Weak measurements and quantum weak values for NOON states

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Weak measurements and quantum weak values for NOON states

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Quantum weak values arise when the mean outcome of a weak measurement made on certain preselected and postselected quantum systems goes beyond the eigenvalue range for a quantum observable. Here, we propose how to determine quantum weak values for superpositions of states with macroscopically or mesoscopically distinct mode number, that might be realised as two-mode Bose-Einstein condensate and photonic NOON states. Specifically, we give a model for a weak measurement of the Schwinger spin of a two-mode NOON state, for arbitrary N. The weak measurement arises from a quantum nondemolition measurement of number difference, which for atomic NOON states can be realised via the ac Stark effect using an optical meter prepared in a coherent state. The meter-system coupling results in an entangled cat-state. By subsequently evolving the system under the action of a nonlinear Josephson Hamiltonian, we show how postselection results in quantum weak values, for arbitrary N. Since the weak measurement can be shown to be minimally invasive, the weak values provide a useful strategy for a Leggett-Garg test of N-scopic realism.

I. INTRODUCTION

It would seem impossible that the outcome of a measurement of a quantum observable could yield an average that is outside the eigenvalue range associated with the observable spectra. Yet such a paradoxical situation was predicted by Aharonov, Albert and Vaidman in their paper entitled "How the result of a measurement of a component of the spin of a spin 1/2 particle can turn out to be 100" [1]. The situation arises for the outcomes of so-called *weak measurements* [1–15]. Weak measurements are measurements that couple weakly to the quantum system being measured, so as to give a minimal disturbance to that system. In their paper, Aharonov, Albert and Vaidman explained how one can perform a weak measurement of the spin σ_z of a spin 1/2 particle and obtain a result where the average $\langle\sigma_z\rangle$ exceeds 100.

The paradoxical measurement outcomes that lead to the strange predictions are called quantum *weak values* [1–4, 6–15]. Weak values arise as the outcomes of weak measurements on systems prepared by preselection and postselection. The weak values are created by the phenomenon of quantum interference and have been used to interpret quantum mechanics in scenarios where quantum interference leads to counter-intuitive predictions [3, 11, 12, 16]. The weak measurements and weak values also have practical application, in providing a means to monitor a quantum system with a demonstrably minimal disturbance to that system [3]. For this reason, weak measurements have been used to test Leggett-Garg's form of macro- and micro-realism in experiments that show violation of Leggett-Garg inequalities [3, 7–15].

The topic of weak values has attracted much interest. The experimental prediction of Aharonov, Albert and Vaidman has been realised at the level of a spin 1/2 system by Pryde et al, who demonstrated weak values for a photonic qubit [4]. Their weak measurement scheme involved a single photon interacting with the photonic qubit in a process that created an entangled state. The experiment detected weak values outside the eigenvalue range for the spin σ_z defined by the polarisation of the photon.

Goggin et al applied the weak measurement of Pryde et al in an experiment that demonstrated failure of the Leggett-Garg premises for the microscopic photonic system [9]. The Leggett-Garg premises are: firstly, that the system must be, prior to the measurement, in one spin state or the other ("up" *or* "down"); and, secondly, that a measurement can in principle be performed on the system to determine which spin state the system is in, without interfering with the subsequent two-state spin dynamics [17]. The measurement perceived by Leggett and Garg is called the non-invasive measurement. The connection between quantum weak values and the violation of Leggett-Garg inequalities was formalised by Ruskov et al, Jordan et al, and Williams and Jordan, who showed that if a weak measurement is used as the non-invasive measurement, then the violation of the inequalities is associated with the appearance of weak values [7, 8].

While there has been much progress and insight gained into quantum mechanics using weak values, to date this has not been fully extended to mesoscopic or macroscopic systems. Weak measurements have been used to probe quantum states and to demonstrate violation of Leggett-Garg inequalities in superconducting circuits [10, 14]. Williams and Jordan proposed the implementation of a weak measurement with quantum weak values for solid-state qubits, that could be generalised to macroscopic superconducting systems based on the assumption of a macroscopic qubit [8]. This was followed by an experimental observation of weak values for a superconducting circuit [6]. However, to our knowledge, there has been no experimental report of quantum weak values for superposition states involving even moderate numbers of photons or atoms. The potential for weak values in mesoscopic atomic systems was illustrated by Huang and Agarwal [20], who studied the quantum interference arising from two close-lying atomic coherent states, and showed how the phase shift due to the quantum interference can be amplified using weak measurements.

In this paper, we consider a quantum weak value gedanken experiment that applies to NOON states, given as [21, 22]

$$|NOON\rangle = d_N|N\rangle_a|0\rangle_b + d_0|0\rangle_a|N\rangle_b \quad (1)$$

Here $|N\rangle_a$ and $|N\rangle_b$ are number states for two modes (that we denote by a and b) and d_0, d_N are probability amplitudes. We give a model for a quantum measurement of the spin of the two-mode quantum system, the spin being defined as $\hat{S}_z/2N$ where $\hat{S}_z$ is the two-mode Schwinger operator for the number difference between the two modes. The interaction due to the measurement couples a meter system to the quantum system, with a coupling strength γ, creating an entanglement between the meter and quantum system [23, 24]. In the limit of large γ, a final homodyne detection would collapse the quantum system into a state of definite spin, thus completing the von Neumann measurement process. For weak coupling γ, this collapse does not take place and the system is minimally disturbed by the measurement. For all γ, however, the average spin $\langle\hat{S}_z\rangle$ can be correctly evaluated.

Similar to Ref. [8], we demonstrate weak values by considering a unitary evolution from a time t_2 to a time t_3 that rotates the probability amplitudes associated with the NOON state, while retaining the two-state nature of the system. The weak values are obtained by postselecting the result for the measurement at time t_2 given a result at time t_3. We show that the two-state NOON unitary evolution can be realised to an excellent approximation by the Hamiltonian used to model two trapped Bose-Einstein condensates with a Josephson coupling, in certain parameter regimes that include nonlinear effects [25–32]. In fact, by solving the two-mode nonlinear Josephson Hamiltonian, we find that weak values are predicted over a range of parameter values, including where the system is not the ideal NOON state at time t_3, but rather a superposition of two mesoscopically distinguishable states with a range of outcomes for $\hat{S}_z$. This suggests an experimental realisation to be feasible.

The proposed weak measurement opens a way to test mesoscopic realism using weak values and NOON states. This is because a measurement can be constructed for the system at a time t_2 that can be justified as noninvasive for the test of the Leggett-Garg inequality. We give details of how one can experimentally demonstrate the non-invasiveness of the weak measurement, and give predictions for such an experiment, confirming the connection between the observation of weak values and the violation of the Leggett-Garg inequalities for a macroscopic superposition. A preliminary description of this proposal for a Leggett-Garg test of macro-realism has been presented in a Letter [33].

The paper is organised as follows. In Sections II and III we give details of the weak measurement model of the spin $\hat{S}_z$. In Section IV we show how the weak values emerge for the postselected spin at time t_2. The Leggett-Garg test of meso-realism and the Josephson Hamiltonian is explained in Section V. In Section VI, we give predictions for weak values and violation of Leggett-Garg inequalities using the Josephson model in the non-ideal case.

II. A QND MEASUREMENT FOR SPIN

We consider a two-mode system. The boson creation and destruction operators for the modes are $\hat{a}^\dagger, \hat{a}$ and $\hat{b}^\dagger, \hat{b}$ respectively and we will denote the modes by the symbols a and b. The operators $\hat{J}_z = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$, $\hat{J}_x = (\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})/2$, $\hat{J}_y = (\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/2i$, $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$ are the Schwinger spin operators (we take $\hbar = 1$). For convenience, we introduce the population (number) difference operator $\hat{S}_z = 2\hat{J}_Z$. Thus,

$$\hat{S}_z = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} = \hat{n}_a - \hat{n}_b \quad (2)$$

where $\hat{n}_a = \hat{a}^\dagger\hat{a}$ and $\hat{n}_b = \hat{b}^\dagger\hat{b}$. The objective is to give a (non-invasive) measurement of the spin J_z (or $\hat{S}_z$) of the two-mode system. The two-mode system could be a Bose-Einstein condensate (BEC) in a double well potential [25–29, 32], a two-component BEC where each component is associated with a distinct atomic levels and a distinct mode [31], or a two-mode photonic state [21]. In the weak value gedanken experiment that we discuss in Section III, a weak measurement is to be performed at a time t_2 (Figure 1).

We consider the quantum non-demolition (QND) measurement M for $\hat{J}_z = \hat{S}_z/2$ described by the measurement Hamiltonian [23, 24]

$$H_M = \hbar G\hat{S}_z\hat{n}_c/2 \quad (3)$$

The measurement is performed by coupling the two-mode system to an optical field. The field is modelled as a single mode with boson operator $\hat{c}$ and number operator $\hat{n}_c = \hat{c}^\dagger\hat{c}$. The optical "meter" field is prepared in a coherent state $|\gamma\rangle$ and coupled to the two-mode system for a time τ. The measurement interaction is modelled by the Hamiltonian H_M where G is the coupling constant.

In this paper, we consider systems that are eigenstates of $\hat{N} = \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}$. We denote the total number of particles (atoms or photons) as N. Assuming a pure state, the general form of the two-mode state immediately prior to measurement is

$$|\psi\rangle_{in} = \sum_{m=0}^{N} d_m|m\rangle_a|N-m\rangle_b \quad (4)$$

where d_m are probability amplitudes. As a first step, we consider how to measure $\langle S_z\rangle$ of this state. The output *after* the measurement is given by

$$\begin{aligned}|\psi\rangle_{out} &= e^{-iH\tau/\hbar}\sum_m d_m|m\rangle_a|N-m\rangle_b|\gamma\rangle_c \\ &= \sum_{m=0}^{N} d_m|m\rangle_a|N-m\rangle_b|\gamma e^{iG\tau(N-2m)}\rangle_c \quad (5)\end{aligned}$$

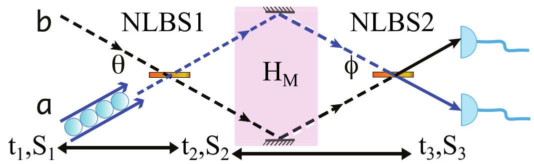

Figure 1. *Schematic of an experiment to detect weak values for NOON states:* A system is prepared in a NOON state at the time t_2. Here, the preparation involves N bosons in a mode a incident on a nonlinear medium at time t_1 as described in Sections IV and V. A second mode b is initially in a vacuum state. The nonlinear interaction is modelled by the Hamiltonian H_I. This interaction is symbolised by the nonlinear beam splitter $NLBS1$, where θ denotes the time of interaction $t_2 - t_1$ in scaled units (see main text). The spin S_i is defined to be $+1$ or -1 according to the sign of the mode number difference at time t_i. Once the NOON state is prepared by the first nonlinear beam splitter, a weak QND measurement M of the spin S_2 takes place at time t_2, as depicted by the purple shading. The measurement interaction is described by the Hamiltonian H_M. After the measurement, the system evolves under the action of H_I for a time denoted ϕ (in scaled units), as symbolised by the second nonlinear beam splitter $NBS2$. Assuming an near-instantaneous measurement, the time ϕ is $t_3 - t_2$ in scaled units. After the second interaction, a strong measurement of S_3 takes place at time t_3. Weak values are observed when the value $\langle S_2\rangle_{S_3=1}$ of the spin S_2 conditional on the result $S_3 = 1$ exceeds the eigenvalue bounds given by $|\langle S_2\rangle| \leq 1$.

The state after an interaction time $\tau = \pi/2NG$ is:

$$|\psi\rangle = \sum_{m=0}^{N} d_m|m\rangle_a|N-m\rangle_b|\gamma e^{i\pi(N-2m)/2N}\rangle_c \qquad (6)$$

Homodyne detection on the optical system enables measurement of the meter quadrature phase amplitude $\hat{p} = (\hat{c} - \hat{c}^\dagger)/i$. For γ large, the different values of $\hat{S}_z$ are measurable by outcomes for $\hat{p}$ and the two-mode system after the homodyne measurement collapses to a state of definite $\hat{S}_z$. This is the limit of a strong or projective measurement. More generally, for all values of γ, it has been shown by Ilo-Okeke and Byrnes [24] that

$$\langle \hat{S}_z\rangle = -\frac{1}{2\gamma}\langle \hat{p}\rangle \qquad (7)$$

The average of $\hat{p}$ gives the value for the average of the Schwinger spin $\hat{S}_z$ of the incident two-mode state. This relation is true for all values of γ including the limit where $\gamma \to 0$, called the weak measurement limit.

III. A WEAK MEASUREMENT ON NOON STATES

Our interest in this paper is where the incident state (4) before the measurement M is a macro- or mesoscopic superposition state. Specifically, we consider the case where the two-mode system (4) is the ideal NOON state given by

$$|\psi\rangle_{in} = d_0|0\rangle_a|N\rangle_b + d_N|N\rangle|0\rangle_b \qquad (8)$$

In this case, the outcome of the measurement $\hat{S}_z$ is either N or $-N$. For later convenience, we suppose the measurement is made on the system at the time t_2, so that the state $|\psi_{in}\rangle$ before the measurement is created at time t_2 (Figure 1). We define S_2 to be the outcome of the normalised measurement $\hat{S}$ defined as $\hat{S} = \hat{S}_z/N$ at this time t_2. More generally, we define the outcome of the measurement of $\hat{S}$ at time t_i to be S_i.

Where the input state incident (4) on the measurement device is the NOON state (8), the final state (6) after measurement M can be written:

$$|\psi\rangle = d_0|0\rangle_a|N\rangle_b|i\gamma\rangle_c + d_N|N\rangle_a|0\rangle_b|-i\gamma\rangle_c \qquad (9)$$

This state describes an entanglement of the two-mode quantum system with the meter field. The two outputs (the two-mode state and the meter field) are next spatially separated, and a measurement is then made of the quadrature $\hat{p} = \frac{1}{i}(\hat{c} - \hat{c}^\dagger)$ of the meter field. For γ large, the two different values $\pm N$ for $\hat{S}_z$ (and hence of S_2) are measurable by the different sign of the outcomes for $\hat{p}$. The two-mode system after the homodyne measurement collapses to a state of definite $\hat{S}_z$, either the eigenstate $|N\rangle_a|0\rangle_b$ with eigenvalue N or the eigenstate $|0\rangle_a|N\rangle_b$ with eigenvalue $-N$.

A measurement of the meter quadrature p thus yields a measurement of the spin S_2. We can evaluate $\langle\hat{p}\rangle$ directly from (9) to give the relationship

$$\begin{aligned}\langle\hat{p}\rangle &= 2\gamma(|d_0|^2 - |d_N|^2) \\ &= -2\gamma\langle S_2\rangle \end{aligned} \qquad (10)$$

Details are given in the Appendix. Here we have used $\langle S_2\rangle = |d_N|^2 - |d_0|^2$, which is the expectation value of S_2 for the initial two-mode state (8). We see that $\langle S_2\rangle = -\frac{1}{2\gamma}\langle\hat{p}\rangle$ consistent with the general result (7) given in Ref. [24]. The average of $\hat{p}$ will give the value for the average of the Schwinger spin of the incident two-mode state.

We suppose as in Figure 1 that the measurement M takes place at a time t_2. The time t_1 is reserved for earlier events that lead to the preparation of the NOON state at time t_2. We next consider that the two-mode state evolves for a time t under an interaction Hamiltonian H_I, and that a projective measurement is made at the later time t_3. The Hamiltonian is unspecified at this stage, except that it conserves the total particle number N. We will consider that the measurement time τ is small compared to the later evolution time t so that we take $t = t_3 - t_2$. Under the evolution due to H_I, the output state (6) (or (9)) produced immediately *after* the measurement at time t_2 evolves to a new state at the time t_3. The Hamiltonian H_I is such that the two-mode state $|m\rangle|N-m\rangle$ evolves to the state given by

$$\sum_n c_n^{(m)}|n\rangle|N-n\rangle \tag{11}$$

where $c_n^{(m)}$ are probability amplitude constants. The final output state including the meter field is:

$$|\psi(t_3)\rangle = \sum_m d_m|\gamma e^{i\pi(N-2m)/2N}\rangle_c \sum_n c_n^{(m)}|n\rangle_a|N-n\rangle_b \tag{12}$$

An experimentalist can measure S_3 at the final time t_3. The experimentalist can also measure the outcome p of the measurement $\hat{p}$ of the meter field, and obtain the correlation $\langle pS_3\rangle$. We next evaluate $\langle pS_3\rangle$ and compare with $\langle S_2S_3\rangle$.

In this section, we take the case where just prior to the measurement at time t_2 the two-mode system is in the NOON state (8). At time t_3, after measurement and after the subsequent evolution, the overall state is given by Eq. (12) which we simplify as:

$$\begin{aligned}|\psi(t_3)\rangle &= d_0|\gamma e^{+i\pi/2}\rangle_c \sum_n c_n^{(0)}|n\rangle_a|N-n\rangle_b \\ &+ d_N|\gamma e^{-i\pi/2}\rangle_c \sum_n c_n^{(N)}|n\rangle_a|N-n\rangle_b\end{aligned} \tag{13}$$

We evaluate $\langle pS_3\rangle = \langle\psi(t_3)|pS_3|\psi(t_3)\rangle$. Using that

$$\begin{aligned}\langle S_2S_3\rangle &= |d_N|^2\Big(-\sum_{n<N/2}\Big|c_n^{(N)}\Big|^2 + \sum_{n>N/2}\Big|c_n^{(N)}\Big|^2\Big) \\ &- |d_0|^2\Big(-\sum_{n<N/2}\Big|c_n^{(0)}\Big|^2 + \sum_{n>N/2}\Big|c_n^{(0)}\Big|^2\Big)\end{aligned} \tag{14}$$

we find

$$\langle S_2S_3\rangle = -\frac{1}{2\gamma}\langle pS_3\rangle \tag{15}$$

Details of the calculation are given in the Appendix. In summary, if immediately prior to measurement at time t_2 the two-mode system is in the generalised NOON state (8), then we have confirmed the relation (15). This relation is true for all values of the measurement coupling strength γ. The weak measurement result where $\gamma \to 0$ gives the same average as the strong (projective) measurement result (large γ).

The expression (15) enables a weak measurement strategy to be employed for a Leggett-Garg test of macroscopic or mesoscopic realism. The measurement M at time t_2 is made with a very small γ. The measurement can then be demonstrated to be noninvasive in the limit of $\gamma \to 0$. The average $\langle pS_3\rangle$ can be determined accurately by measuring over many trials, to give an accurate value for $\langle S_2S_3\rangle$ that can be used to test the LG inequality. This approach was used in the experiment of Goggin et al, for $N = 1$ [9].

IV. WEAK VALUES

Continuing with the case where we make a weak measurement at time t_2 on a NOON state (8), we now show how weak values emerge from the weak measurement (Figure 1). In the case where the system is in a NOON state, the possible values of S_i at time t_i are $+1$ and -1. Where the values of $\hat{S}_z$ may be different to $\pm N$ at time t_3, as is the case for non-ideal states examined in Section VI, we define the binned measurement $\tilde{S}_3$ made at time t_3 to be $+1$ if the outcome S_z of $\hat{S}_z$ satisfies $S_z \geq 0$, and -1 if $S_z < 0$. To realise quantum weak values, we will evaluate the mean value for S_2, given that the result $+1$ is detected for $\tilde{S}_3$. Weak values are observed when the value $\langle S_2\rangle_{\tilde{S}_3=1}$ of the spin S_2 conditional on the result $\tilde{S}_3 = 1$ exceeds the eigenvalue bounds given by $|\langle S_2\rangle| \leq 1$.

At time t_3, after the weak measurement and after the subsequent evolution, the state is given by Eq. (12). We expand into a superposition of states giving a positive value of $\tilde{S}_3$ and states giving a negative value of $\tilde{S}_3$:

$$\begin{aligned}|\psi(t_3)\rangle &= d_0|\gamma e^{+i\pi/2}\rangle_c \sum_{n\geq N/2} c_n^{(0)}|n\rangle_a|N-n\rangle_b \\ &+ d_N|\gamma e^{-i\pi/2}\rangle_c \sum_{n\geq N/2} c_n^{(N)}|n\rangle_a|N-n\rangle_b \\ &+ d_0|\gamma e^{+i\pi/2}\rangle_c \sum_{n<N/2} c_n^{(0)}|n\rangle_a|N-n\rangle_b \\ &+ d_N|\gamma e^{-i\pi/2}\rangle_c \sum_{n<N/2} c_n^{(N)}|n\rangle_a|N-n\rangle_b\end{aligned} \tag{16}$$

Here we have allowed that a state more general than a NOON state may be generated at time t_3. An experimentalist can measure S_3 at the final time t_3 and postselect for the outcome $\tilde{S}_3 = 1$. Where the system at time t_3 is in a NOON state, the postselection is conditional on $S_3 = 1$. The experimentalist can also measure $\hat{p}$ of the meter field, and obtain the mean value for the outcomes p (and hence the inferred S_2) conditional on the result $\tilde{S}_3 = 1$. We denote these conditional moments as $\langle p\rangle_{\tilde{S}_3=1}$, or $\langle S_2\rangle_{S_3=1}$. We see that

$$\langle p\rangle_{\tilde{S}_3=1} = \frac{\int pP(p,S_3\geq 0)dp}{P(S_3\geq 0)} \tag{17}$$

where

$$P(S_3\geq 0) = \int P(p,S_3\geq 0)dp \tag{18}$$

Using the general state given in Eq. (16) we obtain

$$\begin{aligned}P(p,S_3\geq 0) &= |d_0|^2\left|\langle p|\gamma e^{+i\pi/2}\rangle\right|^2 \sum_{n\geq N/2}\Big|c_n^{(0)}\Big|^2 \\ &+ |d_N|^2\left|\langle p|\gamma e^{-i\pi/2}\rangle\right|^2 \sum_{n\geq N/2}\Big|c_n^{(N)}\Big|^2 \\ &+ Int\end{aligned} \tag{19}$$

where

$$Int = d_0^* d_N \langle \gamma e^{+i\pi/2}|p\rangle_c \langle p|\gamma e^{-i\pi/2}\rangle \sum_{n\geq N/2} c_n^{(0)*} c_n^{(N)}$$
$$d_N^* d_0 \langle \gamma e^{-i\pi/2}|p\rangle_c \langle p|\gamma e^{+i\pi/2}\rangle \sum_{n\geq N/2} c_n^{(N)*} c_n^{(0)} \quad (20)$$

is a quantum interference term. In fact

$$\langle p|\gamma e^{\pm i\frac{\pi}{2}}\rangle = \frac{\exp\left(-\frac{p^2}{4} + \frac{\gamma^2 e^{\pm i\pi}}{2} - \frac{\gamma^2}{2} - ip\gamma e^{\pm i\frac{\pi}{2}}\right)}{(2\pi)^{1/4}} \quad (21)$$

and $\left|\langle p|\gamma e^{\pm i\pi/2}\rangle\right|^2 = \frac{e^{-\left(\frac{p}{\sqrt{2}}-\sqrt{2}\gamma\right)^2}}{\sqrt{2\pi}}$. Hence we can evaluate the conditional moments once we specify the evolution during the time from t_2 to t_3.

In the next Section, we consider an evolution H_I that gives rise to a violation of a Leggett-Garg inequality. We will restrict to this case. We thus consider the interaction Hamiltonian H_I defined in Section V that evolves an initial state $|0\rangle_a|N\rangle_b$ at time t_2 into the state

$$\cos(t_3 - t_2)|0\rangle_a|N\rangle_b + i\sin(t_3 - t_2)|N\rangle_a|0\rangle_b \quad (22)$$

at the later time t_3. Here time t_i is expressed in suitably scaled units, which will be defined in the next section. The interaction also evolves the state $|N\rangle_a|0\rangle_b$ at time t_2 into the state

$$i\sin(t_3 - t_2)|0\rangle_a|N\rangle_b + \cos(t_3 - t_2)|N\rangle_a|0\rangle_b \quad (23)$$

defined at time t_3. Hence we substitute in the expression (16)

$$\begin{aligned} c_0^{(0)} &= \cos(t_3 - t_2) \\ c_N^{(0)} &= i\sin(t_3 - t_2) \\ c_0^{(N)} &= i\sin(t_3 - t_2) \\ c_N^{(N)} &= \cos(t_3 - t_2) \end{aligned} \quad (24)$$

All other coefficients are zero. In this case, $\tilde{S}_3 = S_3$ since an ideal NOON state is created at time t_3. Using Eq. (17), we find

$$\begin{aligned} P(p, S_3 \geq 0) = &\frac{|d_0|^2|c_N^{(0)}|^2}{\sqrt{2\pi}} e^{-\left(\frac{p}{\sqrt{2}}-\sqrt{2}\gamma\right)^2} \\ &+ \frac{|d_N|^2|c_N^{(N)}|^2}{\sqrt{2\pi}} e^{-\left(\frac{p}{\sqrt{2}}+\sqrt{2}\gamma\right)^2} \\ &+ Int \end{aligned} \quad (25)$$

Hence

$$\begin{aligned} \int pP(p, S_3 \geq 0)dp &= 2\gamma\left(|d_0|^2|c_N^{(0)}|^2 - |d_N|^2|c_N^{(N)}|^2\right) \\ &= -\gamma\cos^2\theta \end{aligned} \quad (26)$$

where we note the interference terms do not contribute to this term, since $\int_{-\infty}^{\infty} \exp\left(-\frac{p^2}{2} - 2\gamma^2\right) pdp = 0$. Also,

$$\begin{aligned} P(S_3 \geq 0) = &|d_0|^2|c_N^{(0)}|^2 + d_0^* d_N c_N^{(0)*} c_N^{(N)} e^{-2\gamma^2} \\ &+ d_N^* d_0 c_N^{(N)*} c_N^{(0)} e^{-2\gamma^2} + |d_N|^2|c_N^{(N)}|^2 \end{aligned} \quad (27)$$

which simplifies to

$$P(S_3 \geq 0) = \frac{1}{2}\left(1 - \sin\theta e^{-2\gamma^2}\right) \quad (28)$$

We find

$$\begin{aligned} \langle S_2\rangle_{S_3=1} &= -\frac{1}{2\gamma}\langle p\rangle_{S_3=1} \\ &= \frac{\cos\theta}{1 - \sin\theta e^{-2\gamma^2}} \end{aligned} \quad (29)$$

The form of this result agrees with that derived by Williams and Jordan, based on a similar two-state evolution and assuming a stroboscopic "kicked" weak QND measurements [8, 34].

As one example that is relevant to tests of Leggett-Garg inequalities, we consider where the initial state prepared at time t_2 is a generalised NOON state with amplitudes $d_N = \cos(\theta/2)$ and $d_0 = i\sin(\theta/2)$ and $\theta = \pi/3$, and we select $t_3 - t_2 = \pi/4$. The limits of $\langle S_2\rangle_{S_3=1}$ for $\gamma \to 0$ and $\gamma \to \infty$ are then 3.73 and 0.5 respectively. The threshold for the weak value where $\langle S_2\rangle_{S_3=1} > 1$ is $\gamma < \gamma_0$ given by $\gamma_0 = 0.5241$. In Figure 2 we plot the value $\langle S_2\rangle_{S_3=1}$ versus γ. Weak values that are outside the eigenvalue range of $|\langle S_2\rangle_{S_3=-1}| \leq 1$ are evident for where $\gamma < \gamma_0$.

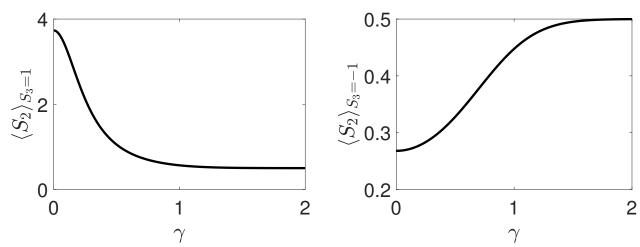

Figure 2. *Weak values for NOON states:* The measured value of $\langle S_2\rangle_{S_3=1}$ and $\langle S_2\rangle_{S_3=-1}$ versus γ for $\theta = t_2 - t_1 = \pi/3$ and $\phi = t_3 - t_2 = \pi/4$. The predictions for $\langle S_2\rangle_{S_3=1}$ show values outside the eigenvalue range bounded by ± 1, for all $\gamma < 0.5241$.

In a similar fashion, we calculate the prediction for $\langle S_2\rangle_{S_3=-1}$. Calculation gives

$$\int pP(p, S_3 < 0)dp = 2\gamma(|d_0|^2|c_0^{(0)}|^2 - |d_N|^2|c_0^{(N)}|^2) \quad (30)$$

and

$$\begin{aligned} P(S_3 < 0) = &|d_0|^2|c_0^{(0)}|^2 + |d_N|^2|c_0^{(N)}|^2 \\ &+ d_0^* d_N c_0^{(0)*} c_0^{(N)} e^{-2\gamma^2} + d_N^* d_0 c_0^{(N)*} c_0^{(0)} e^{-2\gamma^2} \end{aligned} \quad (31)$$

leading to

$$\langle S_2\rangle_{S_3=-1} = -\frac{1}{2\gamma}\langle p\rangle_{S_3=-1} = \frac{\cos\theta}{1+\sin\theta e^{-2\gamma^2}} \quad (32)$$

For the choice $\theta=\pi/3$, we find $\langle S_2\rangle_{S_3=-1}<0.5$ for all γ, implying no weak value prediction for these parameters.

V. LEGGETT-GARG TEST USING WEAK MEASUREMENTS

One may consider a Leggett-Garg test of macroscopic realism using NOON states and the weak measurements proposed in this paper. Leggett and Garg proposed to test macroscopic realism, by considering a two-state system where the two states are in some sense "macroscopically distinct" e.g. a cat that is dead or alive [17]. Leggett and Garg defined two premises that embody the meaning of macroscopic realism. The two premises are summarised in the Introduction.

Leggett and Garg (LG) showed how the two premises (referred to as macro-realism (MR)) constrain the dynamics of a two-state system. Considering three successive times $t_3>t_2>t_1$, the variable S_i denotes which of the two states the system is in at time t_i, the respective states being denoted by $S_i=+1$ or -1. The LG premises imply the LG inequality [7, 17]

$$\langle S_1S_2\rangle+\langle S_2S_3\rangle-\langle S_1S_3\rangle\leq 1 \quad (33)$$

Defining the parameter $LG\equiv\langle S_1S_2\rangle+\langle S_2S_3\rangle-\langle S_1S_3\rangle$ this is also expressed as $-3\leq LG\leq 1$. It is also possible to define the "no disturbance" or "no signalling in time" condition given by the equality

$$d_\sigma=\langle S_3|\sigma\rangle_M-\langle S_3|\sigma\rangle=0$$

that are also implied by the Leggett-Garg macro-realism premises, where M represents the non-invasive measurement [35, 36]. Here $\langle S_3|\sigma\rangle_M$ (and $\langle S_3|\sigma\rangle$) is the expectation value of S_3 given that the measurement $\hat{M}$ is performed (or not performed) at time t_2, conditional on the system being prepared in a state denoted σ at time t_1. The Leggett-Garg inequalities are predicted to be violated for quantum systems [10, 17].

Figure 1 illustrates the proposed LG experiment based on the NOON states and weak measurement. The system at time t_1 is prepared in a state with definite spin $S_1=1$. The system we consider evolves at time t_2 to a NOON state (8). The QND measurement M given by (3) is made on this state at time t_2. The measurement M can be made as a weak measurement, or as a strong projective measurement, depending on the value of γ. In fact because the state at time t_1 is deterministically prepared in the state with positive spin, $\langle S_1S_2\rangle=\langle S_2\rangle$ and $\langle S_1S_3\rangle=\langle S_3\rangle$. In this proposal, the $\langle S_2S_3\rangle$ is measured using a *weak* QND measurement M of S_2 at time t_2 and a strong measurement of S_3 at time t_3.

We consider that between times t_1 and t_2, and from after the measurement at time t_2 until time t_3, the system evolves according to the Josephson two-mode Hamiltonian [25, 28, 30, 31]

$$H_I=2\hbar\kappa\hat{J}_x+\hbar g\hat{J}_z^2 \quad (34)$$

The κ represents the intermode coupling and g the nonlinear self-interaction due to the medium. Regimes exist where a two-state oscillation (of period T_N) takes place (Fig. 3) [28, 32]. If the system is prepared in $|N\rangle_a|0\rangle_b$ at time t_1, then, in this parameter regime, at a later time t_2 the state vector is to a good approximation given by (apart from an overall phase factor)

$$|\psi(t)\rangle=\cos(t_2-t_1)|N\rangle|0\rangle+i\sin(t_2-t_1)|0\rangle|N\rangle \quad (35)$$

Here $t_i=E_\Delta t_i'/\hbar$ is the time defined in scaled units so that t_i' is the actual time in seconds and E_Δ is the energy splitting of the energy eigenstates $|N\rangle|0\rangle\pm|0\rangle|N\rangle$ under H_I. In one state, $|N\rangle_a|0\rangle_b$, all N atoms are in the mode a and in the second state, $|0\rangle_a|N\rangle_b$, all atoms are in the mode b [28]. As in Figure 1, we suppose that the system also evolves under this unitary evolution from t_2 (after the measurement M) to t_3. However, between times t_1 and t_2, we note that the NOON state at time t_2 might be prepared by a different method [21].

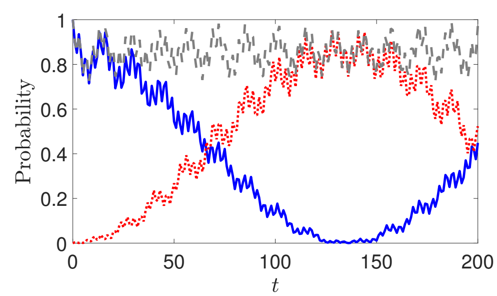

Figure 3. *Near-ideal two-state dynamics:* We show the mesoscopic two-state oscillation predicted for the Hamiltonian H_I with $N=5$ and $g/\kappa=2$ and time t is in units of κ^{-1}.

The two-time correlation for a measurement of spin S_i at time t_i followed by a later measurement of spin S_j at time t_j is $\langle S_iS_j\rangle=\cos[2(t_j-t_i)]$. This is independent of the outcome at time t_i, which determines whether the system is projected into $|N\rangle|0\rangle$ or $|0\rangle|N\rangle$. Choosing $t_1=0$, $t_2=\pi/6$, $t_3=\pi/3$, it is well-known that for this two-time correlation the quantum prediction is $LG=1.5$ which gives a violation of (33) [17]. Alternatively, one can select the values $t_1=0$, $t_2=\pi/6$ and $t_3=5\pi/12$ in units of $E_\Delta/\hbar$, to give a value $LG=1.37$. Figure 3 shows solutions of the Hamiltonian H_I for $N=5$ and $g=1$, confirming the correlation functions that give violations of the LG inequality in this regime.

In any experimental test of the Leggett-Garg inequalities, the question becomes how to perform the ideal non-invasive measurement at the time t_2. For any real measurement made at time t_2, it could be argued that the measurement is not in fact non-invasive, and therefore that the Leggett-Garg inequalities would not apply. The approach taken here, which is well-documented in the literature, is to perform a weak measurement at time t_2 [7, 9, 14]. We will show that the weak measurement in the limit of $\gamma \to 0$ can be justified as noninvasive for the input state at time t_2, and yet yields the required average $\langle S_2 S_3 \rangle$ for the test of the Leggett-Garg inequality. While this provides a convincing test of the Leggett-Garg macro-realism, we point out that alternative approaches are possible. The "clumsiness" of a QND measurement can be accounted for, by performing additional measurements and using a modified Leggett-Garg inequality [36, 37]. This approach is particularly useful for strong QND measurements where γ is large, and has been applied to superconducting qubits [36]. The recent papers of Zhou et al [38] and Formaggio et al [39] demonstrate violation of modified Leggett-Garg inequalities that are based on the assumption of stationarity.

Our proposed experiment is as follows. We assume in this Section that we do indeed generate the ideal statistics of the two-state system, so that the evolution is given by Eq. (35). After preparation in the state $|N\rangle|0\rangle$ at time t_1, the system evolves for a time $t_2 - t_1 = \theta/2$. The state at time t_2 is therefore

$$|\psi(t_2)\rangle = \cos\frac{\theta}{2}|N\rangle|0\rangle + i\sin\frac{\theta}{2}|0\rangle|N\rangle \qquad (36)$$

We then assume the time t_3 is such that $t_3 - t_2 = \pi/4$.

In this gedanken experiment, we distinguish between moments that are measured with the weak measurement M occurring at the time t_2, or not. The former moments are denoted by the subscript M. In Figure 4, the correlation functions $\langle S_1 S_2 \rangle_M = \langle S_2 \rangle_M$, $\langle S_1 S_3 \rangle_M = \langle S_3 \rangle_M$ and $\langle S_2 S_3 \rangle_M$ are plotted versus θ . We note that for all γ,

$$\langle S_2 \rangle_M = \langle S_2 \rangle = -\frac{1}{2\gamma}\langle p \rangle = \cos\theta \qquad (37)$$

This value is independent of γ i.e. whether the measurement at time t_2 is a weak measurement or a strong measurement. The moment

$$\langle S_2 S_3 \rangle_M = -\frac{1}{2\gamma}\langle p S_3 \rangle = \cos(2(t_3 - t_2)) \qquad (38)$$

is also independent γ and is independent of θ. Hence we can write $\langle S_2 S_3 \rangle = \langle S_2 S_3 \rangle_M$, although we note that the measurements made with smaller values of γ will have increased statistical error [9]. In this paper, we examine the case where $t_3 - t_2 = \pi/4$ and hence $\langle S_2 S_3 \rangle = 0$ for all γ.

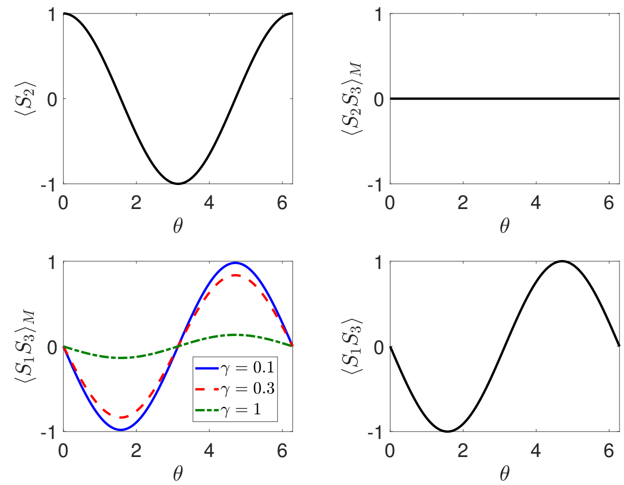

Figure 4. *Correlations associated with the violation of the LG inequality:* The graphs show $\langle S_1 S_2 \rangle_M$, $\langle S_2 S_3 \rangle_M$, $\langle S_1 S_3 \rangle_M$ and $\langle S_1 S_3 \rangle$. The $\langle S_i S_j \rangle_M$ are evaluated with the measurement at time t_2 as illustrated in Figure 1. The $\langle S_1 S_3 \rangle$ is evaluated without a measurement at time t_2. The averages $\langle S_1 S_2 \rangle_M$ and $\langle S_2 S_3 \rangle_M$ are independent of the strength γ of the measurement M. By contrast, $\langle S_1 S_3 \rangle_M \to \langle S_1 S_3 \rangle$ only when $\gamma \to 0$.

A significant difference occurs, however, between $\langle S_1 S_3 \rangle_M$ and $\langle S_1 S_3 \rangle$. We see that without the measurement M at time t_2, the moment is

$$\langle S_1 S_3 \rangle = \cos(2(t_3 - t_1)) = -\sin\theta$$

For a finite γ with the QND measurement M occurring at the intermediate time t_2, we calculate the $\langle S_1 S_3 \rangle_M$ as follows

$$\begin{aligned} \langle S_1 S_3 \rangle_M &= P(S_3 \geq 0) - P(S_3 < 0) \\ &= -\sin\theta e^{-2\gamma^2} \end{aligned}$$

The relevant probabilities were defined and calculated in the previous section. In Figure 4, it is clear that as $\gamma \to 0$, $\langle S_1 S_3 \rangle_M \to \langle S_1 S_3 \rangle$, indicating a minimal disturbance of the system being measured by the weak measurement. This no-disturbance can be measured in a control experiment, and is used to justify the non-invasive nature of the measurement M for the purpose of testing the Leggett-Garg inequality, given as

$$\langle S_1 S_2 \rangle_M + \langle S_2 S_3 \rangle_M - \langle S_1 S_3 \rangle_M \leq 1 \qquad (39)$$

By contrast, there is a distinct difference between $\langle S_1 S_3 \rangle_M$ and $\langle S_1 S_3 \rangle$ for γ large, which corresponds to a strong projective measurement of the spin S_2 at time t_2. In Figure 5 we plot the difference $d_\sigma = \langle S_1 S_3 \rangle_M - \langle S_1 S_3 \rangle$ as the disturbance equality [35, 36].

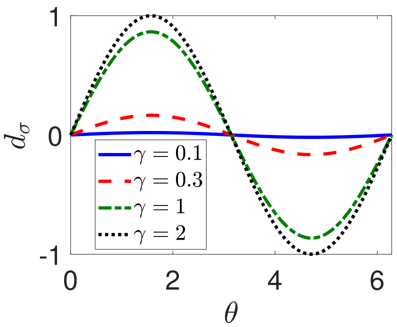

Figure 5. *Measure of disturbance for the weak measurement:* We calculate the value of the moments $\langle S_1S_3\rangle_M$ and $\langle S_1S_3\rangle$. The difference is defined as the disturbance d_σ, plotted here for various measurement strength γ.

In Figures 6 and 7 we plot the violation of the LG inequality by plotting the LG parameter $LG = \langle S_1S_2\rangle_M + \langle S_2S_3\rangle_M - \langle S_1S_3\rangle_M$ versus θ, for different values of weak measurement strength γ. At $\theta/2 = \pi/6$, the optimal value of $LG = 1.37$ is possible for small γ. The violation is possible because for small γ the measurement is non-invasive. For strong γ, violations are not possible using this particular approach with the inequality (39), because the invasive measurement acts on the system at time t_2 causing a collapse of the wave function into a state of definite spin. The violations that occur in the weak measurement regime are directly associated with the presence of weak values [7]. The correlation between the weak values and the violation of the Leggett-Garg inequality is evident in Figures 6 and 7.

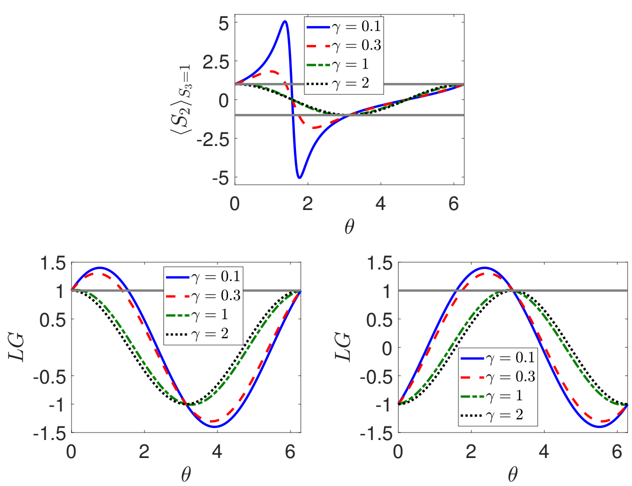

Figure 6. *Correlation between violation of the LG inequality and weak values:* The top graph shows $\langle S_2\rangle_{S_3=1}$ versus θ. In the left lower graph, we plot $LG = \langle S_1S_2\rangle_M + \langle S_2S_3\rangle_M - \langle S_1S_3\rangle_M$. In the right lower graph, we plot $LG = -\langle S_1S_2\rangle_M - \langle S_2S_3\rangle_M - \langle S_1S_3\rangle_M$ defined with the sign of S_2 changed. The Leggett-Garg inequalities are violated when $LG > 1$. This corresponds to a weak value regime, observed when $|\langle S_2\rangle_{S_3=1}| > 1$.

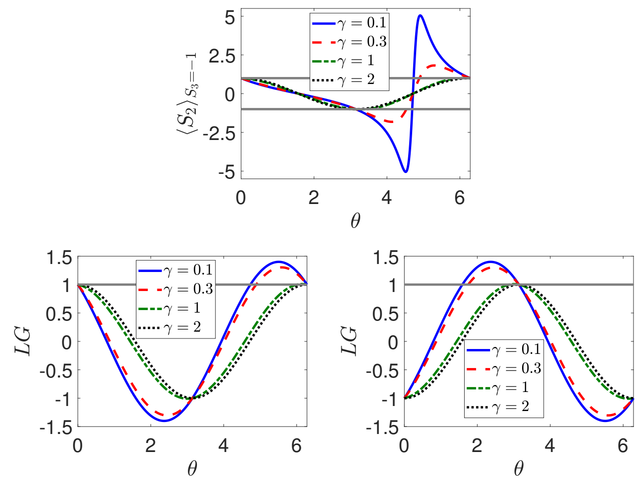

Figure 7. As for Figure 6, but here the top graph shows $\langle S_2\rangle_{S_3=-1}$. In the left graph, we plot $LG = \langle S_1S_2\rangle_M - \langle S_2S_3\rangle_M + \langle S_1S_3\rangle_M$ defined with the sign of S_3 changed. In the right graph, plotted is $LG = -\langle S_1S_2\rangle_M + \langle S_2S_3\rangle_M + \langle S_1S_3\rangle_M$ defined with the signs of S_2 and S_3 changed.

VI. WEAK VALUES WITH NON-IDEAL STATES

A. With an ideal NOON state at time t_2

Let us assume an ideal generalised NOON state has been generated at time t_2. This is not unrealistic for small $N > 1$. For example, for $N = 2$ the Hong-Ou-Mandel effect creates a NOON state [21, 22]. Proposals for more macroscopic NOON states use conditioning on measurements of J_z [40]. However, for the generation of the quantum state according to the dynamics of H_I, the state formed at t_3 is not an ideal NOON state. We examine the effect of this on the weak values and the violation of the LG inequalities.

First, we note that the pure general input state at time t_i is of the form

$$|\psi\rangle_{in} = \sum_{m=0}^{N} d_m|m\rangle_a|N-m\rangle_b \tag{40}$$

given by (4). It is straightforward to show that $\langle S_i\rangle = -\frac{1}{2\gamma}\langle p\rangle$ for all input states of this type where the total number N of bosons is fixed. This means that the expression can be used in the more general case for the evaluation of the spin averages. This is also true of the $\langle S_2S_3\rangle$ where the state at time t_2 is the NOON state.

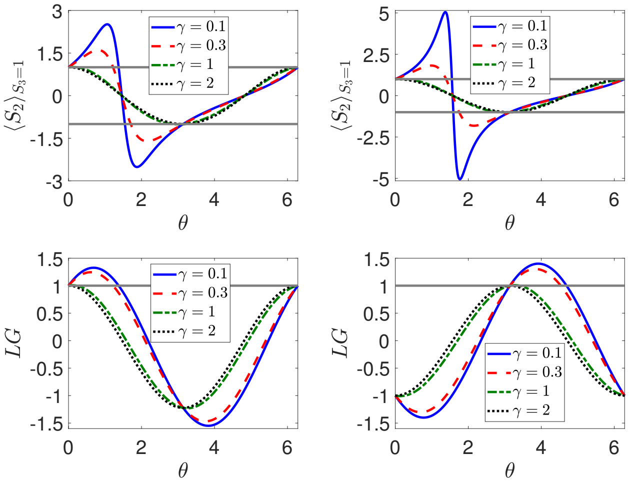

Figure 8. *Violation of the LG inequality and weak values for non-ideal evolution after time t_2:* Plots show the weak values and violation of the Leggett-Garg inequality as for Figure 6, but where the state generated at time t_3 evolves after time t_2 according to the Hamiltonian H_I. Here $N = 5$, $g = 2$ (top and lower left) and $g = 104.43$ (top and lower right). We use $t_3 - t_2 = T_N/4$ where T_N is the oscillation period in dimensionless units (see main text).

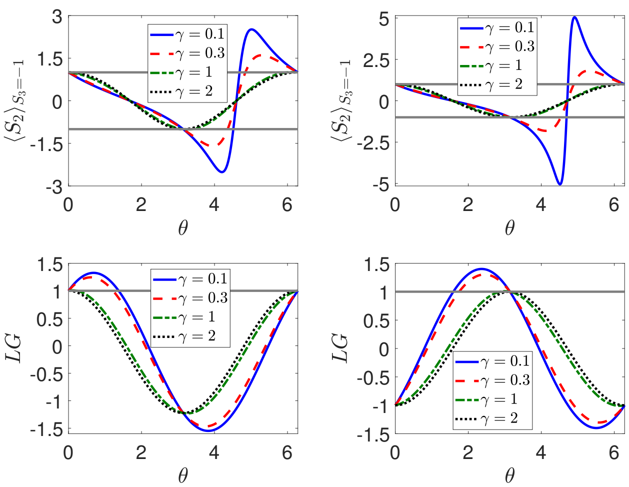

Figure 9. As for Figure 8, but with the parameters defined in Figure 7. Here $N = 5$, $g = 2$ (top and lower left) and $g = 104.43$ (top and lower right).

To evaluate the weak values accounting for the general evolution with H_I, we consider the generalised equations (16-19) that allow for a non-ideal state at time t_3. Specifically, the Hamiltonian H_I is such that the two-mode state $|0\rangle|N\rangle$ evolves to the state given by $\sum_n c_n^{(0)}|n\rangle|N-n\rangle$ and two-mode state $|N\rangle|0\rangle$ evolves to the state given by $\sum_n c_n^{(N)}|n\rangle|N-n\rangle$ where $c_n^{(m)}$ are constants. In Figures 8, 9 and 10, we plot the predictions for $\langle \hat{p} \rangle_{S_3=1}$ where we use the values of the precise coefficients $c_n^{(N)}$ and $c_n^{(0)}$ generated by the evolution with H, for $N = 5$ and $N = 10$. These have been evaluated by the numerical program that yielded the plots of Figure 3. The oscillation time is given by $T_N = \frac{2\pi}{\omega_N}$ where $\omega_N = 2\hbar g \frac{N}{(N-1)!}\left(\frac{\kappa}{g}\right)^N$ [28].

The weak values and Leggett-Garg violations are tolerant to the non-ideal coefficients, at least for smaller N. For larger N corresponding to a BEC, it is known that the parameter regime for oscillation is more difficult to achieve, a phenomenon known as macroscopic quantum self-trapping [26]. This regime may not be impossible however using alternative realisations of the nonlinear Josephson Hamilton [41–43].

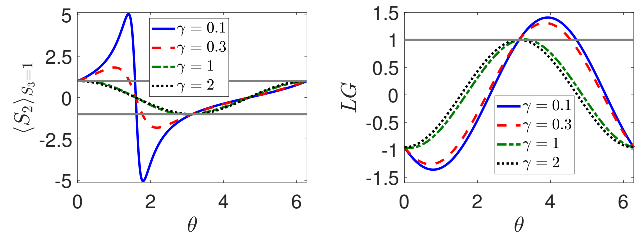

Figure 10. As for Figure 8, with parameters defined as for Figure 6. Here $N = 10$, $g = 6.6$.

B. Non-ideal NOON state at time t_2

We conclude by noting that where the state at time t_2 is not an ideal NOON state, evaluating $\langle S_2 S_3 \rangle$ by way of the measurement given by H_M is more subtle. To illustrate, let us consider where the two-mode state immediately prior to the measurement at time t_2 is

$$|\psi\rangle = d_{n_a}|n_a\rangle_a|N-n_a\rangle_b + d_{N-n_a}|N-n_a\rangle_a|n_a\rangle_b \quad (41)$$

In this case, the state immediately after the measurement at time t_2 is

$$\begin{aligned}|\psi\rangle = d_{n_a}&|n_a\rangle_a|N-n_a\rangle_b|\gamma e^{i\pi(N-2n_a)/2N}\rangle_c \\ &+d_{N-n_a}|N-n_a\rangle_a|n_a\rangle_b|\gamma e^{-i\pi(N-2n_a)/2N}\rangle_c\end{aligned} \quad (42)$$

based on Eq. (6). We consider that the state $|n_a\rangle_a|N-n_a\rangle_b$, $(|N-n_a\rangle_a|n_a\rangle_b)$ evolves as described by a Hamiltonian to the state $\sum_n c_n^{(n_a)}|n\rangle|N-n\rangle$ $\left(\sum_{n'} c_{n'}^{(N-n_a)}|N-n'\rangle|n'\rangle\right)$ in a time $t_3 - t_2$. We then find (see the Appendix)

$$\langle pS_3 \rangle = -2\gamma \sin\left(\frac{\pi}{2N}(N-2n_a)\right)\langle S_2 S_3 \rangle \quad (43)$$

This is similar to the earlier result (15) except that the measurement strength is diminished by the sin factor. The calculation indicates that where a general superposition (4) is prepared at time t_2, the simple weak measurement relation of type (15) does not hold. A more careful analysis is required to place a bound on the value of $\langle S_2 S_3 \rangle$ given the measured $\langle pS_3 \rangle$. This is feasible, but will not be addressed in this paper. The result (43) is useful however. This is because in some cases, the mesoscopic superposition state (41) is easier to prepare than the NOON state. It has been shown that the state (41) is generated over shorter timescales than the traditional NOON state, in BEC systems [26, 28]. Considerations of timescale are important where decoherence effects are significant.

VII. CONCLUSION

In summary, we have demonstrated the possibility of detecting quantum weak values using NOON states. We consider a specific QND measurement of the Schwinger spin, defined as the population difference for two levels with a bosonic occupation. This QND measurement can be realised for atomic systems using an ac Stark shift [24]. The measurement is also applicable to states prepared in polarisation modes, as in polarisation squeezing experiments, where the observables are defined in terms of Stokes operators [44]. The QND measurement in the limit of small coupling corresponds to a weak measurement of the Schwinger spin, meaning that it gives the correct average spin for the prepared quantum state, but with a vanishingly small disturbance of the state. By analysing the case where the measurement is made on a quantum system prepared in a NOON state, we demonstrate how one can detect quantum weak values for the NOON states, for all N. The detection of the weak values is made possible by a unitary evolution of the quantum system after the measurement, as given by the nonlinear two-mode Josephson Hamiltonian. This gives a way to demonstrate the existence of quantum weak values, for mesoscopic and macroscopic superposition states.

The work of this paper suggests a Leggett-Garg test of meso- or macro-realism using NOON states. In this case, the measurement of a two-time correlation involving the weak measurement is required. We have discussed how to demonstrate the non-invasiveness of the weak measurement for the purpose of a Leggett-Garg test, and have examined the feasibility of the experiment using the Josephson model, with finite parameter values.

Finally, we note that regimes associated with more general parameters of the Josephson model do not always lead to a second NOON superposition state being created at the time t_3. The outcomes for the S_3 are not simply $\pm N$ but are spread over all values. We comment that tests of quantum weak values and of the Leggett-Garg inequality may still be possible, using the approach of overlapping regions presented in Refs. [33, 45]

Appendix A: Calculation of $\langle S_2\rangle$

We can evaluate $\langle p\rangle$ from (9), using that $\hat{p}=\frac{1}{i}(\hat{c}-\hat{c}^\dagger)$, thus

$$\langle p\rangle=|d_0|^2\langle\gamma e^{i\pi/2}|p|\gamma e^{i\pi/2}\rangle+|d_N|^2\langle\gamma e^{-i\pi/2}|p|\gamma e^{-i\pi/2}\rangle \tag{A1}$$

Next, we use the result for the inner product of coherent states $\langle\alpha|\beta\rangle=\exp\left[\alpha^*\beta-|\alpha|^2/2-|\beta|^2/2\right]$, to find

$$\langle\alpha|p|\beta\rangle_a=\frac{1}{i}\left(\beta-\alpha^*\right)\exp\left[\alpha^*\beta-|\alpha|^2/2-|\beta|^2/2\right] \tag{A2}$$

and hence

$$\begin{aligned}\langle\gamma e^{-i\pi/2}|p|\gamma e^{-i\pi/2}\rangle&=-2\gamma\\ \langle\gamma e^{i\pi/2}|p|\gamma e^{i\pi/2}\rangle&=2\gamma\end{aligned} \tag{A3}$$

This implies

$$\begin{aligned}\langle p\rangle&=2\gamma(|d_0|^2-|d_N|^2)\\ &=-2\gamma\langle S_2\rangle\end{aligned} \tag{A4}$$

Here we have used that $\langle S_2\rangle=|d_N|^2-|d_0|^2$, which is the expectation value of S_2 for the initial two-mode state (4) for the NOON state. We see that:

$$\langle S_2\rangle=-\frac{1}{2\gamma}\langle p\rangle \tag{A5}$$

The average of p will give the value for the average of the Schwinger spin of the incident two-mode state.

Appendix B: Calculation of $\langle S_2S_3\rangle$

We suppose that the Hamiltonian H_I is such that the two-mode state $|m\rangle|N-m\rangle$ evolves to the state $\sum_n c_n^{(m)}|n\rangle|N-n\rangle$ where $c_n^{(m)}$ are constants. The final output state including the meter field is:

$$|\psi(t_3)\rangle=\sum_m d_m|\gamma e^{i\pi(N-2m)/2N}\rangle_c\sum_n c_n^{(m)}|n\rangle_a|N-n\rangle_b \tag{B1}$$

We next evaluate $\langle pS_3\rangle$ and compare to $\langle S_2S_3\rangle$. We take the case where prior to the measurement at time t_2 the two-mode system is in the NOON state:

$$|\psi\rangle=d_0|0\rangle_a|N\rangle_b+d_N|N\rangle_a|0\rangle_b \tag{B2}$$

At time t_3, after measurement and after the subsequent evolution, the state is given by Eq. (B1) which we simplify as (13). We evaluate $\langle pS_3\rangle=\langle\psi(t_3)|pS_3|\psi(t_3)\rangle$, using (A3). Here it is useful to define $S_3|m\rangle_a|N-m\rangle_b=sgn(2m-N)|m\rangle_a|N-m\rangle_b a$ where $sgn(S)$ is $+1$ if $S>0$, 0 of $S=0$, and -1 otherwise. From (A2) we see that

$$\langle\gamma e^{-i\pi/2}|p|\gamma e^{i\pi/2}\rangle_c=0=\langle\gamma e^{i\pi/2}|p|\gamma e^{-i\pi/2}\rangle_c$$

Hence we obtain

$$\langle pS_3\rangle = |d_0|^2\,2\gamma\Big(-\sum_{n<N/2}\Big|c_n^{(0)}\Big|^2+\sum_{n>N/2}\Big|c_n^{(0)}\Big|^2\Big) - |d_N|^2\,2\gamma\Big(-\sum_{n<N/2}\Big|c_n^{(N)}\Big|^2+\sum_{n>N/2}\Big|c_n^{(N)}\Big|^2\Big) \tag{B3}$$

Using that $\langle S_2S_3\rangle$ is given by Eq. (14), we find

$$\langle S_2S_3\rangle = -\frac{1}{2\gamma}\langle pS_3\rangle \tag{B4}$$

Appendix C: Calculation of non-ideal case

We consider that the state $|n_a\rangle_a|N-n_a\rangle_b$ evolves as described above by a Hamiltonian H_I to the state $\sum_n c_n^{(n_a)}|n\rangle|N-n\rangle$ in a time t_3-t_2. Similarly, the state $|N-n_a\rangle_a|n_a\rangle_b$ evolves to $\sum_{n'} c_{n'}^{(N-n_a)}|N-n'\rangle|n'\rangle$. Hence

$$\begin{aligned} S_3|\psi(t_3)\rangle &= d_{n_a}|\gamma e^{+i\pi(N-2n_a)/2N}\rangle_c \\ &\quad\times\sum_n c_n^{(n_a)} sgn(2n-N)|n\rangle_a|N-n\rangle_b \\ &\quad+d_{N-n_a}|\gamma e^{-i\pi(N-2n_a)/2N}\rangle_c \\ &\quad\times\sum_{n'} c_{n'}^{(N-n_a)} sgn(N-2n')|N-n'\rangle_a|n'\rangle_b \end{aligned} \tag{C1}$$

Therefore

$$\begin{aligned} \langle pS_3\rangle &= 2\gamma\sin\Big(\frac{\pi}{2N}(N-2n_a)\Big)\times \\ &\Big(|d_{n_a}|^2\sum_n\Big|c_n^{(n_a)}\Big|^2 sgn(2n-N) \\ &\quad-|d_{N-n_a}|^2\sum_{n'}\Big|c_{n'}^{(N-n_a)}\Big|^2 sgn(N-2n')\Big) \\ &= -2\gamma\sin\Big(\frac{\pi}{2N}(N-2n_a)\Big)\langle S_2S_3\rangle. \end{aligned} \tag{C2}$$

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