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Phys. Rev. A **97**, 022342 — Published 28 February 2018

DOI: [10.1103/PhysRevA.97.022342](https://doi.org/10.1103/PhysRevA.97.022342)

Axiomatic and operational connections between l_1 -norm of coherence and negativity

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(Dated: February 9, 2018)

Quantum coherence plays a central role in various research areas. The l_1 -norm of coherence is one of the most important coherence measures that are easily computable, but it is not easy to find a simple interpretation. We show that the l_1 -norm of coherence is uniquely characterized by a few simple axioms, which demonstrates in a precise sense that it is the analog of negativity in entanglement theory and sum negativity in the resource theory of magic-state quantum computation. We also provide an operational interpretation of the l_1 -norm of coherence as the maximum entanglement, measured by the negativity, produced by incoherent operations acting on the system and an incoherent ancilla. To achieve this goal, we clarify the relation between the l_1 -norm of coherence and negativity for all bipartite states, which leads to an interesting generalization of maximally correlated states. Surprisingly, all entangled states thus obtained are distillable. Moreover, their entanglement cost and distillable entanglement can be computed explicitly for a qubit-qudit system.

I. INTRODUCTION

Quantum coherence underlies most nonclassical phenomena characteristic of quantum physics, such as entanglement, steering, and Bell nonlocality. It is also a key resource for demonstrating quantum advantages in various information processing tasks, such as quantum metrology, quantum cryptography, and quantum computation. Recently, coherence has attracted unprecedented attention fueled by the resource theoretic formulation proposed in Refs. [1, 2] and further developed in Ref. [3] among many other works; see Ref. [4] for a review.

Although many coherence measures have been proposed, the l_1 -norm of coherence stands out as one of the most important measures that are easily computable [2, 4, 5]. It is useful in studying speedup in quantum computation, such as in Deutsch-Jozsa algorithm [6, 7] and Grover algorithm [8–10]. It features prominently in alternative formulations of uncertainty relations, complementarity relations [5, 11–13], and wave-particle duality in multipath interferometers [5, 14–19]. It plays a crucial role in quantifying the cohering and decohering powers of quantum operations [20–22]. In addition, the l_1 -norm of coherence sets an upper bound for another important coherence measure, the robustness of coherence [23, 24].

Despite its prominent role in the resource theory of quantum coherence, the l_1 -norm of coherence has defied many attempts to find its simple interpretation. Re-

cently, Rana et al. [25, 26] proved that the logarithmic l_1 -norm of coherence is an upper bound for the relative entropy of coherence, which is equal to the distillable coherence [3], and thereby argued that this measure is the analog of the logarithmic negativity in entanglement theory [27–30]. While their observation is quite suggestive, their argument is not so satisfactory, given that there are many different upper bounds for the relative entropy of coherence. It should be pointed out that the logarithmic negativity is essentially the only useful entanglement measure that is easily computable for all states. It sets an upper bound for the distillable entanglement [27, 30] and a lower bound for the asymptotic exact entanglement cost under positive-partial-transpose (PPT) operations [28].

In this paper we corroborate the significance of the l_1 -norm of coherence by two complementary approaches. First, we propose several simple axioms to single out this measure among all competitors. This approach is inspired by a similar characterization of the sum negativity in the resource theory of magic-state quantum computation [31–35]. The roles of negativity in quantum computation and foundational studies are a focus of ongoing research [36–44]. Our study demonstrates, in a precise sense, that the l_1 -norm of coherence is the analog of the negativity in entanglement theory and the sum negativity in the resource theory of magic-state quantum computation, as illustrated in Fig. 1. This analogy is indicative of the deep connections among the three prominent resource theories.

Next, we provide an operational interpretation of the l_1 -norm of coherence as the maximum entanglement, measured by the negativity, created by incoherent operations acting on the system and an incoherent ancilla.

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To this end, we show that the l_1 -norm of coherence is an upper bound for the negativity and determine all states that saturate this bound, which lead to an interesting generalization of maximally correlated states [45]. The asymptotic exact PPT entanglement cost of these states are equal to the logarithmic negativity. Moreover, such states are distillable whenever they are entangled, and their entanglement cost and distillable entanglement can be computed explicitly for a qubit-qudit system.

II. PRELIMINARIES

Both entanglement theory and coherence theory are special examples of resource theories [3, 4, 46–48], which provide a unified framework for studying resource quantification and manipulations under restricted operations that are deemed free. In the case of entanglement, we are restricted to local operations and classical communication (LOCC), so only separable states can be prepared for free. In the case of coherence, we are restricted to incoherent operations, so only incoherent states are free. Recall that a state is incoherent if it is diagonal in the reference basis. A quantum operation with Kraus representation $\{K_j\}$ is incoherent if each Kraus operator K_j is incoherent in the sense that it maps every incoherent state to an incoherent state [2–4]. The operation is strictly incoherent if in addition each K_j^\dagger is incoherent [3]. For example, a unitary is incoherent if it is monomial, that is, if each column (row) has only one nonzero entry, in which case the unitary is also strictly incoherent.

Resource monotones are an essential ingredient of any resource theory. A weak monotone does not increase under nonselective free operations, while a (strong) monotone in addition does not increase on average under selective free operations. Convexity is desirable but not necessary; it is not assumed in this paper. A resource measure is a monotone that is nonnegative and usually vanishes on free states. Although many entanglement measures have been proposed, the negativity is distinguished because of its simplicity and versatility [27–30]. The l_1 -norm of coherence in coherence theory [2, 26] and the sum negativity in the resource theory of magic-state quantum computation [34] are distinguished for the same reason. All three measures have logarithmic versions, which are additive and satisfy all basic requirements of resource measures, although they are not convex.

Consider a finite-dimensional Hilbert space \mathcal{H} with a reference orthonormal basis $\{|j\rangle\}$, the l_1 -norm of coherence [2] of a state $\rho = \sum_{jk} \rho_{jk} |j\rangle\langle k|$ on \mathcal{H} reads

$$C_{l_1}(\rho) := \sum_{j \neq k} |\rho_{jk}| = \sum_{jk} |\rho_{jk}| - 1. \quad (1)$$

It is one of the most important coherence measures that are easily computable [4, 5, 7, 9–26, 49, 50]. In addition, it sets a tight upper bound for another important coherence measure, namely, the robustness of coherence $C_{\mathcal{R}}$

[23, 24]. The two measures C_{l_1} and $C_{\mathcal{R}}$ coincide on pure states and thus share the same convex roof. The logarithmic l_1 -norm of coherence $C_L(\rho) := \log(1 + C_{l_1}(\rho))$ is additive: $C_L(\rho \otimes \sigma) = C_L(\rho) + C_L(\sigma)$ for arbitrary states ρ, σ . It is a universal upper bound for two families of Rényi relative entropies of coherence [51], including the relative entropy of coherence $C_r(\rho)$ and logarithmic robustness of coherence $C_{\mathcal{R}_L}(\rho) := \log(1 + C_{\mathcal{R}}(\rho))$ [26].

Next, consider a bipartite system shared between Alice (A) and Bob (B); we assume that the reference basis is the tensor product of respective reference bases. The coherence measures mentioned above are defined in the same way as before. Now the l_1 -norm of coherence upper bounds not only the robustness of coherence $C_{\mathcal{R}}$ but also the robustness of entanglement $E_{\mathcal{R}}$ [30, 51]. Meanwhile, the logarithmic l_1 -norm of coherence upper bounds the relative entropy of coherence C_r and logarithmic robustness of coherence $C_{\mathcal{R}_L}$ as well as the corresponding entanglement measures E_r and $E_{\mathcal{R}_L}$.

Here we are interested in another useful entanglement measure that has no obvious analog in coherence theory. The *negativity* of ρ [27, 29, 30] is defined as

$$\mathcal{N}(\rho) := \text{tr} |\rho^{\text{T}_A}| - 1 = \|\rho^{\text{T}_A}\|_1 - 1, \quad (2)$$

where ρ^{T_A} denotes the partial transpose of ρ with respect to subsystem A, and $\|\cdot\|_1$ is the Schatten 1-norm (or trace norm). Note that the definition in some literature differs by a factor of two. For a pure state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_j \sqrt{\lambda_j} |jj\rangle$ and $\sum_j \lambda_j = 1$, the negativity reads

$$\mathcal{N}(\rho) = C_{l_1}(\rho) = \sum_{j \neq k} \sqrt{\lambda_j \lambda_k} = \left(\sum_j \sqrt{\lambda_j} \right)^2 - 1, \quad (3)$$

which is equal to the robustness of entanglement, so the two measures share the same convex roof. The logarithmic negativity $\mathcal{N}_L(\rho) := \log(1 + \mathcal{N}(\rho))$ sets an upper bound for the distillable entanglement [27, 29, 30] and a lower bound for the asymptotic exact entanglement cost under PPT operations. Moreover, the latter bound is tight when $|\rho^{\text{T}_A}|^{\text{T}_A} \geq 0$ [28].

III. AXIOMATIC CONNECTION BETWEEN l_1 -NORM OF COHERENCE AND NEGATIVITY

Now we are ready to reveal the axiomatic connections between the l_1 -norm of coherence and negativity as well as sum negativity; see Fig. 1.

Theorem 1 (Uniqueness of l_1 -norm of coherence). Suppose $C(\rho)$ is a continuous weak coherence monotone that is a symmetric function of nonzero off-diagonal entries of ρ . Then $C(\rho)$ is a nondecreasing function of $C_{l_1}(\rho)$. If in addition it is additive, then $C(\rho) = c C_{l_1}(\rho)$ for some constant $c \geq 0$.

Theorem 2 (Uniqueness of negativity). Suppose $E(\rho)$ is a weak entanglement monotone that is a symmetric

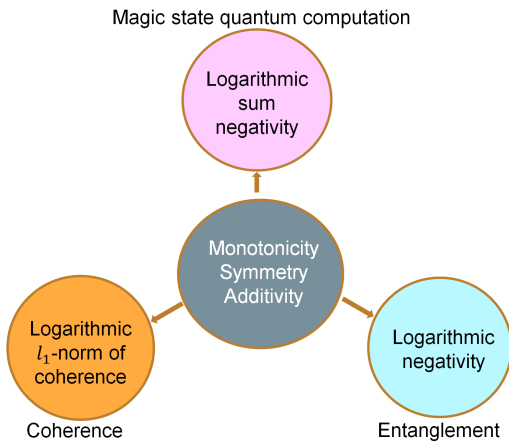


FIG. 1. Common axioms that single out the l_1 -norm of coherence, logarithmic negativity, and logarithmic sum negativity in three prominent resource theories.

function of negative eigenvalues of ρ^{TA} . Then $E(\rho)$ is a nondecreasing function of $\mathcal{N}(\rho)$. If in addition it is additive, then $E(\rho) = c\mathcal{N}_L(\rho)$ for some constant $c \geq 0$.

Recall that a weak monotone does not increase under nonselective free operations. A function is symmetric if it is invariant under arbitrary permutations of its arguments. Here we implicitly assume that $C(\rho \otimes \sigma) = C(\rho)$ for any incoherent state σ , which can be prepared for free; this assumption is more intuitive when C is additive. Similarly, $E(\rho \otimes \sigma) = E(\rho)$ for any separable state σ . Remarkably, both l_1 -norm of coherence and negativity can be uniquely characterized by such simple axioms. In addition, Theorem 1 still holds if free operations are strictly incoherent. We note that a similar uniqueness theorem has been established for the sum negativity in the resource theory of magic-state quantum computation [34]. Our two theorems share the same spirit with Theorem 6 in Ref. [34]. The proof of Theorem 1 is considerably more involved and is relegated to Appendix A; Theorem 2 can be proved following this proof (Lemma 2 in Appendix A in particular). Although the logarithmic sum negativity (referred to as mana in Ref. [34]) was not uniquely characterized as in Theorems 1 and 2 previously, the analogous result follows from a similar argument that underpins Theorem 1.

IV. A TIGHT INEQUALITY BETWEEN l_1 -NORM OF COHERENCE AND NEGATIVITY

In this section, we show that the l_1 -norm of coherence is an upper bound for the negativity and determine all states that saturate this bound, which lead to an interesting generalization of maximally correlated states [45]. The structure and entanglement properties of these states are discussed in detail.

The following theorem clarifies the relation between $\mathcal{N}(\rho)$ and $C_{l_1}(\rho)$ for bipartite states; see Appendix B for a proof.

Theorem 3. Any bipartite state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfies $\mathcal{N}(\rho) \leq C_{l_1}(\rho)$. The inequality is saturated iff ρ^{TA} has the form

$$\rho^{\text{TA}} = \sum_{jk} a_{jk} |jk\rangle\langle\pi(jk)|, \quad (4)$$

where π is a product of disjoint transpositions. We may assume that $\pi(jk) = jk$ whenever $a_{jk} = 0$; each transposition $(jk, j'k')$ satisfies $j \neq j'$ and $k \neq k'$; in addition, jk' and $j'k$ are fixed points of π when $(jk, j'k')$ is such a transposition.

Recall that a transposition is a special permutation that interchanges two elements, while leaving others invariant. A quantum state ρ is a *pairing state* if it saturates the inequality $\mathcal{N}(\rho) \leq C_{l_1}(U\rho U^\dagger)$ for some local unitary $U = U_A \otimes U_B$. It is a *canonical pairing state* if in addition U is the identity, so that ρ^{TA} has the form of Eq. (4). Here the terminology originates from the fact that every transposition is determined by a pair of basis states, and vice versa. Theorem 3 demands that pairing is monogamous; that is, every basis state can be paired with at most one other basis state. Moreover, some basis states must remain as bachelors.

To clarify the structure of pairing states, given a Hermitian operator M acting on $\mathcal{H}_A \otimes \mathcal{H}_B$, define $\mathcal{N}_0(M)$ as the number of negative eigenvalues of M^{TA} ; define $C_{l_0}(M)$ as the number of nonzero off-diagonal entries. The following proposition is proved in Appendix D.

Proposition 1. Any state ρ satisfies $C_{l_0}(\rho) \geq 2\mathcal{N}_0(\rho)$.

When ρ is a pairing state, $\mathcal{N}_0(\rho)$ will be referred to as the *pairing number*. According to Theorem 3 and Proposition 1, $\mathcal{N}_0(\rho)$ is equal to the minimum of $C_{l_0}(\rho)$ under local unitary transformations. If ρ is a canonical pairing state as in Eq. (4), then $\mathcal{N}_0(\rho) = C_{l_0}(\rho)/2$ is equal to the number of transpositions in the disjoint cycle decomposition of π . A pairing state is entangled iff its pairing number is nonzero and is thus negative partial transpose (NPT). Actually, we have a stronger conclusion, as shown in the following theorem and proved in Appendix E.

Theorem 4. Any entangled pairing state ρ is NPT and distillable.

A family of lower bounds for the distillable entanglement of pairing states is provided in Appendix J.

Pairing states are interesting for many other reasons.

Theorem 5. The asymptotic exact PPT entanglement cost $E_{\text{PPT}}(\rho)$ of any pairing state ρ is equal to the logarithmic negativity, that is, $E_{\text{PPT}}(\rho) = \mathcal{N}_L(\rho)$.

Proof. Without loss of generality, we may assume that ρ is a canonical pairing state, which has the form of Eq. (4). Then $|\rho^{\text{TA}}| = \sum_{jk} |a_{jk}| (|jk\rangle\langle jk|)$ is diagonal, so that $|\rho^{\text{TA}}|^{\text{TA}} = |\rho^{\text{TA}}|$ is positive semidefinite, which implies the theorem according to Ref. [28]. \square

Proposition 2. Any pairing state ρ satisfies $E_r(\rho) \leq E_{\mathcal{R}_L}(\rho) \leq \mathcal{N}_L(\rho)$, where $E_r(\rho)$ and $E_{\mathcal{R}_L}(\rho)$ are the relative entropy of entanglement and logarithmic robustness of entanglement.

Proof. The first inequality applies to all states [51]. To prove the second one, we may assume that ρ is a canonical pairing state as in Eq. (4), so that $C_{l_1}(\rho) = \mathcal{N}(\rho)$. Now the conclusion follows from the following inequality $E_{\mathcal{R}}(\rho) \leq C_{\mathcal{R}}(\rho) \leq C_{l_1}(\rho)$ [23, 24]. \square

Pairing states include all bipartite pure states and maximally correlated states. Any bipartite pure state with Schmidt rank r is a pairing state with pairing number $r(r-1)/2$. A *canonical maximally correlated state* [3, 45, 52] has the form

$$\rho_{\text{MC}} := \sum_{r,s=0} c_{rs} |j_r k_r\rangle \langle j_s k_s|, \quad (5)$$

where $j_s \neq j_r$ and $k_s \neq k_r$ whenever $s \neq r$. Note that $\mathcal{N}(\rho_{\text{MC}}) = C_{l_1}(\rho_{\text{MC}}) = \sum_{r \neq s} |c_{rs}|$. A maximally correlated state is any state that is equivalent to ρ_{MC} under a local unitary transformation. Here our definition is more general than the definition in some literature [45, 51, 53].

In general, pairing states do not have to be maximally correlated. Nevertheless, those with the maximal pairing number are when $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$. The following theorem is proved in Appendix F.

Theorem 6. Let ρ be a pairing state on $\mathcal{H}_A \otimes \mathcal{H}_B$ with $d_A := \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$. Then the pairing number of ρ is at most $d_A(d_A - 1)/2$, and the upper bound is saturated only if ρ is maximally correlated.

Corollary 1. A two-qubit state ρ is a canonical pairing state iff it is diagonal or canonically maximally correlated.

Corollary 2. A bipartite state ρ on $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_A) = 2$ is a canonical pairing state iff ρ has the form

$$\rho = \bigoplus_{j \geq 0}^t p_j \rho_j, \quad (6)$$

where $t \leq \dim(\mathcal{H}_B)/2$, $p_j \geq 0$, $\sum_j p_j = 1$, $\text{tr}_A(\rho_j)$ have mutually orthogonal supports, ρ_0 is diagonal, and ρ_j for each $j \geq 1$ is a canonical maximally correlated state.

Corollary 1 is an immediate consequence of Theorems 3 and 6. It is a special case of Corollary 2, which is proved in Appendix G. Remarkably, several key entanglement (coherence) measures of the canonical pairing state in Eq. (6) can be computed explicitly. Note that the distillable coherence C_D and coherence cost C_C are equal to the relative entropy of coherence C_r and coherence of formation C_F , respectively [3], while the counterparts E_D and E_C for entanglement are in general very difficult to compute. The following theorem is proved in Appendix H.

Theorem 7. The distillable entanglement E_D (coherence C_D) and entanglement cost E_C (coherence cost C_C) of the canonical pairing state ρ in Eq. (6) read

$$\begin{aligned} C_D(\rho) &= C_r(\rho) = E_D(\rho) = E_r(\rho) = S(\rho^{\text{diag}}) - S(\rho), \quad (7) \\ C_C(\rho) &= C_F(\rho) = E_C(\rho) = E_F(\rho) = \sum_j p_j E_F(\rho_j), \quad (8) \end{aligned}$$

where ρ^{diag} is the diagonal matrix with the same diagonal as ρ and $E_F(\rho_j) = H\left(\frac{1 + \sqrt{1 - \mathcal{N}(\rho_j)^2}}{2}\right)$ with binary entropy function $H(x) := -x \log x - (1-x) \log(1-x)$.

V. OPERATIONAL CONNECTION BETWEEN l_1 -NORM OF COHERENCE AND NEGATIVITY

Recently Streltsov et al. introduced a general framework for constructing coherence measures based on the maximum entanglement produced by incoherent operations acting on the system and an incoherent ancilla [52]. More precisely, given any entanglement measure E , one can define a coherence measure C_E as follows,

$$C_E(\rho) := \lim_{d_A \rightarrow \infty} \left\{ \sup_{\Lambda_i} E(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) \right\}. \quad (9)$$

Here d_A denotes the dimension of the ancilla, and the supremum is over all operations Λ_i that are incoherent. It turns out that $C_E(\rho)$ coincides with the relative entropy of coherence when E is the relative entropy of entanglement [52]. Moreover, a one-to-one mapping can be established between coherence measures and entanglement measures that are based on the convex roof [53]. However, the situation is not clear for many other induced coherence measures.

Here we are interested in the coherence measure induced from the negativity

$$C_{\mathcal{N}}(\rho) := \lim_{d_A \rightarrow \infty} \left\{ \sup_{\Lambda_i} \mathcal{N}(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) \right\} \quad (10)$$

and the analog $C_{\mathcal{N}_L}(\rho)$ from the logarithmic negativity.

Theorem 8.

$$C_{\mathcal{N}}(\rho) = C_{l_1}(\rho) \quad \forall \rho, \quad (11)$$

$$C_{\mathcal{N}_L}(\rho) = C_L(\rho) = \log(1 + C_{l_1}(\rho)) \quad \forall \rho. \quad (12)$$

Theorem 8 offers an operational interpretation of the l_1 -norm of coherence as the maximum entanglement, measured by the negativity, produced by incoherent operations acting on the system and an incoherent ancilla. A similar interpretation applies to the logarithmic l_1 -norm of coherence. Compared with known connections between the l_1 -norm of coherence and negativity, the one established here is much more precise. In addition, Eq. (11) still holds if C_{l_1} and \mathcal{N} are replaced by their respective convex roofs \hat{C}_{l_1} and $\hat{\mathcal{N}}$ according to Ref. [53], that is, $C_{\hat{\mathcal{N}}}(\rho) = \hat{C}_{l_1}(\rho)$. Similarly, $C_{\hat{\mathcal{N}}_L}(\rho) = \hat{C}_L$. These results further corroborate the strong connections between the l_1 -norm of coherence and negativity.

Proof. Note that Eq. (12) is a direct consequence of Eq. (11), so it suffices to prove Eq. (11). The inequality $C_{\mathcal{N}}(\rho) \leq C_{l_1}(\rho)$ follows from the equation

$$\begin{aligned} \mathcal{N}(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) &\leq C_{l_1}(\Lambda_i[\rho \otimes |0\rangle\langle 0|]) \\ &\leq C_{l_1}(\rho \otimes |0\rangle\langle 0|) = C_{l_1}(\rho) \end{aligned} \quad (13)$$

for any incoherent operation Λ_i , where the two inequalities follow from Theorem 3 and the monotonicity of C_{l_1} , respectively.

To prove the converse inequality $C_{\mathcal{N}}(\rho) \geq C_{l_1}(\rho)$, assume that the ancilla has the same dimension as the system under consideration, so we can construct the generalized CNOT gate $U_{\text{CN}}|jk\rangle = |j(j+k)\rangle$, which is strictly incoherent. Given $\rho = \sum_{jk} \rho_{jk}|j\rangle\langle k|$, the state

$$\tilde{\rho}_{\text{MC}} := U_{\text{CN}}(\rho \otimes |0\rangle\langle 0|)U_{\text{CN}}^\dagger = \sum_{jk} \rho_{jk}|jj\rangle\langle kk| \quad (14)$$

is canonically maximally correlated. Since $\mathcal{N}(\tilde{\rho}_{\text{MC}}) = C_{l_1}(\tilde{\rho}_{\text{MC}}) = C_{l_1}(\rho)$, we conclude that $C_{\mathcal{N}}(\rho) \geq C_{l_1}(\rho)$, which confirms Eq. (11) given the opposite inequality. \square

VI. SUMMARY

We proposed an axiomatic characterization of the l_1 -norm of coherence, which reveals a precise analogy to the negativity in entanglement theory and the sum negativity in the resource theory of magic-state quantum computation. In addition, we provided an operational interpretation of the l_1 -norm of coherence as the maximum entanglement, measured by the negativity, created by (strictly) incoherent operations acting on the system and an incoherent ancilla. Similar interpretations apply to the logarithmic l_1 -norm of coherence and its convex roof. In the course of study, we clarified the relation between the l_1 -norm of coherence and negativity for bipartite states. We also proposed pairing states as a generalization of maximally correlated states and as a bridge for connecting coherence theory and entanglement theory. Surprisingly, the asymptotic exact PPT entanglement cost of any pairing state is equal to the logarithmic negativity. Moreover, entangled pairing states are distillable, and their entanglement cost and distillable entanglement can be computed explicitly for a qubit-qudit system.

Our study not only clarified the meaning of the l_1 -norm of coherence, but also provided a unified perspective for understanding three prominent resource theories, namely, entanglement, coherence, and magic-state quantum computation, which are of interest to a wide spectrum of researchers.

ACKNOWLEDGMENTS

HZ acknowledges financial support from the Excellence Initiative of the German Federal and State Governments (ZUK 81) and the DFG. MH was supported in part by a JSPS Grant-in-Aid for Scientific Research (A) No. 17H01280, (B) No. 16KT0017, and Kayamori Foundation of Informational Science Advancement. LC was supported by Beijing Natural Science Foundation (4173076), the National Natural Science Foundation (NNSF) of China (Grant No. 11501024), and the Fundamental Research Funds for the Central Universities (Grants No. KG12001101, No. ZG216S1760, and No. ZG226S17J6).

APPENDIX

In this appendix, we prove Theorems 1, 3, 4, 6, 7, Proposition 1, and Corollary 2 in the main text. In order to prove Theorem 3, we show that the l_1 -norm is an upper bound for the trace norm and determine the condition for saturating this bound. In addition, we provide additional examples and nonexamples of pairing states and derive a family of lower bounds for the distillable entanglement of pairing states.

Appendix A: Proof of Theorem 1

Theorem 1 is an immediate consequence of Lemmas 1 and 2 below.

Lemma 1. Suppose $C(\rho)$ is a continuous weak coherence monotone that is a symmetric function of nonzero off-diagonal entries of ρ . Then $C(\rho)$ is a symmetric function of the absolute values of nonzero off-diagonal entries of ρ .

Remark 1. According to the following proof, the assumption in Lemma 1 actually can be relaxed as follows: $C(\rho)$ is a continuous symmetric function of nonzero off-diagonal entries of ρ that is invariant under incoherent unitary transformations and satisfies $C(\rho \otimes \sigma) = C(\rho)$ whenever σ is an incoherent state.

Remark 2. Given a density matrix $\rho = \sum_{jk} \rho_{jk}|j\rangle\langle k|$, the operator $\tau(\rho) := \sum_{jk} |\rho_{jk}|(|j\rangle\langle k|)$ is not necessarily positive semidefinite. One counterexample is

$$\rho \hat{=} \frac{1}{4} \begin{pmatrix} 1 & a & 0 & -a \\ a & 1 & a & 0 \\ 0 & a & 1 & a \\ -a & 0 & a & 1 \end{pmatrix}, \quad \tau(\rho) \hat{=} \frac{1}{4} \begin{pmatrix} 1 & a & 0 & a \\ a & 1 & a & 0 \\ 0 & a & 1 & a \\ a & 0 & a & 1 \end{pmatrix}, \quad (\text{A1})$$

where $a = \frac{1}{\sqrt{2}}$. Note that ρ has eigenvalues $\frac{1}{2}, \frac{1}{2}, 0, 0$, while τ has eigenvalues $\frac{1}{4}(1 \pm \sqrt{2}), \frac{1}{4}, \frac{1}{4}$.

In Lemma 1 we do not assume that the monotone C is also defined at $\tau(\rho)$ given a density matrix ρ except when

$\tau(\rho)$ is positive semidefinite. In addition, the following proof does not resort to $\tau(\rho)$, but converts ρ to another density matrix.

Proof. To prove Lemma 1, we need to prove the equality $C(\tilde{\rho}) = C(\rho)$ whenever the set of absolute values of off-diagonal entries of $\tilde{\rho}$ is the same as that of ρ given any pair of density matrices ρ and $\tilde{\rho}$. We may assume that ρ and $\tilde{\rho}$ act on the d -dimensional Hilbert space \mathcal{H} with $d \geq 2$, so that $C_{l_1}(\tilde{\rho}) = C_{l_1}(\rho) \leq d - 1$ [2, 4].

Since $C(\rho)$ is continuous, without loss of generality, we may assume that all phase factors $\rho_{jk}/|\rho_{jk}|$ are L th roots of unity for some positive integer L , that is,

$$\rho_{jk} = |\rho_{jk}|e^{2\pi i m_{jk}/L}, \quad (\text{A2})$$

where m_{jk} are integers. When $\rho_{jk} = 0$, the value of $\rho_{jk}/|\rho_{jk}|$ is irrelevant to the following discussion. Similarly, we assume that all $\tilde{\rho}_{jk}/|\tilde{\rho}_{jk}|$ are \tilde{L} th roots of unity. To prove the lemma, we shall construct a quantum state that has the same coherence measure as both ρ and $\tilde{\rho}$.

Let K be the smallest integer that is divisible by L, \tilde{L} and satisfies $K \geq 2d$. Let $\omega = e^{2\pi i/K}$ be a primitive K th root of unity. Define G as the group composed of all $d \times d$ diagonal matrices whose diagonal entries are K th roots of unity (that is, integer powers of ω). Note that G has order K^d . Define

$$\rho_2 := \frac{1}{K^d} U_G (\mathbb{1}_{K^d} \otimes \rho) U_G^\dagger, \quad (\text{A3})$$

where $\mathbb{1}_{K^d}$ denotes the identity on the K^d -dimensional Hilbert space and

$$U_G := \bigoplus_{U \in G} U. \quad (\text{A4})$$

Then $C(\rho_2) = C(\rho)$ given that $\mathbb{1}_{K^d}/K^d$ is an incoherent density matrix and U_G is a diagonal incoherent unitary. Moreover, ρ_2 has off-diagonal entry $|\rho_{jk}|\omega^a/K^d$ with multiplicity K^{d-1} for each $j, k = 0, 1, \dots, d-1$ with $j \neq k$ and $a = 0, 1, \dots, K-1$.

Define

$$|\psi\rangle := \frac{1}{\sqrt{K}} \sum_{a=0}^{K-1} \omega^a |a\rangle, \quad |\varphi\rangle := \frac{1}{\sqrt{K}} \sum_{a=0}^{K-1} |a\rangle. \quad (\text{A5})$$

Note that $|\psi\rangle\langle\psi|$ has K off-diagonal entries equal to ω^a/K for each $a = 1, 2, \dots, K-1$. In addition, $|\psi\rangle$ can be mapped to $|\varphi\rangle$ by the diagonal incoherent unitary transformation

$$V := \sum_{a=0}^{K-1} \omega^{-a} |a\rangle\langle a|. \quad (\text{A6})$$

In order to construct the desired density matrix, we need to introduce the positive operator

$$M := \frac{1}{K^{d-1}} \bigoplus_{j < k} |\rho_{jk}| (M_1 \oplus M_2 \oplus M_3), \quad (\text{A7})$$

with

$$\begin{aligned} M_1 &:= \mathbb{1}_{2K^{d-2}} \otimes (|\psi\rangle\langle\psi|), \\ M_2 &:= \mathbb{1}_{\frac{2(K^{d-2}-1)}{K-1}} \otimes (|\varphi\rangle\langle\varphi|), \\ M_3 &:= \frac{1}{K} \mathbb{1}_K \otimes (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|). \end{aligned} \quad (\text{A8})$$

When all ρ_{jk} are nonzero, M acts on a Hilbert space of dimension

$$d(d-1) \left(K^{d-1} + \frac{K(K^{d-2}-1)}{K-1} + K \right), \quad (\text{A9})$$

and it has rank

$$d(d-1) \left(K^{d-2} + \frac{K^{d-2}-1}{K-1} + \frac{K}{2} \right). \quad (\text{A10})$$

In general, the trace of M is upper bounded by 1 according to the following equation

$$\begin{aligned} \text{tr}(M) &= \left(\frac{1}{K} + \frac{K^{d-2}-1}{(K-1)K^{d-1}} + \frac{1}{K^{d-1}} \right) \sum_{jk} |\rho_{jk}| \\ &= \left(\frac{1}{K} + \frac{K^{d-2}-1}{(K-1)K^{d-1}} + \frac{1}{K^{d-1}} \right) C_{l_1}(\rho) \\ &\leq \left(\frac{1}{K} + \frac{K^{d-2}-1}{(K-1)K^{d-1}} + \frac{1}{K^{d-1}} \right) (d-1) < 1, \end{aligned} \quad (\text{A11})$$

where we have employed the inequality $C_{l_1}(\rho) \leq d-1$ [2, 4] as well as the assumptions $d \geq 2$ and $K \geq 2d$.

Calculation shows that the set of off-diagonal entries of M is the same as that of ρ_2 . To verify this claim, here we assume that all ρ_{jk} with $j < k$ have different nonzero absolute values, although this assumption is not essential. Note that every nonzero off-diagonal entry of ρ_2 is equal to $|\rho_{jk}|\omega^a/K^d$ for $a = 0, 1, \dots, K-1$ and $j, k = 0, 1, \dots, d-1$ with $j < k$. The same is true for M . In addition, the number of off-diagonal entries of ρ_2 that are equal to $|\rho_{jk}|\omega^a/K^d$ for given j, k, a is $2K^{d-1}$ (note that $|\rho_{jk}| = |\rho_{kj}|$). The counterpart for M is

$$\frac{2(K^{d-2}-1)}{K-1} \times K(K-1) + 2K = 2K^{d-1} \quad (\text{A12})$$

when $a = 0$ and

$$2K^{d-2} \times K = 2K^{d-1} \quad (\text{A13})$$

when $a = 1, 2, \dots, K-1$. The numbers are equal in both cases, from which our claim follows.

Define the density matrix

$$\rho_3 := M \oplus (1 - \text{tr } M). \quad (\text{A14})$$

Then the set of off-diagonal entries of ρ_3 is the same as that of M and that of ρ_2 . It follows from the assumption that $C(\rho_3) = C(\rho_2) = C(\rho)$.

Now we are ready to construct the desired density matrix. Let

$$\begin{aligned} M'_1 &:= \mathbb{1}_{2K^{d-2}} \otimes (|\varphi\rangle\langle\varphi|), \\ M' &:= \frac{1}{K^{d-1}} \bigoplus_{j < k} |\rho_{jk}\rangle (M'_1 \oplus M_2 \oplus M_3), \\ \rho_4 &:= M' \oplus (1 - \text{tr } M'). \end{aligned} \quad (\text{A15})$$

Then ρ_4 and ρ_3 can be mapped to each other by a diagonal unitary transformation given that $|\psi\rangle$ and $|\varphi\rangle$ can be mapped to each other by the diagonal unitary V and that $\text{tr } M = \text{tr } M'$. Incidentally, each matrix element of ρ_4 is the absolute value of the corresponding matrix element of ρ_3 . Therefore, $C(\rho_4) = C(\rho_3) = C(\rho_2) = C(\rho)$.

By the same reasoning, we have $C(\rho_4) = C(\tilde{\rho})$, given that the set of absolute values of off-diagonal entries of $\tilde{\rho}$ is the same as that of ρ . It follows that $C(\tilde{\rho}) = C(\rho)$, which completes the proof of the lemma. \square

Lemma 2. Suppose $C(\rho)$ is a weak coherence monotone that is a symmetric function of the absolute values of nonzero off-diagonal entries of ρ . Then $C(\rho)$ is a nondecreasing function of $C_{l_1}(\rho)$. If in addition it is additive, then $C(\rho) = c C_L(\rho)$ for some constant $c \geq 0$.

Remark 3. The continuity assumption appearing in Lemma 1 is not required in Lemma 2.

Proof. Let ρ and $\tilde{\rho}$ be two arbitrary quantum states that satisfy $C_{l_1}(\rho) = C_{l_1}(\tilde{\rho})$. If $C_{l_1}(\rho) = 0$, then $\rho, \tilde{\rho}$ have no nonzero off-diagonal entries, so $C(\rho) = C(\tilde{\rho})$ by assumption. Otherwise, suppose the absolute values of nonzero off-diagonal entries of ρ are given by a_0, a_1, \dots, a_r , and that of $\tilde{\rho}$ are given by b_0, b_1, \dots, b_s . Let

$$\begin{aligned} \sigma &\hat{=} \frac{1}{C_{l_1}(\rho)} \text{diag}(a_0, a_1, \dots, a_r), \\ \tilde{\sigma} &\hat{=} \frac{1}{C_{l_1}(\rho)} \text{diag}(b_0, b_1, \dots, b_s); \end{aligned} \quad (\text{A16})$$

then $\sigma, \tilde{\sigma}$ represent incoherent quantum states. In addition, the set of absolute values of nonzero off-diagonal entries of $\rho \otimes \tilde{\sigma}$ is the same as that of $\tilde{\rho} \otimes \sigma$. Therefore, $C(\rho \otimes \tilde{\sigma}) = C(\tilde{\rho} \otimes \sigma)$ by assumption, which further implies that $C(\rho) = C(\tilde{\rho})$ given that $\sigma, \tilde{\sigma}$ are incoherent. This conclusion shows that $C(\rho)$ is a function of $C_{l_1}(\rho)$ and thus also a function of $C_L(\rho)$.

Suppose $C_{l_1}(\tilde{\rho}) \geq C_{l_1}(\rho)$, then we can find a state σ' such that $C_{l_1}(\rho \otimes \sigma') = C_{l_1}(\tilde{\rho}) \geq C_{l_1}(\rho)$. It follows that $C(\tilde{\rho}) = C(\rho \otimes \sigma') \geq C(\rho)$, given that C is a function of C_{l_1} and that it is monotonic under incoherent operations. Therefore, $C(\rho)$ is a nondecreasing function of $C_{l_1}(\rho)$.

If C is also additive, then $C(\rho^{\otimes k}) = kC(\rho)$ for any ρ and any positive integer k , which implies that $C(\rho) \geq 0$. If ρ is incoherent, then we have $2C(\rho) = C(\rho^{\otimes 2}) = C(\rho)$, which implies that $C(\rho) = 0$. Choose a density matrix $\tilde{\rho}$ with $C_L(\tilde{\rho}) = 1$; if C is not identically zero, multiply C by a positive constant if necessary, we may assume

that $C(\tilde{\rho}) = 1$. Consequently $C(\tilde{\rho}^{\otimes k}) = k = C_L(\tilde{\rho}^{\otimes k})$. Therefore, $C(\rho) = C_L(\rho)$ whenever $C_L(\rho)$ is an integer.

Given any positive rational number n/m as the ratio of two positive integers, we can construct a quantum state ϱ with $C_L(\varrho) = n/m$, so that $C_L(\varrho^{\otimes m}) = n$. Therefore, $mC(\varrho) = C(\varrho^{\otimes m}) = C_L(\varrho^{\otimes m}) = n$. It follows that $C(\rho) = C_L(\rho)$ whenever $C_L(\rho)$ is a rational number. Finally, the restriction to the set of rational numbers can be eliminated given that $C(\rho)$ is nondecreasing in $C_L(\rho)$ and coincides with the latter on a dense subset of the ray of nonnegative real numbers. Interestingly, C is automatically continuous and strongly monotonic given the extra additivity assumption. This observation completes the proof of Lemma 2. \square

Appendix B: Proof of Theorem 3

Proof. The bound $\mathcal{N}(\rho) \leq C_{l_1}(\rho)$ follows from the equation below,

$$\begin{aligned} \mathcal{N}(\rho) + 1 &= \text{tr } |\rho^{\text{T}_A}| = \|\rho^{\text{T}_A}\|_1 \leq \|\rho^{\text{T}_A}\|_{l_1} = \|\rho\|_{l_1} \\ &= C_{l_1}(\rho) + 1, \end{aligned} \quad (\text{B1})$$

where the inequality follows from Lemma 3 in Appendix C.

Suppose ρ^{T_A} has the form in Eq. (4) as reproduced here,

$$\rho^{\text{T}_A} = \sum_{jk} a_{jk} |jk\rangle\langle\pi(jk)|, \quad (\text{B2})$$

where π is a product of disjoint transpositions. In addition, $\pi(jk) = jk$ whenever $a_{jk} = 0$; each transposition $(jk, j'k')$ satisfies $j \neq j'$ and $k \neq k'$; meanwhile, jk and $j'k'$ are fixed points of π when $(jk, j'k')$ is such a transposition. Then the inequality $\mathcal{N}(\rho) \leq C_{l_1}(\rho)$ is saturated according to Lemma 3, which can also be verified directly.

Conversely, suppose $\mathcal{N}(\rho) = C_{l_1}(\rho)$; then we have $\|\rho^{\text{T}_A}\|_1 = \|\rho^{\text{T}_A}\|_{l_1}$, so that ρ^{T_A} has the form in Eq. (4) according to Lemma 3, where π is a permutation. If $a_{jk} = 0$, then all entries of ρ^{T_A} in the row labeled by jk are zero, and so are all entries in the column labeled by jk , given that ρ and ρ^{T_A} are Hermitian. Therefore, we may assume $\pi(jk) = jk$ without loss of generality. Let τ be the restriction of π on those basis states with $a_{jk} \neq 0$. Recall that every permutation can be written as a product of disjoint cycles. If τ has a cycle of length at least 3, then ρ^{T_A} cannot be Hermitian. Therefore, each cycle in the disjoint cycle decomposition has length at most 2; in other words, τ is a product of disjoint transpositions (including the identity, which corresponds to no transposition), and the same holds for π .

Suppose $(jk, j'k')$ is a transposition in the disjoint cycle decomposition of τ . Then $a_{j'k'} = a_{jk}^* \neq 0$, and ρ^{T_A} has the form

$$\rho^{\text{T}_A} = a_{jk} |jk\rangle\langle j'k'| + a_{j'k'}^* |j'k'\rangle\langle jk| + M, \quad (\text{B3})$$

where M is a Hermitian operator whose support is orthogonal to $|jk\rangle$ and $|j'k'\rangle$. Therefore,

$$\text{tr}[\rho(|jk'\rangle\langle j'k|)] = \text{tr}[\rho^{\text{T}\Lambda}(|j'k'\rangle\langle jk|)] = a_{jk} \neq 0, \quad (\text{B4})$$

which implies that $\langle j'k|\rho^{\text{T}\Lambda}|j'k\rangle = \langle j'k|\rho|j'k\rangle > 0$ given that ρ is positive semidefinite. Consequently, $a_{j'k} > 0$ and $\tau(j'k) = j'k$; by the same token $a_{jk'} > 0$ and $\tau(jk') = jk'$. In particular, jk' and $j'k$ are fixed points of τ . If $j = j'$ or $k = k'$, then $\tau(jk) = jk$, which contradicts the assumption that τ exchanges jk and $j'k'$. Therefore, $j \neq j'$ and $k \neq k'$ whenever $(jk, j'k')$ is a transposition in the disjoint cycle decomposition of τ or π . \square

Appendix C: Connection between the trace norm and l_1 -norm

Here we prove that the l_1 -norm is an upper bound for the trace norm and determine the condition for saturating this bound. This result is needed to prove Theorem 3.

Lemma 3. Every matrix X satisfies $\|X\|_1 \leq \|X\|_{l_1}$; the inequality is saturated iff X has the following form

$$X = \sum_j s_j |j\rangle\langle \pi(j)|, \quad (\text{C1})$$

where s_j are complex numbers, and π is a permutation.

Note that the permutation π in Eq. (C1) may be subjected to additional constraints so as to comply with the numbers of rows and columns of X . The equality condition in Eq. (C1) is closely related to the singular-value decomposition of X . After deriving Lemma 3, we realized that the inequality $\|X\|_1 \leq \|X\|_{l_1}$ follows from Theorem 3.32 in Ref. [54], but the equality condition has not been discussed before as far as we know.

Proof. Our proof is based on the theory of majorization [53, 55–57]. Let $x = (x_0, x_1, \dots, x_{d-1})^{\text{T}}$ and $y = (y_0, y_1, \dots, y_{d-1})^{\text{T}}$ be two d -dimensional real vectors. The vector x is majorized by y if

$$\sum_{j=0}^k x_j^\downarrow \leq \sum_{j=0}^k y_j^\downarrow \quad \forall k = 0, 1, \dots, d-1, \quad (\text{C2})$$

and the inequality is saturated when $k = d-1$. Here x^\downarrow is constructed from x by arranging its components in decreasing order. This relation is written as $x \prec y$ or $y \succ x$ henceforth. It can also be defined for vectors of different dimensions, in which case we implicitly add a number of “0” to the vector with few components so as to match the other vector.

Suppose $X = \sum_{j,k} X_{jk} |j\rangle\langle k|$. Define column vectors $u = (|X_{jk}|^2)_{jk}$, $v = \text{diag}(X^\dagger X)$, and $w = \text{eig}(X^\dagger X)$, where $\text{eig}(X^\dagger X)$ denotes the vector composed of the eigenvalues of $X^\dagger X$. Then it is straightforward to verify that

$$u \prec v \prec w, \quad (\text{C3})$$

where the second inequality is well known in matrix analysis [56]. Consequently,

$$\begin{aligned} \|X\|_1 &= \sum_j \sqrt{w_j} \leq \sum_j \sqrt{v_j} \leq \sum_{jk} \sqrt{u_{jk}} \\ &= \sum_{jk} |X_{jk}| = \|X\|_{l_1}. \end{aligned} \quad (\text{C4})$$

Here the two inequalities follow from Eq. (C3) and the fact that the function $\sum_j \sqrt{w_j}$ is Schur concave in w . Actually, the function is strictly Schur concave, so $w \simeq v \simeq u$ whenever the two inequalities are saturated, which amounts to the equality $\|X\|_1 = \|X\|_{l_1}$. Here $v \simeq u$ means $v \prec u$ and $u \prec v$; that is, u and v have the same nonzero components up to a permutation. In that case, the number of nonzero entries of X is equal to the number of nonzero singular values, that is, the rank of X . In addition, each row of X has at most one nonzero entry, and so does each column. Consequently, X has the form of Eq. (C1), in which case the inequality $\|X\|_1 \leq \|X\|_{l_1}$ is indeed saturated. \square

1. Alternative proof of the inequality $\|X\|_1 \leq \|X\|_{l_1}$

Proof. Let $X = \sum_j s_j |u_j\rangle\langle v_j|$ be the singular-value decomposition of X , where $s_j > 0$ are singular values of X , $|u_j\rangle$ are orthonormal, and so are $|v_j\rangle$. Then

$$\begin{aligned} \|X\|_1 &= \sum_j s_j = \sum_j |\langle u_j | X | v_j \rangle| = \sum_j \left| \sum_{k,l} u_{jk}^* X_{kl} v_{jl} \right| \\ &\leq \sum_{j,k,l} |X_{kl}| |u_{jk}^* v_{jl}| \leq \frac{1}{2} \sum_{j,k,l} |X_{kl}| (|u_{jk}|^2 + |v_{jl}|^2) \\ &\leq \sum_{k,l} |X_{kl}| = \|X\|_{l_1}. \end{aligned} \quad (\text{C5})$$

Here the third inequality follows from the fact that $\sum_j |u_{jk}|^2 \leq 1$ and $\sum_j |v_{jl}|^2 \leq 1$, given that $|u_j\rangle$ and $|v_j\rangle$ are respectively orthonormal. \square

Appendix D: Proof of Proposition 1

Proof. Note that $C_{l_0}(\rho^{\text{T}\Lambda}) = C_{l_0}(\rho)$ is an even number and that the diagonal entries of $\rho^{\text{T}\Lambda}$ are nonnegative. Let $r = C_{l_0}(\rho)/2$, then $\rho^{\text{T}\Lambda}$ can be written as follows

$$\rho^{\text{T}\Lambda} = M_0 + \sum_{j=1}^r M_j, \quad (\text{D1})$$

where M_0 is diagonal and positive semidefinite, while each M_j for $j = 1, 2, \dots, r$ is a Hermitian matrix with rank 2 and one negative eigenvalue. Therefore, $\rho^{\text{T}\Lambda}$ has at most r negative eigenvalues, which implies the desired inequality $C_{l_0}(\rho) \geq 2\mathcal{N}_0(\rho)$. \square

Appendix E: Proof of Theorem 4

Proof. Suppose ρ is an entangled pairing state, which has the form of Eq. (4). Then ρ cannot be diagonal and is thus NPT according to Theorem 3. Suppose $(jk, j'k')$ is a transposition in the disjoint cycle decomposition of π . Then $j \neq j', k \neq k', a_{j'k'} = a_{jk}^* \neq 0, a_{j'k}a_{jk'} \neq 0$, and

$$\rho^{\text{TA}} = a_{jk}|jk\rangle\langle j'k'| + a_{jk}^*|j'k'\rangle\langle jk| + a_{j'k}|j'k\rangle\langle j'k'| + a_{jk'}|jk'\rangle\langle jk'| + M, \quad (\text{E1})$$

where M is a Hermitian operator whose support is orthogonal to the four kets $|jk\rangle, |j'k'\rangle, |j'k\rangle$, and $|jk'\rangle$; cf. the proof of Theorem 3. Let

$$P = (|j\rangle\langle j| + |j'\rangle\langle j'|) \otimes (|k\rangle\langle k| + |k'\rangle\langle k'|) \quad (\text{E2})$$

be a local projector. Then

$$(P\rho P)^{\text{TA}} = P\rho^{\text{TA}}P = a_{jk}|jk\rangle\langle j'k'| + a_{jk}^*|j'k'\rangle\langle jk| + a_{j'k}|j'k\rangle\langle j'k'| + a_{jk'}|jk'\rangle\langle jk'|, \quad (\text{E3})$$

so $P\rho P$ is a subnormalized two-qubit entangled maximally correlated state, which is distillable according to Refs. [45, 58]. It follows that any entangled pairing state is both NPT and distillable. \square

Appendix F: Proof of Theorem 6

Proof. Note that ρ and ρ^{TA} have the same numbers of nonzero diagonal entries and off-diagonal entries. Suppose ρ is a canonical pairing state with pairing number m (without loss of generality). Then both ρ and ρ^{TA} have $2m$ nonzero off-diagonal entries; meanwhile, the number of nonzero entries is at most d_A^2 according to Theorem 3. Suppose ρ has n nonzero diagonal entries. Then $2m \leq n(n-1)$ and $2m+n \leq d_A^2$, which imply that $m \leq d_A(d_A-1)/2$. If the inequality is saturated, then $n = d_A$, and ρ has the following form

$$\rho = \sum_{r,s=0}^{d_A-1} c_{rs}|j_r k_r\rangle\langle j_s k_s|, \quad (\text{F1})$$

where $c_{rs} \neq 0$ for all r, s , and $(j_s, k_s) \neq (j_r, k_r)$ whenever $s \neq r$. Now Theorem 3 further implies that $j_s \neq j_r$ and $k_s \neq k_r$ whenever $s \neq r$, so that ρ is a canonical maximally correlated state. \square

Appendix G: Proof of Corollary 2

Proof. If ρ is given in Eq. (6), then $\rho^{\text{TA}} = \bigoplus_{j \geq 0}^t p_j \rho_j^{\text{TA}}$, so that $\mathcal{N}(\rho) = \sum_j p_j \mathcal{N}(\rho_j) = \sum_j p_j C_{l_1}(\rho_j) = C_{l_1}(\rho)$.

If ρ is a canonical pairing state, which saturates the inequality $\mathcal{N}(\rho) = C_{l_1}(\rho)$, then ρ can be written as follows

according to Theorem 3,

$$\rho^{\text{TA}} = \sum_s (a_s |0j_s\rangle\langle 0j_s| + b_s |0k_s\rangle\langle 1j_s| + b_s^* |1j_s\rangle\langle 0k_s| + c_s |1k_s\rangle\langle 1k_s|) + \sum_r (a'_r |0j'_r\rangle\langle 0j'_r| + c'_r |1k'_r\rangle\langle 1k'_r|), \quad (\text{G1})$$

where all j_s, k_s, j'_r, k'_r are distinct. Therefore, ρ has the form of Eq. (6) with constraints as specified. \square

Corollary 2 fails when $\dim(\mathcal{H}_A) \geq 3$. One counterexample is

$$\rho = \frac{1}{4}(|02\rangle\langle 02| + |20\rangle\langle 20| + |\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|), \quad (\text{G2})$$

where $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|\varphi\rangle = (|11\rangle + |22\rangle)/\sqrt{2}$.

Appendix H: Proof of Theorem 7

Suppose ρ is a canonical pairing state on $\mathcal{H}_A \otimes \mathcal{H}_B$, which has the form of Eq. (6). Let n be an arbitrary positive integer and $\tilde{\rho}$ an arbitrary bipartite state shared by Alice and Bob. Then Theorem 7 follows from Eqs. (H1) to (H6) below.

$$E_r(\rho) = C_r(\rho) = S(\rho^{\text{diag}}) - S(\rho), \quad (\text{H1})$$

$$E_r^\infty(\rho) = \frac{1}{n} E_r(\rho^{\otimes n}) = E_r(\rho), \quad (\text{H2})$$

$$C_D(\rho) = E_D(\rho) = E_r(\rho), \quad (\text{H3})$$

$$\begin{aligned} C_F(\rho) &= E_F(\rho) = \sum_j p_j E_F(\rho_j) \\ &= \sum_j p_j H\left(\frac{1 + \sqrt{1 - \mathcal{N}(\rho_j)^2}}{2}\right), \end{aligned} \quad (\text{H4})$$

$$C_C(\rho) = E_C(\rho) = \frac{1}{n} E_F(\rho^{\otimes n}) = E_F(\rho), \quad (\text{H5})$$

$$E_F(\rho \otimes \tilde{\rho}) = E_F(\rho) + E_F(\tilde{\rho}). \quad (\text{H6})$$

Here ρ^{diag} is the diagonal matrix with the same diagonal as ρ . Note that the formula $C_r(\rho) = S(\rho^{\text{diag}}) - S(\rho)$ applies to all quantum states [2]. In addition,

$$E_r^\infty(\rho) := \lim_{k \rightarrow \infty} \frac{1}{k} E_r(\rho^{\otimes k}), \quad E_C(\rho) := \lim_{k \rightarrow \infty} \frac{1}{k} E_F(\rho^{\otimes k}), \quad (\text{H7})$$

and $E_r^\infty(\rho)$ is known as the regularized relative entropy of entanglement.

If the parameter p_0 characterizing ρ in Eq. (6) vanishes, then ρ is a direct sum of maximally correlated states, so that $S(\rho^{\text{diag}}) = S(\rho_B)$, where $\rho_B = \text{tr}_A(\rho)$. Consequently,

$$C_D(\rho) = C_r(\rho) = E_D(\rho) = E_r(\rho) = S(\rho_B) - S(\rho). \quad (\text{H8})$$

This formula has the same form as that for a canonical maximally correlated state. Recall that any canonical maximally correlated state σ satisfies [3, 45, 51, 59, 60]

$$C_D(\sigma) = C_r(\sigma) = E_D(\sigma) = E_r(\sigma) = S(\sigma_B) - S(\sigma), \quad (\text{H9})$$

$$C_C(\sigma) = C_F(\sigma) = E_C(\sigma) = E_F(\sigma). \quad (\text{H10})$$

Before proving Eqs. (H1) to (H6), we recall that the three entanglement measures E_r, E_F, E_D do not increase on average under selective LOCC. Meanwhile, E_r, E_F, E_C are convex [61, 62]. By contrast, the coherence measures $C_D = C_r$ and $C_C = C_F$ are convex and additive, and do not increase on average under selective incoherent operations [1–3]. All eight entanglement and coherence measures mentioned above are additive for maximally correlated states.

Proof of Eq. (H1). Suppose the density matrix ρ has the form of Eq. (6). Denote by \tilde{P}_j for $j > 0$ the projector onto the support of $\text{tr}_A(\rho_j)$, which has rank at most 2. Let $\tilde{P}_0 = \mathbb{1}_B - \sum_{j>0} \tilde{P}_j$, where $\mathbb{1}_B$ denotes the identity on \mathcal{H}_B . Let $P_j = \mathbb{1}_A \otimes \tilde{P}_j$ for $j \geq 0$. Then the CPTP map defined by the set of Kraus operators $\{P_j\}$ is local and also incoherent. This map turns ρ into ρ_j with probability p_j . Therefore,

$$E_r(\rho) \geq \sum_j p_j E_r(\rho_j), \quad (\text{H11})$$

because E_r does not increase on average under LOCC. On the other hand, the convexity of E_r implies that

$$E_r(\rho) \leq \sum_j p_j E_r(\rho_j). \quad (\text{H12})$$

The above reasoning still applies if E_r is replaced by C_r . In conjunction with Eq. (H9), we deduce that

$$\begin{aligned} E_r(\rho) &= \sum_j p_j E_r(\rho_j) = \sum_j p_j C_r(\rho_j) = C_r(\rho) \\ &= S(\rho^{\text{diag}}) - S(\rho), \end{aligned} \quad (\text{H13})$$

where the last equality applies to all quantum states [2, 4]. This conclusion confirms Eq. (H1). \square

Proof of Eq. (H2). To prove Eq. (H2), it suffices to prove the equality $E_r(\rho^{\otimes n}) = nE_r(\rho)$. To simplify the notation, here we illustrate the argument in the case $n = 2$, which admits straightforward generalization.

$$\begin{aligned} E_r(\rho^{\otimes 2}) &= E_r\left(\sum_{j_1, j_2} p_{j_1} p_{j_2} \rho_{j_1} \otimes \rho_{j_2}\right) \\ &= \sum_{j_1, j_2} p_{j_1} p_{j_2} E_r(\rho_{j_1} \otimes \rho_{j_2}) \\ &= \sum_{j_1, j_2} p_{j_1} p_{j_2} [E_r(\rho_{j_1}) + E_r(\rho_{j_2})] = 2E_r(\rho). \end{aligned} \quad (\text{H14})$$

Here the second equality follows from a similar reasoning that leads to Eq. (H13). The third equality follows from the fact that ρ_j is either maximally correlated or separable and that the relative entropy of entanglement is additive for maximally correlated states [45, 51]. \square

Proof of Eq. (H3). By the same reasoning that leads to Eq. (H11), we have

$$E_D(\rho) \geq \sum_j p_j E_D(\rho_j) = \sum_j p_j E_r(\rho_j) = E_r(\rho), \quad (\text{H15})$$

note that $E_D = E_r$ for maximally correlated states [60] and that $E_r(\rho) = \sum_j p_j E_r(\rho_j)$ according to Eq. (H13). Since the opposite inequality $E_D(\rho) \leq E_r(\rho)$ holds in general, we conclude that $E_D(\rho) = E_r(\rho)$. Similar reasoning implies that $C_D(\rho) = C_r(\rho)$, which confirms Eq. (H3) given (H1). Incidentally, the equality $C_D(\rho) = C_r(\rho)$ holds for all quantum states according to Ref. [3]. \square

Proof of Eq. (H4). According to a similar reasoning that leads to Eq. (H13),

$$\begin{aligned} C_F(\rho) &= E_F(\rho) = \sum_j p_j E_F(\rho_j) \\ &= \sum_j p_j H\left(\frac{1 + \sqrt{1 - \mathcal{N}(\rho_j)^2}}{2}\right). \end{aligned} \quad (\text{H16})$$

Here the last equality follows from Wootters' formula for the entanglement of formation of two-qubit states [63] and the fact that the concurrence is equal to the negativity for any two-qubit maximally correlated state and separable state. \square

Proof of Eq. (H5). This equation follows from a similar argument that leads to (H2) and is also a corollary of Eq. (H6) proved below. Note that $C_C = C_F$ is additive in general [3]. \square

Proof of Eq. (H6). According to a similar reasoning that leads to Eq. (H13), we have

$$\begin{aligned} E_F(\rho \otimes \tilde{\rho}) &= \sum_j p_j E_F(\rho_j \otimes \tilde{\rho}) = \sum_j p_j [E_F(\rho_j) + E_F(\tilde{\rho})] \\ &= E_F(\rho) + E_F(\tilde{\rho}), \end{aligned} \quad (\text{H17})$$

where the second equality follows from Refs. [59, 64] since each ρ_j is either maximally correlated or separable. \square

Appendix I: More examples and nonexamples of pairing states

In this section we provide additional examples and nonexamples of pairing states by considering mixtures of a bipartite entangled pure state and the completely mixed state.

Proposition 3. Let $|\psi\rangle$ be a bipartite entangled pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho = p|\psi\rangle\langle\psi| + (1-p)/(d_A d_B)$ with $0 \leq p \leq 1$. Then ρ is a pairing state iff $p = 0$ or $p = 1$.

Proof. When $p = 0$, ρ is the completely mixed state and a separable pairing state. When $p = 1$, ρ is a bipartite pure state, which saturates the inequality $C_{l_1}(\rho) \leq \mathcal{N}(\rho)$ in the Schmidt basis, so ρ is also a pairing state.

When $0 < p < 1$, we have

$$\mathcal{N}(\rho) < p\mathcal{N}(|\psi\rangle\langle\psi|) \leq pC_{l_1}(|\psi\rangle\langle\psi|) = C_{l_1}(\rho). \quad (\text{I1})$$

Therefore, ρ cannot be a pairing state. \square

Appendix J: Lower bounds for the distillable entanglement of pairing states

Here we provide a family of lower bounds for the distillable entanglement of canonical pairing states.

Suppose ρ is a canonical pairing state that has the form of Eq. (B2). Let A_1, A_2, \dots, A_k be disjoint two-element subsets of $\{0, 1, \dots, d_A - 1\}$ and

$$P_j := \left(\sum_{m \in A_j} |m\rangle\langle m| \right) \otimes \mathbb{1}_B, \quad j = 1, 2, \dots, k. \quad (\text{J1})$$

Then P_1, P_2, \dots, P_k define a (possibly incomplete) local projective measurement. Let

$$\tilde{\rho}_j := \frac{P_j \rho P_j}{p_j}, \quad p_j = \text{tr}(P_j \rho P_j). \quad (\text{J2})$$

Then each $\tilde{\rho}_j$ with $p_j > 0$ is a $2 \times d_B$ pairing state. If ρ is entangled, then it is possible to choose at least one subset, say A_1 , such that $\tilde{\rho}_1$ is entangled.

Suppose A_1, A_2, \dots, A_k are chosen such that all states $\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_k$ are entangled. Then the distillable entanglement of ρ can be lower bounded as follows,

$$\begin{aligned} E_D(\rho) &\geq \sum_{j=1}^k p_j E_D(\tilde{\rho}_j) = \sum_{j=1}^k p_j E_r(\tilde{\rho}_j) \\ &= \sum_{j=1}^k p_j [S(\tilde{\rho}_j^{\text{diag}}) - S(\tilde{\rho}_j)], \end{aligned} \quad (\text{J3})$$

where the inequality follows from the monotonicity of E_D under LOCC, and the two equalities follow from Eqs. (H3) and (H1), respectively.

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