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# Signature of charge migration in modulation of double ionization

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We present a theoretical investigation of charge migration following strong-field ionization in a multi-electron system. We study a model homo-nuclear molecule with two electrons, each restricted to one dimension (1+1D), interacting with a strong, static electric field. We show that in this system charge migration results from the interplay between multiple ionization channels that overlap in space, creating a coherent electron-hole wave packet in the cation. We also find that, in our case, charge migration following the first ionization manifests as a modulation of the subsequent double ionization signal. We derive a parametrized semiclassical model from the full multi-electron system and we discuss the importance of the choice of cation electronic-structure basis for the efficacy of the semiclassical representation. We use the ab-initio solution of the full 1+1D system as a reference for the qualitative and quantitative results of the parametrized semiclassical model. We discuss the extension of our model to long wavelength time-dependent fields with full dimension, many-electron targets.

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## I. INTRODUCTION

With the progress in ultrafast laser technology, strong-field physics is now able to probe the structure and dynamics of matter at the space- and time-scale of the electron [1–7]. For time-resolved analyses, the temporal coherence of the laser electric field defines an intrinsic clock, with a period of a few femtoseconds in the near- to mid-infrared (IR) regime [8–15]. For instance, in a linearly polarized laser field, the recollision events [16, 17] – where a previously ionized electron revisits its parent cation – define a sub-cycle reference associated with the different recollision trajectories [18, 19], therefore with sub-femtosecond resolution in the IR. Bringing together high temporal and spatial resolution for a “real time” observation of electron dynamics has long been a goal of ultrafast physics. In this context, the observation and analysis of the phenomena of charge migration have attracted a lot of attention recently [20–27]. Charge migration is the coherent, correlation-driven, purely electronic dynamic immediately following excitation or ionization of a molecule, before the nuclei have had time to move. It has been widely speculated that the attosecond electron dynamics can play a crucial role in the longer time-scale phenomena like charge transfer or bond rearrangement, which implies that the control of attosecond dynamics can determine such chemical processes [21, 28].

Coherence and correlation are at the core of strong-field physics, especially given that the initial systems are usually atoms or molecules in their ground state. For instance, nonsequential double ionization (NSDI) [29–32] – and its “knee” signature that exceeds simple sequential ionization predictions by orders of magnitudes – spotlights the central role of electron correlation in strong-field processes [33–35]. Further analyses of the dynamics have revealed a rich variety in the pathways lead-

ing to NSDI [36–38]. Laser-induced electron diffraction (LIED) [39] – exploiting the in situ coherent probe of a pre-ionized electron scattering on its parent ion – has been used to retrieve the nuclear geometric structure of molecular compounds [40, 41]. Alternatively, still using the intrinsic coherence of strong-field laser-matter interaction, direct imaging of the electronic structure has been performed with tomographic reconstruction of molecular orbitals [42–44]. Even without recollision, strong-field ionization studies have proven a powerful probe of the electronic structure and properties of matter [9, 45, 46].

In this Article, we present a theoretical and numerical investigation of charge migration following strong-field multichannel ionization. We consider a model homo-nuclear molecule with two electrons, each restricted to one dimension (1+1D), interacting with a strong, static electric field. Somewhat counter-intuitively we find the dynamics in such a static field physically enlightening, despite the absence of any explicit time dependence. Indeed, the removal of a first electron by the field starts a molecular clock in the cation which can be later probed, *e.g.*, with the ionization of a second electron. We show that the charge migration results from the interplay between multiple ionization channels that overlap in space, *i.e.*, a coherent electron-hole wave packet in the cation. In our 1+1D system, we identify a signature of the cation charge migration dynamics, a modulation of the subsequent double ionization signal. From the full multi-electron system, we derive a parametrized semiclassical model where the hypotheses and approximations leading to the analytical prediction are identified. Of all the ionization and molecular parameters in the model, we highlight the importance of the cation electronic-structure basis for the efficacy of the semiclassical representation. We validate the model, qualitatively and quantitatively, against reference ab-initio solutions of the full 1+1D sys-

tem. We discuss the extension of our parametrized semiclassical model to long wavelength oscillating laser fields with full dimension, many-electron targets.

For realistic polyatomic molecules, brute-force ab-initio solutions of the full time-dependent Schrödinger equation (TDSE) are out of analytical and computational reach. As an alternative, one could turn to quantum-chemical techniques like time-dependent density functional theory (TDDFT) [47–52]. Such an approach, however, faces two fundamental difficulties: (1) The validity of chosen exchange-correlation functional in the nonlinear regime; (2) The interpretation of correlation-driven charge-migration dynamics in a framework that treats electrons as virtually independent particles. As a second alternative, one could stick to the TDSE framework and break the problem into smaller components, corresponding to the key elements of the charge migration, and piece them together with relevant system parameters [53, 54]. Such approaches are built to facilitate the analysis and interpretation of the process, but they also suffer from fundamental difficulties of their own: (1) The determination of the model parameters and (2) even assuming these are perfectly known, the intrinsic precision of the semiclassical approximation. Here we take advantage of the simplicity of our 1+1D model to address the second difficulty as one can perform brute-force integration of the full TDSE, for reference, and all ionization and molecular parameters can be computed with high precision.

The Paper is organized as follows: Section II introduces some key concepts of our analysis of charge migration, and its signature in the modulation of double ionization, in a static field from the point of view of classical mechanics. Section III introduces our quantum model and theoretical treatment of multichannel ionization. Section IV details the derivation of the corresponding semiclassical model. Section V applies it to the analysis of our 1+1D model and the signature of charge migration in double ionization. Finally, in section VI we conclude the paper, summarizing the key findings and discussing perspectives for real systems and real laser fields. Unless otherwise specified, we use atomic units (a.u.) throughout the paper.

## II. MODULATION OF DOUBLE IONIZATION AS A SIGNATURE OF CHARGE MIGRATION

Although we are ultimately interested in the full quantum ionization dynamics, it is useful to first look at the classical limit, where the Coulomb interaction with the core and other electrons is negligible (strong-field approximation – SFA). Following ionization, the dynamics of an electron in a static electric field of amplitude  $\mathcal{E}$  can be solved for analytically

$$r(\tau_{\text{cl}}) = -\frac{\mathcal{E}}{2}\tau_{\text{cl}}^2 \Leftrightarrow \tau_{\text{cl}} = \sqrt{\left|\frac{2r}{\mathcal{E}}\right|}, \quad (1)$$

assuming the electron is initially at the origin and with zero velocity. Here we are most interested in the reciprocal part of the equation: Static fields spatially separate electrons based on the delay since ionization  $\tau_{\text{cl}}$ , and the ionization distance  $r$  can be seen as a time axis. The use of static DC fields elides *two* types of interferences that would be present in their AC counterparts: (1) The cycle-to-cycle interference responsible for individual peaks in the above-threshold ionization (ATI) and high-harmonic generation (HHG) spectra, and (2) the sub-cycle interference of quantum paths leading to the same observable [19], *e.g.*, direct and back-scattered photoelectrons in ATI, and short and long trajectories in HHG.

Now consider (sequential) double ionization in the same static field. The arguments of Eq. (1) can be applied to each electron coordinate  $r_1$  and  $r_2$  individually. This means that the analysis of the *spatially* correlated two-electron density gives us information about the *dynamics* following the first ionization event, and eventually leading to the second ionization. An illustration of that analysis is displayed in the upper panel of Fig. 1, where the density has been obtained from our 1+1D model wave function  $|\psi(r_1, r_2)|^2$  (to be defined later). In the lower part of the panel we observe diagonal “stripes” in the density (highlighted with dashed arrows), which we therefore identify as a modulation of the double ionization signal with respect to  $\tau_{\text{cl}}$ . Interestingly, in the upper part of the panel, we notice that this modulation is synchronized with another type of density motion, between the two centers of the molecule (solid diagonal arrows). Later we will show that this corresponds to a migration of the charge in the cation following multichannel ionization, and that the modulation of the double ionization signal is a signature of that charge migration dynamics.

## III. MODEL

We consider the quantum-mechanical dynamics of a  $N$ -active electron system, as described by the time-dependent Schrödinger equation (TDSE)

$$i\partial_t\psi(\mathbf{r}^N; t) = \hat{\mathcal{H}}_N(\mathbf{r}^N)\psi(\mathbf{r}^N; t), \quad (2)$$

in the presence of a static electric field of amplitude  $\mathcal{E}$ . Here  $\hat{\mathcal{H}}_N$  and  $|\psi\rangle$  are the Hamiltonian operator and total  $N$ -electron wave function, respectively. For simplicity we use  $\mathbf{r}^N$  as a short-hand notation for the multi-electron coordinates  $r_1, \dots, r_N$ .

### A. Molecular model for numerical simulations

For numerical simulations we consider the simplest system that fulfills the requirements for charge migration following ionization discussed in the introduction. It is

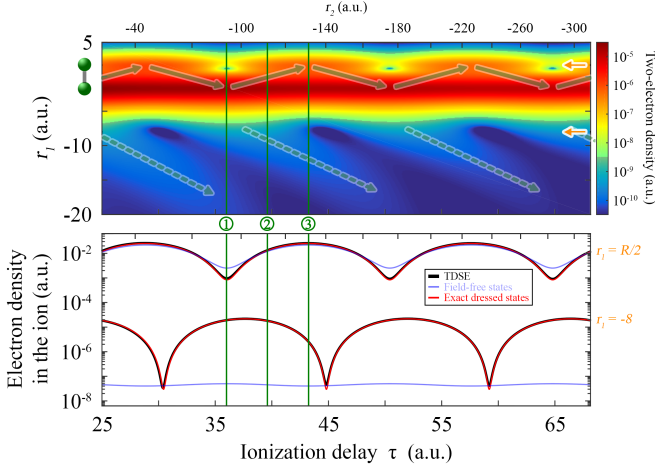


FIG. 1. Upper panel: Two-electron density  $|\psi(r_1, r_2)|^2$  for our full 1+1D TDSE simulations. In the molecular model we set the internuclear distance  $R = 3.5$  a.u., the electron-nucleus/-electron softening parameter  $a_{en}/a_{ee} = 0.5/1$ , the effective charge  $Z = 1$  and static electric field amplitude  $\mathcal{E} = 0.13$  a.u.. Solid line arrows label the charge migration between the two molecular centers, following the first multi-channel ionization event. Dashed arrows represent the subsequent double ionization bursts, as signatures of the charge migration in the cation. Lower panel: Comparison of the electron density in the cation (15) for selected  $r_1$  sections (labeled with small arrows in the upper panel) of the TDSE and semiclassical models (13). The ionization delay  $\tau$  used here is defined in Eq. (14). For a qualitative understanding, its classical limit introduced in Eq. (1) can be used instead.

given by the 1+1D soft-Coulomb potential [55]

$$\hat{V}(r_1, r_2) = -\frac{Z}{\sqrt{(r_1 \pm \frac{R}{2})^2 + a_{en}^2}} - \frac{Z}{\sqrt{(r_2 \pm \frac{R}{2})^2 + a_{ee}^2}} + \frac{1}{\sqrt{(r_1 - r_2)^2 + a_{ee}^2}}, \quad (3)$$

where  $a_{en}$  and  $a_{ee}$  are the electron-nucleus and electron-electron softening parameters, respectively. This model is simple enough to allow for full ab-initio TDSE computations, to serve as a reference in quantitative analysis and yet, it is complex enough to exhibit multichannel ionization and subsequent charge migration in the cation. For all figures shown here, we take the internuclear distance  $R = 3.5$  a.u. (1.9 Å), effective charge  $Z = 1$ ,  $a_{en}/a_{ee} = 0.5/1$  and a static field amplitude  $\mathcal{E} = 0.13$  a.u. (the same amplitude as an AC field with an intensity of  $6 \times 10^{14}$  W/cm<sup>2</sup>). For reference, with these parameters the energy difference between the (dressed) ionization channels is about 12 eV, which gives the period of about 14.5 a.u. (about 350 as) for the charge migration observed in the cation. We note that full dimensional molecular systems with closely spaced energy

states generally can be dipole coupled with much lower field amplitudes than considered here.

In order to discriminate between the channels, we consider a system with two spatially different orbitals and so use the lowest-energy triplet state as the initial condition. In the dipole approximation the 1+1D Hamiltonian operator  $\mathcal{H}_2$  in Eq. (2) preserves the (anti)symmetry of the initial condition, which prevents any leakage to a lower-lying symmetric state. From the practical point of view, it also means that one needs only to record the ionization wave function in one of the two electron coordinates. We choose  $r_2$  as this coordinate and absorb the outgoing wave function along the other electronic coordinate.

In numerical simulations we first smoothly ramp up the field from the (field-free) triplet initial state to reach the desired static-field amplitude. All numerical analyses are performed after sufficient static-field duration such that all transient effects associated with the ramp-up have moved outside of the simulation box. We have checked the robustness of our results with the ramp-up and static field durations, field strength, molecular and discretization parameters. Later we will use the result of these ab-initio full TDSE simulations as a reference against which parametrized semiclassical models can be quantitatively tested.

## B. Single-ionization effective model

In the physical picture of the TDSE (2), single ionization corresponds to portions of the wave function extending in exactly one electronic coordinate, *e.g.*,  $r_N$ , while the other ( $\mathbf{r}^{N-1}$ ) remain localized. In such regions, the dynamics between the ionized and the cation electrons becomes decoupled and the total Hamiltonian operator splits into two effective ones [13, 54]

$$\hat{\mathcal{H}}_N(\mathbf{r}^N) \approx \hat{\mathcal{H}}_{N-1}(\mathbf{r}^{N-1}) + \hat{\mathcal{H}}_1(r_N). \quad (4)$$

In this formulation, we identify three different components to the total wave function, which should be clearly distinguished. First is the neutral component – later labeled with “*n*” superscript – with all  $N$  electron coordinates close to the core region, and from which originates ionization. Next are the cation and ionized electron components, respectively with  $N-1$  and 1 electrons and described with the effective operators  $\hat{\mathcal{H}}_{N-1}$  and  $\hat{\mathcal{H}}_1$ . Although they are *decoupled*, it is important to keep in mind that the dynamics of the cation and ionized electron components are still *correlated* (*entangled*), through the ionization condition: Dynamically, they are both born out of the same neutral component at the time of ionization.

The decomposition (4) is most interesting in that it naturally defines orthonormal representation bases for each subspace: The cation component can be expanded in a basis of  $N-1$  electron states, which we generically denote  $|\tilde{\psi}_k\rangle$  (here the tilde is used to discriminate the neu-

tral, with  $N$  electrons, and the cation, with  $N - 1$ , components). Throughout the paper, these correspond to our ionization channels. The most intuitive basis corresponds to the ionic field-free eigenstates, labeled with “0” superscript in what follows. In a single Slater determinant approximation, such states can be labeled by the orbital of the electron-hole. As we shall see, other “smarter” choices of basis are also possible. On the other hand, the ionized electron component is described with a one-electron continuum, which we generically label with  $|v\rangle$ . Depending on the degree of precision required, one can use, *e.g.*, plane waves/Volkov states, Coulomb waves [56], exact one-electron continuum states [57].

In the product basis, the total  $N$ -electron wave function, with ionization along the electronic coordinate  $r_N$ , therefore reads [22, 58, 59]

$$|\psi\rangle = e^{iI_p^n t} |\psi^n\rangle + \sum_{k=1}^{\tilde{N}} e^{iI_{p_k} t} \int \beta_k(v, t) |\tilde{\psi}_k\rangle \otimes |v\rangle dv. \quad (5)$$

Analytically, finding the coefficients  $\beta_k$  is equivalent to solving the problem. Here  $|\psi^n\rangle$  is the neutral component, from which the single ionization originates, with total ionization potential  $I_p^n$ . Although it vanishes in the region of interest, this term is kept as a reminder of the correlation (entanglement) between the cation and ionized electron components, through the initial ionization condition.  $I_{p_k}$  is the total ionization potential of the corresponding cation basis set component  $|\tilde{\psi}_k\rangle$ . In this representation  $I_p^n - I_{p_k}$  is the vertical ionization energy to the channel  $k$ .  $\tilde{N}$  is the truncation order in the cation basis expansion. Its actual value is largely system specific and depends, *e.g.*, on the number of ionization channels, the cation component basis, if excited states are involved in the process of interest, etc.

### C. Retrieving channel populations

To obtain the population in a given ionization channel  $k$  we project the total wave function onto the corresponding cation state with

$$\begin{aligned} \rho_k(r_N) &= N \left| \langle \tilde{\psi}_k | \langle r_N | \psi \rangle \right|^2 \\ &= N \left| \int \beta_k(v) \langle r_N | v \rangle dv \right|^2, \end{aligned} \quad (6)$$

from Eq. (5). In practice, this corresponds to taking a slice of the total  $N$ -electron wave function at a given ionization coordinate  $r_N$ , or equivalently a given ionization delay  $\tau$ , and projecting the result onto the cation basis state element. Here the prefactor  $N$  is introduced to cast the problem in a form similar to TDDFT, where all properties are functionals of the single-particle density [49]

$$\rho(r_N) = N \int |\psi(\mathbf{r}^{N-1}, r_N)|^2 d\mathbf{r}^{N-1}. \quad (7)$$

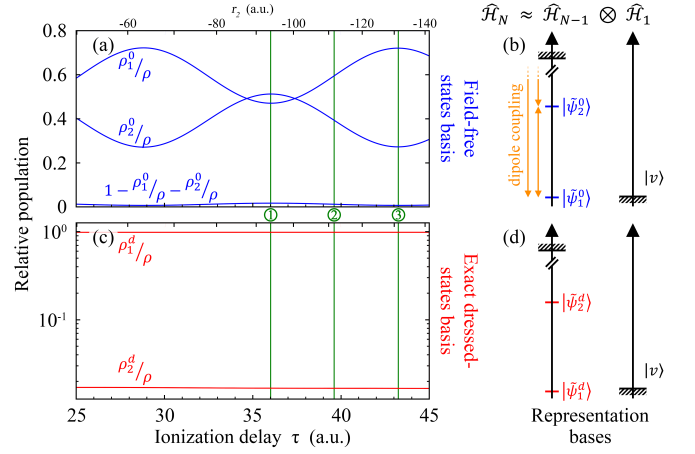


FIG. 2. Charge-migration dynamics and modeling following multichannel ionization by a strong static laser field. Panels (a) and (c) show the relative population  $\rho_k/\rho$ , as defined in Eqs. (6) and (7), extracted from full TDSE simulations for field-free and exact dressed state cation bases, respectively. The ionization delay  $\tau$  is defined with Eq. (14). For a qualitative understanding, its classical limit introduced in Eq. (1) can be used instead. Panels (b) and (d) illustrate the differences between the two representation bases.

In figure 2 (a) we display the relative populations,  $\rho_k(r_N)/\rho(r_N)$ , in the first two field-free states of the cation for our 1+1D molecular model of potential (3). Results are shown as a function of the ionization coordinate, or equivalently the delay since the first ionization (see section II). Because of the dipole coupling between the cation states [see panel (b)] we observe population transfer between them, which manifests through the oscillations in the curves. Alternatively, panels (c) and (d) display the populations in a “smart” exact dressed states cation basis set [60–62]. In the dressed basis, the coupling between channels is effectively removed and the population in each of them is conserved over time.

The cation component basis closure, together with Eqs. (6) and (7), imposes the charge conservation condition

$$\rho = \sum_{k \geq 1} \rho_k.$$

Alternatively, one can use the charge conservation condition to determine the number of cation states to include in the expansion (5), *i.e.*,  $\tilde{N}$ , by tracking the residual population proportion  $1 - \sum_{k=1}^{\tilde{N}} \rho_k/\rho$ .

### IV. SEMICLASSICAL MODEL

With static field and truncation of the cation component basis to order  $\tilde{N}$  in Eq. (5),  $\hat{\mathcal{H}}_{N-1}$  takes the form of a constant finite-dimensional Hermitian matrix, which

we denote  $\tilde{\mathcal{H}}_{N-1}$ . We first diagonalize the matrix

$$\tilde{\mathcal{H}}_{N-1} \left| \tilde{\psi}_k^d \right\rangle = -I_{p_k}^d \left| \tilde{\psi}_k^d \right\rangle$$

and rewrite the problem in an orthonormal basis of eigenstates. Immediately, we see that the  $I_{p_k}^d$  and  $\left| \tilde{\psi}_k^d \right\rangle$  correspond to the *dressed* ionization potential and cation component state. Effectively, this decouples the dynamics of the dressed channels, which can be treated as virtually independent systems – with population conserved in time – while fully retaining the dipole couplings with the external field. Here as well, it is important to keep in mind that, although decoupled, the dressed ionization channels remain *correlated* (*entangled*), through the multichannel ionization condition. In what follows, dressed states and associated parameters/quantities are labeled with “*d*” superscript.

### A. Approximate solution

With each dressed state behaving as an independent channel we follow the procedure of the Lewenstein model [18] and combine the wave function decomposition (5) with the TDSE (2) and effective Hamiltonian (4). For illustration and similarly to Ref. [18], we consider a plane wave and SFA description of the continuum electron such that, in the length gauge, the dynamics is described by

$$i\dot{\beta}_k^d - \dot{v}r_N\beta_k^d = e^{i(I_p^n - I_{p_k}^d)t} d_k^d(v) + \frac{v^2}{2}\beta_k^d + \mathcal{E}r_N\beta_k^d,$$

for each dressed channel. Here  $d_k^d$  is the (complex) transition element describing ionization from the neutral to the dressed ionization channel  $k$ ,  $|\psi^n\rangle \rightarrow \left| \tilde{\psi}_k^d \right\rangle \otimes |v\rangle$ . The SFA imposes  $\dot{v} = -\mathcal{E}$ , and we solve the differential equation analytically [18]

$$\begin{aligned} & e^{iI_{p_k}^d t} \int \beta_k^d(v, t) \left| \tilde{\psi}_k^d \right\rangle \otimes |v\rangle dv = \\ & -i \int \int d_k^d(v(t_0)) e^{iS_k^d[t_0, t, r_N, v]} dt_0 dv \left| \tilde{\psi}_k^d \right\rangle \otimes |\mathbb{I}_d\rangle \end{aligned} \quad (8)$$

where  $|\mathbb{I}_d\rangle$  is the identity – the contribution from the continuum (plane-wave)  $|v\rangle$  is included in the global phase term – and  $S_k^d$  is given by

$$S_k^d = I_{p_k}^d t + \left( I_p^n - I_{p_k}^d \right) t_0 - \int_{t_0}^t \frac{v(s)^2}{2} ds + vr_N, \quad (9)$$

with  $v(s) = v + \mathcal{E}(t_0 - s)$ . Beyond the present case of DC ionization, we notice the similarity in the analytical solution above with other semiclassical treatments of multichannel/active electron systems in strong-field physics [7, 23, 25, 54].

Physically, Eqs. (8-9) can be interpreted as the ionization time  $t_0$  and initial velocity  $v$  required for a classical

electron to reach the coordinate  $r_N$  at time  $t$ . Similarly,  $I_{p_k}^d(t - t_0)$  and  $-\int_{t_0}^t \frac{v(s)^2}{2} ds$  are the phases accumulated by the dressed cation component  $k$  and ionized electron, respectively, following ionization. The term  $I_p^n t_0$  ensures the synchronization of phases between the neutral and ionization channel, at the instant of ionization, and the term  $vr_N$  is due to the spatial dependence of the continuum state.

More interestingly, in Eq. (8) we recognize the temporal factorization of the wave function in terms of the two steps of the ionization process – ionization, with  $d_k^d(v(t_0))$ , and propagation, with  $e^{iS_k^d[t_0, t, r_N, v]}$ . Here, for each dressed channel  $k$ , the result reads as a single active electron (SAE) system would. Assuming one can generalize the notion of such dressed states to oscillating fields (see section VI), it opens a clear perspective for extending quantitative rescattering (QRS) results [65] and similar HHG spectrum factorization [66] to multichannel processes. This would be done by performing the temporal to frequency factorization [57] for each dressed channel independently, and coherently summing all contributions. We stress that this is possible only because the dressed channels are virtually independent systems. Indeed, QRS factorization relies on the generality of the propagation component of the factorization, which is found to be very similar across targets. This property breaks down when population transfer (charge migration) happens between the different cationic states, *e.g.*, as it does in the field-free cation state basis.

### B. Stationary phase approximation

With its multidimensional integrals, Eq. (8) is rather cumbersome for analyses of the charge migration dynamics. To simplify its expression we consider the familiar stationary phase approximation (SPA). Without loss of generality we assume the static field is fully on at time  $t = 0$  and select a position  $r_N$  along the ionization direction and time  $t \gtrsim \sqrt{|2r_N/\mathcal{E}|}$  (1), to avoid transient effects of the laser ramp-up. For each dressed channel, the SPA condition  $\nabla S_k^d = 0$  is defined by the two equations

$$I_p^n - I_{p_k}^d + \frac{v^2}{2} = 0 \quad \text{and} \quad \int_{t_0}^t v + \mathcal{E}(t_0 - s) ds - r_N = 0. \quad (10)$$

By definition, the neutral ionization potential is larger than that of any of the cations, such that the SPA velocity is a purely imaginary number

$$v = -i\sqrt{2(I_p^n - I_{p_k}^d)},$$

which is sometimes interpreted as a generalized energy conservation condition for ionization, extended to the complex plane. Then, the second equation defines the

corresponding SPA ionization time

$$t_0 = t - \frac{v + \sqrt{2(-\mathcal{E}r_N - I_p^n + I_{p_k}^d)}}{\mathcal{E}}.$$

This equation defines the classically forbidden region  $r_N > -(I_p^n - I_{p_k}^d)/\mathcal{E}$ , assuming  $\mathcal{E} > 0$ , and corresponds to the picture of tunnel ionization through a barrier corresponding to the vertical ionization energy to channel  $k$ .

After additional calculations and simplifications, we find that the real part of the stationary phase is given by

$$S_k^s(r_N, t) = I_p^n t + \frac{\left[2(-\mathcal{E}r_N - I_p^n + I_{p_k}^d)\right]^{\frac{3}{2}}}{3\mathcal{E}}. \quad (11)$$

Alternatively, similar results can be obtained in parabolic coordinates, where the Coulomb potential plus static-field problem is separable [63, 64]. Finally, following QRS-type arguments [65], we assume the spatial profile along the ionization channel is generic and further factorize the wave-function

$$|\psi\rangle = \sqrt{\frac{\rho(r_N)}{N}} \sum_{k=1}^{\tilde{N}} \gamma_k^d e^{i(S_k^s(r_N, t) + \Phi_k^d)} |\tilde{\psi}_k^d\rangle, \quad (12)$$

where  $\gamma_k^d$  is the proportion of ionization to, and  $\Phi_k^d$  is the absolute ionization phase out of, the dressed channel  $k$ .

Interestingly, we notice that Eq. (12) corresponds to a fully parametrized semiclassical (ionization) model, where the parameters are related to the physical/chemical ionization properties to each cation channel. Quantum-chemical methods such as TDDFT or Hartree-Fock theory provide means to obtain such parameters [23]. A detailed discussion about how these should be computed goes beyond the scope of the present paper.

## V. NUMERICAL SIMULATIONS

We now turn to the 1+1D molecular system introduced in section III A for a quantitative analysis of the parametrized semiclassical model of Eq. (12). In our 1+1D model, in Fig. 2 (c) we see that most of the population is captured by the first two field-free cation states. For reference, in the exact dressed-state basis, the residual population is about three orders of magnitude smaller than the populations in the first two dressed states. We therefore take  $\tilde{N} = 2$  in the parametrized semiclassical model (12), which leads to the two-electron density

$$||\psi\rangle|^2 = \frac{\rho(r_2)}{2} \left| \gamma_1^d \tilde{\psi}_1^d(r_1) + \gamma_2^d e^{i(S_{12}^s(r_2) + \Phi_{12}^d)} \tilde{\psi}_2^d(r_1) \right|^2, \quad (13)$$

with  $S_{12}^s$  and  $\Phi_{12}^d = \Phi_2^d - \Phi_1^d$  the stationary and ionization phase differences between the two (dressed) channels.

## A. Ionization delay

From the expression of the stationary phase (11), we notice that the phase difference between the two ionization channels

$$S_{12}^s(r_2) = \frac{\left[2(-\mathcal{E}r_N - I_p^n + I_{p_2}^d)\right]^{\frac{3}{2}} - \left[2(-\mathcal{E}r_N - I_p^n + I_{p_1}^d)\right]^{\frac{3}{2}}}{3\mathcal{E}},$$

is independent of the absolute time  $t$ . We have confirmed numerically that once the static-field-ionized wave-packet has reached a given ionization coordinate  $r_2$ , the two-electron density remains constant for all later times. This ensures that we can reconstruct the ionization delay out of the coordinate  $r_2$ , by providing a consistent reference from full TDSE computations: The charge migration dynamics in the cation, and its signature in the modulation of subsequent ionization, only depends on the time since the first ionization, *i.e.*, how far the ionized electron has traveled. Throughout the remainder of the paper we therefore omit the absolute time variable  $t$  in most equations.

At the leading order expansion, the phase difference between the two channels becomes

$$S_{12}^s(r_2) \approx_{|r_2| \gg 1} (I_{p_2}^d - I_{p_1}^d) \tau_{\text{cl}}.$$

This factorizes as the energy difference between the two dressed states multiplied by the classical ionization delay (1). It corresponds to the fully classical picture for the ionized electron, where the two ionization channels have the same continuum dynamics and therefore the same accumulated phase. These cancel and the phase modulation is fully determined by the energy difference between the two ionization channels.

For higher-order results, we keep the full semiclassical expression and define the ionization delay as

$$\tau(r_2) = \frac{S_2^s(r_2) - S_1^s(r_2)}{I_{p_2}^d - I_{p_1}^d}. \quad (14)$$

In all figures, Eq. (14) is used to perform the conversion from the ionization coordinate  $r_2$  to delay  $\tau$  in the analysis of our TDSE simulations. Physically, compared to the fully classical limit (1), higher-order terms can be interpreted as accounting for channel-specific ionization condition due to the different ionization barriers  $I_p^n - I_{p_{1,2}}^d$  (see section IV B).

We now have all the ingredients necessary to study charge migration following multichannel ionization in our 1+1D molecular model. More specifically, as a function of the ionization delay  $\tau$  (14), the electron density in the cation is given by

$$|\tilde{\psi}(r_1, \tau)|^2 = \frac{|\psi(r_1, r_2(\tau))|^2}{\rho(r_2(\tau))/2}. \quad (15)$$

We display the result for various delays in Fig. 3 (thick black curves – each panel corresponds to a section ①–③)



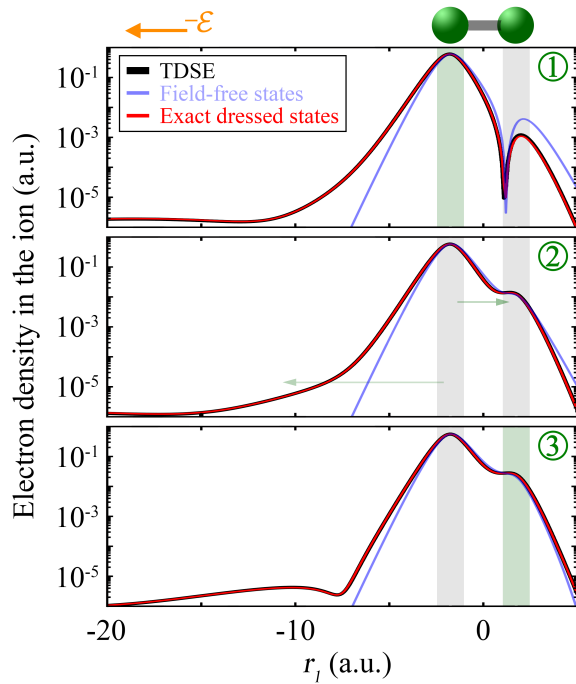


FIG. 3. Comparison between the electron density in the cation  $2|\psi|^2/\rho$  from the TDSE and our semiclassical model, with field-free and exact dressed ion states (see legend), for the section labeled in Figs 1 and 2. The associated movie can be found in the supplemental material [67].

in other figures); a movie of the charge migration is also available in the supplementary material [67]. It confirms the observations of section II: Following multichannel ionization a migration of the charge is observed between the two molecular centers (vertical stripes in the panels). Looking closely at the double ionization dynamics ( $r_1 \ll -1$ ) we see that a burst of ionization is generated following the full localization of the electron density on the down-field center (panel ①), and the double ionization channel is suppressed when the electron is delocalized over the two centers (panel ③). This is a clear signature of the charge migration dynamics in the double ionization oscillations, through the control it exercises on the release of the second electron (localization on the down-field molecular center).

### B. Cation component representation basis

One could fit the results of the TDSE calculation as a means of extracting the parameters of the semiclassical model (13). This all-at-once-fit strategy is unreliable, yielding unstable and sometimes unphysical results. Alternatively, and similarly to what would be done for systems where brute-force TDSE is not feasible, we have determined these parameters independently through physically reduced models, *e.g.*, corresponding to the electronic structure of the cation for the different ionization

energies  $I_{p_k}^d$ . This method has proven successful, leading to well-defined (stable) parameters. We have checked the robustness of our ability to get these with changing electric field and molecular parameters.

As discussed above, the most natural representation basis for the cation component is built from field-free eigenstates, as they are unambiguously defined by the cation component effective Hamiltonian  $\hat{\mathcal{H}}_{N-1}$ . The corresponding semiclassical electron density, with truncation order  $\tilde{N} = 2$ , is displayed with light blue curves on Fig. 3. Compared with the TDSE reference (thick black), we see that this model reproduces very well the charge migration dynamics in the core region, around the molecular centers (vertical stripes). Away from the core, however, the model fails to describe the (double) ionization dynamics. These observations are confirmed on the bottom panel of Fig. 1, where the semiclassical density (light blue) reproduces very well the TDSE reference (thick black) at the up-field center ( $r_1 = R/2$  set of curves) but completely fails to capture the modulation in the density away from the core, in the double ionization region ( $r_1 = -8$  a.u.). The failure to account for subsequent double ionization is hardly a surprise given the chosen basis of field-free states, which represent bound electrons. In order to capture double ionization one would probably have to expand drastically the representation basis with  $\tilde{N} \gg 2$  and include, *e.g.*, many Rydberg states.

The aforementioned failure to reproduce all of the full 1+1D TDSE dynamics is not to be put on the semiclassical model (13), but solely on the choice for the cation component basis representation. Indeed, for comparison, we have performed a precise computation of the DC field dressed states, including the distant – double ionization – region from the core. We refer to these as the “exact” dressed states throughout the paper. The resulting semiclassical prediction is displayed with thin red curves on the bottom panel of Fig. 1 and in Fig. 3. The results fall almost perfectly on top of the TDSE reference (thick black curves), and captures both the charge migration dynamics, following multichannel ionization, and the modulations it induces in subsequent (sequential) double ionization. Compared to the “natural” field-free state representation, the exact dressed states can be heuristically interpreted as follows: First one builds the effective cation Hamiltonian operator matrix  $\hat{\mathcal{H}}_{N-1}$  including many (Rydberg) states, such as to span distant regions from the core, to include subsequent ionization routes. Then one performs the diagonalization of the extended matrix to find the dressed states (as described in the introduction of section IV). Finally, one performs the truncation to order  $\tilde{N} = 2$ , *e.g.*, noticing that the first two exact dressed states capture virtually almost all of the electron population in the cation [see panel (d) of Fig. 2]. Compared to the field-free model described above, the inversion of the “Hamiltonian matrix diagonalization” and “truncation” steps has two main advantages: First it offers a much more precise modeling – it quantitatively reproduces the full TDSE results, including the modula-



tion in double ionization. Second it keeps the number of dressed states involved in the model minimal, with only two effective channels here.

## VI. CONCLUSION – BEYOND STATIC FIELDS

In conclusion we have shown that even in the simplest case of a static (DC) field, ionization to more than one channel can lead to subsequent charge migration in the cation. A signature of this charge migration can be found in the modulation of subsequent double ionization, even though it happens through a sequential process – with a static field, recollision is not possible. Mathematically, the charge migration can be linked to the different rates at which each ionization channel accumulates phase and the resulting constructive and destructive interferences in the coherent superposition of involved cation states. Physically, the modulation can be understood in terms of the ionization delay-dependent localization of the electron density on the cation. In our case only the full localization of the electron on the down-field molecular center can lead to double ionization, while the double ionization channel is suppressed when the electron is delocalized over the two centers.

By varying the internuclear distance  $R$  in our potential model of Eq. (3) (not shown) – this amounts to tuning the molecular properties of the target – we identify the two necessary ingredients for charge migration in the cation: (i) Ionization to more than one dressed channel, and (ii) spatial overlap between the dressed cation states, *i.e.*, a coherent electron-hole wave packet in the cation. For small internuclear distances, dressed states strongly overlap spatially but ionization occurs to only one channel. On the other hand, with large internuclear distances significant ionization to both dressed channels is observed but they do not overlap spatially. In both cases, it leads to the disappearance of the charge migration dynamics described above. Here, it is important to note that charge migration is a coherent evolution of charge from one region of space to another. Charge migration can be distinguished from typical excitations due to longer timescales needed for the larger body of charge to move. In a larger molecule, for example, charge migration would describe the motion of charge from one end to the other, whereas excitations would be dominated by transitions between nearby atoms.

Our analysis also provides a quantitative validation of semiclassical models for multichannel ionization and subsequent charge migration dynamics. This is particularly relevant given that such approaches have flourished in the past few years with the attempt to model, analyze and identify charge migration in complex multi-active electron systems [7, 11, 22–25, 43, 54], where full ab-initio TDSE computations are out of reach. We stress that the key to the success of the semiclassical model derivation was to rewrite the problem in a basis of dressed states that effectively decouples the different ionization

channels. Each can then be treated independently, similarly to well-known single-active-electron systems, and the overall coherence is only located in the ionization condition (as an initial-condition entanglement). Another byproduct of the dressed-state formulation is that, after diagonalization of the matrix  $\tilde{\mathcal{H}}_{N-1}$ , the complexity of computations grows linearly with the number of dressed states. On the other hand, the complexity scales quadratically if one stays, *e.g.*, in the field-free basis (the time propagator is a full matrix).

The comparison between models using reduced field-free states and exact dressed states with the 1+1D TDSE reference in Figs. 1 and 3 is also instructive for semiclassical models and analyses. For processes involving only a single ionized electron we see in our simulations that a reduced basis of field-free cation states is sufficient to describe the charge migration dynamics in the core region, with very good accuracy. On the other hand, if subsequent ionization or the dynamics in regions distant from the core are of interest, more care/refined states should be considered in order to account for them in the semiclassical approach.

Our main motivation for considering a static (DC) field is that it spatially separates ionization delays, and a single semiclassical trajectory is associated with each ionized electron coordinate (see also section II). We used this property to reconstruct the ionization delay and, from there, the charge-migration dynamics in the cation following multichannel ionization from full ab-initio TDSE computations. This allowed us to quantitatively validate our semiclassical model (13) and interpret the signature of charge migration in modulation of the double ionization. Almost all experiments of strong-field physics and attosecond science, however, use oscillating (AC) laser fields. Looking back at Fig. 1, we notice that the charge migration period is about 14.5 a.u. (about 350 as). At this time scale, mid-infrared lasers can be seen as a slowly varying electric field – in our example the period of the dynamics is about 1/20 of that of a 2  $\mu\text{m}$  laser. In the sequential regime, *i.e.*, neglecting recollision-induced double ionization, the wave function expansion (5) is not specific to DC fields and can equally well be applied to AC ones. From there, the main difference is that the cation component effective Hamiltonian operator  $\tilde{\mathcal{H}}_{N-1}$  now has an explicit time-dependence. As a result the associated matrix  $\tilde{\mathcal{H}}_{N-1}$  is finite dimensional with *periodic* time-dependent coefficients. For such systems, the Floquet framework extends the notion of dressed eigenstates [68], and can decouple them. For the ionized electron component, the dynamics of a continuum electron in an oscillating field has been repeatedly shown to be very well described by semiclassical models. This means that working in the Floquet basis offers the possibility to describe the system as a coherent superposition of decoupled – effectively single-active-ionized electron – channels. For such single active electron processes, QRS [65] has been shown to produce qualitatively and quantitatively more accurate results than their semiclassical counterparts. It

therefore opens avenues to extending the technique to multichannel processes with oscillating fields.

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