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Phys. Rev. A **96**, 053858 — Published 27 November 2017

DOI: 10.1103/PhysRevA.96.053858

Topological Protection of Coherence in a Dissipative Environment

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(Dated: November 5, 2017)

One dimensional topological insulators are characterized by edge states with exponentially small energies. According to one generalization of topological phase to non-Hermitian systems, a finite system in a non-trivial topological phase displays surface states with exponentially long life times. In this work we explore the possibility of exploiting such non-Hermitian topological phases to enhance the quantum coherence of a fiducial qubit embedded in a dissipative environment. We first show that a network of qubits interacting with lossy cavities can be represented, in a suitable super-one-particle sector, by a non-Hermitian "Hamiltonian" of the desired form. We then study, both analytically and numerically, one-dimensional geometries with up to three sites per unit cell, and up to a topological winding number $W = 2$. For finite-size systems the number of edge modes is a complicated function of W and the system size N. However we find that there are precisely W modes localized at one end of the chain. In such topological phases the quibt's coherence lifetime is exponentially large in the system size. We verify that, for $W > 1$, at large times, the Lindbladian evolution is approximately a non-trivial unitary. For $W = 2$ this results in Rabi-like oscillations of the qubit's coherence measure.

I. INTRODUCTION

There is a growing interest in the study of non-Hermitian generalizations of topological phases of matter [1–10] which can be observed in dissipative systems. Topological features are potentially useful, as they tend to be robust with respect to small perturbations and local noise sources. In this work we explore the possibility of exploiting such, non-trivial, non-Hermitian topological phases to protect the coherence of a preferential qubit in a network of dissipative cavities.

Since eigenvalues of non-Hermitian matrices are complex there are at least two possible definitions of topological phases in non-Hermitian systems [3, 9]. These definitions differ in how one generalizes the Hermitian notion of gap: namely one can consider either the real or the imaginary part of the eigenvalues. According to the imaginary-part classification of Ref. [9], as a consequence of a generalized *bulk-edge* correspondence, a non-trivial topological dissipative phase is characterized at finite size by the presence of quasi-dark states localized at the boundary of the system. By quasi-dark states we mean eigenstates of the system that have a decay time exponentially large in the system size. It is natural to expect that this feature may be useful to protect quantum coherence. Indeed, as we will show, if a fiducial qubit is placed at one end of a linear system, both these features, localization and *dark*ness, conspire to preserve its coherence in a well defined way.

In recent experiments such non-Hermitian systems – in fact essentially non-Hermitian quantum walks – can be observed in classical waveguides using the analogy between Helmoltz and Schrödinger equation [6]. In Ref. [1] it was proposed that a non-Hermitian version of the Su-Schrieffer-Heeger (SSH) model [11] could emerge from a single resonator described by a Jaynes-Cummings model in the semi-classical, large-photon number regime.

Here we consider a network of dissipative cavity resonators interacting á-la Jaynes-Cummings. This model is known to describe the physics of many experimental quantum platforms, ranging from superconducting qubits to arrays of microcavities [12]. We show that, in an appropriate *super*-one-particle sector, the Lindbladian is precisely given by a non-Hermitian quantum walk determined by the network geometry. Moreover, the coherence of a preferential qubit in the network is *exactly* described by the Schrödinger evolution with such a "non-Hermitian Hamiltonian".

Having in mind the goal of prolonging the coherence, we analyze analytically, and confirm numerically, the behavior of the coherence for various finite size networks. The simplest of such a networks is a non-Hermitian tight-binding model with a single, both diagonal and off-diagonal, impurity. We then consider *topologically non-trivial models*, such as a non-Hermitian SSH model, that can have topological charge zero or one. In finite size, there are always two dark modes for N odd while there is one quasi-dark mode in the topologically non-trivial sector for N even. However there is always (irrespective of N) a dark or quasi-dark mode localized at one end of the chain. An analogous situation is found in models with three sites per unit cell, were the topological winding number W can be zero, one or two. The exact number of quasi-dark modes is not a simple function of W alone. However we find precisely W dark or quasi-dark modes localized at one end of the chain. In the case $W = 2$, the long-time dynamics of the dissipative network becomes unitary, spanning a two-dimensional space were the coherence shows Rabi-like oscillations.

II. SETTING THE STAGE

Our model is a network of dissipative cavities (modes) interacting with two-level systems (qubit) in a Jaynes-Cummings fashion. To make it more general, we allow qubits to interact with more than one cavity, although this may be experimentally challenging to realize. We imagine a network of M qubits interacting with K cavity modes. Excitations can hop from mode to mode and also from qubit to mode. At this stage we don't include hopping from qubit to qubit, as this is definitely harder to realize. Our goal will be to monitor, and possibly enhance, the coherence of a fiducial qubit in this network.

We assume the standard rotating-wave approximation, such that the coherent part of the evolution is given by the following Hamiltonian:

$$H = \sum_{i=1}^{M} \omega_i^0 \sigma_i^z + \sum_{l,m=1}^{K} J_{l,m}(a_l^\dagger a_m + \text{h.c.}) \tag{1}$$

$$+ \sum_{l=1}^{K} \omega_l a_l^\dagger a_l + \sum_{i=1}^{M}\sum_{l=1}^{K} \kappa_{l,i}(a_l^\dagger \sigma_i^- + \text{h.c.}), \tag{2}$$

where $a_l^\dagger$ and a_l are the creation and annihilation operators for the cavity mode l and $\sigma_i^\pm$ are the ladder operators for qubit i. On top of this, cavities leak photons at rate Γ_l. A Lindblad master equation for the system can be written as $\dot\rho = \mathcal{L}[\rho]$ with $\mathcal{L} = \mathcal{K} + \mathcal{D}$. The coherent term is $\mathcal{K} = -i\,[H, \bullet]$ and the dissipative part reads

$$\mathcal{D}[\rho] = \sum_{l=1}^{K} \Gamma_l [a_l \rho a_l^\dagger - \frac{1}{2}\{a_l^\dagger a_l, \rho\}], \tag{3}$$

i.e., we assume sufficiently low temperatures such that no photons are excited via interaction with the bath. This form of the dissipation is consistent with the cavity physics whereby essentially only the cavity modes decay whereas the two level-systems (corresponding to some hyperfine level of an atom in the cavity) are extremely long-lived and decay only indirectly through interaction with the cavity. An example of such a dissipative network with $M = 4$ and $K = 5$ is schematically depicted in Fig. 1. Let $i = 1$ indicate the fiducial qubit. In order to study the evolution of the qubit's coherence, we initialize it in a pure state $\alpha_0|\uparrow\rangle + \beta_0|\downarrow\rangle$, while we require that all cavities be empty and all other qubits in the $|\downarrow\rangle$ state. We denote with $|0\rangle$ the overall vacuum (cavities with no photons and qubits in the $|\downarrow\rangle$ state) and $|j\rangle$, $j = 1, \ldots, N \equiv M + K$ the state with an excitation, either bosonic or spin-like, at position j, with $j = 1$ denoting the fiducial qubit and $j = 2, 3, \ldots, N$ the remaining cavities/qubits. With this initial condition the relevant Hilbert space is $\mathcal{H} = \text{Span}\{|0\rangle, |j\rangle, j = 1, \ldots, N\}$, and the dynamics are restricted to the space $\mathcal{V} = L(\mathcal{H})$. A density matrix in $\mathcal{V}$ has the form

$$\rho = \rho_{0,0}|0\rangle\langle 0| + \left(\sum_{j=1}^{N} \rho_{0,j}|0\rangle\langle j| + \text{h.c.}\right) \tag{4}$$

$$+ \sum_{i,j=1}^{N} \rho_{i,j}|i\rangle\langle j|. \tag{5}$$

After tracing out all but the fiducial qubit degrees of freedom, the reduced qubit density matrix reads

$$\rho^{\text{qubit}} = \left(\rho_{0,0} + \sum_{i=2}^{N} \rho_{i,i}\right)|\downarrow\rangle\langle\downarrow|$$

$$+ (\rho_{0,1}|\downarrow\rangle\langle\uparrow| + \text{h.c.}) + \rho_{1,1}|\uparrow\rangle\langle\uparrow|. \tag{6}$$

A coherence measure of the qubit can be defined as [13]

$$\mathcal{C}(t) = \sum_{i,j\,(i\neq j)} \left|\rho_{i,j}^{\text{qubit}}(t)\right|. \tag{7}$$

Using equation (6) we obtain $\mathcal{C} = 2\,|\rho_{0,1}|$.

III. MAPPING TO A NON-HERMITIAN TIGHT-BINDING MODEL

If we initialize the system with at most one excitation, the Lindbladian generates states with at most one excitation and the dynamics are contained in the sector $\mathcal{V}$. We are then led to consider the following linear spaces $\mathcal{V}_{0,0} = \text{Span}(|0\rangle\langle 0|)$, $\mathcal{V}_{0,1} = \text{Span}(\{|0\rangle\langle j|, j = 1, \ldots, N\})$, $\mathcal{V}_{1,0} = \text{Span}(\{|j\rangle\langle 0|, j = 1, \ldots, N\})$ and $\mathcal{V}_{1,1} = \text{Span}(\{|i\rangle\langle j|,\, i, j = 1, \ldots, N\})$. The Hamiltonian conserves the number of excitations so the coherent part $\mathcal{K}$ is block diagonal in the reduced space $\mathcal{V} = \mathcal{V}_{0,0} \oplus \mathcal{V}_{0,1} \oplus \mathcal{V}_{1,0} \oplus \mathcal{V}_{1,1}$. Moreover

$$\mathcal{D}(|0\rangle\langle 0|) = 0 \tag{8}$$

$$\mathcal{D}(|0\rangle\langle j|) = -\frac{\Gamma_j}{2}|0\rangle\langle j| \tag{9}$$

$$\mathcal{D}(|i\rangle\langle j|) = \Gamma_i \delta_{i,j}|0\rangle\langle 0| - \frac{1}{2}(\Gamma_i + \Gamma_j)\,|i\rangle\langle j|. \tag{10}$$

Note that $\Gamma_i = 0$ for $i =$ qubit site, as we are ignoring the spontaneous decay of the qubits (typically much smaller than cavity loss rate). This implies that on $\mathcal{V}$ the Lindbladian has the following block-structure (asterisks denote the only non-

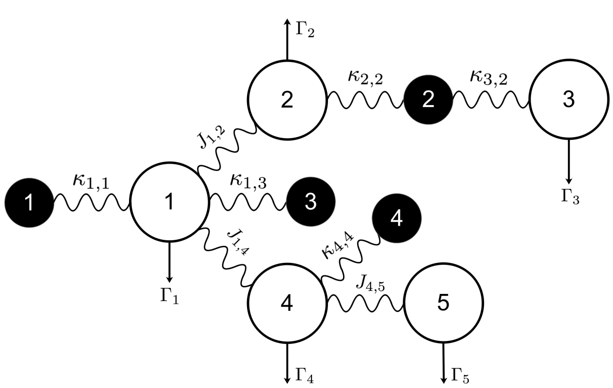


FIG. 1. A general network of qubits interacting with lossy cavities. Wavy lines indicate coherent hopping and straight arrows incoherent decay. White dots represent (leaky) cavities while black dots are (long-lived) two-level systems (qubit).

zero elements) in $\mathcal{V} = \mathcal{V}_{0,0} \oplus \mathcal{V}_{0,1} \oplus \mathcal{V}_{1,0} \oplus \mathcal{V}_{1,1}$

$$\mathcal{L}|_{\mathcal{V}} = \begin{pmatrix} 0 & & & & & * & * & * \\ & * & * & & & & & \\ & * & * & & & & & \\ & & & * & * & & & \\ & & & * & * & & & \\ * & & & & & * & * & * \\ * & & & & & * & * & * \\ * & & & & & * & * & * \end{pmatrix}. \tag{11}$$

We also call $\tilde{\mathcal{L}} = \mathcal{L}|_{\mathcal{V}_{0,1}}$ the restriction of $\mathcal{L}$ to $\mathcal{V}_{0,1}$ and, in this basis, one has $\mathcal{L}|_{\mathcal{V}_{1,0}} = \overline{\tilde{\mathcal{L}}}$ (overline indicates complex conjugate). Clearly the vacuum $|0\rangle\langle 0|$ is a steady state (with eigenvalue zero). We use the following notation for the Hilbert-Schmidt scalar product in $\mathcal{V}$: $\langle\!\langle x|y\rangle\!\rangle = \mathrm{Tr}(x^\dagger y)$ and use the identification $|j\rangle\!\rangle \leftrightarrow |0\rangle\langle j|$ for $j = 1, \ldots, N$ which defines a basis of $\mathcal{V}_{0,1}$.

According to Eq. (7) we need the matrix element $[\rho(t)]_{0,1} = \langle 0|\rho(t)|1\rangle = \langle\!\langle 1|\rho(t)\rangle\!\rangle$. Because of the block-structure of the Lindbladian one obtains $[\rho(t)]_{0,1} = \langle\!\langle 1|e^{t\mathcal{L}}|\rho(0)\rangle\!\rangle = \langle\!\langle 1|e^{t\tilde{\mathcal{L}}}|\tilde{\rho}(0)\rangle\!\rangle$,where we indicated with $\tilde{\rho}(0)$ the projection of $\rho(0)$ to $\mathcal{V}_{0,1}$ according to the above direct sum decomposition of $\mathcal{V}$. Note that if the qubit is initialized in the state $\alpha_0|\uparrow\rangle + \beta_0|\downarrow\rangle$, we have $\tilde{\rho}(0) = \overline{\alpha_0}\beta_0|0\rangle\langle 1|$ or equivalently $|\tilde{\rho}(0)\rangle\!\rangle = \overline{\alpha_0}\beta_0|1\rangle\!\rangle$. In the following we will always consider $\overline{\alpha_0}\beta_0 = 1/2$, i.e. maximal initial coherence, such that

$$\mathcal{C}(t) = \left|\langle\!\langle 1|e^{t\tilde{\mathcal{L}}}|1\rangle\!\rangle\right|. \tag{12}$$

As usual we can identify $\mathcal{V}_{0,1} \simeq \mathbb{C}^N$, and the Hilbert-Schmidt scalar product carries over to the ℓ^2 scalar product. We also use the the norm $\|x\| = \sqrt{\langle\!\langle x|x\rangle\!\rangle}$ for $x \in \mathcal{V}_{0,1}$ and the induced norm for elements of $L(\mathcal{V}_{0,1})$. Since the basis $|j\rangle\!\rangle$ is orthonormal, Hilbert-Schimdt adjoint simply corresponds to transposition and complex conjugation in this basis. With these identifications the setting resembles that of standard one-particle quantum mechanics, with the important difference that operators are not Hermitian. For example, for the case of a single qubit, $M = 1$, interacting with a single cavity and cavities connected on a linear geometry $J_i = J_{i,i+1}$ (see Figure 2 for a schematic picture), the matrix $\tilde{\mathcal{L}}$ becomes

$$\tilde{\mathcal{L}} = -i \begin{pmatrix} \omega_1^0 & \kappa & 0 & \cdots & 0 \\ \kappa & \omega_1 - i\frac{\Gamma_1}{2} & J_1 & \cdots & 0 \\ 0 & J_1 & \omega_2 - i\frac{\Gamma_2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_K \\ 0 & 0 & 0 & J_K & \omega_K - i\frac{\Gamma_K}{2} \end{pmatrix} \equiv -i\mathsf{H}, \tag{13}$$

where we also defined the matrix H which is a non-Hermitian generalization of a tight-binding chain.

Remark. The ℓ^2 scalar product (and corresponding norm) in $\mathcal{V}_{0,1}$ is natural in that, via Hilbert-Schmidt, allows to move from Schrödinger to Heisenberg representation. However in this setting, the ℓ^2 moduli square *are not probabilities*. Conservation of quantum-mechanical probabilities is enforced by the complete positivity and trace preserving property of the full map $e^{t\mathcal{L}}$ for $t \geq 0$. Trace conservation in turn implies $\langle\!\langle \mathbb{I}|\mathcal{L} = 0$, where $\langle\!\langle \mathbb{I}|$ corresponds to the identity operator on the Hilbert space $\mathcal{V}$. This property, however, does not carry over to the restricted generator $\tilde{\mathcal{L}}$. What can still be said is that the eigenvalues of $\tilde{\mathcal{L}}$, since they are a subset of those of $\mathcal{L}$, fulfill $\mathrm{Re}(\lambda) \leq 0$.

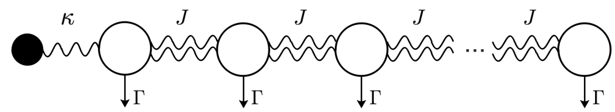


FIG. 2. The "single impurity model": a qubit in a cavity connected to a linear array of cavities.

In general $\mathcal{C}(t)$ will decay in time starting form its maximum value 1 at $t = 0$. From Eq. (12) we realize that our goal is to make a particular matrix element of the restricted evolution $e^{t\tilde{\mathcal{L}}}$, have large absolute value for possibly large times. In fact, ideally we would like: i) $\tilde{\mathcal{L}}|1\rangle\!\rangle = \lambda_1|1\rangle\!\rangle$ and ii) $\mathrm{Re}(-\lambda_1) = 0$. Both of these conditions can be trivially achieved simply setting $\kappa_{l,1} = 0$, $\forall l$. However this entirely decouples the qubit from the rest of the network which means one does not have a way to address the qubit anymore - in fact experimenters generally try to *increase* the qubit-mode coupling. In view of this we replace the two conditions above with the more physical requirements, i') $\tilde{\mathcal{L}}|1\rangle\!\rangle \approx \lambda_1|1\rangle\!\rangle$ and ii') $\mathrm{Re}(-\lambda_1)$ as small as possible.

Condition ii') (that there exist an eigenvalue of $\tilde{\mathcal{L}}$ with almost zero real part) resembles the condition for having an approximate zero mode familiar in (Hermitian) topological insulators. More generally, in a linear geometry, a way to fulfill conditions i') and ii') is to find, approximate, non-Hermitian, topological zero mode of $\tilde{\mathcal{L}}$. Non-Hermitian generalization of topological insulators have been studied to some extent (see e.g., [1, 3, 6, 14]). In particular we will be concerned with finite size systems which have not been discussed in the literature so-far. Before turning to topological models let us first consider what seems to be the simplest geometry.

IV. SINGLE IMPURITY

The simplest case is that of linear geometry with a single impurity (see Fig. (2)), i.e. we set $J_i = J$, $\Gamma_i = \Gamma$ and also $\omega_i = \omega_1^0$ for all i (no detuning) in Eq. (13):

$$\mathsf{H} = \begin{pmatrix} 0 & \kappa & 0 & \cdots & 0 \\ \kappa & \frac{i\Gamma}{2} & J & \cdots & 0 \\ 0 & J & \frac{i\Gamma}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J \\ 0 & 0 & 0 & J & \frac{i\Gamma}{2} \end{pmatrix}, \tag{14}$$

where H has been transformed to the rotating frame of frequency ω_1^0. This is a non-Hermitian generalization of a single impurity in a tight binding chain [15]. For $N = 3$ this model has been investigated in [16, 17], where it was established that adding one auxiliary cavity to a dissipative optical cavity coupled to a qubit can significantly increase the coherence time of the qubit. An equation for the eigenvalues can be found using the techniques to diagonalize tridiagonal matrices. The eigenvalues of the matrix (14) $\tilde{\mathcal{L}}$ can be written as $\lambda_k = -i2J\cos(k) - \Gamma/2$, where k is a (possibly complex) quasi-momentum that satisfies the following equation

$$[2\cos(k) + ia]\sin(kN) - \beta^2 \sin(k(N-1)) = 0, \qquad (15)$$

where $a = \Gamma/(2J)$, $\beta = \kappa/J$. In order to look for a localized state we look for a solution of the above equation with complex $k = x + iy$. Essentially the localization length is given by $\zeta = y^{-1}$. More details are provided in Appendix A. Neglecting terms of order $O\left(e^{-N|y|}\right)$ the eigenvalues of $\tilde{\mathcal{L}}$ of such localized modes are given by

$$\lambda_\pm = -\frac{4\kappa^2}{\Gamma \pm \sqrt{16(J^2-\kappa^2)+\Gamma^2}} + O\left(e^{-N|y|}\right). \qquad (16)$$

This formula is valid in regions where $\mathrm{Re}(\lambda_\pm) < 0$. Because the wave vector k is complex, a plane wave trial solution will decay like $e^{-yn} = e^{-n/\zeta}$ which defines the localization length ζ. In such cases the localization length is given by

$$\zeta = 1/\ln\left|\frac{4J}{\Gamma \pm \sqrt{16(J^2-\kappa^2)+\Gamma^2}}\right|. \qquad (17)$$

For κ/Γ small (strong dissipative regime), using a perturbative argument (more details in Appendix (C)), one can show that the coherence has approximately the form of a single exponential decay e^{-t/τ_0}, with $\tau_0^{-1} = 2\kappa^2/\Gamma$. Using Eq. (16) the eigenvalue connected with τ_0^{-1} is λ_+. By continuity, we can now we can use the expression for the localized mode outside from the strict perturbative region. In other words we have

$$\mathcal{C}(t) \approx e^{-t/\tau} \qquad (18)$$

$$\tau = \mathrm{Re}\Big[\frac{\Gamma + \sqrt{16(J^2-\kappa^2)+\Gamma^2}}{4\kappa^2}\Big]. \qquad (19)$$

The above equations are extremely accurate in the region of small κ but surprisingly are quite accurate also for large κ. Increasing κ one starts observing non-Markovian oscillations[18] in the coherence also noted in [17] at an energy scale of the order of $J^2 + \Gamma^2/16$ (when the square root term in Eq. (19) becomes imaginary). In this regime Eq. (18) describes well the envelope of the coherence. See Fig. 3 for comparisons with numerics.

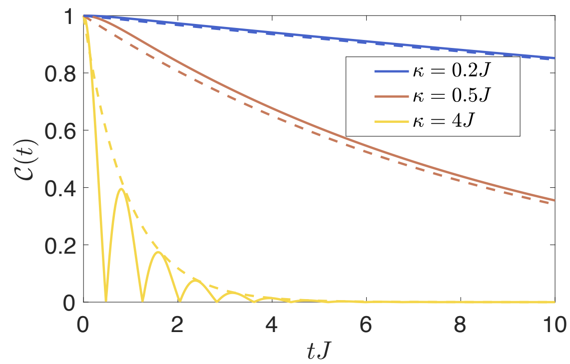


FIG. 3. (Color online) Behavior of the coherence for the single impurity model. Here and in the following we compute Eq. (12) by numerical diagonalization of the corresponding reduced Lindbladian. Continuous lines are numerical simulation and dashed lines are analytical approximation of Eqns. (18-19). Dissipation is fixed to $\Gamma = 4J$. The results for $N = 4$ are indistinguishable from those at $N = 400$.

V. TOPOLOGICAL CLASSIFICATION OF DISSIPATIVE SYSTEMS

We recall here for completeness the basics of the topological classification of models of Ref. [9] (see also [3]). Since eigenvalues are now complex, there are at least two ways to generalize this notion to the non-Hermitian world. Namely one may extend the role played by the Hermitian gap to either the imaginary or the real part of the eigenvalues. Two points in parameter space are defined to be in the same phase if the corresponding (non-Hermitian) Hamiltonians can be smoothly connected without closing the imaginary (resp. real) part of the eigenvalues. For the "imaginary-gap" classification of Ref. [9], according to a generalized bulk-edge correspondence, a non-trivial phase at finite size would have edge modes with infinite or exponentially large life-time. Clearly this is the relevant classification in our context.

We assume a periodic linear chain with n sites per unit cell such that, in the thermodynamic limit, the Hamiltonian is given by $\mathsf{H} = \oint dk/(2\pi) \sum_{\alpha,\beta} \mathsf{H}_{\alpha,\beta}(k)|k,\alpha\rangle\!\rangle\langle\!\langle k,\beta|$ and we simply need to focus on the $n \times n$ Bloch matrix $\mathsf{H}(k)$. The dissipation has the special form shown in Sec. III which consists of imaginary terms on the diagonal (of negative imaginary part). Without constraint such models are topologically trivial if the number of leaky sites per cell is greater than one [9]. We then focus on the case where there is only one leaky site per cell. As shown in [9], any such $\mathsf{H}(k)$ that does not admit a dark state can be written in the following way

$$\mathsf{H}(k) = \begin{pmatrix} U(k) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{h}(k) & \tilde{v}_k \\ \tilde{v}_k^\dagger & \Delta(k) - i\Gamma \end{pmatrix} \begin{pmatrix} U(k)^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \qquad (20)$$

where $\tilde{h}(k)$ is an $(n-1)\times(n-1)$ diagonal matrix with real eigenvalues, $U(k)$ is an $(n-1)\times(n-1)$ unitary matrix that diagonalizes $\tilde{h}(k)$ and also makes the $(n-1)$ dimensional vector $\tilde{v}_k$ real and positive. Any $U(k)$ satisfying the above criteria can be chosen without affecting the following result. In Ref. [9] it is further shown that the winding number of H then reduces to the winding number of $U(k)$ which is given by

$$W = \oint \frac{dk}{2\pi i}\partial_k \ln\det(U(k)). \qquad (21)$$

From what we have said, in a non-trivial topological phase, at finite size one expects to observe dark states localized at the edges. Such a dark (or quasi-dark) state $|\xi\rangle\!\rangle$ fulfills $\tilde{\mathcal{L}}|\xi\rangle\!\rangle = \lambda|\xi\rangle\!\rangle$ with $\mathrm{Re}(\lambda) \simeq 0$. However, given the structure of the space $\mathcal{V}_{0,1}$ all such states are e.g. traceless. Hence these are not strictly quantum states, they are in fact *off-diagonal elements* of a quantum state. In the quantum-chemistry community these are sometimes called *coherences*.

We would like to conclude this section by reminding a general result for completely positive maps/semigroups. We assume here finite dimensionality. Let the Jordan decomposition of $\mathcal{L}$ be $\mathcal{L} = \sum_k \lambda_k P_k + D$ where D is the nilpotent part. Define the projector onto the dark states sector as

$$P_{\mathrm{ds}} = \sum_{k,\mathrm{Re}(\lambda_k)=0} P_k. \tag{22}$$

Decomposing the Liouville space as $\mathbb{1} = P_{\mathrm{ds}} \oplus (\mathbb{1} - P_{\mathrm{ds}})$ one has $e^{t\mathcal{L}} = \mathcal{W}_t \oplus \mathcal{R}_t$ where $\mathcal{W}_t$ is the part of the evolution inside the dark-state sector: $\mathcal{W}_t = P_{\mathrm{ds}}\mathcal{W}_t = \mathcal{W}_t P_{\mathrm{ds}}$ and the remaining term $\mathcal{R}_t$ can be made as small as one wishes in norm, by taking larger t. It can be shown (see Theorem 6.16 of [19]) that $\mathcal{W}_t$ is a unitary evolution, more precisely $\mathcal{W}_t[\rho_0] = U_t\tilde{\rho}_0 U_t^\dagger$ where the state $\tilde{\rho}_0$ is partly determined by the initial state ρ_0. In other words, the time evolution inside the dark state sector is unitary.

VI. NON-HERMITIAN SSH MODEL

To start we consider the model given by the following non-Hermitian generalization of the SSH Hamiltonian (for simplicity we rename all hopping constants J_i both for qubit-mode and mode-mode hopping)

$$\mathsf{H} = \begin{pmatrix} 0 & J_1 & 0 & 0 & 0 \\ J_1 & -i\Gamma & J_2 & 0 & 0 \\ 0 & J_2 & 0 & J_1 & 0 \\ 0 & 0 & J_1 & -i\Gamma & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}. \tag{23}$$

One may obtain an intuitive understanding of the model by considering the periodic boundary conditions version of the above. In that case it suffices to consider the 2×2 Bloch Hamiltonian

$$\mathsf{H}(k) = \begin{pmatrix} 0 & v_k \\ \overline{v}_k & -i\Gamma \end{pmatrix}, \tag{24}$$

with $v_k = J_1 + J_2 e^{ik}$. Model (24) is, up to a constant term, pseudo-anti-Hermitian, in that $\tilde{\mathsf{H}}(k) := \mathsf{H}(k) + i(\Gamma/2)\mathbb{1}$ satisfies $\sigma^z \left[\tilde{\mathsf{H}}(k)\right]^\dagger \sigma^z = -\tilde{\mathsf{H}}(k)$. Moreover $\tilde{\mathsf{H}}(k)$ is a linear combination of the matrices $\{-\sigma^x, -\sigma^y, i\sigma^z\}$ which span the Lie algebra of $SU(1,1)$ ($S(1,1)$ in turn is the group of 2×2 complex matrices U satisfying $U^\dagger\sigma^z U = \sigma^z$ and $\det(U) = 1$). Model (24) is then also referred to as $SU(1,1)$ model [3]. The more familiar, Hermitian, SSH model being a $SU(2)$ model.

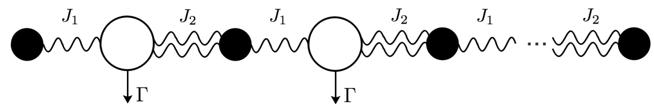


FIG. 4. Non-Hermitian SSH model Eq. (23) for N odd.

The eigenvalues of Eq. (24) are simply

$$\lambda_{k,\pm} = -i\frac{\Gamma}{2} \pm \sqrt{|v_k|^2 - \frac{\Gamma^2}{4}} \tag{25}$$

$$= -i\frac{\Gamma}{2} \pm \sqrt{J_1^2 + J_2^2 + 2J_1J_2\cos(k) - \frac{\Gamma^2}{4}}, \tag{26}$$

with momenta given by $k = 4\pi n/N$ (N even). For example, if $\Gamma^2/4 < v_{\mathrm{min}}^2 \equiv (J_1^2 + J_2^2 - 2|J_1J_2|)$, the square root term above is real and all the modes decay at a rate $\Gamma/2$. This model admits a topological phase characterized by a winding number according to the "imaginary gap" classification of [9]. The winding number W Eq. ((21)) turns out to be analogous to that of the Hermitian SSH model, and it simply counts the number of times the vector $J_1 + J_2e^{ik}$ winds around the origin as k moves around the Brillouin zone $[0, 2\pi)$. Consequently $W = 1$ for $|J_2| > |J_1|$ while $W = 0$ for $|J_2| < |J_1|$ [20].

This picture gets modified for an open chain. Most importantly, as a consequence of the topological character of the model and the so-called bulk-edge correspondence, there will appear edge state(s) localized at the boundary of the chain. The calculations are different depending on whether N is even or odd. We fix the geometry by fixing the dissipation to act only on the even sites as in Eq. (23).

A. N odd

For N odd the configuration of the bonds is given in Fig. 4. For N odd there is always one edge state irrespective of the values of J_1, J_2. In this case the edge-mode has exactly zero eigenvalue i.e., is a dark state. The edge mode is localized at the site where the weak link is (whether it is J_1 or J_2). Clearly the transition is at $J_1 = J_2$. If J_1 is the weak link we can write such an edge mode as

$$|\xi_L\rangle\!\rangle = A \begin{pmatrix} e^{ik} \\ 0 \\ e^{3ik} \\ 0 \\ e^{5ik} \\ \vdots \\ e^{Nik} \end{pmatrix} \tag{27}$$

where A is a normalization factor. One finds that $\mathsf{H}|\xi_L\rangle\!\rangle = 0$ provided $J_1 + J_2e^{2ik} = 0$. Under this condition $|\xi_L\rangle\!\rangle$ is a dark state. From this equation we see that

$$|\langle\!\langle n|\xi_L\rangle\!\rangle|^2 = A^2 e^{-n\delta}$$

for n odd, where $\delta \equiv \ln(|J_2/J_1|) > 0$ was assumed to be positive. Hence we call $\ell \equiv 1/\ln(|J_2/J_1|)$ the localization length of the edge mode. Fixing the normalization one finds

$$A^2 = \frac{1-x^2}{x - x^{N+2}}, \tag{28}$$

with $x = |J_1/J_2| < 1$.

The case $|J_2| < |J_1|$ can be reduced to the previous one by a left-right symmetry transformation. Under this transformation the dark state is mapped onto $|\xi_R\rangle\!\rangle$ which is localized at the opposite end of the chain.

Recalling the result for the periodic case one sees that, in general, the other, non-localized, modes decay on a relaxation time-scale given by $\tau_{\text{relax}} \approx \Gamma^{-1}O(1)$. Coming to the behavior of the coherence we see that, after a time τ_{relax} all but the mode $|\xi_L\rangle\!\rangle$ will have decayed. Hence the coherence, for $t > \tau_{\text{relax}}$, is approximately given by

$$\begin{aligned}\mathcal{C}(t) &= \left|\sum_k e^{\lambda_k t}\langle\!\langle 1|P_k 1\rangle\!\rangle\right| \\ &\approx |\langle\!\langle 1|\xi_L\rangle\!\rangle\langle\!\langle \xi_L|1\rangle\!\rangle| = |\langle\!\langle \xi_L|1\rangle\!\rangle|^2 \\ &= \frac{1-x^2}{1-x^{N+1}}.\end{aligned} \tag{29}$$

Note that, since $x < 1$, this is a decreasing function of N. The largest value with $N > 1$, odd, is obtained for $N = 3$.

For $|J_2| < |J_1|$ the role of $|\xi_L\rangle\!\rangle$ and $|\xi_R\rangle\!\rangle$ are reversed. Hence now the dark state is localized at the end of the chain. After a time τ_{relax} the coherence drops to a value $\mathcal{C}(t) \simeq |\langle\!\langle \xi_R|1\rangle\!\rangle|^2 = |\langle\!\langle \xi_L|N\rangle\!\rangle|^2 = z^{N-1}(1-z^2)(1-z^{N+1})^{-1}$, where z is now $z = |J_2/J_1|$, i.e. an exponentially small value. The two asymptotic expressions are in fact the same and can be combined in a single expression valid for all J_1, J_2

$$\mathcal{C}(t) \simeq \begin{cases} J_2^{N-1}\frac{J_2^2 - J_1^2}{J_2^{N+1} - J_1^{N+1}} & J_1 \neq J_2 \\ \frac{2}{N+1} & J_1 = J_2 \end{cases}. \tag{30}$$

To summarize, for N odd there is always an exact localized dark state for all values of parameters and consequently an infinite lifetime of the coherence's qubit. However, in the topologically trivial phase $W = 0$ ($|J_1| > |J_2|$) the edge mode is localized at the opposite end of the chain, and the asymptotic value of the coherence is exponentially small. The numerical simulations confirm that a non-trivial topological winding number has a strong effect on the coherence time of the qubit, as illustrated on Fig. 5.

To connect with the previous discussion we see that, in general we satisfy the requirement ii') (there is an eigenmode with $\text{Re}(\lambda) = 0$), but not necessarily i') . In other words, in general $|\xi_L\rangle\!\rangle\langle\!\langle \xi_L|$ is not close to $|1\rangle\!\rangle\langle\!\langle 1|$. We progressively enter this regime when the localization length becomes very short (or δ very large). Clearly this happens when $|J_2| \gg |J_1|$.

B. N even

For N is even the configuration of the links is depicted in Fig. 6. When N is even, $|\xi_L\rangle\!\rangle$ of Eq. ((27)) does not satisfy the last row of the eigenvalue equation but rather one has $\mathsf{H}|\xi_L\rangle\!\rangle = J_1 e^{ik(N-1)}|N\rangle\!\rangle$. This is consistent with our expectation of an exponentially small eigenvalue. The exact diagonalization of the model can be found in [21] (see also [22, 23]). For N even edge modes appear for $d \equiv J_2/J_1 > 1 + 2/N$. This is an interesting effect as one can in principle enter the topologically non-trivial phase for fixed values of the parameters by only changing N. The eigenvalues of the edge modes are given by [21]

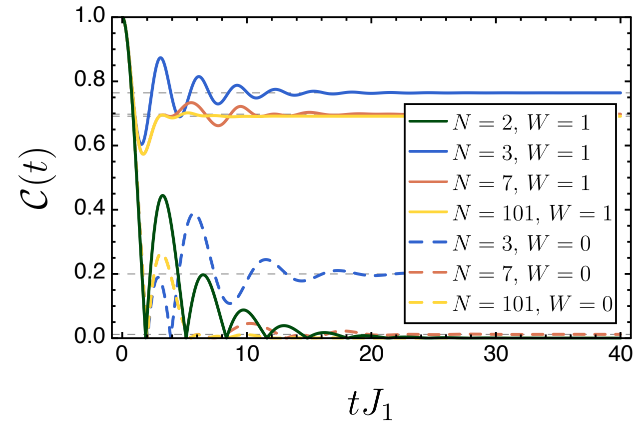


FIG. 5. (Color online) Behavior of the coherence in the non-Hermitian SSH model with an odd number of sites. Continuous lines are results in the topological phase ($W = 1$) with parameters $J_1 = 1$, $J_2 = 1.8$ and $\Gamma = 0.5$. Dashed lines are for the topologically trivial phase ($W = 0$, $J_1 = 1$, $J_2 = 0.5$ $\Gamma = 0.5$). The thin dashed lines are the asymptotic values given by Eq. (30). The qubit has infinite lifetime for all values of parameters, but the asymptotic coherence is exponentially small in the topologically trivial region. The intrinsic coherence lifetime of the qubit ($N = 2$) is added for comparison. We observe that the lattice of cavities with $W = 1$ vastly improves the lifetime of the coherence.

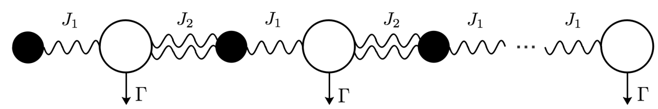


FIG. 6. Non-Hermitian SSH model Eq. (23) for N even.

$$\lambda_\pm = -i\frac{\Gamma}{2} \pm \sqrt{J_1^2 + J_2^2 + 2J_1J_2\cosh(y) - \frac{\Gamma^2}{4}} \tag{31}$$

where y satisfies

$$\sinh(\frac{N}{2}y) = x\sinh\left[(\frac{N}{2}+1)y\right]. \tag{32}$$

For N large the solution of Eq. (32) approaches $e^y = d$. Up to first order in d^{-N} one obtains that the solution of Eq. (32) is

$$e^y = d + d^{-N}\left(d^{-1} - d\right) + O(d^{-2N}). \tag{33}$$

N	$\tau_{\rm coh}$		$\lvert\langle\!\langle \xi_+\rvert 1\rangle\!\rangle\rvert^2$	
	Exact	Theory	Exact	Theory
6	6.9367	10.9813	0.5355	0.6638
8	31.8117	35.5794	0.6715	0.6915
10	111.1859	115.2774	0.6888	0.6941
20	4.1153×10^4	4.1159×10^4	0.6914	0.6914

TABLE I. Comparison of exact numerics with the approximate theoretical formulae. Parameters are $J_1 = 1$, $J_2 = 1.8$ and $\Gamma = 0.5$.

Plugging the above into Eq. (32) one finds

$$\lambda_+ = -i\frac{J_1^2}{\Gamma} d^{-N}\left(d^{-1} - d\right)^2 \tag{34}$$

$$\lambda_- = -i\Gamma + i\frac{J_1^2}{\Gamma} d^{-N}\left(d^{-1} - d\right)^2. \tag{35}$$

The λ_+ eigenvalue corresponds to the mode localized at the first site of the chain. Moreover, even if there are two localized modes, only one of them has exponentially large life-time in the system size. So for N even the the left edge mode has a coherence time of $\tau_{\rm coh} = \Gamma J_1^{-2} d^N (d^{-1} - d)^{-2}$. The λ_- eigenvalue corresponds to edge mode localized at the end of the chain, with fastest decay time.

In order to compute the coherence we need the first component of the edge mode $|\xi_+\rangle\!\rangle$. It turns out that (see [21])

$$|\langle\!\langle 1|\xi_+\rangle\!\rangle|^2 = \frac{4\sinh^2(Ny/2)}{\left[\frac{\sinh[(N+1)y]}{\sinh(y)} - (N+1)\right]}\frac{\lambda_+ + i\Gamma}{2\lambda_+ + i\Gamma}. \tag{36}$$

Since λ_+ is exponentially small, the last fraction is exponentially close to 1 and can be evaluated up to d^{-N} using Eq. (34). For the remaining terms we plug in the asymptotic value $y = \ln(d)$ and obtain

$$\begin{aligned} |\langle\!\langle 1|\xi_+\rangle\!\rangle|^2 &= \frac{\left(1 + \frac{J_1^2}{\Gamma^2} x^N \left(x - x^{-1}\right)^2\right)}{(1-x^2)^{-1} - x^N(N+1)} + O(x^{2N}) \\ &= 1 - x^2 + x^N\left(1 - x^2\right)^2 \times \\ &\times \left((N+1) - \frac{1-x^2}{x^2}\frac{J_1^2}{\Gamma^2}\right) + O(x^{2N}) \end{aligned} \tag{37}$$

In this case the state $|\xi_+\rangle\!\rangle$ is not an exact dark state and it will start decaying at a time around $\tau_{\rm coh}$. As for the odd case, the other states decay after a time $\tau_{\rm relax} = \Gamma^{-1}O(1)$. Hence, whenever there is a separation of time-scales $\tau_{\rm coh} > \tau_{\rm relax}$, one will observe a coherence of $\mathcal{C}(t) \approx |\langle\!\langle 1|\xi_+\rangle\!\rangle|^2$ for times roughly in the window $t \in [\tau_{\rm relax}, \tau_{\rm coh}]$. Numerical experiments for the even case are shown in Fig. 7. In table I we show comparisons of the numerics with the analytic expressions. For comparison, the $W = 0$ case is also shown in Fig. 7, where the coherence is from bulk modes only, and the decay is given by the bulk relaxation time Γ^{-1}.

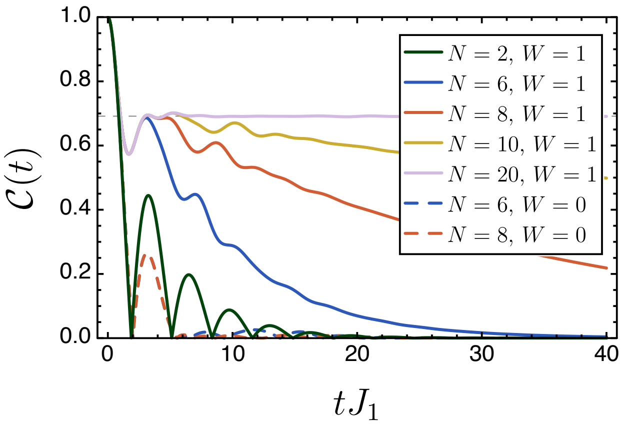


FIG. 7. (Color online) Behavior of the coherence in the non-Hermitian SSH model with an even number of sites. Continuous lines are results in the topological phase ($W = 1$) with parameters $J_1 = 1$, $J_2 = 1.8$ and $\Gamma = 0.5$. Increasing N has the effect of exponentially increasing the (coherence) time-scale $\tau_{\rm coh}$ at which the approximate dark state starts decaying. Dashed lines are for the topologically trivial phase ($W = 0$, $J_1 = 1$, $J_2 = 0.5$ $\Gamma = 0.5$). For $N = 10$, 20 the plot is indistinguishable from that of $N = 8$. The thin dashed lines is the asymptotic value given by Eq. (37). The intrinsic coherence lifetime of the qubit ($N = 2$) is added for comparison. We observe that the lattice of cavities with $W = 1$ vastly improves the lifetime of the coherence.

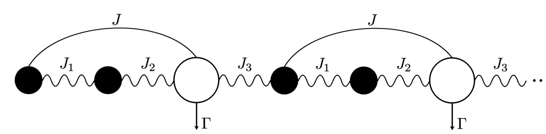


FIG. 8. Model (38) with a three-site unit cell.

VII. THREE-SITE UNIT CELL

We now turn to a case where the unit cell consists of three sites. According to the prescription of Ref. [9] for the existence of a topological phase we consider only one leaking site per cell. We allow for nearest neighbor hopping and also between the first and third site in the cell (see Fig. (8)). As we will see, this geometry will allow us to have topological number of 0, 1 and 2. The Hamiltonian is

$$\begin{aligned} \mathsf{H} = \sum_x &\Big(J_1|x,1\rangle\!\rangle\langle\!\langle x,2| + J_2|x,2\rangle\!\rangle\langle\!\langle x,3| + J_3|x,3\rangle\!\rangle\langle\!\langle x+1,1| \\ &+ J|x,1\rangle\!\rangle\langle\!\langle x,3| + \text{h.c.}\Big) + \\ &\sum_x \Big(\epsilon_1|x,1\rangle\!\rangle\langle\!\langle x,1| + \epsilon_2|x,2\rangle\!\rangle\langle\!\langle x,2| - i\Gamma|x,3\rangle\!\rangle\langle\!\langle x,3|\Big). \end{aligned} \tag{38}$$

For periodic boundary conditions the corresponding Bloch Hamiltonian reads

$$\mathsf{H}(k) = \begin{pmatrix} \epsilon_1 & J_1 & J_3 e^{ik} + J \\ J_1 & \epsilon_2 & J_2 \\ J_3 e^{-ik} + J & J_2 & -i\Gamma \end{pmatrix}.$$

Using Eq. (21) it can be shown that the winding number is given by

$$W = \Theta(|J_3| > |J + J_2 \tan(\vartheta/2)|) + \Theta(|J_3| > |J - J_2 \cot(\vartheta/2)|), \quad (39)$$

where $\Theta(\text{true}) = 1$, $\Theta(\text{false}) = 0$ and $\vartheta = \arccos[(\epsilon_1 - \epsilon_2)/\sqrt{4J_1^2 + (\epsilon_1 - \epsilon_2)}]$. The above quantity can assume the values $W = 0, 1, 2$. The value $W = 2$ can be obtained, for example, by taking J_3 sufficiently large. When $W = 2$ the open, finite size chain has two edge modes per end. This gives the possibility to encode a qubit in the dark state manifold of the model. In the following we restrict to the case $\epsilon_2 = \epsilon_1 = \epsilon$ for which

$$W = \Theta(|J_3| > |J + J_2|) + \Theta(|J_3| > |J - J_2|). \quad (40)$$

As we can see from the above the presence of the two-sites hopping J is not necessary for having $W = 2$ but it allows to have $W = 1$.

As we have seen in section VI, at finite size the exact number of edge modes can be a complicated function of N and the other parameters of the models. For the model of Eq. (38), we have verified numerically that for N mod $3 = 2$ there are always (irrespective of W) two edge modes with imaginary part of the eigenvalues exactly equal to zero. In other words there are always two exact dark states. However, we have also checked that essentially only W of them are localized on the qubit site. For N mod $3 \neq 2$ our simulations suggest that there are W edge modes with life-time exponentially large in the system size (see Fig. 9). Moreover, precisely W of them are localized at the qubit site. This picture is consistent with what we have found analytically in sec. VI. In other words, there are always (for all N) W dark or quasi-dark modes localized at the qubit site. Since, as we have seen, the behavior of the coherence is not only dictated by the number of localized modes, but rather by the modes *localized at the qubit*, the value of W has a strong impact on the coherence.

From what we have said so far, the behavior of the coherence of the first qubit is now clear. For $W = 0$ the coherence decays to zero after a time $\tau_{\text{relax}} = \Gamma^{-1}O(1)$ or it saturates to an exponentially small value in N if $N = 3p + 2$. For times $\tau_{\text{relax}} \lesssim t \lesssim \tau_{\text{coh}}$, for $W = 1$ it saturates to an amount given by $\mathcal{C}(t) \simeq |\langle\!\langle 1|\xi_1\rangle\!\rangle|^2$ where $|\xi_1\rangle\!\rangle$ is the dark state localized at the left of the chain. For $W = 2$ the coherence will oscillate between two values in a similar way as in Rabi oscillations, $\mathcal{C}(t) \simeq \left| e^{-i\omega_1 t} |\langle\!\langle 1|\xi_1\rangle\!\rangle|^2 + e^{-i\omega_2 t} |\langle\!\langle 1|\xi_2\rangle\!\rangle|^2 \right|$ where $|\xi_{1,2}\rangle\!\rangle$ are the two dark states localized at the left with (real) eigenvalues $\omega_{1,2}$. The time-scale τ_{coh} is infinite for $N = 3p+2$ and exponentially large in N otherwise. A plot of the behavior of the coherence in different topological sectors is shown in Fig. 10.

Finally, let us comment on the long-time behavior of the full Lindbladian evolution. For $N = 3p + 2$ there is an exact, non-trivial dark space and so, for what we have said at the end of section V, the evolution inside this dark space is unitary. When $N \mod 3 \neq 2$ and $W = 2$ there are two modes with life-time τ_{coh} exponentially large in N. In this case an exact dark space sector cannot be defined, however we have verified that the dynamics are approximately unitary for times t in the window $\tau_{\text{relax}} \lesssim t \lesssim \tau_{\text{coh}}$. In this sense the term Rabi oscillations is accurate.

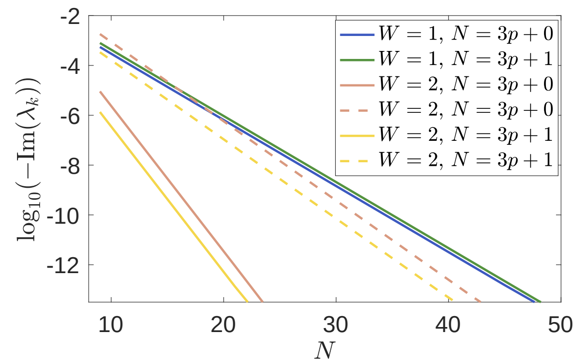


FIG. 9. (Color online) Scaling of the imaginary part of the eigenvalues of the edge modes, for different topological sectors and different values of $N \mod 3$. For $N = 3p+2$ we have observed always two exact dark states($\mathrm{Im}(\lambda_k) = 0$) for all parameters values. This simulations suggest that, for $N \mod 3 \neq 2$ there are W edge modes with exponentially large life-time. Parameters are $\epsilon_1 = \epsilon_2 = 0$, $J_1 = 1.4$, $J_2 = 0.3$, $J = 0.7$, $\Gamma = 1.5$ and $J_3 = 1$ for $W = 1$ while $J_3 = 3$ for $W = 2$.

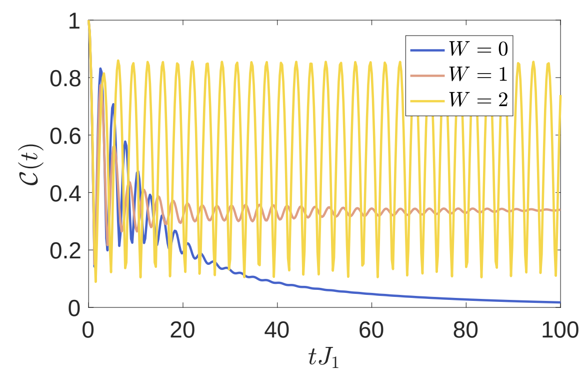


FIG. 10. (Color online) Behavior of the coherence in the linear chain with three sites per cell Eq. (38). We chose $N = 8$, but the numerical results are not sensitive to N mod 3. The winding number can assume values $W = 0, 1, 2$. W also counts the number of edge modes localized near the first qubit. For $W = 2$ the dark state manifold is a physical qubit and one sees Rabi oscillations in the coherence. Parameters are $J_1 = 1, \epsilon = 0, \Gamma = 0.5, J_2 = 0.3, J = 0.7$ and J_3 fixes the value of W: $J_3 = 0.2$ ($W = 0$), $J_3 = 0.7$, ($W = 1$), and $J_3 = 2$, ($W = 2$).

VIII. EFFECT OF NOISE ON COHERENCE DECAY

In this section we explore the effect of disorder on the coherence time of our topologically protected systems. Specifically we consider random (real) detuning of the qubits with respect to the cavity modes. This amounts to add a diagonal

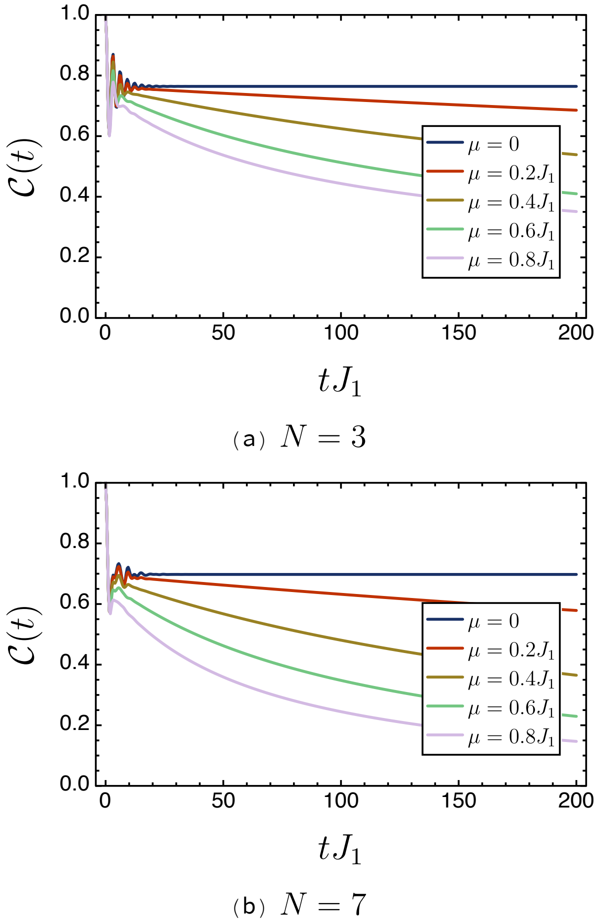


FIG. 11. (Color online) The effect of noise on the coherence in the non-Hermitian SSH model with (a) $N = 3$ and (b) $N = 7$ respectively. We take the $W = 1$ phase, where $J_1 = 1$, $J_2 = 1.8$, $\Gamma = 0.5$, and the noise rate μ is taken between 0 and $0.8J_1$. The final result is averaged over 1000 runs of randomly generated systems with the respective noise rates.

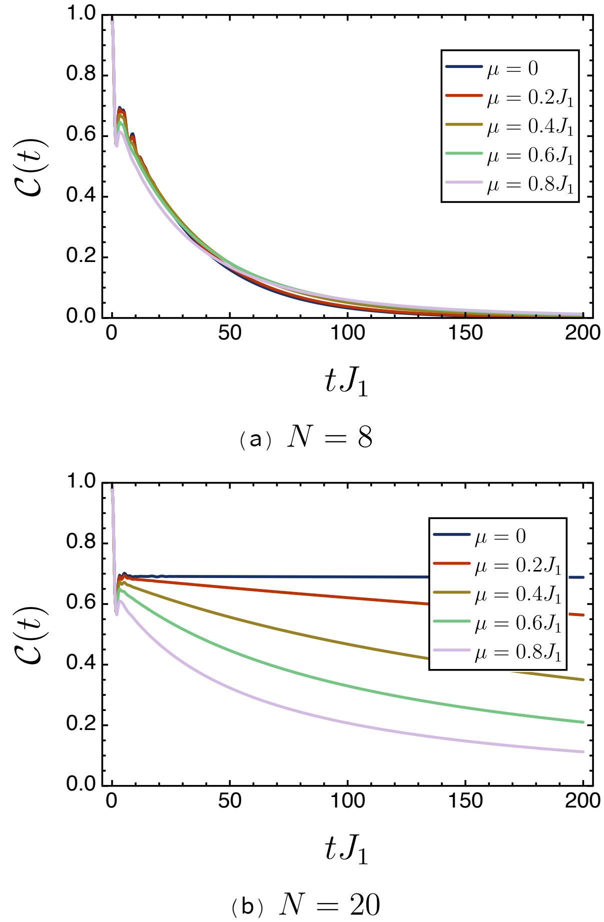


FIG. 12. (Color online) The effect of noise on the coherence in the non-Hermitian SSH model with (a) $N = 8$ and (b) $N = 20$ respectively. We take the $W = 1$ phase, where $J_1 = 1$, $J_2 = 1.8$, $\Gamma = 0.5$, and the noise rate μ is taken between 0 and $0.8J_1$. The final result is averaged over 1000 runs of randomly generated systems with the respective noise rates.

term to our one-particle effective "Hamiltonians" with on-site "chemical potentials" μ_i where μ_i are i.i.d. random variables with zero mean and uniform, distribution in $[-\mu, \mu]$. We compute the corresponding coherence averaging over many (1000 in numerical simulations) realization.

A. Noise Effect on the Non-Hermitian SSH Model

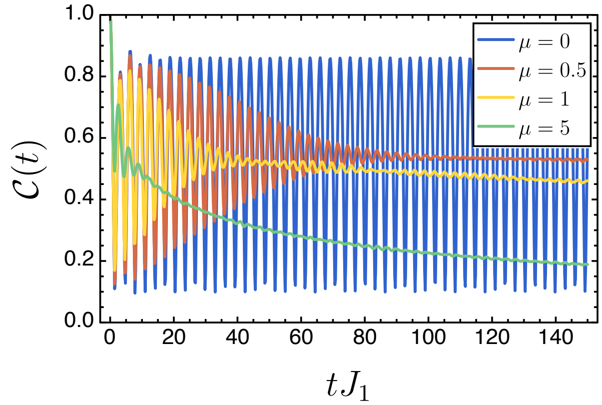
We first consider the topological model of Eq. (23). For odd system size N, the topologically protected $W = 1$ phase has a true dark state, and the coherence of the qubit saturates to a finite value (Fig. 5). We observe that the system is quite robust against disorder (see Fig. 11). Even for a noise strength μ comparable to the tunneling strength J_1, the qubit's coherence remains significant over a long period of time. In addition, a shorter chain of cavities better protects the system against noise.

For even system size N, the imaginary part of the dark state in the $W = 1$ phase is not exactly 0, but for a large enough N, the qubit's coherence still saturates to a finite value for times in a an exponentially large (in N) window. (Fig. 7) We observe that for $N = 8$ (Fig. 12 (a)), where the coherence of the clean system itself decays to 0, noise does not change the time evolution much. On the other hand, for $N = 20$ (Fig. 12 (b)), where the coherence saturates, the effect of noise is similar to odd N, and again, when the chain of cavities gets longer, the disruptive effect of noise gets more pronounced.

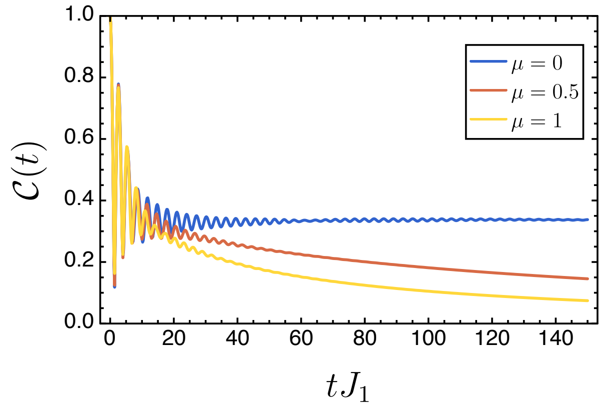
B. Noise Effect on the Three-Site Unit Cell System

Here we consider the topological model of Eq. (38) where we set $\epsilon_1 = \epsilon_2 = 0$. For this model (Fig. 8) there are three distinct topological phases, $W = 0$, $W = 1$, and $W = 2$, and the latter two protect the qubit's coherence from decaying.

For the $W = 1$ phase (Fig. 13 (b)), the effect of noise is similar to that of the $W = 1$ phase of the non-Hermitian SSH



(a)



(b)

FIG. 13. (Color online) The effect of noise on the coherence in the three-site unit cell model with $N = 8$. (a) Phase $W = 2$ with $J_1 = 1$, $J_2 = 0.3$, $J_3 = 2$, $J = 0.7$, and $\Gamma = 0.5$. (b) Phase $W = 1$ with $J_1 = 1$, $J_2 = 0.3$, $J_3 = 0.7$, $J = 0.7$, and $\Gamma = 0.5$. The noise rate μ is taken to be 0, $0.5J_1$ and J_1. The final result is averaged over 1000 runs of randomly generated systems with the respective noise rates.

model. The coherence of the qubit no longer saturates to a finite value, but decays to 0. We again note that the topologically protected system is quite robust against the introduction of noise. A noise strength comparable to the first tunneling rate ($\mu = J_1$) does not decrease the coherence much even over a long time.

For the $W = 2$ phase (Fig. 13 (a)), adding detuning noise vastly alters the oscillatory behavior of the clean system. In this case the time evolution of the coherence under noise resembles that of the $W = 1$ case, in other words the coherence saturates to a finite value. Yet larger noise strength seems to be able to eventually drive the system to a $W = 0$-like phase where the coherence decays to zero (see Fig. 13 (a), $\mu = 5J_1$). Further investigations are needed to clarify the nature of this noise-induced, dissipative, topological phase transition [24].

IX. CONCLUSIONS

Non-Hermitian topological phases in a finite system permit the construction of states whose decay time is either infinite or exponentially large in the system size. This feature is extremely appealing from the point of view of creating long-lived quantum bits. In this work we have shown that networks of qubits interacting with lossy cavities may be configured to possess non-trivial topological structure. For networks with a simple linear geometry, we have found that localization and long-livedness of the topological edge modes both concur to increase dramatically the coherence of a qubit sitting at the end of the chain. Specifically, a non-zero topological winding number W results in an exponentially long lived qubit. Although at finite size the exact number of edge modes is a complicated function of W and N, there are always W edge modes localized at one end of the chain. For $W = 2$ we find that the long-time dissipative, Lindbladian evolution becomes approximately unitary, and the coherence of the qubit displays long-lived Rabi-oscillations. In general, such long-lived, topological edge modes, are not legitimate quantum states, but rather they are off-diagonal elements of quantum a states or, *coherences*. The possibility of using such long-lived coherences for quantum computation is an interesting and challenging task for future studies.

ACKNOWLEDGMENTS

This work was partially supported by the ARO MURI grant W911NF-11-1-0268 the DOE Grant Number ER46240 and the ERC Advanced Grant program (H.S.). L.C.V. would like to thank Mark Rudner for useful correspondence.

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[20] It turns out that this model is topological also according to the (real-gap) classification proposed in [3]. More precisely the shifted matrix $\tilde{\mathsf{H}}(k)$ has exactly the same (real-gap) classification [3].
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[24] L. Campos Venuti, M. Zhengzhi, and S. Haas, In preparation.
[25] The meaning of $\tau_{\max}$ is that for times $t \gtrsim \tau_{\max}$ the coherence is guaranteed to be essentially zero for any initial state. But we may have lost interest in $\mathcal{C}$ long before.
[26] This is not a serious limitation as the set of non-diagonalizable $\tilde{\mathcal{L}}$ has measure zero in the space of parameters, $J_i, \Gamma_i, \ldots$.
[27] T. Kato, *Perturbation Theory for Linear Operators* (Springer, 1995).

Appendix A: "Single impurity" case

Through shift and rescaling $\tilde{\mathcal{L}} = iJ\mathsf{H}' - (\Gamma/2)\mathbb{1}$ we are led to consider the following matrix

$$\mathsf{H}' = \begin{pmatrix} -ia & \beta & 0 & \cdots & 0 \\ \beta & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A1}$$

We write the eigenvalues as $2\cos(k)$. It can be shown that k satisfies the following equation (both for N even and odd)

$$[2\cos(k) + ia]\sin(kN) - \beta^2 \sin(k(N-1)) = 0, \tag{A2}$$

and one can restrict oneself to $0 < \mathrm{Re}(k) < \pi$. In order to look for localized states we look for a complex root of Eq. ((A2)). Hence we set $k = x + iy$. Plugging this in the above and forgetting terms $e^{-N|y|}$ we obtain

$$(2\cos(k) + ia) = \beta^2 \frac{\sin(k(N-2))}{\sin(k(N-1))} \simeq \beta^2 \begin{cases} e^{ix}e^{-y} & y > 0 \\ e^{-ix}e^{y} & y < 0 \end{cases} \tag{A3}$$

We also set $ik = q$.

Case $y > 0$. The equation is

$$2\cosh(q) + ia = \beta^2 e^q$$

setting $z = e^q$ one finds

$$q = \ln\left((-i)\frac{a \pm \sqrt{a^2 + 4(1-\beta^2)}}{2(1-\beta^2)}\right)$$

and the corresponding eigenvalues

$$\lambda = -i\frac{\beta^2}{2(1-\beta^2)}\left(a \pm \sqrt{a^2 + 4(1-\beta^2)}\right) - ia. \tag{A4}$$

We need to make sure that $y = -\mathrm{Re}(q) > 0$. From this we obtain $\mathrm{Re}(\ln(z)) = \ln|z| < 0$ or $|z| < 1$.

Case $y < 0$. Now the equation is

$$2\cosh(q) + ia = \beta^2 e^{-q}$$

setting $z = e^{-q}$ one finds the same equation as for $y > 0$. This means that the eigenvalues have the same from (A4), but now $y < 0$ implies $|z| > 1$.

Going back to the eigenvalues of $\tilde{\mathcal{L}} = iJ\mathsf{H}' - (\Gamma/2)\mathbb{1}$, remembering $a = \Gamma/(2J)$ and $\beta = \kappa/J$ we get finally

$$\lambda_{\pm} = -\frac{4\kappa^2}{\Gamma \pm \sqrt{16(J^2 - \kappa^2) + \Gamma^2}} + O\left(e^{-N|y|}\right), \tag{A5}$$

as shown in the main text.

Appendix B: A note on timescales

Here we would like to define a time-scale associated to the coherence decay. This time scale should measure the time after which the coherence has degraded to an unacceptable value. Several definition of such (de-)coherence time are possible. For example one may take the smallest τ such that $\mathcal{C}(\tau) = \mathcal{C}(0) - \epsilon$. According to Eq. (12) $\mathcal{C}(t)$ has the form $\mathcal{C}(t) = \left|\sum_j c_j e^{\lambda_j t}\right|$, where λ_j are (a subset of) Lindbladian eigenvalues satisfying $\mathrm{Re}(\lambda_j) \le 0$. Let us say that one is interested in very small ϵ. In this limit the coherence time τ becomes proportional to ϵ. A meaningful definition then would be $\tau_{\mathrm{lin}} = \epsilon / \mathrm{Re}\left(-\sum_j c_j \lambda_j\right)$ (the name stemming from the fact that $\mathcal{C}(t)$ is approximately linear for $t \lesssim \tau_{\mathrm{lin}}$). A more conservative definition is given by the shortest time-scale associated with the set $\{\mathrm{Re}(-\lambda_j)\}$ i.e. the timescale defined as $\tau_{\min}^{-1} = \max_j \mathrm{Re}(-\lambda_j)$. If $\tau_{\min}$ is large one is *guaranteed* that the coherence will be close to maximal for all $0 \le t \lesssim \tau_{\min}$ for *any initial state*. This is a very pleasant feature which makes $\tau_{\min}$ quite attractive. Let us also define the the *slowest* decay time of $\mathcal{C}(t)$ by $\tau_{\max}^{-1} = \min_j \mathrm{Re}(-\lambda_j)$. Clearly $\tau_{\max}$ can be much larger than any meaningful definition of coherence time [25]. Obviously all these time-scales agree if the coherence decays as a single exponential. Quite surprisingly in all the situation we considered in the text, we verified that indeed $\mathcal{C}(t)$ can be well approximated with a single exponential over a wide range of J/Γ (Γ dissipation scale and J coherent energy scale, see main text). Some arguments why this is so will be given in the next section. In all the cases considered the coherence timescale $\tau_{\rm coh}$ defined in the main text coincides with what is commonly called Purcell rate in the cavity QED community. Through topological protection we are able to exponentially increase the Purcell rate.

Appendix C: Conditions to optimize the coherence decay

In the following we will identify *sufficient* conditions for the requirements i') and ii'). For simplicity we assume that $\tilde{\mathcal{L}}$ can be diagonalized [26] with spectral resolution $\tilde{\mathcal{L}} = \sum_j \lambda_j P_j$. We start analyzing the following consequence of i'):

Fact 1. Assume i'), i.e. $\tilde{\mathcal{L}}|1\rangle\!\rangle = \lambda_1|1\rangle\!\rangle + \epsilon|e\rangle\!\rangle$ with $\|e\| = O(1)$. Then, up to an error ϵ, the evolution of the coherence is governed by a single exponential, in particular

$$\mathcal{C}(t) = \left|e^{t\lambda_1}\right| + O(\epsilon). \tag{C1}$$

Proof. We start with the identity

$$e^{t\tilde{\mathcal{L}}}|1\rangle\!\rangle = e^{t\lambda_1}|1\rangle\!\rangle + \epsilon\frac{e^{t\tilde{\mathcal{L}}} - e^{t\lambda_1}}{\tilde{\mathcal{L}} - \lambda_1}|e\rangle\!\rangle.$$

We then obtain $\langle\!\langle 1|e^{t\tilde{\mathcal{L}}}1\rangle\!\rangle = e^{t\lambda_1} + \epsilon\eta$, and taking the modulus $\left|\langle\!\langle 1|e^{t\tilde{\mathcal{L}}}1\rangle\!\rangle\right| = \left|e^{t\lambda_1}\right| + \epsilon\eta' + O(\epsilon^2)$ with $|\eta'| \le \left|\langle\!\langle 1|\frac{e^{t\tilde{\mathcal{L}}}-e^{t\lambda_1}}{\tilde{\mathcal{L}}-\lambda_1}e\rangle\!\rangle\right| = O(1)$. Moreover, assuming that $\tilde{\mathcal{L}}$ can be diagonalized, $|\eta'|$ does not blows up with t, rather

$$\begin{aligned}|\eta'| &\le \|e\|\left\|(\tilde{\mathcal{L}}-\lambda_1)^{-1}\right\|\left(\sum_j\left|e^{t\lambda_j}\right|\|P_j\| + \left|e^{t\lambda_1}\right|\right)\\ &= (c+1)\,\|e\|\left\|(\tilde{\mathcal{L}}-\lambda_1)^{-1}\right\|,\end{aligned} \tag{C2}$$

having set $c = \sum_j \|P_j\|$, since $\mathrm{Re}(\lambda_j) \le 0$ and $t \ge 0$.

The same conclusion holds, not surprisingly, using a slightly relaxed assumption $P_1 = |1\rangle\!\rangle\langle\!\langle 1| + \epsilon X$. Using the normalization of the projectors $P_iP_j = \delta_{i,j}P_j$ one obtains $\langle\!\langle 1|P_j|1\rangle\!\rangle = \mathrm{Tr}(P_j|1\rangle\!\rangle\langle\!\langle 1|) = \mathrm{Tr}[P_j(P_1 - \epsilon X)]$. The latter expression equals 1 for $j=1$ and $O(\epsilon)$ otherwise. Hence

$$\langle\!\langle 1|e^{t\tilde{\mathcal{L}}}1\rangle\!\rangle = \sum_j e^{t\lambda_j}\,\mathrm{Tr}(P_j|1\rangle\!\rangle\langle\!\langle 1|) \tag{C3}$$

$$= e^{t\lambda_1} - \epsilon\Big[\mathrm{Tr}(P_1X) + \sum_{j\neq 1} e^{t\lambda_j}\,\mathrm{Tr}(P_jX)\Big], \tag{C4}$$

and the result holds with $|\eta'| \le \sum_j|\mathrm{Tr}(P_jX)|$. As can be seen from the absence of the resolvent term, in this case the error can be made significantly smaller.

Note that if $\mathrm{Re}(\lambda) = 0$ the leading term of the coherence does not decay. This fact will be important when discussing dark or quasi-dark states in topological models.

To gain further insight we analyze the weak and strong dissipative limits.

Fact 2 *(strong dissipative limit).* Assume a linear geometry and a hopping to dissipation ratio $|J_1/\Gamma_2| = \epsilon$ sufficiently small. Then conditions i') and ii') are satisfied and in particular $\mathcal{C}(t) = e^{-t/\tau_{\rm coh}} + O(\epsilon)$ with $\tau_{\rm coh}^{-1} = 2J_1^2/\Gamma_2 + J_1O(\epsilon^2)$.

Proof. We consider the off-diagonal terms of Eq. (13) as a perturbation. The spectrum of the unperturbed system is $\{0, -\Gamma_i/2, i = 2, \ldots, N\}$, and the zero eigenvalue has eigenprojector $|1\rangle\!\rangle\langle\!\langle 1|$. Using (non-Hermitian) perturbation theory, the first correction to the zero eigenvalue occurs at second order and is given by $\lambda_1^{(2)} = -2J_1^2/\Gamma_2$. The corresponding eigenprojector is given by $P_1 = |1\rangle\!\rangle\langle\!\langle 1| + O(\epsilon)$ so that we are in the condition for fact 1 and the result follows. Note that, since eigenvalues are continuous in their parameters, as long as there is no level crossing, λ_1 is the eigenvalue with real part closest to zero. In other words the coherence time is given by the *slowest* time-scale of $\tilde{\mathcal{L}}$.

We define $\mathsf{H} = \mathsf{H}_0 + \mathsf{D}$ where H_0 (D) is the Hermitian (anti-Hermitian) part of H. Note that the matrix D is diagonal in the "position" basis $|j\rangle\!\rangle$. Since H_0 is Hermitian it can be written as $\mathsf{H}_0 = \sum_k \epsilon_k^{(0)}|k\rangle\!\rangle\langle\!\langle k|$, where $|k\rangle\!\rangle$ are the unperturbed eigenvectors. Up to first order, the eigenvalues of $\tilde{\mathcal{L}}$ become

$$\lambda_k = -i\epsilon_k^{(0)} - i\langle\!\langle k|\mathsf{D}|k\rangle\!\rangle \tag{C5}$$

$$= -i\epsilon_k^{(0)} - \frac{1}{2}\sum_{j=2}^{N}\Gamma_j\left|\langle\!\langle k|j\rangle\!\rangle\right|^2. \tag{C6}$$

In the isotropic case where all the cavities are equal $\Gamma_j = \Gamma$ and the above becomes

$$\lambda_k = -i\epsilon_k^{(0)} - \frac{\Gamma}{2}\left(1 - |\langle\!\langle k|1\rangle\!\rangle|^2\right).$$

Moreover, assume now that the Hamiltonian H_0 has a state localized at the first site: $\exists k_0 : |k_0\rangle\!\rangle \approx |1\rangle\!\rangle$. This means that $|1\rangle\!\rangle\langle\!\langle 1|$ is an approximate eigenprojector of H_0: $P_{k_0} \equiv |k_0\rangle\!\rangle\langle\!\langle k_0| = |1\rangle\!\rangle\langle\!\langle 1| + \epsilon Y$ with a small ϵ. Surprisingly $|1\rangle\!\rangle\langle\!\langle 1|$ is also an eigenprojector of H up to the same order. In fact the first correction to the eigenprojectors of H is $P^{(1)} = -P_{k_0}\mathsf{D}S - S\mathsf{D}P_{k_0}$ where S is the reduce resolvent [27]. Plugging in $\mathsf{D} = -i(\Gamma/2)\left(\mathbb{1} - |1\rangle\!\rangle\langle\!\langle 1|\right)$ we obtain $P^{(1)} = i(\Gamma/2)\epsilon\left[Y\left(\left(\mathbb{1} - |1\rangle\!\rangle\langle\!\langle 1|\right)S + S\left(\mathbb{1} - |1\rangle\!\rangle\langle\!\langle 1|\right)Y\right)\right]$. Then $|1\rangle\!\rangle\langle\!\langle 1|$ is a projector of $\tilde{\mathcal{L}}$ up to an error $O(\Gamma\epsilon)$. In other words we have the following

Fact 3 *(weak dissipative limit).* Assume that the Hamiltonian H_0 has a state localized at the first site: $P_{k_0} = |1\rangle\!\rangle\langle\!\langle 1| + \epsilon' Y$ with a small ϵ'. For small Γ both hypothesis i') and ii') hold. Moreover Fact 1 holds with $\epsilon = \Gamma\epsilon'$: $\mathcal{C}(t) = e^{-t/\tau_{\rm coh}} + O(\epsilon'\Gamma)$. The coherence time is given in this case by $\tau_{\rm coh}^{-1} = (\Gamma/2)(1 - |\langle\!\langle k_0|1\rangle\!\rangle|^2) + O(\Gamma^2)$.