \partial_x \psi_2 - \partial_x \psi_1^* \psi_2$, is continuous across $z = 0$. This condition implies that

$$\mathcal{A}^* \mathcal{D} - \mathcal{C}^* \mathcal{B} = 1, \tag{A2a}$$

$$\mathcal{A}^* \mathcal{C} - \mathcal{C}^* \mathcal{A} = 0, \tag{A2b}$$

$$\mathcal{B}^* \mathcal{D} - \mathcal{D}^* \mathcal{B} = 0. \tag{A2c}$$

The last two equations imply that $\mathcal{A}^* \mathcal{C}$ and $\mathcal{B}^* \mathcal{D}$ are real.

Time-reversal invariance, with exchanging the type of particles, implies that $\tilde{\psi}(z) = \psi^*(-z)$ should also satisfy the boundary condition (A1). This implies that

$$\begin{pmatrix} \mathcal{A}^* & -\mathcal{B}^* \\ -\mathcal{C}^* & \mathcal{D}^* \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^{-1}. \tag{A3}$$

From this condition and Eqs. (A2) one finds $\mathcal{A} = \mathcal{D}$ and that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ have the same phase, i.e.,

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} A & B \\ C & A \end{pmatrix} e^{i\theta}, \tag{A4}$$

where A, B, and C are real numbers satisfying the equation

$$A^2 - BC = 1. \tag{A5}$$

and θ is also real. Thus, a short-range interaction is characterized by two real numbers and one phase.

The transmission and reflection amplitudes can be computed by requiring the wavefunction

$$\psi(z) = \begin{cases} e^{ipz} + Re^{-ipz}, & z < 0, \\ Te^{ipz}, & z > 0, \end{cases} \tag{A6}$$

to satisfy the boundary condition (A1), which yields the result

$$T = \frac{-2ipe^{i\delta}}{C - 2iAp - Bp^2}\,, \qquad R = -\frac{C + Bp^2}{C - 2iAp - Bp^2}\,. \tag{A7}$$

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