Comment on “Many-body localization in Ising models with random long-range interactions”

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Phys. Rev. A **96**, 057601 — Published 27 November 2017

DOI: 10.1103/PhysRevA.96.057601

Comment on "Many-body localization in Ising models with random long-range interactions"

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(Dated: October 30, 2017)

This comment is dedicated to the investigation of many-body localization in a quantum Ising model with long-range power law interactions, $r^{-\alpha}$, relevant for a variety of systems ranging from electrons in Anderson insulators to spin excitations in chains of cold atoms. It has been earlier argued [1, 2] that this model obeys the dimensional constraint suggesting the delocalization of all finite temperature states in thermodynamic limit for $\alpha \leq 2d$ in a d-dimensional system. This expectation conflicts with the recent numerical studies of the specific interacting spin model in Ref. [3]. To resolve this controversy we reexamine the model of Ref. [3] and demonstrate that the infinite temperature states there obey the dimensional constraint. The earlier developed scaling theory for the critical system size required for delocalization [2] is extended to small exponents $0 \leq \alpha \leq d$. Disagreements between two works are explained by the non-standard selection of investigated states in the ordered phase in Ref. [3].

PACS numbers: 73.23.-b 72.70.+m 71.55.Jv 73.61.Jc 73.50.-h 73.50.Td

I. INTRODUCTION

The many-body localization (MBL) transition separates two distinguishable thermodynamic behaviors. The delocalized system acts as a thermal bath for each small part of it [4, 5] while in the localized system its different parts are approximately independent and can be characterized by related local integrals of motion [6]. Recent experimental investigations of many-body localization [7, 8] are carried out in systems of interacting spins coupled by the long-range interaction decreasing with distance according to power law $U(r) \propto r^{-\alpha}$. The interaction exponent α can be modified experimentally [9, 10] and it is important to understand the effect of power-law interactions on localization.

According to the previous work of one of the authors (with coworkers) [1, 11] (see also more recent work, Refs. [2, 10, 12, 13]) the MBL problem in systems with power law interactions is subject to a dimensional constraint. This constraint suggests that localization is not possible in the thermodynamic limit of an infinite system at a finite temperature if the interaction decreases with the distance slower than $1/r^{2d}$ (where d is a system's dimension in the case of mixed Ising-Heisenberg interactions). Dimensional constraints have been derived assuming that $\alpha \geq d$ to avoid the single-particle delocalization.

However, recently a disagreement with this dimensional constraint has been reported in Ref. [3] for the quantum Ising model with the long-range interactions for $\alpha = 0.5$, 1 and 1.5. In this paper the spin chain of N spins described by the Hamiltonian

$$\widehat{H} = J \sum_{1 \leq i < j \leq L} \frac{(1 + h_i h_j)}{|j - i|^{\alpha}} \sigma_i^z \sigma_j^z + B \sum_{i=1}^{L} \sigma_i^x \qquad (1)$$

has been investigated, where spins are represented by Pauli matrices σ and disorder is introduced using random parameters h_i uncorrelated in different sites i and uniformly distributed within the domain $(-W, W)$ while the parameter W describes the effective disorder. In all studies the transverse field, B, has been set to $0.6J$.

The localization transition has been investigated in Ref. [3] using level statistics and entanglement entropy. The level statistics have been characterized using the averaged ratio of successive gaps, $< r >$, defined according to [14]. The consideration has been limited to the eigenstates with energies close to the middle energy between the minimum and maximum energies $E_{\rm mid} = (E_{\rm min} + E_{\rm max})/2$ represented by the dimensionless parameter $\epsilon = (E - E_{\rm min}) / (E_{\rm max} - E_{\rm min})$ (see Ref. [3] for detail).

Based on the analysis of the level statistics parameter $< r >$ at $\epsilon \approx 59/120$ the authors found the localization at any disordering W for the smallest interaction exponents $\alpha = 0.5$, 1 and size-independent localization-delocalization transition for $\alpha = 1.5$ in contrast to the earlier suggested dimensional constraint [1].

Since systems with long-range interactions are of both acute fundamental and experimental [7, 8] interest due to the ubiquity of charge, dipole, magnetic and elastic forces [1, 11], it is important to understand and interpret the conflict between the qualitative analysis [1, 2, 10–12] leading to the aforementioned dimensional constraint and the numerical results of Ref. [3]. The consideration of the earlier work [2] has been limited to the interaction exponents $\alpha \geq d$ and its extension to smaller exponents $0 \leq \alpha < d$ is another problem of interest. These problems are investigated in the present work.

Below we show that the discrepancy between the two approaches originates from the specifics of the model, Eq. (1), considered in Ref. [3] and the associated choice of the representative energy $E_{\rm mid}$ (see Fig. 2). The dimensional constraint for the states with energies close to zero (corresponding to infinite temperature) is investigated in Sec. II while the analysis of the middle energy states is performed in Sec. III.

II. ZERO ENERGY STATES ($T = \infty$)

The states with energy E can be formally characterized by the temeprature defined from the standard thermodynamic average $\langle E(T)\rangle = \frac{1}{Q}\sum_i E_i e^{-E_i/(k_B T)}$. The infinite temperature states approximately correspond to most probable states at a nearly zero energy where different spins are uncorrelated, while the states considered in Ref. [3] correspond to small negative temperatures where the system is ferromagnetically ordered as discussed below. The use of the term "temperature" is well justified in the delocalized regime where the system equilibrates while it is more formal in the localized regime.

For zero energy states the consideration of Ref. [15] can be applied to the present problem. Strong or weak interaction regimes are applicable, in either case the critical randomness parameter, W, should increase to infinity in the thermodynamic limit $N \to \infty$. In the limit of interest of a large number of spins, N, one can neglect the unity term compared to the product $h_i h_i$ in the definition of the interaction, Eq. (1), provided that $W \gg 1$.

Here we briefly repeat some qualitative arguments from Ref. [15]. Each spin i is subjected to a longitudinal field $\Phi_i = \sum_j J_{ij}\sigma_j^z$, where $J_{ij} = J h_i h_j/|i-j|^\alpha$. For fully random spin projections (infinite temperature limit) the longitudinal field Φ_i is zero in average and it is distributed nearly uniformly within the domain $(-\sigma, \sigma)$ where the size of the domain can be estimated as

$$\sigma \sim \sqrt{\sum_j J_{ij}^2} \sim \begin{cases} JW^2, & \text{if } \alpha > d/2, \\ JW^2\sqrt{\ln(N)}, & \text{if } \alpha = d/2, \\ JW^2 N^{1/2-\alpha/d}, & \text{if } \alpha < d/2. \end{cases} \quad (2)$$

The localization-delocalization transition is associated with resonant spins i satisfying the condition $|\Phi_i| < B$ and the probability of such resonance can be estimated as $P_{res} \sim B/\sigma$. The total number of spin resonances per state is given by $N_{res} = N P_{res}$. According to Ref. [15] the delocalization transition can be determined by the condition $N_{res}\ln(J_N/B) \sim 1$ in the case of a strong interaction, $J_N = J/N^\alpha > B$. It turns out that the interaction is indeed strong for $\alpha \le d$ and the localization delocalization transition is determined as

$$1 \sim \begin{cases} N\frac{B}{JW_c^2}\ln(N^{1-\alpha/d}), & \text{if } d \ge \alpha > d/2, \\ N\frac{B}{JW_c^2}\sqrt{\ln(N)}, & \text{if } \alpha = d/2, \\ N^{1/2+\alpha/d}\frac{B}{JW_c^2}\ln(N), & \text{if } \alpha < d/2, \end{cases} \quad (3)$$

where the parameter W_c estimates the critical randomness corresponding to the localization transition.

The case of $\alpha > d$ corresponding to the weak interaction regime has been considered in Ref. [15]. In that case it has been found in accordance with earlier studies [1] that the delocalization inevitably takes place in the thermodynamic limit for $\alpha < 2d$ and the critical disordering, W_c at a finite number of spins N is determined as

$$N^{2-\alpha/d} B \sim J W_c^2. \quad (4)$$

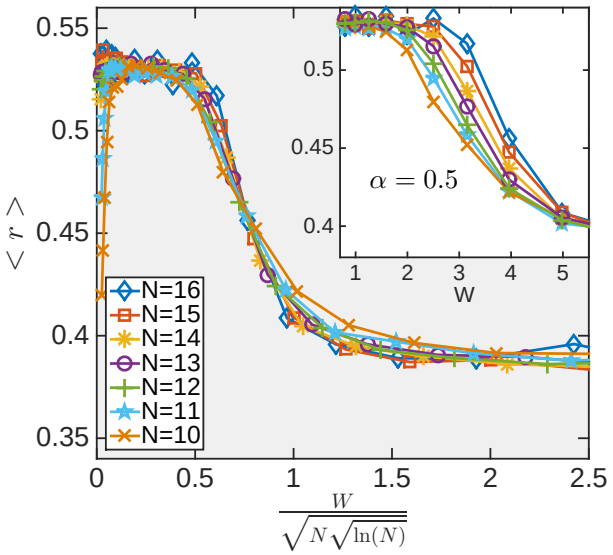

FIG. 1. The level statistics ($< r >$) vs. disordering W (inset) or rescaled disordering according to Eq. (3) for interaction exponent $\alpha = 0.5$ and different numbers of spins $N = 10$, 11, 12, 13, 14, 15 and 16.

Both estimates in Eqs. (3) and (4) are valid assuming $W_c \gg 1$ which takes place at a sufficiently large number of spins (N).

To verify the theoretical predictions, Eqs. (3) and (4), we analyzed numerically the level statistics performing exact diagonalization of the Hamiltonian, Eq. (1), for the states at zero energy, corresponding to the infinite temperature limit and for a number of spins $10 \le N \le 16$. The results are presented in Fig. 1, for power law interaction exponent $\alpha = 0.5$ and for the states of even parity with respect to the symmetry transformation $\sigma^z \to -\sigma^z$ of the Hamiltonian, Eq. (1), as in Ref. [3]. All curves are averaged over 1000 realizations of random interactions as in Ref. [3].

Delocalization clearly takes place at a sufficiently small disordering, W, where the average ratio parameter has a plateau at $< r > \approx 0.53$. This contrasts with the statements of Ref. [3] for small interaction exponent $\alpha = 0.5$ and the reason for this discrepancy is the difference in the energy of the considered system states as detailed below in Sec. III.

At very small disordering, $W \le 0.1$, the ratio parameter deviates from the plateau. This is due to the system reflection symmetry at $W = 0$, which breaks the Hamiltonian into two non-interacting blocks causing level degeneracy. There is no localization for $W \to 0$ as was verified by an analysis of the participation ratios of eigenstates, which is comparable to the total number of states and does not even decrease at $W \to 0$ (not shown here). This symmetry can be broken by increased disordering or by alternating the spin-spin $zz - yy$ interaction similarly to Ref. [16] where the phase factor is proposed for interact-

ing particles in a closed one-dimensional chain.

We also analyzed the level statistics in the model considered in Ref. [17] and found the localization threshold scaling for small transverse fields consistent with Eq. (3) for $\alpha = 0.5$ for states with zero energy. The details will be published separately.

It is expected that the localization threshold W_c approaches infinity in the thermodynamic limit as $W_c \sim \sqrt{N \ln(N)}$ for $\alpha = 0.5$ (see Eqs. (3)). The shifts of the transition between delocalization and localization regimes towards larger disordering W is seen in the inset of Fig. 1. To examine the relevance of the transition point dependence on size we rescaled the parameter of disorder as indicated in x-axis of Fig. 1 similarly to Ref. [2]. For small interaction exponent, $\alpha = 0.5$, this rescaling places the transitions to nearly the same curve confirming the qualitative predictions of Eq. (3).

The qualitative agreement with the theory predictions also takes place for $\alpha = 1$ and 1.5 (not shown) though at $\alpha = 1.5$ the dependence on the number of spins is relatively weak, $W_c \sim N^{1/4}$, to be completely conclusive.

III. MIDDLE ENERGY STATES.

According to Ref. [3] the middle energy states have energies around $E_{\rm mid} = (E_{\rm max} + E_{\rm min})/2$. In the Sherrington-Kirkpatrick model [18] different interactions are of a random sign so the middle energy $E_{\rm mid} = 0$ corresponds to the infinite temperature which is not true for the long range interactions of identical signs in Eq. (1).

The absolute values of the minimum and maximum energies are different for the model given by Eq. (1). The maximum energy corresponds to ferromagnetic ordering determined as $\sigma_i^z = 1$ (or -1) for small disordering $W \ll 1$, or as $\sigma_i^z = {\rm sign}(h_i)$ (or $-{\rm sign}(h_i)$) otherwise. The ferromagnetic ordering can take place in spin chains with ferromagnetic interaction decreasing with distance slower than $1/r^2$ [20–24]. In the model described by Eq. (1) the transition is relevant for the states with large energy corresponding to negative temperatures. For small interaction exponent $\alpha < d$ the energy of this state increases with the number of spins superlinearly as $E_{\rm max} \propto N^{\frac{d-\alpha}{d}}$.

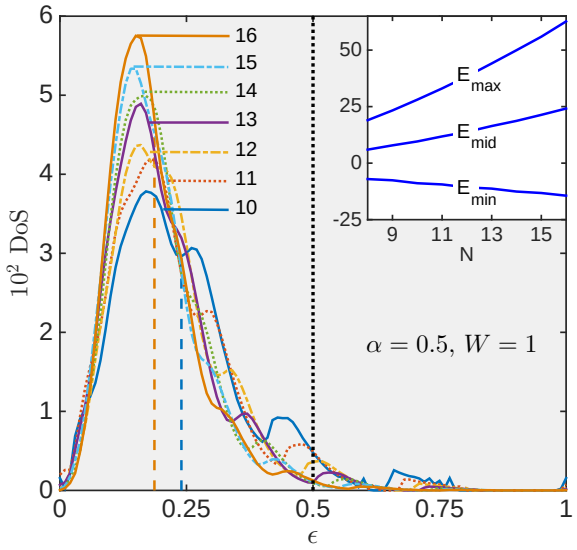

FIG. 2. Density of states (DoS) normalized to 1 for $\alpha = 0.5$ vs. relative energy $\epsilon = (E - E_{\rm min})/(E_{\rm max} - E_{\rm min})$ for typical disordering parameter $W = 1$ and different numbers of spins shown near each graph. Middle energy states with $\epsilon = 0.5$ are indicated by dotted line, while two dashed lines restrict the domain corresponding to zero energies at different numbers of particles. The inset shows dependencies of the maximum, minimum and middle energies on the number of spins.

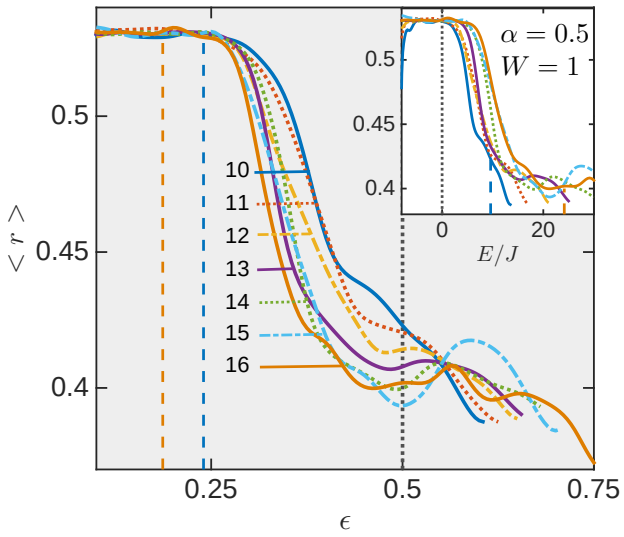

FIG. 3. Level statistics, $< r >$, for $\alpha = 0.5$ vs. relative energy $\epsilon = (E - E_{\rm min})/(E_{\rm max} - E_{\rm min})$ and absolute energy (inset) for typical disordering parameter $W = 1$ and different numbers of spins shown near each line. The middle energy states and zero energy states are denoted similarly to Fig. 2 in the main graph and in the reversed way in the inset.

The absolute value of the minimum (ground state) energy, however, increases with the number of spins only proportionally to this number since the ground state energy is determined by the negative Fourier transform of spin-spin interaction that does not diverge with the number of spins, N [25]. Consequently, for $\alpha \leq d$ the middle energy increases superlinearly with the number of spins as $E_{\rm mid} \approx E_{\rm max}/2$. This trend is illustrated for $\alpha = 0.5$ in the inset of Fig. 2 which means that the middle energy states considered in Ref. [3] correspond to the many-body density of states remarkably smaller than its maximum at energies close to zero. This is illustrated by numerical calculations of the density of states for $\alpha = 0.5$ shown in Fig. 2 as a convolution of the actual density of states with a normalized gaussian function

with $\sigma = 0.005$ in units of ϵ.

It is natural to expect delocalization to be suppressed for states with reduced density [15, 26] in agreement with the observations of Ref. [3]. This is illustrated in Fig. 3, where the level statistics parameter, $< r >$, is shown as the function of energy under the same conditions $\alpha = 0.5$ and $W = 1$ as in Fig. 2. The plateau at energies close to 0 corresponding to delocalization ($< r > \approx 0.53$) does not extend to the middle energy states. The average ratio, $< r >$, remains significantly smaller than the delocalization limit, 0.53, for other disordering parameters W as shown in Fig. 4, where level statistics for $\alpha = 0.5$ are plotted together with the same dependencies from Ref. [3]. However, in Ref. [3] similar graphs are plotted versus the disorder W^2 of squared random parameters h_i (see Eq. (1)), which falls into $(-W^2, W^2)$ range for h_i generated in $(-W, W)$ domain. The explanation of the change in notation from W to W^2 in Ref. [3] appears to have been overlooked.

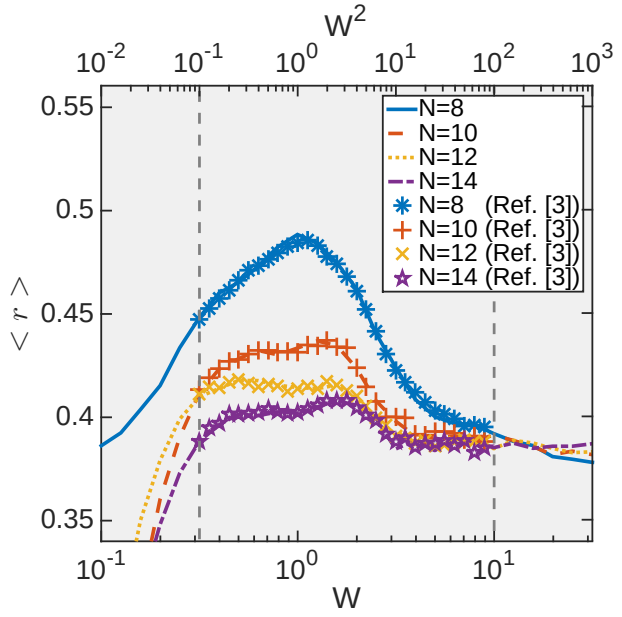

FIG. 4. The dependence of level statistics, $< r >$, for middle energy states ($\alpha = 0.5$) on a disordering parameter W for different numbers of spins. The results of the present work are shown vs. W (bottom axis), while the results of Ref. [3] are shown vs. the disorder of squared random parameters h (see Eq. (1)), which is equivalent to W^2 (top axis). Vertical lines mark the data range in Ref. [3] (see Fig. 3a there).

How does one understand and interpret the delocalization transition for middle energy states having reduced density? The enhancement of the localization can be naturally expected because of the reduction in the number of accessible states. A very strong enhancement of localization under similar conditions has been found in a random energy model [26] while in a more realistic spin glass model with binary interactions strong enhancement takes place only below the spin glass phase transition point [15], where the majority of spins is substantially frozen out.

The middle energies most probably correspond to ordered phases in the cases of power-law interactions for $\alpha = 0.5$ and 1. Indeed, the superlinear scaling of middle energies with the number of spins suggests that the majority of spins (around 3/4) are ferromagnetically ordered for weak disordering, $W \ll 1$, or along random fields h_i for strong disordering. Accordingly, the analysis of MBL based on level statistics should be performed with caution since the states with opposite average spin projections are separated by macroscopic barriers. Consequently, the localization criterion based on level statistics [14] may not be applicable. This can explain why the delocalization limit $< r > \approx 0.53$ is not reached for the middle energy states for $\alpha \leq 1$.

Another reason for strong enhancement of localization in the ordered phase is large typical spin-flip energies increasing as $N^{1-\alpha}$ ($\sqrt{N}$ for $\alpha = 0.5$). This can suppress the delocalization of eigenstates in the Fock space as observed in Ref. [3] though the delocalization is still possible (cf. Ref. [15]). Its analysis is beyond the scope of the present comment.

IV. CONCLUSION

It is demonstrated that systems with long range power law interactions ($\alpha < d$) inevitably delocalize in the thermodynamic limit, similarly to the previously investigated case $d \leq \alpha < 2d$. The modified scaling of the critical system size needed for the delocalization is suggested, Eq. (3), for the case of infinite temperature.

The contradiction of this dimensional constraint to the results of the recent paper [3] for the specific interacting spin model, Eq. (1) has been examined. The results conflict with each other for slowly decreasing interaction $\alpha < 1$ because the eigenstates of the problem investigated in Ref. [3] belong to an ordered phase.

The investigated model demonstrates a clear suppression of delocalization in the ordered phase that can be an interesting subject for future experimental and theoretical studies.

ACKNOWLEDGMENTS

This work is partially supported by the National Science Foundation (CHE-1462075) and by Tulane University Bridge Fund. Authors acknowledge stimulating discussions with Alexander Mirlin and Marcus Heyl.

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