



This is the accepted manuscript made available via CHORUS. The article has been published as:

Information measures for a local quantum phase transition: Lattice fermions in a one-dimensional harmonic trap

Yicheng Zhang, Lev Vidmar, and Marcos Rigol

Phys. Rev. A **97**, 023605 — Published 5 February 2018

DOI: [10.1103/PhysRevA.97.023605](https://doi.org/10.1103/PhysRevA.97.023605)

Information measures for a local quantum phase transition: Lattice fermions in a one-dimensional harmonic trap

Yicheng Zhang, Lev Vidmar, and Marcos Rigol

Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802, USA

We use quantum information measures to study the local quantum phase transition that occurs for trapped spinless fermions in one-dimensional lattices. We focus on the case of a harmonic confinement. The transition occurs upon increasing the characteristic density and results in the formation of a band insulating domain in the center of the trap. We show that the ground-state bipartite entanglement entropy can be used as an order parameter to characterize this local quantum phase transition. We also study excited eigenstates by calculating the average von Neumann and second Renyi eigenstate entanglement entropies, and compare the results to the thermodynamic entropy and the mutual information of thermal states at the same energy density. While at low temperatures we observe a linear increase of the thermodynamic entropy with temperature at all characteristic densities, the average eigenstate entanglement entropies exhibit a strikingly different behavior as functions of the temperature below and above the transition. They are linear in temperature below the transition but exhibit activated behavior above it. Hence, at nonvanishing energy densities above the ground state, the average eigenstate entanglement entropies carry fingerprints of the local quantum phase transition.

I. INTRODUCTION

Since the early days of the exploration of strongly correlated many-body quantum systems with ultracold atoms in optical lattices [1–3], paradigmatic lattice models have been realized with the presence of additional inhomogeneous trapping potentials [4, 5]. Over the years [6–27], one of the central goals has been to understand how the results of measurements in inhomogeneous systems can be related to the phases and quantum phase transitions that occur in their homogeneous counterparts [28]. In this work, we revisit this question in the context of recent progress in measuring the second Renyi entanglement entropy with ultracold atoms in optical lattices [29, 30]. This has opened a new window for the exploration of many-body physics and quantum phase transitions [31–33] at the interface with quantum information theory [34–43].

Several properties of trapped systems are noticeably different from their homogeneous counterparts. For example, the quasimomentum distribution function of trapped noninteracting fermions at zero temperature does not exhibit the traditional step function shape [44]. For lattice bosons (fermions), Mott-insulating and superfluid (metallic) phases can coexist space separated in a trap [6–9]. More importantly, the emergence of Mott-insulating domains in a trapped system does not follow the traditional quantum phase transition paradigm, in which a global order parameter (the compressibility) vanishes in the insulating phase. In trapped systems, the global compressibility is nonzero no matter whether Mott domains are absent or present [6]. Instead, local order parameters, such as the fluctuations of the site occupation or local compressibilities [6, 8, 12], need to be used to characterize the Mott domains. Hence, the term *local* quantum phase transition is more fitting to describe the formation of a Mott-insulating domain in a trapped sys-

tem. Also, rather than phase diagrams, state diagrams are the proper way to describe trapped systems in the thermodynamic limit [8, 17, 20, 26]. In the state diagrams, the characteristic density ρ , which is the ratio between the particle number N and the characteristic trap length R [to be defined in Eq. (1)], replaces the density used in the homogeneous case.

In this work, we study the behavior of quantum information measures when a band-insulating domain emerges in systems with noninteracting spinless fermions in the presence of a harmonic trap in one-dimensional lattices. We characterize both the ground state and excited states using the von Neumann and second Renyi eigenstate entanglement entropies. In the ground state, a local quantum phase transition occurs at a critical characteristic density ρ_c . Below ρ_c , the site occupations are smaller than one in the entire lattice, while above ρ_c a band insulating domain with site occupation one forms in the center of the trap. We show that the entanglement entropy, upon a bipartition of the lattice in two equal parts, can be used as an order parameter for the local quantum phase transition. The parameter regime we focus on, namely $\rho \sim \rho_c$, is complementary to the one accessed recently in a conformal field theory study [40].

For excited eigenstates, we study the average von Neumann and second Renyi entropies and compare them to the mutual information and thermodynamic entropies of thermal states. It was recently proved that, for translationally-invariant fermionic quadratic models, the half-system bipartite entanglement entropy of typical eigenstates is smaller than the thermodynamic entropy of thermal states at the same mean energy (the difference is extensive in the system size) [45]. Here we not only show that the same holds for harmonically trapped quadratic systems (i.e., in systems that are not translationally invariant), but also that the average eigenstate entanglement entropy (with Gibbs weights) may exhibit a completely different functional dependence on the char-

acteristic density and the temperature than the thermodynamic entropy. We note that the nature of the inhomogeneity considered here is different from that in previous studies in which translational invariance was broken by means of diagonal disorder [46] and random long-range hoppings [47].

Our work reveals a particularly interesting behavior of the average eigenstate entanglement entropy as a function of the temperature. At low temperatures, it increases linearly with temperature below ρ_c (similarly to the thermodynamic entropy of the translationally-invariant metallic counterpart), but exhibits an activated behavior above ρ_c . Hence, at nonvanishing energy densities above the ground state, the average eigenstate entanglement entropy allows one to identify the presence of the ground-state local quantum phase transition. This is in stark contrast to the thermodynamic entropy, which is linear in temperature independently of the value of ρ . The latter is the result of the many-body energy spectrum being gapless irrespective of the value of ρ .

The presentation is organized as follows. In Sec. II, we introduce the model and the entanglement measures used to characterize it. In Sec. III, we study the properties of the ground state, while Sec. IV is devoted to the study of properties at nonvanishing energy densities above the ground state. A summary of the results is presented in Sec. V. We discuss details of the numerical calculations in Appendices A–E.

II. MODEL AND INFORMATION MEASURES

We consider spinless fermions in one-dimensional lattice with L (even) sites, described by the Hamiltonian

$$\hat{H} = -t \sum_{l=0}^{L-1} (\hat{f}_l^\dagger \hat{f}_{l+1} + \text{H.c.}) + \frac{t}{R^2} \sum_{l=0}^{L-1} \left(l - \frac{L-1}{2} \right)^2 \hat{n}_l, \quad (1)$$

where \hat{f}_l is the fermion annihilation operator at site l , and $\hat{n}_l = \hat{f}_l^\dagger \hat{f}_l$. We set the unit of energy $t = 1$ and the lattice spacing to one. The strength of the harmonic confining potential is determined by the parameter R , which is a characteristic length for the trapped lattice system. We therefore study the properties of the system as a function of the characteristic density $\rho = N/R$ [44]. In order to have a vanishing density at the edges of the lattice, in the ground state calculations we take $L \gtrsim 4R$ [see Fig. 1(b)], while in the excited state calculations we take $L \gtrsim 8R$.

Even though the model is quadratic, an analytic solution for the single-particle energy eigenstates is only available at low and high energies [48]. At intermediate energies, relevant to the formation of the band-insulating domain in the center of the trap, the properties of the single-particle eigenstates can be studied numerically. Using local observables, such as the site occupations and their fluctuations, the critical characteristic density for the formation of the band-insulating domain

in the center of the trap was shown to be $\rho_c \approx 2.6$ [44]. At the corresponding Fermi energy, a semiclassical WKB approximation showed that the single-particle density of states exhibits a logarithmic singularity [49]. These findings will be further discussed in this work.

For single-particle eigenenergies greater than the Fermi energy corresponding to ρ_c , the eigenstates become doubly degenerate (they are even or odd upon lattice reflection). We weakly break this degeneracy by modifying $R^{-2} \rightarrow R^{-2}(1 \mp \eta)$ in Eq. (1) for sites $l \leq (L-1)/2$, with $\eta \ll 1$. We choose $\eta = 1/N_c = 1/(R\rho_c)$, with the value of ρ_c to be determined later [see Eq. 10].

We study the entanglement entropy of the many-body eigenstates of \hat{H} in Eq. (1) for bipartitions of the system into equal subsystems A and \bar{A} . For a many-body eigenstate $|m\rangle$, the reduced density matrix of the subsystem A is $\hat{\rho}(m) = \text{Tr}_{\bar{A}}(|m\rangle\langle m|)$. We are interested in the second Renyi entropy, $S_n(m) = \frac{1}{1-n} \ln \text{Tr}[\hat{\rho}(m)^n]$ for $n = 2$, and in the von Neumann entropy, $S_{\text{vN}}(m) = -\text{Tr}[\hat{\rho}(m) \ln \hat{\rho}(m)]$, which is the limit $n \rightarrow 1$ of S_n . For many-body eigenstates of noninteracting fermions, the reduced density matrix $\hat{\rho}(m)$ has a Gaussian form. $\hat{\rho}(m)$ can be obtained from the one-body (covariance) correlation matrix $F_{ij}(m) = \langle m | \hat{f}_i^\dagger \hat{f}_j | m \rangle$ defined on the sites i, j of subsystem A [50]. The entanglement entropies are then unique functions of the eigenvalues λ_j of $F(m)$, which we diagonalize numerically. Specifically, the von Neumann entanglement entropy is

$$S_{\text{vN}}(m) = - \sum_{j=1}^{L/2} \left[\lambda_j \ln \lambda_j + (1 - \lambda_j) \ln(1 - \lambda_j) \right], \quad (2)$$

and the second Renyi entanglement entropy is

$$S_2(m) = - \sum_{j=1}^{L/2} \ln \left[(1 - \lambda_j)^2 + \lambda_j^2 \right]. \quad (3)$$

In the continuum, the limit $\rho \rightarrow 0$ in the lattice, the behavior of the Renyi entanglement entropies of the ground state of trapped noninteracting fermions is well understood. The leading term scales as $\ln N$ [35, 36]. It has been shown that the prefactor in the leading term is $(1 + n^{-1})/12$, where n is the order of the Renyi entanglement entropy, i.e., it is identical to the one in homogeneous systems with open boundaries [37]. In lattice systems with nonvanishing values of ρ , the systems of interest here, such a scaling is in general no longer valid.

We also study entanglement properties of excited eigenstates. We calculate the average eigenstate entanglement entropy $\bar{S}_n(T)$ at a given mean energy density as

$$\bar{S}_n(T) = \frac{\sum_m S_n(m) e^{-E_m/T}}{\sum_m e^{-E_m/T}}, \quad (4)$$

where $n = \text{vN}$ or $n = 2$, and E_m is the energy of eigenstate $|m\rangle$. The temperature T sets the mean energy density of the eigenstates involved in the average, and the

summation runs over eigenstates with a fixed N , i.e., it corresponds to a canonical ensemble average. In the numerical calculation of Eq. (4), we discard eigenstates with a relative weight $\exp[-(E_m - E_{\text{GS}})/T]$, where E_{GS} is the ground-state energy, below a threshold value $\exp[-\Lambda]$. We use $\Lambda = 30$, which yields a negligibly small numerical error, as shown in Fig. B1 of Appendix B. In Eq. (C1) of Appendix C, we also extend Eq. (4) to compute grand canonical ensemble averages. We find that the results for the canonical and grand canonical ensemble averages are very similar, see Fig. C1 in Appendix C. Therefore, in what follows we only discuss the results for the canonical ensemble average.

We compare the average eigenstate entanglement entropy in Eq. (4) to properties of (mixed) thermal states $\hat{\rho}(T) = e^{-(\hat{H} - \mu\hat{N})/T} / \text{Tr}\{e^{-(\hat{H} - \mu\hat{N})/T}\}$ at the same temperature and average particle number N (used to determine μ). In particular, we study the quantum mutual information

$$I(T) = 2S_{\text{vN}}(T) - S_{\text{GE}}(T), \quad (5)$$

where $S_{\text{vN}}(T)$ is the von Neumann entropy of the reduced density matrix of $\hat{\rho}(T)$, and $S_{\text{GE}}(T)$ is the thermodynamic (grand canonical ensemble) entropy. The calculation of $S_{\text{vN}}(T)$ can be done using Eq. (2), with λ_j being the eigenvalues of the one-body correlations matrix $F(T)$ with matrix elements $F_{ij}(T) = \text{Tr}\{\hat{\rho}(T)\hat{f}_i^\dagger\hat{f}_j\}$. The thermodynamic entropy $S_{\text{GE}}(T) = -\text{Tr}[\hat{\rho}(T)\ln\hat{\rho}(T)]$ can be obtained using the expression

$$S_{\text{GE}}(T) = -\sum_{j=1}^L \left[n_j \ln n_j + (1 - n_j) \ln(1 - n_j) \right], \quad (6)$$

where $n_j = [1 + e^{(\varepsilon_j - \mu)/T}]^{-1}$, and ε_j is the single-particle eigenenergy. Note that the quantum mutual information is not an entanglement measure, but it quantifies the amount of correlations between the two subsystems. For thermal states, it was proved that $I(T)$ follows an area law with the system size [51, 52].

III. LOCAL QUANTUM PHASE TRANSITION

We first focus on the properties of the ground state of \hat{H} in Eq. (1). We define the total energy density as $\bar{E} = E/R = \sum_{j=1}^N \varepsilon_j/R$. The discrete second derivative of \bar{E} with respect to ρ is $\bar{E}'' = R\Delta\varepsilon_N$, where $\Delta\varepsilon_N = \varepsilon_{N+1} - \varepsilon_N$ is the level spacing of the single-particle spectrum of \hat{H} . Results for $\bar{E}''(\rho)$ vs ρ are shown in Fig. 1(a) for two values of R (i.e., for two system sizes). They exhibit a perfect collapse. These results provide two important insights. First, the scaling $\Delta\varepsilon_N \approx \text{const}/R$ indicates that the system is gapless in the thermodynamic limit ($R \rightarrow \infty$) for all values of ρ under investigation. Second, \bar{E}'' vs ρ at $\rho \approx 2.6$ appears to be nonanalytic, a behavior that is typical of a quantum phase transition [28]. We

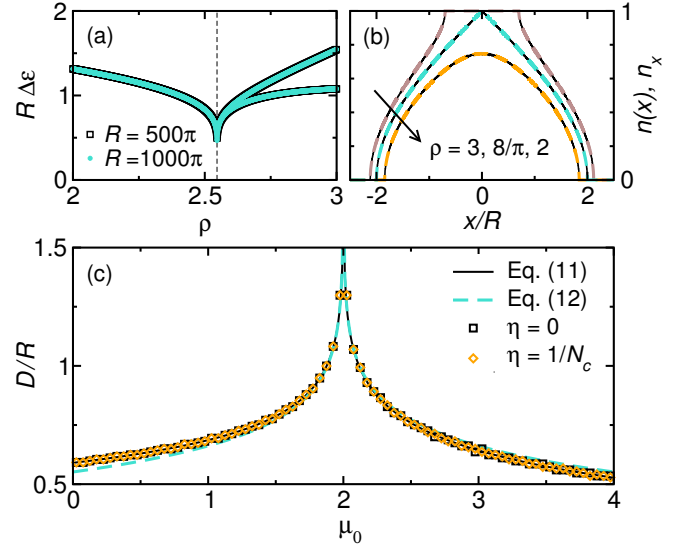


FIG. 1. Properties of one-dimensional lattice fermions in a harmonic trap. (a) Second derivative of the total energy vs ρ , $\bar{E}''(\rho) = R\Delta\varepsilon_N$, which is proportional to the level spacing $\Delta\varepsilon_N = \varepsilon_{N+1} - \varepsilon_N$ of the single-particle energy spectrum of \hat{H} in Eq. (1). The dashed line denotes the critical characteristic density $\rho_c = 8/\pi$ [see Eq. (10)]. We choose the parity breaking parameter $\eta = 1/N_c$ throughout the paper [see text after Eq. (1)]. (b) Site occupations at (from top to bottom) $\rho = 3$, $\rho = \rho_c = 8/\pi$, and $\rho = 2$. Solid lines are exact numerical results n_x (with integer x) and the overlapping dashed lines are the LDA results $n(x)$ from Eq. (9). (c) Single-particle density of states $D(\mu_0)/R$, where μ_0 is the single-particle energy (the Fermi energy of the many-body ground state). The solid and dashed lines are the LDA results from Eq. (11) and the corresponding leading term from Eq. (12), respectively. The symbols are the numerical exact diagonalization results with ($\eta = 1/N_c$) and without ($\eta = 0$) parity breaking. The results in (b) and (c) were obtained for $R = 500\pi$.

show in what follows that $\bar{E}''(\rho)$ is indeed nonanalytic at $\rho = \rho_c$, a point at which a band-insulating domain forms in the center of the system [see Fig. 1(b)]. The transition in this case can be regarded as a local quantum phase transition as argued earlier.

A. Local density approximation

Here, we use the local density approximation (LDA) to describe two features: (i) the formation of the band insulating domain in the center of the trap [26, 44], and (ii) the divergence of the single-particle density of states [49].

Within the LDA, one constructs an effective local chemical potential at the (continuous) position x , which is the distance from the center of the trap

$$\mu(x) = \mu_0 - \frac{x^2}{R^2}, \quad (7)$$

where μ_0 is the chemical potential (Fermi energy), with

$\mu_0 > -2$ to have nonzero site occupations. Then, for each value of x , the system is treated as a homogeneous one with chemical potential $\mu(x)$, e.g., $n(x) = k_F(x)/\pi$ with $k_F(x)$ and $\mu(x)$ related through $\mu(x) = -2 \cos[k_F(x)]$.

The site occupations $n(x)$ vanish for all $|x| > x_0$, where $x_0 = R\sqrt{\mu_0 + 2}$. For $\mu_0 < 2$, $n(x) < 1$ for all x since

$$n(x) = \frac{1}{\pi} \arccos \left[-\frac{\mu(x)}{2} \right]. \quad (8)$$

On the other hand, for $\mu_0 \geq 2$, a band insulating domain is present for $|x| < x_1$, where $x_1 = R\sqrt{\mu_0 - 2}$. The site-occupation distribution in the trap is then

$$n(x) = \begin{cases} 1 & |x| < x_1 \\ \frac{1}{\pi} \arccos \left(-\frac{\mu(x)}{2} \right) & x_1 < |x| < x_0 \end{cases}. \quad (9)$$

Next, we relate the total number of particles in the trap, $N = \int_{-x_0}^{x_0} n(x) dx$, to the chemical potential μ_0 . The integration results in $N = Rg(\mu_0)$ [16, 27], where $g(\mu_0)$ is given by Eq. (A1) in Appendix A, for $\mu_0 \leq 2$. This explains why the characteristic density $\rho = N/R$ needs to be kept constant when taking the thermodynamic limit in a trapped system [8, 16, 17, 26, 27]. Figure 1(b) shows the site-occupation distribution $n(x)$ predicted by the LDA at three characteristic densities ρ as a function of x/R . They are indistinguishable from the exact numerical results $n_x = \langle \hat{n}_x \rangle$. Figure A1 in Appendix A shows site-occupation profiles for small values of ρ compared against the predictions of the LDA in the continuum limit.

The band insulating domain emerges in the center of the trap when the occupation there becomes one, i.e., when $\mu_0 = 2$. This allows us to obtain the critical particle number $N_c = \int_{-x_0}^{x_0} n_c(x) dx = 8R/\pi$, so that the critical characteristic density is

$$\rho_c = \frac{N_c}{R} = \frac{8}{\pi}. \quad (10)$$

This value matches the point at which the second derivative of the total energy density appears to behave non-analytically in Fig. 1(a).

The LDA also allows one to calculate the single-particle density of states, $D(\mu_0) = dN/d\mu_0$. It gives

$$D(\mu_0) = \frac{2R}{\pi\sqrt{2-\mu_0}} \begin{cases} K(t) & -2 \leq \mu_0 < 2 \\ K(t) - F(t^{-1/2}|t) & \mu_0 > 2 \end{cases}, \quad (11)$$

where $t = (\mu_0 + 2)/(\mu_0 - 2)$, and $F(t|m)$ [$K(t)$] is the elliptic integral [complete elliptic integral] of the first kind [53]. A divergence of the density of states occurs at $\mu_0 = 2$, which corresponds to ρ_c in Eq. (10). Beyond this point ($\varepsilon_j > 2$), single-particle states become localized at either the left or the right side of the center of the trap, due to the Bragg reflection [44, 49]. The exact numerical results for the density of states shown in Fig. 1(c) are in perfect agreement with the ones obtained

evaluating Eq. (11). Close to $\mu_0 = 2$, one can expand Eq. (11) to obtain the leading order term

$$\lim_{\mu_0 \rightarrow 2} D(\mu_0) = \frac{R}{2\pi} (6 \ln 2 - \ln |\mu_0 - 2|), \quad (12)$$

which confirms that the divergence is logarithmic, as advanced in Ref. [49].

B. Order parameter: Entanglement entropies

Our analysis so far has revealed that, for $\rho > \rho_c$ in the thermodynamic limit, there are lattice sites in the center of the trap with site occupation one. The correlations $\langle \hat{f}_i^\dagger \hat{f}_j \rangle$ across those sites are therefore zero (in the absence of degeneracies in the single-particle spectrum, which is our case). Those sites with occupation one split the L by L one-body correlation matrix F into two disconnected blocks. The eigenvalues within each block (relevant to obtain S_{vN} and S_n) are those of a pure state. The entanglement entropies S_{vN} and S_n therefore must vanish for $\rho > \rho_c$, which make them suitable candidates for the order parameter of the local quantum phase transition.

The main panel of Fig. 2(a) shows S_{vN} as a function of ρ across the local quantum phase transition, for four values of R . In finite systems, S_{vN} vanishes when $\rho > \rho_c$. Moreover, the curves for different values of R

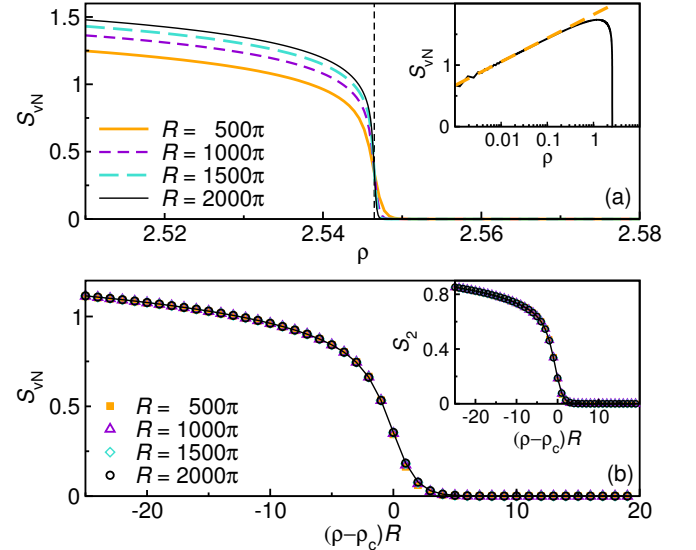


FIG. 2. Ground state entanglement entropy. (a) von Neumann entanglement entropy S_{vN} vs ρ about the critical point (main panel), and in a wider range (inset, results for $R = 500\pi$). The vertical dashed line in the main panel denotes the critical point $\rho_c = 8/\pi$. The dashed line in the inset is $S_{vN} = (1/6) [\ln N + \ln 8 + 1.485]$ from Ref. [37]. (b) Rescaling of the x -axis, $\rho \rightarrow (\rho - \rho_c)R$, which results in data collapse about the critical point. We plot S_{vN} (main panel) and S_2 (inset) for different values of R , as indicated in the legend.

(for sufficiently large R) cross at ρ_c [see the vertical line in Fig. 2(a)], and they become sharper with increasing R . This observation is consistent with a vanishing S_{vN} for any $\rho > \rho_c$ in the thermodynamic limit. We note that S_{vN} is not an extensive quantity [54, 55], so it is fitting that this is the kind of order parameter that one needs to characterize the local quantum phase transition undergone by these systems.

The inset in Fig. 2(a) shows that S_{vN} is nonzero for all nonzero $\rho < \rho_c$ (the same holds true for S_2 , not shown). The dashed line depicts the analytical result in the continuum limit [37], $S_{\text{vN}} = (1/6) [\ln N + \ln 8 + y_1]$, with $y_1 \approx 1.485$. It provides an excellent description for the results in the lattice for small values of the characteristic density, $\rho \lesssim 0.5$.

Finally we use the value of ρ_c predicted by the LDA, Eq. (10), to study the scaling of the entanglement entropies across the transition. The main panel (inset) of Fig. 2(b) shows a perfect data collapse for S_{vN} (S_2) vs $(\rho - \rho_c)R$ for four values of R . This suggest that

$$S_n = \mathcal{F}_n [(\rho - \rho_c)R] = \mathcal{F}_n [N - N_c] \quad (13)$$

is a universal scaling function that describes the corresponding entanglement entropy across the local quantum phase transition.

From our results in this section we conclude that, through a scaling analysis, experimental measurements of entanglement entropies in finite systems can enable the location of the local quantum phase transition in the thermodynamic limit. The result obtained from such an analysis will be much more accurate than those obtained using local observables such as the site occupations and their fluctuations.

IV. FINITE TEMPERATURE PROPERTIES

A. Thermal states: Mutual information and thermodynamic entropy

Here we study the properties of the trapped system in thermal equilibrium. The density plots in Figs. 3(a) and 3(b) show the thermodynamic entropy density $S_{\text{GE}}(T, \rho)/R$ and the mutual information $I(T, \rho)$, respectively, as a function of ρ and the temperature.

The most prominent characteristic of S_{GE} is that it is maximal about ρ_c . This is better seen in Fig. 4(a), where we plot S_{GE} vs ρ for six temperatures (notice the log scale in the y-axis). At low temperatures, a sharp peak can be seen about ρ_c . With increasing T , S_{GE} increases for all values of ρ , and the peak broadens and eventually disappears. The presence of a sharp peak at low temperatures is a direct consequence of the divergence of the density of states at the Fermi energy corresponding to ρ_c [see Fig. 1(c) and Eq. (12)], and could also be used to locate the local quantum phase transition in our trapped one-dimensional systems at low temperatures.

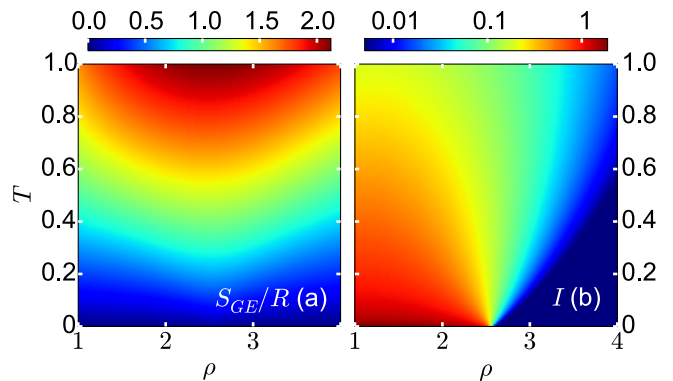


FIG. 3. Numerical results for: (a) the thermodynamic entropy density, S_{GE}/R , and (b) the mutual information, I , as a function of ρ and T for trapped systems with $R = 125\pi$.

We should, however, mention that a similar peak can be seen in the entropy of noninteracting fermions in the translationally invariant square lattice with Hamiltonian

$$H_{2D} = -t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (\hat{f}_{\mathbf{i}}^\dagger \hat{f}_{\mathbf{j}} + \text{H.c.}) , \quad (14)$$

where $\langle \mathbf{i}, \mathbf{j} \rangle$ stands for nearest neighbor sites. Such a system also exhibits a logarithmic divergence of the density of states at half filling ($\mu_0 = 0$). The entropy density for noninteracting fermion in a square lattice (with L^2 sites) is depicted in Fig. 4(b). It can be seen to be qualitatively, and quantitatively, similar to that of the one-dimensional trapped system [whose number of occupied sites is about $4R$, see Fig. 1(b)]. Hence, while a sharp peak in the entropy density does not necessarily indicate a local quantum phase transition (there is none in the two-dimensional system), it does for our trapped one-dimensional system.

On the other hand, at low temperatures, the behav-

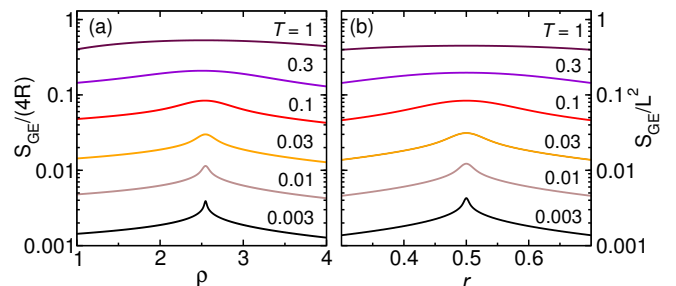


FIG. 4. Thermodynamic entropy density $S_{\text{GE}}/\mathcal{N}$. (a) Entropy density of harmonically trapped fermions in a one-dimensional lattice, Eq. (1), with $\mathcal{N} = 4R$, as a function of the characteristic density $\rho = \langle N \rangle / R$ for $R = 250\pi$. (b) Entropy density of fermions in a translationally invariant square lattice, Eq. (14), with $\mathcal{N} = L^2$, as a function of the site occupation $r = \langle N \rangle / L$ for $L = 2000$.

ior of the mutual information I [Fig. 3(b)] is qualitatively similar to that of the ground-state entanglement entropy. Namely, it is nonvanishing for $\rho < \rho_c$, and it is vanishingly small for $\rho > \rho_c$. Upon increasing the T , the fingerprints of the local quantum phase transition disappear because $I(\rho < \rho_c)$ decreases (correlations between the two halves of the system decrease, an expected effect of the temperature) and $I(\rho > \rho_c)$ slightly increases (correlations between the two halves slightly increase because of the “melting” of the band-insulating domain), leading to a featureless structure at large temperatures. This is better seen in the inset of Fig. 5, in which we plot the mutual information vs ρ for five temperatures.

We note that, unlike S_{GE} , the mutual information exhibits an area-law scaling (it is not divided by R). This is the result of the prefactor in the extensive part of $S_{\text{vN}}(T)$ being identical to that of $S_{\text{GE}}(T)/2$. One of the main goals of the next section is to show that this observation does not hold for the entanglement entropy of excited eigenstates at the same energy density. In particular, the prefactor in the extensive part of $S_{\text{vN}}(m)$ for the overwhelming majority of the many-body eigenstates $|m\rangle$ at a given energy density can be considerably smaller than that in $S_{\text{vN}}(T)$ at the same energy density.

B. Average eigenstate entanglement entropy

We now turn our focus to the average entanglement entropies of excited many-body eigenstates of the Hamiltonian (1), at mean energy densities that correspond to nonzero temperatures T . We perform large-scale numerical calculations to compute the weighted averages \bar{S}_{vN} and \bar{S}_2 , defined in Eq. (4). Note that since there are exponentially large numbers of eigenstates involved in the calculations, the values of R accessible to us in this section are much smaller than in the previous sections. Still, they allow us to extract low-temperature properties

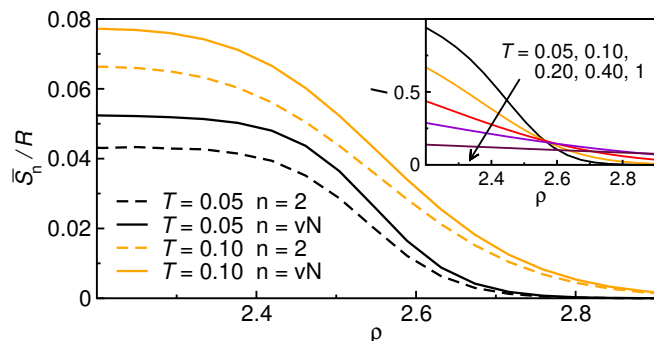


FIG. 5. (Main panel) Average von Neumann entanglement entropy \bar{S}_{vN}/R (solid lines) and second Renyi entanglement entropy \bar{S}_2/R (dashed lines) in eigenstates of the Hamiltonian plotted as functions of ρ for two temperatures. (Inset) Mutual information I vs ρ for five temperatures. All results were obtained for $R = 7.5\pi$.

in the thermodynamic limit.

The main panel of Fig. 5(a) shows results for \bar{S}_{vN} and \bar{S}_2 as functions of ρ for two temperatures. In contrast to the mutual information I , plotted in the inset of Fig. 5 and in Fig. 3, \bar{S}_{vN} and \bar{S}_2 increase with increasing temperature for all values of ρ . This is understandable as, at nonzero mean energy densities above the ground state, the overwhelming majority of the many-body eigenstates are expected to exhibit a volume-law scaling with the system size, with a prefactor that increases with temperature. We explicitly verify the volume law scaling in Appendix D for $\rho = 2$ and 3. Hence, at a fixed temperature T and characteristic density ρ , we fit the numerical results with the ansatz

$$\bar{S}_{\text{n}}(T, \rho) = s_{\text{n}}(T, \rho) R + \delta_{\text{n}}(T, \rho), \quad (15)$$

and extract the prefactor in the extensive part s_{n} , i.e., the average eigenstate entanglement entropy density. Since ρ needs to be kept fixed when taking the thermodynamic limit, the leading term in Eq. (15) can also be written as $\bar{S}_{\text{n}}(T, \rho) \approx [s_{\text{n}}(T, \rho)/\rho] N$, making explicit the linear dependence on the number of particles. The numerical results in Fig. D1 of Appendix D agree perfectly with the functional form given in Eq. (15). As expected, the von Neumann entropy density s_{vN} is always above the second Renyi entropy density s_2 .

Figures 6(a) and 6(b) show the average eigenstate entanglement entropy densities s_{n} , together with the corresponding thermodynamic entropy density $s_{\text{GE}} = \lim_{R \rightarrow \infty} S_{\text{GE}}/(2R)$, as functions of temperature at low temperatures. The thermodynamic entropy density exhibits a linear increase with T for all values of ρ , analogous to the one observed for free fermions in a homogeneous lattice. This is a consequence of the system being gapless independently of the value of ρ .

On the other hand, the average eigenstate entanglement entropy densities exhibit a qualitatively different behavior depending on whether ρ is smaller or larger than ρ_c . For $\rho < \rho_c$, as s_{GE} , they exhibit a linear increase with temperature

$$s_{\text{n}}(T, \rho) = \alpha_{\text{n}}(\rho) T, \quad s_{\text{GE}}(T, \rho) = \alpha_{\text{GE}}(\rho) T, \quad (16)$$

with $\alpha_{\text{n}}(\rho) < \alpha_{\text{GE}}(\rho)$. For $\rho > \rho_c$, however, the average eigenstate entanglement entropy densities exhibit an activated-like behavior that is absent in s_{GE} . We fit the results for the von Neumann entanglement entropy density with the ansatz

$$s_{\text{vN}}(T, \rho) = a T^\zeta \exp \left[-b \left(\frac{\Delta(\rho)}{T} \right)^\gamma \right], \quad (17)$$

where $\Delta(\rho) = \rho - \rho_c$. The analysis described in Appendix E reveals that the optimal exponent of the temperature dependent prefactor is $\zeta = 0.28$. We then evaluate the rescaled function $\tilde{s}_{\text{vN}} = -\ln[s_{\text{vN}}/T^\zeta]$, see the symbols in Fig. 6(c), which results in excellent data collapse for different values of ρ when $T/\Delta \lesssim 0.5$. The

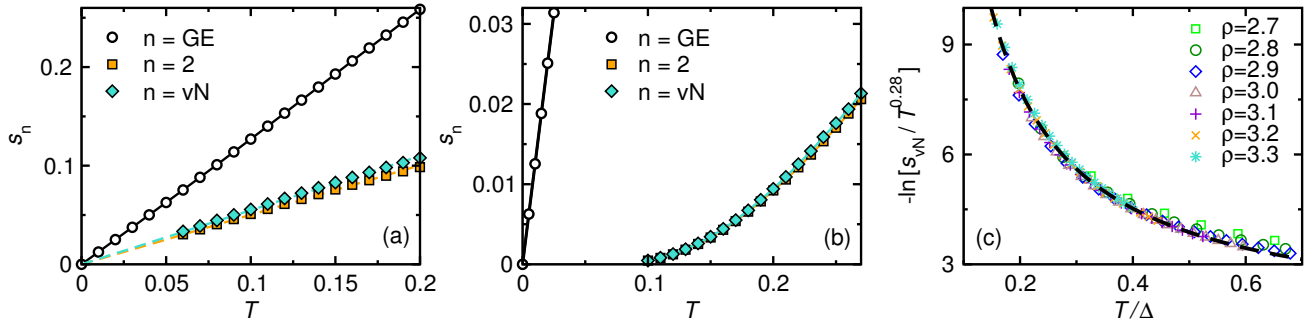


FIG. 6. Average eigenstate entanglement entropy density s_n , as obtained from fits to Eq. (15), and the thermodynamic entropy density $s_{GE} = \lim_{R \rightarrow \infty} S_{GE}/(2R)$. The entropy densities are plotted as functions of temperature for: (a) $\rho = 2$ and (b) $\rho = 3$. (c) Data collapse of the von Neumann entanglement entropy density for different values of $\rho \gtrsim \rho_c$. The symbols depict the numerical results. The dashed lines in (a) are linear fits according to Eq. (16), while the solid lines in (a) and (b) are guides to the eye. The dashed line in (c) is the function $1.295(\Delta/T) + 1.288$.

dashed line in Fig. 6(c) is a simple algebraic function $b(\Delta/T) - \ln a$ [i.e., setting $\gamma = 1$ in Eq. (17)], which is an excellent match to the numerical results.

It is expected that, in general, s_n is smaller than s_{GE} at all temperatures and characteristic densities. This follows after a recent study [45], in which it was proved that $s_{vN} < s_{GE}$ for identical bipartitions of translationally-invariant fermionic quadratic models at infinite temperature. Our results for $\rho > \rho_c$ in trapped systems show that the average eigenstate entanglement entropy density s_n and the thermodynamic entropy density s_{GE} can actually exhibit qualitatively different behavior (which was not the case in Ref. [45]). The former exhibits activated-like behavior while the latter is linear in temperature. Furthermore, the data collapse in Fig. 6(c) shows that Δ acts as a sort of gap for the eigenstate entanglement entropies in trapped systems (it determines the width of the band insulating domain in the center of the trap). This despite the fact that there is no energy gap in the single-particle spectrum of the Hamiltonian (1), as shown in Fig. 1(a), and, consequently, in the many-body energy spectrum. Hence, at low temperatures, the average eigenstate entanglement entropies exhibit fingerprints of the formation of the band insulating domain, i.e., of the local quantum phase transition, and could be used to locate it.

V. SUMMARY

We used quantum information measures to study spinless fermions in one-dimensional lattices in the presence of a harmonic trap. In contrast to most studies of entanglement entropies, which focus on small subsystems of ground states, here we considered a bipartition of the system in two equal parts and studied ground states, excited states, and finite-temperature mixed states.

In the ground state, we showed that the entanglement entropies scale as $\ln N$ at small characteristic den-

sities [37], while they vanish after the emergence of the band insulating domain (as expected from the fact that one has two disconnected metallic domains). More importantly, we showed that a scaling analysis of the entanglement entropies in finite systems allows one to accurately determine the critical characteristic density for the formation of the band insulating domain. Hence, ground-state bipartite entanglement entropies are excellent order parameters to describe the characteristic-density driven local quantum phase transition in those systems.

For low-temperature thermal states, we showed that: (i) a sharp peak in the thermodynamic entropy, associated to a divergence of the single-particle density of states at the Fermi energy corresponding to ρ_c , signals the critical point, and (ii) the mutual information exhibits a behavior that is qualitatively similar to that of the entanglement entropies in the ground state, namely, it is nonzero below the local quantum phase transition and very small above it. Increasing the temperature destroys this contrast.

Finally, we systematically studied excited eigenstates. We showed that their average entanglement entropy scales linearly with N , i.e., the overwhelming majority of the excited states exhibit a volume law, but the prefactor is always smaller than that of the thermodynamic entropy. This complements many recent works that have studied entanglement entropies for lattice bipartitions in which the smaller subsystem is not a vanishing fraction of the system [56–67], and, in particular, the works dealing with quadratic models [45, 58, 61]. Our main finding for excited states is that, as functions of the temperature, the average eigenstate entanglement entropy densities behave very differently for $\rho < \rho_c$ (linear increase with the temperature) and $\rho > \rho_c$ (activated like behavior). Hence, at low temperatures, they carry the fingerprints of the local quantum phase transition.

VI. ACKNOWLEDGMENTS

We acknowledge discussions with R. Modak and V. Alba. This work was supported by the NSF, Grant No. PHY-1707482.

Appendix A: Site occupations for $\rho \leq \rho_c$

For $\rho \leq \rho_c$, the site occupations $n(x)$ within the LDA are given by Eq. (8). We obtain the relation between ρ and μ_0 from the equation $\rho = \int_{-x_0}^{x_0} n(x) dx / R$. It yields

$$\rho = \frac{4\sqrt{2-\mu_0}}{\pi} \left(E \left[\frac{\mu_0+2}{\mu_0-2} \right] - K \left[\frac{\mu_0+2}{\mu_0-2} \right] \right), \quad (\text{A1})$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. For any given ρ , we calculate μ_0 numerically from Eq. (A1). μ_0 is needed to determine $n(x)$. The agreement between $n(x)$ (dashed lines) and the numerically exact site occupations n_x (solid lines), shown in Fig. 1(b) and Fig. A1 as a function of x/R , is excellent. We also compare both results to the ones in the continuum limit (see, e.g., Refs. [27, 40]), which can be obtained from Eq. (A1) by expanding μ_0 around -2 . This yields a simple relation $\rho = 1 + \mu_0/2$ and the semicircle particle distribution

$$n_{\text{sc}}(x) = \frac{1}{\pi} \sqrt{2\rho - \left(\frac{x}{R}\right)^2}. \quad (\text{A2})$$

The results for $n_{\text{sc}}(x)$ are shown as dashed-dotted lines in Fig. A1. They are nearly indistinguishable from the exact ones up to $\rho \lesssim 0.5$.

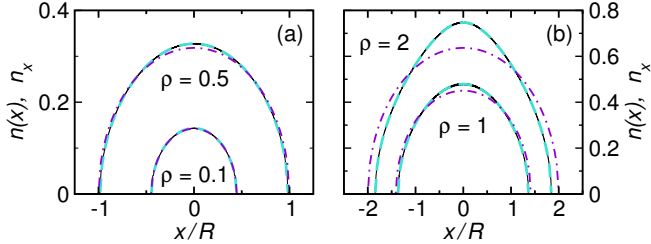


FIG. A1. Site occupations as functions of x/R for different values of ρ . Solid lines are exact numerical results $n_x = \langle \hat{n}_x \rangle$ (with integer x and $R = 500\pi$), and the overlapping dashed lines are the LDA results $n(x)$ from Eq. (8). The dashed-dotted lines are the results for the continuum (limit $\rho \rightarrow 0$ in a lattice), given by Eq. (A2).

Appendix B: Numerical evaluation of Eq. (4)

We here describe how the average eigenstate entanglement entropy \bar{S}_n , defined in Eq. (4), is evaluated numerically. We focus on systems with a fixed particle number N (canonical ensemble averages), while a comparison

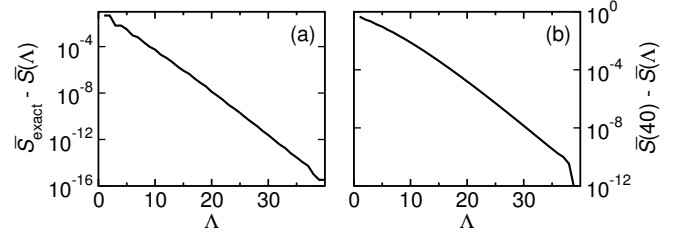


FIG. B1. Accuracy of the truncation used to evaluate the average von Neumann eigenstate entanglement entropy in Eq. (4), for $\rho = 2$ and $T = 0.1$. (a) Difference between the exact average \bar{S}_{exact} and the result after the truncation $\bar{S}(\Lambda)$, for $R = 5$ and $L = 24$ lattice sites. (b) $\bar{S}(40) - \bar{S}(\Lambda)$ for $R = 20$ and $L = 150$.

with grand canonical ensemble averages is presented in Appendix C.

In a system of N spinless fermions in a one-dimensional lattice with L sites, the total number of many-body energy eigenstates is $\binom{L}{N}$. Among all the energy eigenstates, we systematically find the ones $|m\rangle$ with a relative weight $\exp[-(E_m - E_{\text{GS}})/T] > \exp[-\Lambda]$, where E_{GS} is the ground state energy, $E_{\text{GS}} = \sum_{i=1}^N \varepsilon_i$, and Λ sets the numerical accuracy. Note that the single-particle eigenenergies ε_i are ordered such that $\varepsilon_i < \varepsilon_{i+1}$.

The accuracy of such a truncation scheme is studied in Fig. B1(a) for the von Neumann entanglement entropy \bar{S}_{vN} , for $\rho = 2$, $T = 0.1$, and $L = 24$. The results show that the difference between the exact average \bar{S}_{exact} and the average after truncating the sum $\bar{S}(\Lambda)$ decreases exponentially with Λ . For larger systems ($L \gtrsim 100$), however, it is numerically impossible to obtain the exact averages. In Fig. B1(b), we report the difference between $\bar{S}(\Lambda)$ and $\bar{S}(\Lambda = 40)$, for $\rho = 2$, $T = 0.1$, and $L = 150$ ($\Lambda = 40$ is about the largest cut off we can consider for that system size). The differences can again be seen to decay nearly exponentially, and to be very small ($\sim 10^{-8}$) for $\Lambda = 30$. That is value of Λ used to obtain the results reported in Figs. 5, 6, C1, and D1.

Appendix C: Canonical vs grand canonical averages

We extend the average eigenstate entanglement entropy in Eq. (4), which corresponds to a canonical calculation, to a grand canonical average

$$\bar{S}_n(T) = \frac{\sum_N \sum_{m(N)} S_n(m) e^{-(E_m - \mu N)/T}}{\sum_N \sum_{m(N)} e^{-(E_m - \mu N)/T}}, \quad (\text{C1})$$

where the chemical potential μ is determined so that the average particle number is N . In contrast to the canonical average, one needs to perform an additional sum over sectors with different particle numbers N and many-body eigenstates $\{|m(N)\rangle\}$. This increases significantly the computation time.

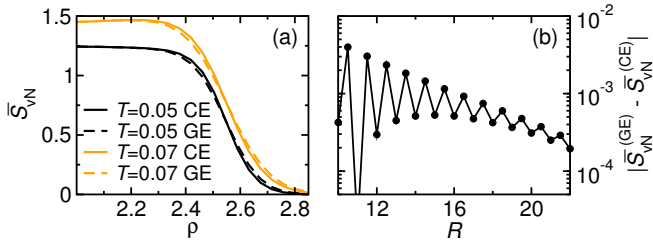


FIG. C1. Comparison between the canonical and grand canonical averages of the von Neumann eigenstate entanglement entropy \bar{S}_{vN} . The former is calculated using Eq. (4), while the latter one is calculated using Eq. (C1). (a) Results for $R = 7.5\pi$ as a function of ρ , for two temperatures. (b) Absolute difference between $\bar{S}_{\text{vN}}^{(\text{CE})}$ and $\bar{S}_{\text{vN}}^{(\text{GE})}$, at $\rho = 2$ and $T = 0.1$, as a function of R .

Figure C1(a) compares the canonical and grand canonical averages of the von Neumann entanglement entropy \bar{S}_{vN} , plotted as functions of ρ . The results are very close for all ρ . Figure C1(b) shows the difference between the two averages upon increasing the system size (or, equivalently, R) for $\rho = 2$. As expected, the differences decrease with increasing system size, and are expected to vanish in the thermodynamic limit ($R \rightarrow \infty$ while keeping ρ constant). Because of this, we only report results for canonical averages in Figs. 5, 6, and D1.

Appendix D: Volume-law scaling of the average eigenstate entanglement entropy

Figure D1 shows that \bar{S}_n increases linearly with R at fixed temperature, as stated in Sec. IV B. Note that re-

sults are reported for $\rho = 2$ ($\rho < \rho_c$) and $\rho = 3$ ($\rho > \rho_c$) for different temperatures. This justifies the use of the ansatz in Eq. (15) to extract the average eigenstate entanglement entropy density s_n reported in Fig. 6.

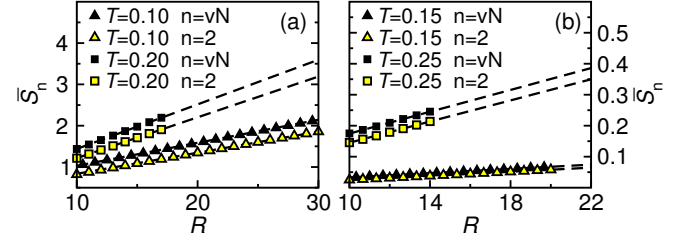


FIG. D1. Volume-law scaling of the average eigenstate entanglement entropies. (a) $\rho = 2$ and (b) $\rho = 3$. Symbols display the von Neumann entropy \bar{S}_{vN} and the second Renyi entropy \bar{S}_2 for different temperatures. Dashed lines are linear fits using the ansatz in Eq. (15).

Appendix E: Average eigenstate entanglement entropy for $\rho > \rho_c$

Figure 6(b) shows an activated-like behavior of $s_n(T, \rho)$ as a function of T , for $\rho > \rho_c$. We fitted the numerical results to the ansatz in Eq. (17) with $\zeta = 0.28$. To determine this optimal value of ζ , we did as follows. We calculated $\tilde{s}_{\text{vN}} = -\ln[s_{\text{vN}}/T^\zeta]$ vs Δ/T for different values of ζ in the interval $\zeta \in [-0.5, 1]$. For a fixed value of ζ , we fitted the results for \tilde{s}_{vN} (obtained for seven values of $\rho = \{2.7, 2.8, \dots, 3.3\}$) with a high-order polynomial $\sum_{n=0}^6 a_n(\Delta/T)^n$ in the regime $\Delta/T \leq 0.5$. We then calculated the sum of squares of the differences between \tilde{s}_{vN} and the fitted polynomial. This sum was found to have a minimum at $\zeta = 0.28$.

-
- [1] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Cold bosonic atoms in optical lattices, *Phys. Rev. Lett.* **81**, 3108 (1998).
 - [2] M. Greiner, O. Mandel, T. Esslinger, T. Hänsch, and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms, *Nature (London)* **415**, 39 (2002).
 - [3] M. Greiner, O. Mandel, T. W. Hänsch, and I. Bloch, Collapse and revival of the matter wave field of a Bose-Einstein condensate, *Nature* **419**, 51 (2002).
 - [4] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [5] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, One dimensional bosons: From condensed matter systems to ultracold gases, *Rev. Mod. Phys.* **83**, 1405 (2011).
 - [6] G. G. Batrouni, V. Rousseau, R. T. Scalettar, M. Rigol, A. Muramatsu, P. J. H. Denteneer, and M. Troyer, Mott domains of bosons confined on optical lattices, *Phys. Rev. Lett.* **89**, 117203 (2002).
 - [7] V. A. Kashurnikov, N. V. Prokof'ev, and B. V. Svistunov, Revealing the superfluid–Mott-insulator transition in an optical lattice, *Phys. Rev. A* **66**, 031601 (2002).
 - [8] M. Rigol, A. Muramatsu, G. G. Batrouni, and R. T. Scalettar, Local quantum criticality in confined fermions on optical lattices, *Phys. Rev. Lett.* **91**, 130403 (2003).
 - [9] M. Rigol and A. Muramatsu, Quantum Monte Carlo study of confined fermions in one-dimensional optical lattices, *Phys. Rev. A* **69**, 053612 (2004).
 - [10] C. Kollath, U. Schollwöck, J. von Delft, and W. Zwerger, Spatial correlations of trapped one-dimensional bosons in an optical lattice, *Phys. Rev. A* **69**, 031601 (2004).
 - [11] S. Bergkvist, P. Henelius, and A. Rosengren, Local-density approximation for confined bosons in an optical lattice, *Phys. Rev. A* **70**, 053601 (2004).
 - [12] S. Wessel, F. Alet, M. Troyer, and G. G. Batrouni, Quantum Monte Carlo simulations of confined bosonic atoms in optical lattices, *Phys. Rev. A* **70**, 053615 (2004).

- [13] M. Rigol and A. Muramatsu, Universal properties of hard-core bosons confined on one-dimensional lattices, *Phys. Rev. A* **70**, 031603 (2004).
- [14] M. Rigol and A. Muramatsu, Ground-state properties of hard-core bosons confined on one-dimensional optical lattices, *Phys. Rev. A* **72**, 013604 (2005).
- [15] P. N. Ma, K. Y. Yang, L. Pollet, J. V. Porto, M. Troyer, and F. C. Zhang, Influence of the trap shape on the detection of the superfluid–Mott-insulator transition, *Phys. Rev. A* **78**, 023605 (2008).
- [16] G. G. Batrouni, H. R. Krishnamurthy, K. W. Mahmud, V. G. Rousseau, and R. T. Scalettar, Canonical trajectories and critical coupling of the Bose-Hubbard Hamiltonian in a harmonic trap, *Phys. Rev. A* **78**, 023627 (2008).
- [17] M. Rigol, G. G. Batrouni, V. G. Rousseau, and R. T. Scalettar, State diagrams for harmonically trapped bosons in optical lattices, *Phys. Rev. A* **79**, 053605 (2009).
- [18] M. Campostrini and E. Vicari, Critical behavior and scaling in trapped systems, *Phys. Rev. Lett.* **102**, 240601 (2009).
- [19] L. Pollet, N. V. Prokof'ev, and B. V. Svistunov, Criticality in trapped atomic systems, *Phys. Rev. Lett.* **104**, 245705 (2010).
- [20] K. Jiménez-García, R. L. Compton, Y.-J. Lin, W. D. Phillips, J. V. Porto, and I. B. Spielman, Phases of a two-dimensional Bose gas in an optical lattice, *Phys. Rev. Lett.* **105**, 110401 (2010).
- [21] M. Campostrini and E. Vicari, Trap-size scaling in confined-particle systems at quantum transitions, *Phys. Rev. A* **81**, 023606 (2010).
- [22] M. Campostrini and E. Vicari, Quantum critical behavior and trap-size scaling of trapped bosons in a one-dimensional optical lattice, *Phys. Rev. A* **81**, 063614 (2010).
- [23] K. W. Mahmud, E. N. Duchon, Y. Kato, N. Kawashima, R. T. Scalettar, and N. Trivedi, Finite-temperature study of bosons in a two-dimensional optical lattice, *Phys. Rev. B* **84**, 054302 (2011).
- [24] L. Pollet, Recent developments in quantum Monte Carlo simulations with applications for cold gases, *Rep. Prog. Phys.* **75**, 094501 (2012).
- [25] G. Ceccarelli, C. Torrero, and E. Vicari, Critical parameters from trap-size scaling in systems of trapped particles, *Phys. Rev. B* **87**, 024513 (2013).
- [26] A. Angelone, M. Campostrini, and E. Vicari, Universal quantum behavior of interacting fermions in one-dimensional traps: From few particles to the trap thermodynamic limit, *Phys. Rev. A* **89**, 023635 (2014).
- [27] W. Xu and M. Rigol, Universal scaling of density and momentum distributions in Lieb-Liniger gases, *Phys. Rev. A* **92**, 063623 (2015).
- [28] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, New York, 2011).
- [29] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, Measuring entanglement entropy in a quantum many-body system, *Nature* **528**, 77 (2015).
- [30] A. M. Kaufman, M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, P. M. Preiss, and M. Greiner, Quantum thermalization through entanglement in an isolated many-body system, *Science* **353**, 794 (2016).
- [31] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Scaling of entanglement close to a quantum phase transition, *Nature* **416**, 608 (2002).
- [32] T. J. Osborne and M. A. Nielsen, Entanglement in a simple quantum phase transition, *Phys. Rev. A* **66**, 032110 (2002).
- [33] S.-J. Gu, S.-S. Deng, Y.-Q. Li, and H.-Q. Lin, Entanglement and quantum phase transition in the extended Hubbard model, *Phys. Rev. Lett.* **93**, 086402 (2004).
- [34] M. Campostrini and E. Vicari, Scaling of bipartite entanglement in one-dimensional lattice systems with a trapping potential, *J. Stat. Mech.* (2010), P08020.
- [35] P. Calabrese, M. Mintchev, and E. Vicari, Entanglement entropy of one-dimensional gases, *Phys. Rev. Lett.* **107**, 020601 (2011).
- [36] P. Calabrese, M. Mintchev, and E. Vicari, Entanglement entropies in free-fermion gases for arbitrary dimension, *EPL (Europhysics Letters)* **97**, 20009 (2012).
- [37] E. Vicari, Entanglement and particle correlations of Fermi gases in harmonic traps, *Phys. Rev. A* **85**, 062104 (2012).
- [38] P. Calabrese, P. Le Doussal, and S. N. Majumdar, Random matrices and entanglement entropy of trapped Fermi gases, *Phys. Rev. A* **91**, 012303 (2015).
- [39] V. Alba, Out-of-equilibrium protocol for Rényi entropies via the Jarzynski equality, *Phys. Rev. E* **95**, 062132 (2017).
- [40] J. Dubail, J.-M. Stéphan, J. Viti, and P. Calabrese, Conformal Field Theory for Inhomogeneous One-dimensional Quantum Systems: the Example of Non-Interacting Fermi Gases, *SciPost Phys.* **2**, 002 (2017).
- [41] V. Alba and P. Calabrese, Entanglement and thermodynamics after a quantum quench in integrable systems, *Proc. Natl. Acad. Sci.* **114**, 7947 (2017).
- [42] V. Alba and P. Calabrese, Quench action and Rényi entropies in integrable systems, *Phys. Rev. B* **96**, 115421 (2017).
- [43] G. De Chiara and A. Sanpera, Genuine quantum correlations of quantum many-body systems, [arXiv:1711.07824](https://arxiv.org/abs/1711.07824).
- [44] M. Rigol and A. Muramatsu, Confinement control by optical lattices, *Phys. Rev. A* **70**, 043627 (2004).
- [45] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol, Entanglement entropy of eigenstates of quadratic fermionic Hamiltonians, *Phys. Rev. Lett.* **119**, 020601 (2017).
- [46] Y. Huang and J. E. Moore, Excited-state entanglement and thermal mutual information in random spin chains, *Phys. Rev. B* **90**, 220202 (2014).
- [47] C. Liu, X. Chen, and L. Balents, Quantum entanglement of the Sachdev-Ye-Kitaev models, [arXiv:1709.06259](https://arxiv.org/abs/1709.06259).
- [48] A. M. Rey, G. Pupillo, C. W. Clark, and C. J. Williams, Ultracold atoms confined in an optical lattice plus parabolic potential: A closed-form approach, *Phys. Rev. A* **72**, 033616 (2005).
- [49] C. Hooley and J. Quintanilla, Single-atom density of states of an optical lattice, *Phys. Rev. Lett.* **93**, 080404 (2004).
- [50] I. Peschel, Calculation of reduced density matrices from correlation functions, *J. Phys. A: Math. Gen.* **36**, L205 (2003).
- [51] B. Groisman, S. Popescu, and A. Winter, Quantum, classical, and total amount of correlations in a quantum state, *Phys. Rev. A* **72**, 032317 (2005).
- [52] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, Area laws in quantum systems: Mutual information and correlations, *Phys. Rev. Lett.* **100**, 070502 (2008).

- [53] G. Arfken and H. Weber, *Mathematical methods for physicists* (Elsevier Acad. Press, 2008).
- [54] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Rev. Mod. Phys.* **80**, 517 (2008).
- [55] J. Eisert, M. Cramer, and M. B. Plenio, Colloquium: Area laws for the entanglement entropy, *Rev. Mod. Phys.* **82**, 277 (2010).
- [56] L. F. Santos, A. Polkovnikov, and M. Rigol, Weak and strong typicality in quantum systems, *Phys. Rev. E* **86**, 010102 (2012).
- [57] J. M. Deutsch, H. Li, and A. Sharma, Microscopic origin of thermodynamic entropy in isolated systems, *Phys. Rev. E* **87**, 042135 (2013).
- [58] M. Storms and R. R. P. Singh, Entanglement in ground and excited states of gapped free-fermion systems and their relationship with fermi surface and thermodynamic equilibrium properties, *Phys. Rev. E* **89**, 012125 (2014).
- [59] W. Beugeling, A. Andreanov, and M. Haque, Global characteristics of all eigenstates of local many-body Hamiltonians: participation ratio and entanglement entropy, *J. Stat. Mech.* (2015), P02002.
- [60] J. R. Garrison and T. Grover, Does a single eigenstate encode the full Hamiltonian?, [arXiv:1503.00729](#).
- [61] S. Nandy, A. Sen, A. Das, and A. Dhar, Eigenstate Gibbs ensemble in integrable quantum systems, *Phys. Rev. B* **94**, 245131 (2016).
- [62] A. Dymarsky, N. Lashkari, and H. Liu, Subsystem ETH, [arXiv:1611.08764v1](#).
- [63] H. Fujita, Y. O. Nakagawa, S. Sugiura, and M. Watanabe, Universality in volume law entanglement of pure quantum states, [arXiv:1703.02993v2](#).
- [64] L. Vidmar and M. Rigol, Entanglement entropy of eigenstates of quantum chaotic Hamiltonians, *Phys. Rev. Lett.* **119**, 220603 (2017).
- [65] Y. Huang, Universal eigenstate entanglement of chaotic local hamiltonians, [arXiv:1708.08607](#).
- [66] T.-C. Lu and T. Grover, Renyi entropy of chaotic eigenstates, [arXiv:1709.08784](#).
- [67] X. Chen and A. W. W. Ludwig, Universal spectral correlations in the chaotic wave function, and the development of quantum chaos, [arXiv:1710.02686](#).