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Floquet Protocols of Adiabatic State-Flips and Re-Allocation of Exceptional Points

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We introduce the notion of adiabatic state-flip of a Floquet Hamiltonian associated with a non-Hermitian system that it is subjected to two driving schemes with clear separation of time scales. The fast (Floquet) modulation scheme is utilized to re-allocate the exceptional points in the parameter space of the system and re-define the topological features of an adiabatic cyclic modulation associated with the slow driving scheme. Such topological re-organization can be used in order to control the adiabatic transport between two eigenmodes of the Floquet Hamiltonian. The proposed scheme provides a degree of reconfigurability of adiabatic state transfer which can find applications in system control in photonics and microwave domains.

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I. INTRODUCTION

The adiabatic theorem of Hermitian physics is at the heart of many phenomena with far reaching technological applications. In simple terms it states that when a wave system described by a (sufficiently) slowly varying Hamiltonian $H(t)$ is initially prepared at a non-degenerate normal mode of $H(t = t_0)$, it will remain in the corresponding normal mode of the instantaneous $H(t)$ throughout the evolution. Consequently a cyclic adiabatic change in a multi-parameter space will return the system to its initial state, with possible an overall phase modification – the famous Berry phase. The latter turns out to be insensitive to the specifics of the adiabatic motion and depends only on the choice of the path in the parameter space [1]. Although the theorem has been originally derived in the framework of quantum mechanics, its applications were far reaching, covering areas ranging from acoustics, to microwaves and optics [2–4].

The situation is richer when a non-Hermitian system is driven adiabatically. In this case, the existence of non-Hermitian spectral singularities known as exceptional points (EP) (simultaneous coalesce of eigenvalues and eigenvectors) [5] can lead to a completely different physics than the one predicted by the adiabatic theorem of a Hermitian system. If the parametric variation of the Hamiltonian around an EP occurs quasi-statically, the instantaneous eigenstates transform into each other at the end of the cycle with only one of them acquiring a geometric phase [6–8]. If, however, the adiabatic evolution around the EP is dynamical (but still slow) then only one state dominates the output and what determines this preferred eigenstate is the sense of rotation in the parameter space. This surprising effect has been recently confirmed in microwave and optomechanical systems [9–11]. The growing attention to this chiral mode switching (state-flip) has roots in its potential technological implications; specifically the robustness of the associated adi-

abatic transfer against small fluctuations in the control path [12] is an asset for many practical applications. In this work we focus on applications of the non-Hermitian state-flip in the framework of classical waves.

Less effort has been devoted in proposing driving protocols that alter the dynamics by changing the topological features of a *fixed* (both in position and direction) adiabatic control path by re-allocating the EP in the parameter space. Here, we explore this viewpoint and provide *reconfigurable* protocols that manipulate the relative position of an EP singularity, by placing it inside or outside a fixed closed adiabatic control path. Then, we harvest such topological re-organization in order to control adiabatic state transfer between two states of a non-Hermitian system. The scheme involves a Floquet driving with a (fast) frequency which is rationally related to the inverse period needed for a cyclic adiabatic variation associated with two other control parameters. First we show that the Floquet driving re-allocates or even creates/annihilates EPs in the parameter space in a controllable manner. Then we introduce the *Floquet scenario of adiabatic state-flip*. We show that the “Floquet” EPs have the same topological features as the ones associated with a static non-Hermitian Hamiltonian. Finally we control an adiabatic state-flip from one Floquet eigenmode to another by re-allocating (inside or outside the cyclic adiabatic path) the Floquet EP via management of the Floquet driving.

The structure of the paper is as follows. In the next Section II we discuss the general considerations and statement of the Adiabatic State-Flip for Floquet systems. Special attention is given to the preparation of the initial state and to the representation/observation of the evolved state. Then in Section III we give two specific examples of Adiabatic State-Flip for Floquet systems. The first example (subsection III.A) is analytically treated and thus provides a clear understanding of the Adiabatic State-Flip. The second example (subsection III.B) is physically more appealing but it is not analytically solvable and has been treated only numerically. Nevertheless, it reconfirms our statements for Adiabatic State-Flip. Finally our conclusions are given at the last section IV.

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II. GENERAL CONSIDERATIONS AND STATEMENT OF THE PROBLEM

We consider the following time-dependent non-Hermitian system described by a Schrödinger-like equation:

$$i\frac{d}{dt}|\Psi(t)\rangle = H(b_\omega(t), b_\Omega(t))|\Psi(t)\rangle \quad (1)$$

where $b_\omega(t)$ and $b_\Omega(t)$ are two (complex) periodic time-varying parameters with frequencies ω and Ω respectively. Below for simplicity we shall always assume that the Hamiltonian H is a 2×2 matrix. Equation (1) can be understood, in the framework of coupled-mode theory, as a way to describe two periodically-driven coupled modes. Such system describes two single-mode coupled resonators or the (quasi-degenerate) modes of a single resonator – as long as they are well separated from all other modes. In our considerations below we assume that the resonators are of mechanical (e.g. acoustic) or electromagnetic nature. **All parameters in Eq. (1) are dimensionless and written in arbitrary units.**

Furthermore, we impose separation of time scales between the two driving schemes by requesting that $\omega = N\Omega$ where $N \gg 1$ is a positive integer. The adiabatic variation of $b_\Omega(t)$ is ensured by the condition $\Omega \rightarrow 0$ while the *Floquet* (fast) driving of $H(t)$ is controlled by ω and shall be used to manipulate the position of the EP in the parameter space. During the time interval $t \in [0, T_\Omega = \frac{2\pi}{\Omega}]$ the parameter b_Ω defines a close path in the $\mathcal{Re}(b_\Omega) - \mathcal{Im}(b_\Omega)$ parameter space. A clockwise (CW) motion along the parametric path occurs whenever $\Omega > 0$. The case of counter-clockwise (CCW) evolution corresponds to $\Omega < 0$. At the end of the close path we have that $H(t + T_\Omega) = H(t)$.

A. Preparation, Evolution and Representation for a Floquet Adiabatic State-Flip Scenario

In the previous adiabatic cyclic schemes the emphasis of the analysis was given to the notion of instantaneous Hamiltonians and their corresponding eigenvalues and eigenvectors [13, 14]. In contrast, in the presence of Floquet (fast) driving, like in Eq. (1), the appropriate description of the dynamics is done using the notion of the Floquet Hamiltonian. In this case we shall assume that the preparation and the representation of our evolve state(s) occurs in the corresponding Floquet basis. Therefore, any notion of state-flip via adiabatic encircle of EPs has to be analyzed and examined with respect to this basis. At the same time any notion of EPs has to involve information about the Floquet quasi-energies.

To be specific, we consider a time period of $H(t)$ (see Eq. (1)) from $t = 0$ to $T_\Omega = \frac{2\pi}{\Omega}$. We define the observation times t to be integer multiples of $2\pi/\omega$. We further assume that, between two consequent observation times, the slow varying parameter b_Ω is approximately

a constant. Then the corresponding evolution operator $\tilde{U}(t + \frac{2\pi}{\omega}, t)$ is [15]

$$\tilde{U}\left(t + \frac{2\pi}{\omega}, t\right) \equiv e^{-i\frac{2\pi}{\omega}H_F}; \quad (2)$$

where H_F is the “instantaneous” Floquet Hamiltonian. Its quasi-energies and eigenvectors will be defined as λ_\pm and $|\lambda_\pm\rangle$ respectively.

The dynamics, in the limit of distinct time scales $N \gg 1$, can be evaluated numerically by applying the Floquet evolution matrix Eq. (2) to the initial preparation $|\Psi(t = T_\Omega)\rangle = \prod_{l=0}^{N-1} \tilde{U}(2\pi(l+1)/\omega, 2\pi l/\omega) |\Psi(0)\rangle$ [16]. Note that during the numerical evaluation one needs to consider the variations in b_Ω between subsequent time steps.

We define our adiabatic state-flip control protocols in the following way. Assume that at $t = 0$ we prepare the system at a Floquet eigenstate $|\Psi(t = 0)\rangle = |\lambda_\pm\rangle$. Then we evolve this initial state under Eq. (1) where b_Ω is a slow-varying complex parameter that performs a *fixed* close path in the complex b_Ω parameter space. We want to demonstrate that a controlled variation of ω can lead to evolved states $|\Psi(t)\rangle$ at $t = T_\Omega$ which are proportional to the complementary (or the same) Floquet eigenstates $|\lambda_\mp\rangle$ ($|\lambda_\pm\rangle$).

B. Observables for a Floquet Adiabatic State-Flip Scenario

For our analysis we have introduced the measure $p(t)$ which quantifies the relative weight with which each instantaneous eigenvalue /eigenvector participate in the evolution. Specifically $p(t)$ is defined as

$$p(t) = \frac{|a_+(t)|^2 \lambda_+(t) + |a_-(t)|^2 \lambda_-(t)}{|a_+(t)|^2 + |a_-(t)|^2} \quad (3)$$

where the eigenvector populations $a_\pm(t)$ are evaluated via the decomposition of the evolved state $|\Psi(t)\rangle$ in the instantaneous Floquet basis $|\lambda_\pm\rangle$.

III. THEORETICAL ANALYSIS AND SPECIFIC EXAMPLES OF ADIABATIC STATE-FLIP IN FLOQUET SYSTEMS

We provide below two example cases of Adiabatic State-Flip in Floquet systems. The first example, although artificial, is analytically treatable and provides us with appropriate insight of the Adiabatic State-Flip. Although it can be solved exactly by employing the rotating frame, we have instead used the Floquet language (see Section II) in order to make transparent the general statements for Floquet Adiabatic State-flip. The second example, is treated numerically but it is physically more appealing. For example it can describe, in the frame of coupled mode theory, two periodically driven modes of an

optomechanical system [9, 10]. Alternatively it can describe two resonant modes of two coupled LC resonators in the UHF range [25], whose capacitive coupling and LC impedance of the individual resonators are periodically modulated in time.

A. Example 1

We start our analysis with an analytically solvable system described by the time-dependent Hamiltonian

$$H(t) = \begin{bmatrix} a & b_\omega(t) \\ b_\omega^*(t)b_\Omega(t) & -a \end{bmatrix} \quad (4)$$

where $b_\Omega(t) = e^{i\Omega t}$ and $b_\omega(t) = e^{i\omega t}$ are the two complex time-varying parameters. During the time-interval $t \in [0, T_\Omega]$ the parameter b_Ω defines a unit circle centered around the origin of the $\text{Re}(b_\Omega) - \text{Im}(b_\Omega)$ parameter space. For simplicity we assume that the constant a is a real number. The associated instantaneous Floquet Hamiltonian is

$$H_F = \begin{bmatrix} a + \omega & 1 \\ b_\Omega & -a \end{bmatrix} \quad (5)$$

with corresponding quasi-energies and eigenvectors given by

$$\lambda_\pm = \frac{\omega}{2} \pm \sqrt{b_\Omega + n^2}; \quad |\lambda_\pm\rangle = \begin{bmatrix} n \pm \sqrt{b_\Omega + n^2} \\ b_\Omega \end{bmatrix} \quad (6)$$

where $n \equiv a + \frac{\omega}{2}$. When b_Ω is considered time-independent, the eigensystem Eq. (6) has an EP degeneracy at $b_\Omega^{EP} = -n^2 = -(a + \omega/2)^2$. Depending on the value of the Floquet frequency ω , the *Floquet EP* can be inside ($|n| < 1$), close ($|n| \gtrsim 1$) or far ($|n| \gg 1$) away from the adiabatic unit circle that is defined by the variation of the complex parameter $b_\Omega(t)$.

In the limit of distinct time scales $N \gg 1$, the evaluation of $|\Psi(t = T_\Omega)\rangle$ can be done, in principle, using Eq. (2) and following the general description presented above. However, for the specific Hamiltonian Eq. (5) the evolved state $|\Psi(t)\rangle$ can be analytically evaluated. This analytical treatment will allow us to get some confidence on the general statements about adiabatic state-flip in Floquet systems.

1. Dynamics

We want to evaluate the evolution of the state $|\Psi(t)\rangle$ at integer multiples of $2\pi/\omega$ (within the adiabatic cycle $t \in [0, T_\Omega]$), and specifically its form at the end of the adiabatic circle. First, using a time-dependent transformation $U(t)$, we eliminate the Floquet driving from Eqs. (1,4) and simplify the original Eqs. (1,4) dramatically. The (non-unitary) transformation $U(t)$ is:

$$U(t) = \begin{bmatrix} e^{i\frac{\omega t}{2}} \left(1 + \frac{n^2}{b_\Omega}\right) & e^{i\frac{\omega t}{2}} \frac{n}{b_\Omega} \\ 0 & e^{-i\frac{\omega t}{2}} \end{bmatrix} \quad (7)$$

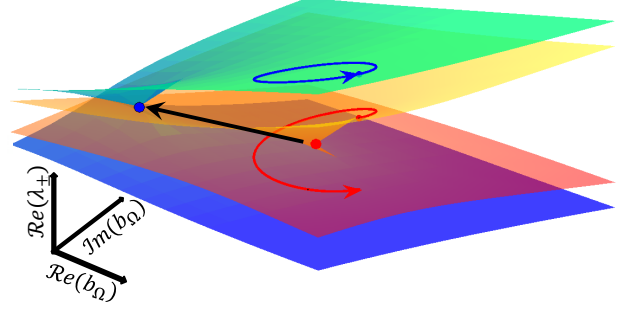


FIG. 1: Overview of the instantaneous Floquet eigenvalue surfaces λ_\pm versus the adiabatic complex parameter b_Ω for the system of Eq. (1). The eigenvalue surfaces correspond to the real part of λ_\pm for two different Floquet driving frequencies $\omega = 0$ (inner red and orange surfaces) and $\omega = 6$ (outer blue and green surfaces). The solid red line represents the real part of $p(t)$, see Eq. (3). The direction of the *fixed* adiabatic cyclic variation of b_Ω is chosen in a way that we have state-flip for $\omega = 0$. When $\omega = 6$ the EP is re-allocated (see black arrow) outside the adiabatic cycle and the system remains at the same instantaneous Floquet state as the initial preparation. In both case $|\Psi(t=0)\rangle = |\lambda_+(t=0)\rangle$ and the parameter $a = 0$. In addition, we shift λ_\pm in Eq. (6) by a real constant so that $\lambda_\pm = 0$ at the EPs.

Substituting $|\Psi(t)\rangle = U(t)|\varphi(t)\rangle$ in Eqs. (1,4) we get [17]

$$i\frac{d}{dt}|\varphi(t)\rangle = H_{eff}|\varphi(t)\rangle, \quad H_{eff} = \begin{bmatrix} 0 & 1 \\ b_\Omega + n^2 & 0 \end{bmatrix} \quad (8)$$

where H_{eff} involves the Floquet (fast) frequency ω as a parameter. Below, without loss of generality, we assume that $n > 0$. For completeness, we also find the instantaneous eigenvalues and eigenvectors of H_{eff} :

$$E_{eff}^\pm(t) = \pm \sqrt{b_\Omega + n^2}; \quad |E_{eff}^\pm(t)\rangle = \begin{bmatrix} 1 \\ E_{eff}^\pm(t) \end{bmatrix}. \quad (9)$$

These eigenvectors are related with the Floquet eigenvectors Eq. (6) via the time-dependent transformation Eq. (7), i.e., $U(t)|E_{eff}^\pm(t)\rangle \propto |\lambda_\pm(t)\rangle$, when t is an integer multiple of $2\pi/\omega$ (observation times). Similarly the instantaneous modes Eq. (9) have the same EP singularity as the instantaneous Floquet eigenvalues.

Let us first discuss the CW variation of b_Ω . The general solution of Eq. (8) can be easily expressed in terms of modified Bessel functions as [18]:

$$|\varphi(t)\rangle = C_1 \begin{bmatrix} I_\nu(\nu z) \\ -e^\tau I'_\nu(\nu z) \end{bmatrix} + C_2 \begin{bmatrix} K_\nu(\nu z) \\ -e^\tau K'_\nu(\nu z) \end{bmatrix} \quad (10)$$

where C_1 and C_2 are arbitrary constants determined by the initial conditions, $I_\nu(\nu z)$ and $K_\nu(\nu z)$ are the ν (=

$\frac{2n}{\Omega}$)-th order modified Bessel function of the first and second kind, $I'_\nu(\nu z)$ ($K'_\nu(\nu z)$) is the derivative of $I_\nu(\nu z)$ ($K_\nu(\nu z)$) with respect to the argument and $z = \sqrt{b_\Omega}/n$, $\tau = i\Omega t/2$. Note that the general solution in Eq. (10) is not a periodic function since the modified Bessel function I_ν and K_ν are multivalued functions.

We are mainly interested in the form of Eqs. (10) at the beginning $t = 0$ and at the end $t = T_\Omega$ of the adiabatic circle. For $t = 0$, and under the adiabaticity condition $\Omega \rightarrow 0^+$ (corresponding to $\nu = \frac{2n}{\Omega} \rightarrow +\infty$), we can easily show that appropriate choice of C_1, C_2 leads to the forms

$$|\varphi(t=0)\rangle \propto \begin{cases} |E_{eff}^-(0)\rangle & \text{when } C_1 = 1; C_2 = 0 \\ |E_{eff}^+(0)\rangle & \text{when } C_1 = 0; C_2 = 1 \end{cases} \quad (11)$$

where we have taken into account that $I'_\nu(\frac{\nu}{n})/I_\nu(\frac{\nu}{n}) \sim \sqrt{1+n^2}$ and that $K'_\nu(\frac{\nu}{n})/K_\nu(\frac{\nu}{n}) \sim -\sqrt{1+n^2}$ [18].

Next we evaluate the evolved state Eq. (10) at the end of the adiabatic circle $t = T_\Omega$. Using the identities $I_\nu(ze^{im\pi}) = e^{im\pi} I_\nu(z)$ and $K_\nu(ze^{im\pi}) = e^{-im\pi} K_\nu(z) - i\pi \sin(\nu m\pi) \csc(\nu\pi) I_\nu(z)$ (m is an arbitrary integer) [18], Eq. (10) can be written, at $t = T_\Omega$, as

$$|\varphi(T_\Omega)\rangle = (C_1 e^{i\nu\pi} - i\pi C_2) \begin{bmatrix} I_\nu(\frac{\nu}{n}) \\ -I'_\nu(\frac{\nu}{n}) \end{bmatrix} + C_2 e^{-i\nu\pi} \begin{bmatrix} K_\nu(\frac{\nu}{n}) \\ -K'_\nu(\frac{\nu}{n}) \end{bmatrix}. \quad (12)$$

In the adiabatic limit $\Omega \rightarrow 0^+$, we can approximate both $I_\nu(\frac{\nu}{n})$ and $I'_\nu(\frac{\nu}{n})$ using their asymptotic forms which are dominated by the exponential factor $e^{\nu\eta}$ [18]. Similarly $K_\nu(\frac{\nu}{n})$ and $K'_\nu(\frac{\nu}{n})$ are dominated by the factor $e^{-\nu\eta}$ [18]. In all cases $\eta \equiv \sqrt{1+1/n^2} - \ln(n + \sqrt{1+n^2})$. Most importantly, its sign is controlled by the magnitude of the Floquet driving frequency ω via the parameter n . At this point, it is important to remind that n also defines the position of the Floquet EPs (see Eq. (9) and discussion below). We find that the transition from positive definite η to negative η -values occurs at $n_C \approx 1.51$.

For $\eta > 0$, corresponding to $n < n_C$, we get that irrespective of the initial conditions Eq. (11) the final state $|\varphi(T_\Omega)\rangle$ is dominated by the first term in Eq. (12) [19]. Taking into account that $I'_\nu(\frac{\nu}{n})/I_\nu(\frac{\nu}{n}) \sim \sqrt{1+n^2}$ [18], we eventually have that $|\varphi(T_\Omega)\rangle \propto |E_{eff}^-(0)\rangle$.

When $\eta < 0$, corresponding to $n > n_C$, one needs to distinguish between two cases for the general solution Eqs. (10, 12). When $C_1 = 1$ and $C_2 = 0$ (corresponding to $|\varphi(t=0)\rangle \propto |E_{eff}^-(0)\rangle$, see Eq. (11)), we have that $|\varphi(t=T_\Omega)\rangle \propto |E_{eff}^-(0)\rangle$. If on the other hand $C_1 = 0$ and $C_2 = 1$ (corresponding to $|\varphi(t=0)\rangle \propto |E_{eff}^+(0)\rangle$, see Eq. (11)), then we have that $|\varphi(T_\Omega)\rangle \propto |E_{eff}^+(0)\rangle$. Comparison with Eqs. (11) lead us to the conclusion that, whenever $n > n_C$, we always come back at the

initial instantaneous state at the end of the adiabatic circle.

The case of counter-clockwise (CCW) variation of b_Ω corresponds to the limit of $\Omega \rightarrow 0^-$ and can be treated in a similar manner. Redefining ν to be $-2n/\Omega$ and performing the same analysis as above we arrive at the following conclusions: when $n < n_C$, irrespective of the initial preparation Eq. (11), we get $|\varphi(T_\Omega)\rangle \propto |E_{eff}^+(0)\rangle$; when $n > n_C$ we always come back, at the end of the circle $t = T_\Omega$, to the initial Floquet eigenstate.

2. Floquet state-flip protocols

The chiral state-flip has been recently predicted theoretically [12, 13, 20] and demonstrated experimentally [9, 11] for adiabatic cyclic variations of non-Hermitian Hamiltonians which encircle an EP— though these investigations have been performed in the absence of any Floquet driving i.e. $\omega = 0$. In fact, these studies underplay (or even disregarded) the fact that a state-flip can also occur in the case that the EP is outside, but still in the vicinity, of an adiabatic cyclic variation – as indicated by our analysis above. Along these lines we also point out the study of Ref. [12] (corresponding to $\omega = 0$) which considered the state-flip scenario for an oscillatory motion along a straight path located in some distance away from the EP (see also recent publication [22]). In fact Hamiltonian Eq. (8) can be considered a starting point for the analysis of a state-flip due to an adiabatic encircling of an EP associated with a non-Hermitian Hamiltonian without any Floquet driving. Therefore all our conclusions apply in the latter framework as well. It is therefore tempting to speculate that the constraint $n^2 \equiv |b_\Omega^{EP}| \leq |b_\Omega| = 1$ is very restrictive (a sufficient condition) i.e. under this condition one *necessarily* has state-flip. In fact a less restrictive condition is that $|n| < |n_C|$, (where in our case $1 < |n_C|$).

Let us finally demonstrate that the ability to engineer the topological features of an adiabatic circle via Floquet frequency ω -variations can be utilized for the controlled manipulation of state-flip between the two “instantaneous” Floquet eigenstates $|\lambda_\pm(t=0)\rangle$. In Fig. 1 we report the real part of the instantaneous Floquet eigenmodes λ_+ (upper two surfaces) and λ_- (lower two surfaces) in the complex b_Ω -parameter space for two different Floquet driving frequencies ω when $a = 0$. Specifically, the red-orange (inner) surfaces correspond to $\omega = 0$ while the blue-green (outer) surfaces correspond to $\omega = 6$. The projection of the real part of $p(t)$ in this space is also shown with red (blue) lines for $\omega = 0$ ($\omega = 6$). Finally the corresponding EP are indicated with filled red (blue) circles respectively. We see that the Floquet driving has re-allocated the position of the EP in the b_Ω parameter space. Specifically, while for $\omega = 0$ the EP is inside the adiabatic cyclic path, it is re-allocated far away from the circle when the Floquet frequency is

$\omega = 6$. In both cases the variation of b_Ω (both variation rate and direction of variation of control parameters $\mathcal{R}e(b_\Omega) - \mathcal{I}m(b_\Omega)$) has been kept fixed. The direction of the adiabatic circle has been chosen in such a way that the system undergoes a state-flip at $\omega = 0$. When the Floquet frequency has been reconfigured to the value $\omega = 6$ the EP has been re-allocated outside the adiabatic circle – thus enforcing the system to evolve to the initial state at the end of the adiabatic cycle $t = T_\Omega$.

B. Example 2

The above scheme is not specific to the toy model Eq. (1). To further confirm its validity, we now perform simulations with a driven Hamiltonian which describes two (evanescently) coupled resonators. We note that the same model Eq. 13 can be also realized in the framework of optical coupler [23]. The Hamiltonian takes the form

$$H(t) = \begin{bmatrix} \epsilon_1 - \frac{\nu\gamma}{2} & \kappa \\ \kappa & \epsilon_2 + \frac{\nu\gamma}{2} \end{bmatrix}, \quad (13)$$

where κ is the coupling strength between the two resonators, ϵ_1, ϵ_2 are the eigenfrequencies of each resonator, and γ is the gain (loss) parameter that describes the loss (gain) at first (second) resonator. We further assume that the resonant frequencies of each of these resonators are periodically modulated as $\epsilon_1(t) = -\epsilon_2(t) = -\frac{F}{2} \sin \omega t - r$. The “fast” (Fourier) variation with period ω can be achieved via modulation of the permittivities (say via a current injection) of the resonators. The parameters r and κ represent two additional variables that vary slowly in time (adiabatic parameters). A possible way to achieve this slow modulation is by bringing in the vicinity of the resonators a mechanical cantilever which oscillates with a slow frequency $\Omega \ll \omega$. In the framework of Eq. (1) one can now identify $b_\omega(t) = \sin(\omega t)$, $\mathcal{R}e(b_\Omega(t)) = r_\Omega(t)$ and $\mathcal{I}m(b_\Omega(t)) = \kappa_\Omega(t)$. In our numerical calculations, below, we shall use $r_\Omega(t) = r_0 \sin(\Omega t)$ and $\kappa_\Omega(t) = \kappa_0 + \kappa_1 \cos(\Omega t)$.

Following the same analysis as previously, we first identify the instantaneous Floquet eigenstates and quasi-energies, that will be used for the preparation and observation of the evolved state. The associated “instantaneous” Floquet Hamiltonian H_F at times which are multiples of $2\pi/\omega$, and in the $\omega \rightarrow \infty$ limit and $F \sim \mathcal{O}(\omega)$, is [24]

$$H_F \approx \begin{bmatrix} -r_\Omega - \frac{\nu\gamma}{2} & \kappa_\Omega J_0\left(\frac{F}{\omega}\right) e^{-i\frac{F}{\omega}} \\ \kappa_\Omega J_0\left(\frac{F}{\omega}\right) e^{i\frac{F}{\omega}} & r_\Omega + \frac{\nu\gamma}{2} \end{bmatrix}, \quad (14)$$

where J_0 is the 0-order Bessel function of the first kind. The instantaneous Floquet eigenvalues are evaluated as $\lambda_\pm = \pm \sqrt{[\kappa_\Omega J_0\left(\frac{F}{\omega}\right)]^2 + [r_\Omega + \nu\gamma/2]^2}$. The instantaneous EPs occur at $(r_\Omega^{EP}, \kappa_\Omega^{EP}) = (0, \pm \frac{\gamma}{2J_0(F/\omega)})$. The corresponding eigenvectors are denoted as $|\lambda_\pm(t)\rangle$. Their

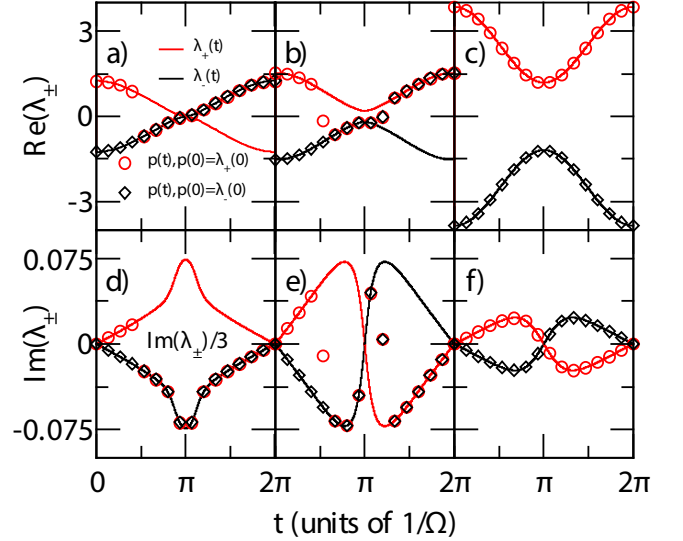


FIG. 2: The evolution, during one adiabatic cycle, of the instantaneous Floquet eigenvalues and the associated $p(t)$ for the system described by Eq. (13). Upper row reports the real part of these quantities while the lower row reports their corresponding imaginary part. (a,d) The Floquet frequency is $\omega = 50$ and it is chosen in a way that the adiabatic cyclic variation of r_Ω and κ_Ω encloses an instantaneous Floquet EP; (b,e) The Floquet frequency is $\omega = 52$ and it is chosen in a way that the EP is in the vicinity of the adiabatic cyclic variation of r_Ω and κ_Ω ; (c,f) The Floquet frequency is $\omega = 80$. In this case the EP is far away from the parameter domain where the adiabatic cyclic variation of r_Ω and κ_Ω occurs. In all cases the fixed adiabatic parameter variations r_Ω and κ_Ω are $r_\Omega(t) = r_0 \sin(\Omega t)$ and $\kappa_\Omega(t) = \kappa_0 + \kappa_1 \cos(\Omega t)$ where $r_0 = 0.1$, $\Omega = 0.01$, $\kappa_0 = 4$ and $\kappa_1 = 2$. The other parameters are $\gamma = 1$ and $F = 100$.

expressions are rather complicated and we do not give them here. Below, we have evaluated them numerically for each observation time t during the evolution, via a direct diagonalization of the Hamiltonian Eq. (14).

In Fig. 2 we report the evolution of eigenvalues $\lambda_+(t)$ (red lines) and $\lambda_-(t)$ (black lines) together with the evolution of the corresponding $p(t)$ (red circles and black diamonds) for three different values of the Floquet driving frequency $\omega = 50, 52$ and 80 . The upper row corresponds to the real part of these quantities while their imaginary part is reported in the lower row. The parameters used in these simulations are $\gamma = 1$ and $F = 100$, while the slow varying parameters r_Ω, κ_Ω have been chosen to change as $r_\Omega(t) = r_0 \sin(\Omega t)$ and $\kappa_\Omega(t) = \kappa_0 + \kappa_1 \cos(\Omega t)$ where $r_0 = 0.1$, $\Omega = 0.01$, $\kappa_0 = 4$ and $\kappa_1 = 2$. For all ω -values presented in Fig. 2 the rate and the direction of evolution of the adiabatic circle remains the same. In Figs. 2a,d the EP is inside the circle while in Figs. 2b,e is in the proximity of it. In both cases, we find a state-flip as predicted from the analysis of the theoretical model Eqs. (1,4).

In contrast, in Figs. 2c,f the Floquet frequency ω is such that it has re-allocate the EP far away from the adiabatic circle. In this case we do not observe a state-flip. Instead the system remains at the same state as the original one at the end of the cycle at $t = T_\Omega$.

IV. CONCLUSIONS

We consider the state evolution of a non-Hermitian system which is exposed to two driving schemes with strong time-scale separation. We have introduced the notion of Floquet state-flip due to adiabatic encircling of instantaneous Floquet EP singularities. Then we have used the extra degree of freedom that the Floquet driving is offering in order to re-organize the position of EP with respect to an adiabatic cycle associated with a slow variation of two additional parameters of the Hamiltonian. This EP re-organization leads to a tailoring of the topological features of the adiabatic cycle and allow us a state-flip reconfigurability.

Let us finally comment on the experimental realization of the proposed Floquet protocol using existing platforms [10, 25]. Specifically, the authors of the recent experimental work [10] have already demonstrated chiral energy transfer between two highly non-degenerate modes

in cavity optomechanical system driven by two drivings schemes with frequency separation. Essentially they realized an effective Hamiltonian, resembling Eq. (4), which describes the dynamics in the presence of both a fast and a slow Floquet driving. These authors have indicated the existence of *virtual* EPs i.e. parameter values for which the system respond *as if* it is in the neighborhood of a parameter domain where eigenvalues and eigenvectors coalesce. However, because they did not use the Floquet picture, they were not able to identify any EP degeneracy which naturally emerges from the analysis of the quasi-modes of the instantaneous Floquet Hamiltonian. In this respect, our work provides the appropriate mathematical framework for the analysis of such type of systems. Instead, they have *speculated* the existence of the EPs by observing the chiral mode flipping effect. Clearly the new concept of Floquet EPs opens up possibilities to understand more complicated driving scenarios. Some of them, will be further investigated in the future using the experimental platform of \mathcal{PT} -circuitry [25].

Note added: During the referee process of this paper we became aware by a referee of the recent paper [26]. This study investigates the effects of a Floquet exceptional point at a multi-period evolution of a *one-frequency* time-periodic Hamiltonian.

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