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# Multi-time correlators in continuous measurement of qubit observables

Juan Atalaya,<sup>1</sup> Shay Hacoheh-Gourgy,<sup>2,3</sup> Leigh S. Martin,<sup>2,3</sup> Irfan Siddiqi,<sup>2,3</sup> and Alexander N. Korotkov<sup>1</sup>

<sup>1</sup>*Department of Electrical and Computer Engineering,  
University of California, Riverside, CA 92521, USA*

<sup>2</sup>*Quantum Nanoelectronics Laboratory, Department of Physics,  
University of California, Berkeley CA 94720, USA*

<sup>3</sup>*Center for Quantum Coherent Science, University of California, Berkeley CA 94720, USA.*

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We consider multi-time correlators for output signals from linear detectors, continuously measuring several qubit observables at the same time. Using the quantum Bayesian formalism, we show that for unital (symmetric) evolution in the absence of phase backaction, an  $N$ -time correlator can be expressed as a product of two-time correlators when  $N$  is even. For odd  $N$ , there is a similar factorization, which also includes a single-time average. Theoretical predictions agree well with experimental results for two detectors, which simultaneously measure non-commuting qubit observables.

*Introduction.*—Partial and continuous quantum measurements (CQMs) have recently attracted a significant attention within the quantum information community. They have been discussed theoretically for a long time [1–11], and renewed interest has been motivated by the rapid progress with superconducting qubits, which are currently the main experimental system for realization of CQMs [12–18]. The main contribution from the theory and experiments on CQMs to fundamental aspects of quantum physics is a clear understanding of quantum evolution in the process of the state collapse due to measurement. Besides that, we now understand quantum dynamics in some peculiar processes like uncollapse [19, 20] and simultaneous measurement of non-commuting observables [21, 22]. There is also a growing interest in the use of CQMs for quantum computing applications, such as quantum feedback [14, 17, 23–26], rapid state purification [27], entanglement by continuous measurement [28–30] and quantum error correction [31, 32].

A recent experiment [22] opened a way to experimental verification of theoretical predictions related to simultaneous measurement of non-commuting observables. In particular, two-time correlators for the output signals from two detectors measuring two observables of a qubit, have been calculated in Ref. [33], and the theoretical results showed a good agreement with experimental data. Similarly, the theoretical results for two-time state correlators have been compared with experiment in Ref. [34].

In this work we again consider temporal correlations for signals from linear detectors, simultaneously measuring non-commuting observables of a qubit. However, now the number of detectors and observables is arbitrary, observables can change in time, and we extend the previous analysis of two-time correlators [33, 35, 36] to multi-time correlators, thus fully describing statistics of the output signals. Our analysis also extends the recent result [37] for correlations between sequential infinitesimally weak measurements in the absence of evolution.

Let us consider a qubit, which is weakly coupled to  $N_d$  linear detectors, measuring simultaneously and continuously an arbitrary set of qubit observables  $\sigma_\ell \equiv$

$\mathbf{n}_\ell \boldsymbol{\sigma}$ , where  $\ell = 1, 2, \dots, N_d$ , the unit vector  $\mathbf{n}_\ell = (n_{x,\ell}, n_{y,\ell}, n_{z,\ell})$  is the  $\ell$ th measurement axis direction on the Bloch sphere, and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is the vector of Pauli matrices. For the output signals  $I_\ell(t)$  from the detectors, the multi-time correlators of interest are

$$K_{\ell_1 \ell_2 \dots \ell_N}(t_1, t_2, \dots, t_N) \equiv \langle I_{\ell_N}(t_N) \cdots I_{\ell_2}(t_2) I_{\ell_1}(t_1) \rangle, \quad (1)$$

where averaging is over the ensemble of realizations, we assume that the time arguments are ordered as  $t_1 < t_2 < \dots < t_N$ , and  $N$  can be smaller, equal, or larger than  $N_d$ . As we show in this paper, for unital evolution (with symmetry between qubit states  $|0\rangle$  and  $|1\rangle$ ) in the absence of phase backaction from measurement, the  $N$ -time correlator (1) has a quite simple form. Rather surprisingly, for even  $N$  it factorizes into a product of  $N/2$  sequential two-time correlators, so that the qubit evolution between  $t_{N-2k}$  and  $t_{N-2k+1}$  does not affect the correlator (1). For odd  $N$ , there is a similar product, which also includes the average signal at the earliest time,  $\langle I_{\ell_1}(t_1) \rangle$ . In this paper we also compare the theoretical predictions with experimental data for a two-detector configuration similar to Ref. [22].

Our results are useful for parameter estimation via correlators (see [33]) and noise characterization as a tool for diagnosing sources of fluctuations in multiqubit systems, with multi-observable correlators probing the dynamics within the whole Hilbert space. Our results are also useful for analysis of error syndromes in quantum error correction codes based on continuous measurements. In particular, the theory presented here has been implicitly used (without any discussion or formulas) in Ref. [32] for error analysis in the four-qubit Bacon-Shor code operated with continuous measurements; the parity operators for that code correspond to non-commuting observables of the gauge qubit.

*The quantum Bayesian formalism.*—A simultaneous continuous measurement of the qubit observables  $\sigma_\ell$  produces the normalized output signals [11, 21, 38]

$$I_\ell(t) = \text{Tr}[\sigma_\ell \rho(t)] + \sqrt{\tau_\ell} \xi_\ell(t) = \mathbf{n}_\ell \mathbf{r}(t) + \sqrt{\tau_\ell} \xi_\ell(t), \quad (2)$$

where the Bloch vector  $\mathbf{r} = (x, y, z)$  for the qubit density matrix  $\rho$  is defined via parametrization  $\rho = (\mathbb{1} + x\sigma_x + y\sigma_y + z\sigma_z)/2$  and  $\tau_\ell$  denotes the measurement (collapse) time needed for informational signal-to-noise ratio of 1 for the  $\ell$ th-measurement channel. We use the Markovian quantum Bayesian theory (assuming the “bad cavity limit”), so the noises  $\xi_\ell(t)$  are assumed to be white, Gaussian and uncorrelated, with two-time correlators

$$\langle \xi_\ell(t) \xi_{\ell'}(t') \rangle = \delta_{\ell\ell'} \delta(t - t'). \quad (3)$$

We also assume that the quantum backaction due to measurement is only of the informational type, so that there is no phase backaction [38]; in a circuit QED experiment this requires that the optimal microwave quadrature is amplified by a phase-sensitive amplifier. Then the quantum Bayesian equation for the qubit state evolution (in Itô interpretation) is [10, 21, 38]

$$\dot{\mathbf{r}} = \Lambda_{\text{ens}}(\mathbf{r} - \mathbf{r}_{\text{st}}) + \sum_{\ell=1}^{N_d} \frac{\mathbf{n}_\ell - (\mathbf{n}_\ell \mathbf{r}) \mathbf{r}}{\sqrt{\tau_\ell}} \xi_\ell(t), \quad (4)$$

where the second term depends on the noisy outputs  $I_\ell(t)$ , while the first term describes ensemble-averaged evolution, characterized by a  $3 \times 3$  matrix  $\Lambda_{\text{ens}}$  and quasistationary state  $\mathbf{r}_{\text{st}}$  ( $\Lambda_{\text{ens}}$ ,  $\mathbf{r}_{\text{st}}$ ,  $\mathbf{n}_\ell$ , and  $\tau_\ell$  can all be time-dependent). Note that

$$\dot{\mathbf{r}}_{\text{ens}} = \Lambda_{\text{ens}}(\mathbf{r}_{\text{ens}} - \mathbf{r}_{\text{st}}), \quad (5)$$

is the most general form of a linear Markovian evolution of the ensemble-averaged qubit state  $\mathbf{r}_{\text{ens}} = \langle \mathbf{r} \rangle$ , which directly corresponds to the Lindblad-form equation [40]  $\dot{\rho}_{\text{ens}} = -(i/\hbar)[H_q, \rho_{\text{ens}}] + \mathcal{L}[\rho_{\text{ens}}]$ , where qubit Hamiltonian  $H_q$  describes Rabi oscillations, while  $\mathcal{L}$  accounts for environmental decoherence and measurement-induced ensemble dephasing. In particular, the measurement contributes  $\mathcal{L}_m[\rho] = \sum_\ell \Gamma_\ell [\sigma_\ell \rho \sigma_\ell - \rho]/2$ , where  $\Gamma_\ell = 1/2\eta_\ell\tau_\ell$  and  $\eta_\ell$  is the quantum efficiency of  $\ell$ th detector [21, 38].

An important special case is unital evolution, for which  $\mathbf{r}_{\text{st}} = 0$ , so that a fully mixed qubit state,  $\mathbf{r}_{\text{ens}} = 0$ , does not evolve. Unital evolution essentially means the symmetry between the qubit states  $|0\rangle$  and  $|1\rangle$ . In experiments with superconducting qubits this symmetry is usually broken by energy relaxation; however, there is an approximate symmetry if Rabi oscillations are much faster than energy relaxation. In particular, in the experiment [22] on simultaneous measurement of non-commuting observables and for the data used in this paper, the evolution of the effective qubit is practically unital [33]. Let us denote the solution of Eq. (5) as  $\mathbf{r}_{\text{ens}}(t|\mathbf{r}_0, t_0)$ , where  $\mathbf{r}_0$  is an initial condition at time  $t_0$ . For unital evolution

$$\mathbf{r}_{\text{ens}}(t|\mathbf{r}_0, t_0) = -\mathbf{r}_{\text{ens}}(t|\mathbf{r}_0, t_0). \quad (6)$$

If phase backaction is included, then we need to add into Eq. (4) the term  $\sum_\ell \tau_\ell^{-1/2} \mathcal{K}_\ell(\mathbf{n}_\ell \times \mathbf{r}) \xi_\ell(t)$ , where the coefficient  $\mathcal{K}_\ell$  parametrizes the relative strength of

phase backaction [38] ( $\mathcal{K}_\ell = \tan \phi_\ell$ , where  $\phi_\ell$  is the angle between the amplified and optimal quadratures). The ensemble dephasing rate is then  $\Gamma_\ell = (1 + \mathcal{K}_\ell^2)/2\eta_\ell\tau_\ell$ . However, as mentioned above, in this paper we assume  $\mathcal{K}_\ell = 0$ .

*Collapse recipe.*—As shown in Section A of Supplemental Material [39], in the absence of phase backaction ( $\mathcal{K}_\ell = 0$ , unitality is not needed), the qubit evolution (4) leads to the following simple recipe for calculation of the multi-time correlators (1). The correct result can be obtained by replacing actual continuous measurement with projective measurement of operators  $\sigma_{\ell_k}$  at time moments  $t_k$  ( $k = 1, 2, \dots, N$ ), while the qubit evolution at  $t \neq t_k$  is replaced with the ensemble-averaged evolution. This “collapse recipe” was proven in Ref. [35] for  $N_d = 1$  and  $N = 2$ , and also in Ref. [33] for  $N = N_d = 2$ . In Section A of Supplemental Material we prove it for arbitrary  $N$  and  $N_d$ .

Using this recipe, in Eq. (1) we have  $2^N$  combinations of discrete outcomes,  $I_{\ell_k} = \pm 1$ , each of them correspondingly collapsing the qubit state to the point  $\pm \mathbf{n}_{\ell_k}$  on the Bloch sphere (an eigenstate of  $\sigma_{\ell_k}$ ). Each combination contributes the value  $\prod_{k=1}^N I_{\ell_k}$  to the  $N$ -time correlator, with the weight equal to the probability of such combination of outcomes. In this way we obtain

$$K_{\ell_1 \dots \ell_N}(t_1, \dots, t_N) = \sum_{\{I_{\ell_k} = \pm 1\}}^{2^N} p(I_{\ell_N} | I_{\ell_{N-1}}) \times p(I_{\ell_{N-1}} | I_{\ell_{N-2}}) \cdots p(I_{\ell_2} | I_{\ell_1}) p(I_{\ell_1}) \prod_{k=1}^N I_{\ell_k}, \quad (7)$$

where  $p(I_{\ell_k} | I_{\ell_{k-1}}) \equiv p(I_{\ell_k}, t_k | I_{\ell_{k-1}}, t_{k-1})$  is the probability to obtain projective result  $I_{\ell_k}$  at time  $t_k$  if at time  $t_{k-1}$  the result was  $I_{\ell_{k-1}}$  (for brevity we omit time moments in the notation), while  $p(I_{\ell_1}) \equiv p(I_{\ell_1}, t_1)$  is the probability to obtain projective result  $I_{\ell_1} = \pm 1$  at time  $t_1$ . It is easy to see that

$$p(I_{\ell_1}) = \frac{1 + I_{\ell_1} \text{Tr}[\sigma_{\ell_1} \rho_{\text{ens}}(t_1)]}{2} = \frac{1 + I_{\ell_1} \mathbf{n}_{\ell_1} \mathbf{r}_{\text{ens}}(t_1)}{2}, \quad (8)$$

where  $\mathbf{r}_{\text{ens}}(t_1)$  is the qubit state at the time  $t_1$ . If the qubit is prepared in a state  $\mathbf{r}_{\text{in}}$  at the time  $t_{\text{in}} < t_1$ , then  $\mathbf{r}_{\text{ens}}(t_1)$  is obtained via Eq. (5). Similarly,

$$p(I_{\ell_k} | I_{\ell_{k-1}}) = \frac{1 + I_{\ell_k} \mathbf{n}_{\ell_k} \mathbf{r}_{\text{ens}}(t_k | I_{\ell_{k-1}} \mathbf{n}_{\ell_{k-1}}, t_{k-1})}{2}, \quad (9)$$

where  $\mathbf{r}_{\text{ens}}(t_k | I_{\ell_{k-1}} \mathbf{n}_{\ell_{k-1}}, t_{k-1})$  is the qubit state at time  $t_k$  obtained from Eq. (5) with the initial condition  $\mathbf{r} = I_{\ell_{k-1}} \mathbf{n}_{\ell_{k-1}}$  at the time  $t_{k-1}$ . This initial condition is due to collapse of the qubit state by projective measurement of  $\sigma_{\ell_{k-1}}$  at the time  $t_{k-1}$  with result  $I_{\ell_{k-1}}$ . Note that while the collapse recipe is applicable to correlators, it is not applicable to the joint probability distribution of the continuous output signals  $I_{\ell_k}(t_k)$ , for which no formula like Eq. (7) is possible.

*Correlator factorization for unital evolution.*—Using Eqs. (7) and (9), we can write the  $N$ -time correlator as

$$\begin{aligned} K_{\ell_1 \dots \ell_N}(t_1, \dots, t_N) &= \sum_{\{I_\ell = \pm 1\}} 2^{N-2} \sum_{I_{\ell_{N-1}} = \pm 1} I_{\ell_{N-1}} \\ &\times \mathbf{n}_{\ell_N} \mathbf{r}_{\text{ens}}(t_N | I_{\ell_{N-1}} \mathbf{n}_{\ell_{N-1}}, t_{N-1}) \\ &\times [1 + I_{\ell_{N-1}} \mathbf{n}_{\ell_{N-1}} \mathbf{r}_{\text{ens}}(t_{N-1} | I_{\ell_{N-2}} \mathbf{n}_{\ell_{N-2}}, t_{N-2})] / 2 \\ &\times p(I_{\ell_{N-2}} | I_{\ell_{N-3}}) \cdots p(I_{\ell_2} | I_{\ell_1}) p(I_{\ell_1}) \prod_{k=1}^{N-2} I_{\ell_k}, \end{aligned} \quad (10)$$

where we separated the factors for the latest pair of measurements and already summed over the result  $I_{\ell_N} = \pm 1$  of the latest measurement. For the summation over  $I_{\ell_{N-1}} = \pm 1$ , let us use the symmetry property (6) of unital evolution, which gives  $\mathbf{r}_{\text{ens}}(t_N | I_{\ell_{N-1}} \mathbf{n}_{\ell_{N-1}}, t_{N-1}) = I_{\ell_{N-1}} \mathbf{r}_{\text{ens}}(t_N | \mathbf{n}_{\ell_{N-1}}, t_{N-1})$  on the first line of Eq. (10) since  $I_{\ell_{N-1}}^2 = 1$ , so summation over  $I_{\ell_{N-1}}$  only affects the second term in the third line, which sums to zero. Therefore, the last two measurements bring only the factor  $\mathbf{n}_{\ell_N} \mathbf{r}_{\text{ens}}(t_N | \mathbf{n}_{\ell_{N-1}}, t_{N-1})$ , which does not depend on the previous measurement results. Moreover, for unital evolution, this factor is exactly the two-time correlator  $K_{\ell_{N-1} \ell_N}(t_{N-1}, t_N)$ , as easy to see using the collapse recipe. We emphasize that for unital evolution the two-time correlators do not depend on the initial state [33]. Thus, for unital evolution we obtain factorization

$$\begin{aligned} K_{\ell_1 \dots \ell_N}(t_1, \dots, t_N) &= K_{\ell_1 \dots \ell_{N-2}}(t_1, \dots, t_{N-2}) \\ &\times K_{\ell_{N-1} \ell_N}(t_{N-1}, t_N). \end{aligned} \quad (11)$$

Continuing this procedure, we see that for even  $N$  the  $N$ -time correlator is a product of two-time correlators,

$$K_{\ell_1 \dots \ell_N}(t_1, \dots, t_N) = \prod_{i=1}^{N/2} K_{\ell_{2i-1} \ell_{2i}}(t_{2i-1}, t_{2i}), \quad (12)$$

while for odd  $N$  the remaining factor is the average output signal at  $t_1$ , which depends on the initial state,

$$K_{\ell_1 \dots \ell_N}(t_1, \dots, t_N) = \langle I_{\ell_1}(t_1) \rangle \prod_{i=1}^{\frac{N-1}{2}} K_{\ell_{2i} \ell_{2i+1}}(t_{2i}, t_{2i+1}). \quad (13)$$

The two-time correlators do not depend on the initial state and can be calculated by integrating Eq. (5) (with  $\mathbf{r}_{\text{st}} = 0$  for unital evolution),

$$K_{\ell_i \ell_k}(t_i, t_k) = \mathbf{n}_{\ell_k} \left[ \exp \left( \int_{t_i}^{t_k} \Lambda_{\text{ens}}(t) dt \right) \mathbf{n}_{\ell_i} \right], \quad (14)$$

where the exponential is time-ordered. For a time-independent  $\Lambda_{\text{ens}}$ , the two-time correlator (14) is obviously a function of the time difference  $t_k - t_i$ , while not depending on  $t_i$ .

The multi-time correlator factorization (12) and (13) in the case of unital evolution is the *main result* of this paper. Rather surprisingly, the  $N$ -time correlator does not depend on the qubit evolution between neighboring

time moments  $t_{N-2i}$  and  $t_{N-2i+1}$ ; in particular, the time duration between them is not important. For even  $N$ , the correlator also does not depend on the initial qubit state.

Our factorization result may seem similar to Wick's theorem in Gaussian field theory [41], indicating a trivial correlation. However, this is not the case, since in our problem only some specific pairwise correlators contribute to Eqs. (12) and (13), while others do not. This is a rather peculiar correlation; for example, the evolution between time moments  $t_{N-2i}$  and  $t_{N-2i+1}$  obviously affects the joint probability distribution but cancels out in the correlator.

Note that the factorization result is somewhat similar to the result of Ref. [37] for sequential infinitesimally weak measurements with no evolution in between. In contrast, we show that the factorization holds for continuous measurements in the presence of an arbitrary unital evolution (but without phase backaction). If the qubit evolution is not unital, then the  $N$ -time correlator still can be calculated via Eq. (7) using the collapse recipe. However, the correlator does not factorize. A recursive relation for the  $N$ -time correlator in this case is presented in Supplemental Material [39].

*Singular contributions at coinciding times.*—So far we assumed that all time moments  $t_k$  in the correlator (1) are different. If a pair of neighboring time moments,  $t_k$  and  $t_{k+1}$ , approach each other and they correspond to the same detector,  $\ell_k = \ell_{k+1}$ , then we also need to add to the correlator a singular contribution,  $\propto \delta(t_{k+1} - t_k)$ , due to white noise in this detector – see Eq. (3). The additional contribution will be  $\tau_{\ell_k} \delta(t_{k+1} - t_k) K_{N-2}$ , where  $K_{N-2}$  denotes the remaining  $(N-2)$ -time correlator with the coinciding pair excluded. Similarly, if there are two pairs of coinciding times (time-separated from each other), which involve the same detectors,  $\ell_k = \ell_{k+1}$ ,  $\ell_i = \ell_{i+1}$ , then there will also be a contribution  $\tau_{\ell_k} \tau_{\ell_i} \delta(t_{k+1} - t_k) \delta(t_{i+1} - t_i) K_{N-4}$ , where  $K_{N-4}$  is the  $(N-4)$ -time correlator without coinciding pairs (if  $\ell_k \neq \ell_i$ , then the formula is the same even if these two pairs coincide in time). These formulas for the singular contributions do not assume unital evolution. Note that there is no singular contributions from three coinciding times (with the same detector) because the noises  $\xi_\ell(t)$  are assumed to be Gaussian.

*Comparison with experiment.*—To check our theoretical results for multi-time correlators, we use the data from the experiment described in detail in Ref. [22] (two-time correlators have been analyzed in Ref. [33]). In this experiment, two linear detectors measure simultaneously and continuously the observables  $\sigma_z$  and  $\sigma_\varphi = \sigma_z \cos \varphi + \sigma_x \sin \varphi$  of a nominally non-evolving qubit, with corresponding normalized output signals  $I_z(t)$  and  $I_\varphi(t)$ . Here  $\varphi$  is the angle between the measurement axes in the Bloch  $xz$  plane of the measured qubit; in the experiment  $\varphi = n\pi/10$  with integer  $n$  from 0 to 10 (we neglect the small correction  $\Delta\varphi = 0.036$  [33]). As shown in Ref. [33], the decoherence-caused evolution of the measured effec-

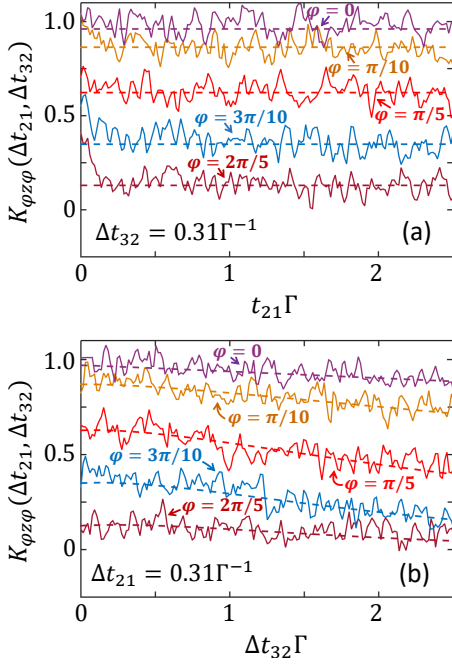


FIG. 1. Comparison between experimental (solid lines) and theoretical (dashed lines) three-time correlators for simultaneous measurement of qubit observables  $\sigma_z$  and  $\sigma_\varphi$ , with  $\varphi$  being the angle between the measurement axes on the Bloch sphere. Upper and lower panels show the correlator  $K_{\varphi z \varphi}(\Delta t_{21}, \Delta t_{32})$  as a function of the time difference  $\Delta t_{21}$  and  $\Delta t_{32}$ , respectively (see text). As predicted by theory, we see practically no dependence on  $\Delta t_{21}$ , in contrast to a significant dependence on  $\Delta t_{32}$ . In both measurement channels,  $\Gamma = (1.3 \mu\text{s})^{-1}$ .

tive qubit is unital even in the presence of energy relaxation of the physical qubit (because of averaging over sufficiently fast Rabi oscillations of the physical qubit, creating the effective rotating-frame qubit). Since in the experiment a Josephson parametric amplifier operated in phase-sensitive mode has been used, amplifying the informational (optimal) quadrature, the phase backaction is nominally absent. Therefore, all conditions for our factorization result for multi-time correlators, Eqs. (12)–(14), are satisfied in the experiment.

Let us first consider the three-time correlator  $K_{\varphi z \varphi}(\Delta t_{21}, \Delta t_{32}) \equiv \int_{t_a}^{t_a+T} dt_1 \langle I_\varphi(t_3) I_z(t_2) I_\varphi(t_1) \rangle / T$ , where  $\Delta t_{21} = t_2 - t_1$ ,  $\Delta t_{32} = t_3 - t_2$ , and additional averaging over time  $t_1$  within the interval  $[t_a, t_a + T]$  is introduced to reduce fluctuations of the experimental correlators. For brevity of notation, we omit dependence of  $K_{\varphi z \varphi}$  on  $t_a$  and  $T$ ; in Fig. 1 we choose  $t_a = 1 \mu\text{s}$  and  $T = 0.2 \mu\text{s}$ . As follows from (13),  $K_{\varphi z \varphi}(\Delta t_{21}, \Delta t_{32}) = K_{z\varphi}(\Delta t_{32}) \int_{t_a}^{t_a+T} dt_1 \langle I_\varphi(t_1) \rangle / T$ , where the two-time correlator  $K_{z\varphi}(\Delta t_{32})$  and the average signal  $\langle I_\varphi(t) \rangle = z(t) \cos \varphi + x(t) \sin \varphi$  can be found using the results of Ref. [33], using the qubit initial state  $\mathbf{r}(0) = \{\sin(\varphi/2), 0, \cos(\varphi/2)\}$  – see Section B of Supplemental Material [39]. Note that each measurement channel produces the measurement-induced ensemble dephasing rate of  $\Gamma = (1.3 \mu\text{s})^{-1}$  in the corresponding basis. Our theory

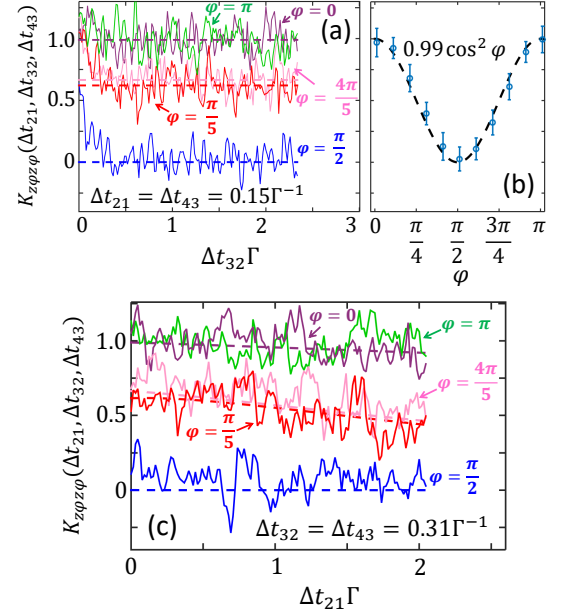


FIG. 2. Comparison between experimental and theoretical four-time correlators. Panels (a) and (c) depict the dependence of the correlator  $K_{\varphi z \varphi z \varphi}(\Delta t_{21}, \Delta t_{32}, \Delta t_{43})$  on  $\Delta t_{32}$  and  $\Delta t_{21}$ , respectively (solid lines for experimental results, dashed lines for theory). Theory predicts no dependence on  $\Delta t_{32}$ . Panel (b) depicts the average (circles) and standard deviation (error bars) of the experimental correlators shown in panel (a), averaged over  $\Delta t_{32}$  within the range  $[0.5/\Gamma, 2.3/\Gamma]$ , for several values of  $\varphi$ . The dashed line in (b) is the theoretical result  $0.99 \cos^2 \varphi$ .

predicts no dependence of  $K_{\varphi z \varphi}$  on  $\Delta t_{21}$ . In agreement with this prediction, the experimental correlators  $K_{\varphi z \varphi}$  shown by solid lines in Fig. 1(a) for several values of  $\varphi$ , do not exhibit a significant dependence on  $\Delta t_{21}$ . Some deviations from the theory (dashed lines) at  $\Delta t_{21} < 0.5 \Gamma^{-1}$  are probably due to slowly fluctuating offsets of the output signals. In contrast, the theory predicts a significant dependence of  $K_{\varphi z \varphi}$  on  $\Delta t_{32}$ , which also agrees with experimental correlators shown in Fig. 1(b). Note that experimental three-time correlators are much noisier than the two-time correlators discussed in Ref. [33], so the ensemble averaging over 200,000 experimental trajectories still produces significant fluctuations in Fig. 1 (see [39] for details of experimental signal processing).

Next, let us consider the similar four-time correlator  $K_{z\varphi z \varphi}(\Delta t_{21}, \Delta t_{32}, \Delta t_{43}) \equiv \int_{t_a}^{t_a+T} dt_1 \langle I_\varphi(t_4) I_z(t_3) \times I_\varphi(t_2) I_z(t_1) \rangle / T$  with  $\Delta t_{ij} = t_i - t_j$ . As follows from Eq. (12),  $K_{z\varphi z \varphi}(\Delta t_{21}, \Delta t_{32}, \Delta t_{43}) = K_{z\varphi}(\Delta t_{21}) K_{z\varphi}(\Delta t_{43})$ , predicting that  $K_{z\varphi z \varphi}$  should not depend on the time difference  $\Delta t_{32}$  (it should also not depend on  $t_a$  and  $T$ ). Figure 2(a) shows the dependence of the experimental correlators  $K_{z\varphi z \varphi}$  on  $\Delta t_{32}$  for several values of  $\varphi$ , with  $\Delta t_{21} = \Delta t_{43} = 0.15 \Gamma^{-1}$ ,  $t_a = 1 \mu\text{s}$ , and  $T = 0.5 \mu\text{s}$ . Indeed, we see that experimental  $K_{z\varphi z \varphi}$  fluctuate around the theoretical constant values (horizontal dashed lines), except for  $\Delta t_{32} < 0.5 \Gamma^{-1}$ , where the solid lines deviate up from the theory [probably be-

cause of the same reason as in Fig. 1(a)]. Figure 2(b) shows the same values of  $K_{z\varphi z\varphi}$  averaged over  $\Delta t_{32}$  (excluding  $\Delta t_{32} < 0.5\Gamma^{-1}$ ) as a function of  $\varphi$ . We see a good agreement with the theoretical result (shown by dashed line),  $K_{z\varphi}^2(\Delta t) = A \cos^2 \varphi$ , where  $A \approx 0.99$  for  $\Delta t = 0.15\Gamma^{-1}$ . Figure 2(c) depicts the dependence of the experimental correlator on  $\Delta t_{21}$  (with fixed  $\Delta t_{32}$  and  $\Delta t_{43}$ ); we again see a good agreement with the theory. Similar results have been obtained for the dependence on  $\Delta t_{43}$  (not shown). Note that the four-time correlators in Fig. 2(a,c) are even noisier than the three-time correlators in Fig. 1; in general, higher-order correlators are increasingly noisier because of multiplication of noise terms.

*Conclusion.*—We have analyzed multi-time correlators for the output signals of linear detectors, continuously measuring arbitrary observables of a qubit at the same

time. We have shown that an  $N$ -time correlator can be expressed as a simple product of two-time correlators for even  $N$  [Eq. (12)], while for odd  $N$  there is also a factor equal to the average signal at the earliest time [Eq. (13)]. This result requires the absence of the phase backaction from continuous measurements and also requires a unital ensemble-averaged evolution. Experimental results for three-time and four-time correlators show good agreement with the theory. Our results can be used in parameter estimation and noise detection protocols in qubit systems, and also for development and analysis of quantum error correction codes operating with continuous measurements; for this purpose a generalization of our theory to multiqubit systems may be needed.

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- [1] K. Kraus, *States, effects and operations: fundamental notions of quantum theory* (Springer, Berlin, 1983).
- [2] C. M. Caves, Phys. Rev. D **33**, 1643 (1986).
- [3] M. B. Menskii, Phys. Usp. **41**, 923 (1998).
- [4] V. P. Belavkin, J. Multivariate Anal. **42**, 171 (1992).
- [5] V. B. Braginsky and F. Ya. Khalili, *Quantum measurement* (Cambridge University Press, Cambridge, UK, 1992).
- [6] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
- [7] L. Diósi, Phys. Lett. A **129**, 419 (1988).
- [8] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
- [9] H. J. Carmichael, *An open systems approach to quantum optics* (Springer, Berlin, 1993).
- [10] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 642 (1993).
- [11] A. N. Korotkov, Phys. Rev. B **60**, 5737 (1999).
- [12] N. Katz, M. Ansmann, R. C. Bialczak, E. Lucero, R. McDermott, M. Neeley, M. Steffen, E. M. Weig, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Science **312**, 1498 (2006).
- [13] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. **6**, 442 (2010).
- [14] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Nature (London) **490**, 77 (2012).
- [15] M. Hatridge, S. Shankar, M. Mirrahimi, F. Schackert, K. Geerlings, T. Brecht, K. M. Sliwa, B. Abdo, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Science **339**, 178 (2013).
- [16] K. W. Murch, S. J. Weber, C. Macklin, and I. Siddiqi, Nature (London) **502**, 211 (2013).
- [17] G. de Lange, D. Riste, M. J. Tiggelman, C. Eichler, L. Tornberg, G. Johansson, A. Wallraff, R. N. Schouten, and L. DiCarlo, Phys. Rev. Lett. **112**, 080501 (2014).
- [18] P. Campagne-Ibarcq, L. Bretheau, E. Flurin, A. Auffeves, F. Mallet, and B. Huard, Phys. Rev. Lett. **112**, 180402 (2014).
- [19] A. N. Korotkov and A. N. Jordan, Phys. Rev. Lett. **97**, 166805 (2006).
- [20] N. Katz, M. Neeley, M. Ansmann, R. C. Bialczak, M. Hofheinz, E. Lucero, A. O’Connell, H. Wang, A. N. Cleland, J. M. Martinis, and A. N. Korotkov, Phys. Rev. Lett. **101**, 200401 (2008).
- [21] R. Ruskov, A. N. Korotkov, and K. Mølmer, Phys. Rev. Lett. **105**, 100506 (2010).
- [22] S. Hacoheh-Gourgy, L. S. Martin, E. Flurin, V. V. Ramasesh, K. B. Whaley, and I. Siddiqi, Nature (London) **538**, 491 (2016).
- [23] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. **70**, 548 (1993).
- [24] R. Ruskov and A. N. Korotkov, Phys. Rev. B **66**, 041401 (2002).
- [25] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, G. Sebastien, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.-M. Raimond, and S. Haroche, Nature (London) **477**, 73 (2011).
- [26] T. L. Patti, A. Chantasri, L. P. García-Pintos, A. N. Jordan, and J. Dressel, Phys. Rev. A **96**, 022311 (2017).
- [27] K. Jacobs, Phys. Rev. A **67**, 030301(R) (2003).
- [28] R. Ruskov and A. N. Korotkov, Phys. Rev. B **67**, 241305(R) (2003).
- [29] D. Risté, M. Dukalski, C. A. Watson, G. de Lange, M. J. Tiggelman, Y. M. Blanter, K. W. Lehnert, R. N. Schouten, and L. DiCarlo, Nature (London) **502**, 350 (2013).
- [30] N. Roch, M. E. Schwartz, F. Motzoi, C. Macklin, R. Vijay, A. W. Eddins, A. N. Korotkov, K. B. Whaley, M. Sarovar, and I. Siddiqi, Phys. Rev. Lett. **112**, 170501 (2014).
- [31] C. Ahn, A. C. Doherty, and A. J. Landahl, Phys. Rev. A **65**, 042301 (2002).
- [32] J. Atalaya, M. Bahrami, L. P. Pryadko, and A. N. Korotkov, Phys. Rev. A **95**, 032317 (2017).
- [33] J. Atalaya, S. Hacoheh-Gourgy, L. S. Martin, I. Siddiqi, and A. N. Korotkov, arXiv:1702.08077.

- [34] A. Chantasri, J. Atalaya, S. Hacoheh-Gourgy, L. S. Martin, I. Siddiqi, and A. N. Jordan, arXiv:1706.09670.
- [35] A. N. Korotkov, Phys. Rev. B **63**, 085312 (2001).
- [36] A. N. Jordan and M. Büttiker, Phys. Rev. Lett. **95**, 220401 (2005).
- [37] L. Diósi, Phys. Rev. A **94**, 010103 (R) (2016).
- [38] A. N. Korotkov, Phys. Rev. B **63**, 115403 (2001); arXiv:1111.4016; Phys. Rev. A **94**, 042326 (2016).
- [39] See Supplemental Material for details.
- [40] C. W. Gardiner and P. Zoller, *Quantum noise* (Springer, Berlin, 2004).
- [41] Alex Kamenev, *Field theory of non-equilibrium systems* (Cambridge, United Kingdom, 2013).