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Masahiro Takeoka, Kaushik P. Seshadreesan, Chenglong You, Shuro Izumi, and Jonathan P. Dowling

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Fundamental precision limit of a Mach-Zehnder interferometric sensor when one of the inputs is the vacuum

Masahiro Takeoka,[1] Kaushik P. Seshadreesan,[2,3,1] Chenglong You,[3] Shuro Izumi,[1,4] and Jonathan P. Dowling[3]

[1] *National Institute of Information and Communications Technology, Koganei, Tokyo 184-8795, Japan*
[2] *Max Planck Institute for the Science of Light, 91058 Erlangen, Germany*
[3] *Hearne Institute for Theoretical Physics and Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA*
[4] *Sophia University, 7-1 Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan*

In the lore of quantum metrology, one often hears (or reads) the following no-go theorem: If you put vacuum into one input port of a balanced Mach-Zehnder Interferometer, then no matter what you put into the other input port, and no matter what your detection scheme, the sensitivity can never be better than the shotnoise limit (SNL). Often the proof of this theorem is cited to be in Ref. [C. Caves, Phys. Rev. D 23, 1693 (1981)], but upon further inspection, no such claim is made there. Quantum-Fisher-information-based arguments suggestive of this no-go theorem appear elsewhere in the literature, but it is not stated in its full generality. Here we thoroughly explore this no-go theorem and give a rigorous statement: the no-go theorem holds whenever the unknown phase shift is split between both of the arms of the interferometer, but remarkably does not hold when only one arm has the unknown phase shift. In the latter scenario, we provide an explicit measurement strategy that beats the SNL. We also point out that these two scenarios are physically different and correspond to different types of sensing applications.

I. INTRODUCTION

In the field of quantum metrology [1–4], a Mach-Zehnder interferometer (MZI) is a tried-and-true workhorses which has the additional advantage that any result obtained for it also applies to a Michelson interferometer (MI) and hence has a potential application to gravitational wave detection. In most current implementations of gravitational wave detectors, the MI is fed with a strong coherent state of light in one input port and vacuum in the other (Fig. 1). It was in this context that Caves in 1981 [5] showed that such a design would always only ever achieve the shotnoise limit (SNL). Then he showed if you put squeezed vacuum into the unused port, you could beat the SNL. Several implementations of this squeezed vacuum scheme have already been demonstrated in the GEO 600 gravitational detector, and plans are underway to utilize this approach in the LIGO and VIRGO detectors in the future [6, 7].

It then appeared, that in the lore of quantum metrology, this result was extended — without proof — to the following no-go theorem: *If you put quantum vacuum into one input port of a balanced MZI, then no matter what quantum state of light you put into the other input port, and no matter what your detection scheme, the sensitivity can never be better than the SNL.* Often the proof of this theorem is cited to be the original 1981 paper by Caves [5], but upon further inspection, no such general claim is made there. A quantum-Fisher-information-based proof of this no-go theorem appeared in Pezzé and Smerzi [8], Lang and Caves [9], and later in Liu et al., [10], but is not explored in full generality.

In this paper, we give the general statement on this issue. The statement proved here is the following: *If the unknown-phase-shifts are in both the arms of the MZI, then the no-go theorem holds no matter whether the MZI is balanced or not. However, if the unknown phase shift is in only one of the two arms, then the no-go theorem does not hold.* The two models for the unknown phase shift unitary operation in the MZI are known to yield different values for the QFI in estimating the phase difference between the two arms [11, 12]. This discrepancy has been thought of as a flaw in the interpretation of the QFI [11], or being related to assumptions made about the input states and the measurements [12]. In contrast, here we point out that the two phase shift unitaries correspond to physically different types of sensors, and that their choice should depend on the concrete application scenario. The model where the unknown phase shift is in both the arms corresponds intrinsically to a two-parameter estimation problem. In this case, we prove that the no-go theorem holds, independently of whether the MZI is balanced or not, by carefully considering the phase-sum parameter (often regarded as the "global phase") along with the phase difference. On the other hand, in the case where the unknown phase shift is in only one arm, we prove that the no-go theorem does not hold by constructing an explicit scheme with a probe and detection that can beat the SNL corresponding to the combined total number of photons used at the input and the detection.

We also point out the pitfalls of using only the quantum Fisher information (QFI), or the closely related quantum Cramér-Rao (QCRB) bound [13], to make claims of a quantum metrological advantage, without explicitly providing a detection scheme that would actually achieve that advantage [11]. The issue is that the optimal positive operator-valued measure (POVM) that achieves the QFI may be difficult to implement or contain hidden resources, such as a strong local oscillator, that are not fairly counted as far as a quantum advantage is concerned [11].

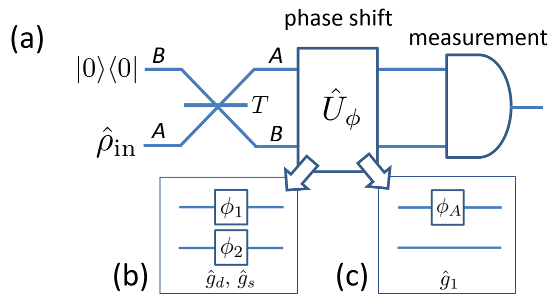

FIG. 1. (a) Mach-Zehnder interferometer phase estimation, and the two different phase shift models: phase shift(s) are applied in (b) two arms, or (c) one arm of the interferometer.

II. MODEL AND PREVIOUS WORK

A schematic of the Mach-Zehnder (MZ) interferometer based sensing setup we consider is illustrated in Fig. 1 (a). Two input modes A and B are interfered via a beam splitter with transmittance T, and then put into the phase shift unitary operation $\hat{U}_\phi$ followed by some measurement. In addition to this standard setting, we restrict one of the input states to always be the quantum vacuum state, whereas the other input can be an arbitrary quantum state (possibly mixed).

A similar setup was considered by Lang and Caves [9] (see also [8, 10]), with inputs in a tensor product of an arbitrary pure state $|\chi\rangle$ and a coherent state $|\alpha\rangle$, and a beam splitter of transmittance $T = 1/2$. They considered the phase shift unitary operator $\hat{U}_\phi = e^{i\hat{g}_s\phi_s}e^{i\hat{g}_d\phi_d}$ as shown in Fig. 1 (b), where ϕ_s and ϕ_d are the phase sum and difference of the two modes, respectively, $\hat{g}_s = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2$, $\hat{g}_d = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$. These two phase shift parameters reflect the unknown phase shifts in the two arms of the MZI, ϕ_1 and ϕ_2, as $\phi_s = \phi_1 + \phi_2$, $\phi_d = \phi_1 - \phi_2$ (see Fig. 1(b)). $\hat{a}^\dagger$ ($\hat{b}^\dagger$) and $\hat{a}$ ($\hat{b}$) are creation and annihilation operators in mode A (B), respectively.

Then the authors showed that for a coherent state input with $\alpha = 0$, i.e. for the vacuum input, the quantum Fisher information (QFI) for the phase difference turns out to be the average photon number of the input:

$$F_Q(|\chi\rangle, \hat{g}_d) = \langle\chi|\hat{n}|\chi\rangle = \bar{n}_\chi, \qquad (1)$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$. This result suggests that the precision of the phase sensing is shotnoise limited when one of the input ports contains only vacuum (and the other mode contains any pure state), since the QCRB is $\Delta^2\phi \geq 1/F_Q$.

The above result still leaves open questions such as, "Does the no-go theorem hold when the interferometer is not balanced?", and "Does it also hold when the phase shift unitary operator is chosen differently?"

No-go theorem extended: Preliminary analysis

Firstly, for the phase shift unitary operator $\hat{g}_d$, when T deviates from $1/2$, the QCRB already appears to beat the SNL. Keeping T as a free parameter, and using the fact that the QFI of a pure state in estimating a phase shift generated by a generator $\hat{g}$ is given by $4\left(\langle\hat{g}^2\rangle - \langle\hat{g}\rangle^2\right)$, we arrive at

$$F_Q(|\chi\rangle, \hat{g}_d, T) = \{1 - (1-2T)^2\}\bar{n}_\chi + (1-2T)^2 V_\chi. \quad (2)$$

(See Appendix A for the derivation.) This beats the SNL for any non-50/50 beam splitter quite spectacularly. For example, with $T \to 0$, the QCRB approaches $\Delta^2\phi = 1/V_x < 1/\bar{n}_\chi$ for some inputs such as squeezed vacuum [14].

Secondly, as pointed out and rigorously discussed in Ref. [11], a different choice of the phase shift unitary can give a different value for the QFI. For example, in lieu of the phase shift operator $\hat{g}_d$, one can instead choose $\hat{U}_\phi = e^{i\hat{g}_1\phi_A}$, where $\hat{g}_1 = \hat{a}^\dagger\hat{a}$, such that phase shift is generated only in one arm. The QFI for the phase shift unitary operator $\hat{g}_1$ is found to be

$$F_Q(|\chi\rangle, \hat{g}_1) = \bar{n}_\chi + V_\chi, \qquad (3)$$

where $V_\chi = \langle\chi|\hat{n}^2|\chi\rangle - \langle\chi|\hat{n}|\chi\rangle^2$ is the photon number variance of $|\chi\rangle$ (see Appendix A for the derivation). This is obviously different from Eq. (1), and again implies a sub-SNL result, since $V_\chi > \bar{n}_\chi$ is possible for some inputs, as mentioned above.

These results, extrapolated from Ref. [9], are thus perplexing [15], since seemingly both Eqs. (2) and (3) suggest the possibility of sub-SNL precision phase sensing even with vacuum input into one of the input ports.

III. PHASE SHIFT IN BOTH ARMS VS. IN ONE ARM OF THE MZI

We point out that the two phase-shift unitary operators (Figs. 1(b) and (c)) have different physical meanings and their choice should depend on what type of application scenario one has in mind. For the gravitational wave detection application, $\hat{g}_d$ and $\hat{g}_s$ should be chosen since the two arms of the MZI both have unknown phase shifts induced by the gravitational waves (Fig. 1(b)). Also some commonly used sensing devices, such as a differential interference contrast microscope [16], should be modeled in the same way (see also its quantum version [17]).

On the other hand, the most primitive use of the Mach-Zehnder interferometer is to put a sample in one of the two arms to measure the corresponding phase shift. This configuration is also widely used as a simple and low-cost technology to measure the sample's density distribution, pressure, temperature, etc. This type of sensor should be modeled by $\hat{g}_1$ (Fig. 1(c)). Since these two models are physically different, they may lead to different outcomes in our problem; the MZI with vacuum in one input port.

That is, they could have different fundamental precision limits with vacuum in one input port. We now rigorously analyze each model in the context of the no-go theorem.

A. Complete proof of no-go theorem for phase shift in both arms

The MZI sensing with the $\hat{g}_s$-$\hat{g}_d$ model in its full generality is a two-parameter estimation problem since there are two unknown parameters, ϕ_s and ϕ_d, in the system. Although only the phase difference ϕ_d is of interest, this two-parameter model allows us to explicitly include the fact that one does not know ϕ_s in prior. This generally limits the precision limit of sensing ϕ_d especially when ϕ_d and ϕ_s are correlated. Therefore, a two-by-two quantum Fisher information matrix (QFIM) is considered. The problem in Eq. (2) is in fact due to the ignorance of the phase sum ϕ_s [18]. In multi-parameter estimation, the QCRB is given by $\Sigma \geq \mathcal{F}_Q^{-1}$, where Σ is the covariance matrix of the estimator including both ϕ_s and ϕ_d and $\mathcal{F}_Q$ is the two-by-two QFIM:

$$\mathcal{F}_Q = \begin{bmatrix} F_{dd} & F_{sd} \\ F_{ds} & F_{ss} \end{bmatrix}, \tag{4}$$

where s and d correspond to ϕ_s and ϕ_d. The first diagonal element of $\mathcal{F}_Q^{-1}$ corresponds to the estimation limit of ϕ_d, which is explicitly given by

$$\frac{F_{ss}}{F_{ss}F_{dd} - F_{sd}F_{ds}}. \tag{5}$$

For an arbitrary mixed quantum state, the QFIM is in general not easy to calculate. However, the optimal input state that maximizes $\mathcal{F}_Q$ is always given by a pure-state input. This is the consequence of the convexity of the QFIM: for $\hat{\rho}_\phi = p\hat{\sigma}_\phi + (1-p)\hat{\tau}_\phi$,

$$\mathcal{F}_Q(\hat{\rho}_\phi) \leq p\mathcal{F}_Q(\hat{\sigma}_\phi) + (1-p)\mathcal{F}_Q(\hat{\tau}_\phi), \tag{6}$$

holds. This can be proved by using the monotonicity of the QFIM under the completely positive trace preserving (CPTP) map [19, 20] and extending the proof of the convexity for the QFI [21] (for completeness, we provide the proof in Appendix B). The statement basically says that a statistical mixture of the input states will never increase the QFIM and thus implies that the QFIM is maximized with a pure state input. The optimal pure state for the QFIM is also optimal for the multi-parameter QCRB since the QFIM is a positive matrix and for positive matrices A and B, $B^{-1} \geq A^{-1}$ holds if and only if $A \geq B$.

Therefore, by considering a pure input state $|\chi\rangle$, the elements of the QFIM are given by

$$F_{ij} = 4\left(\langle \hat{g}_i\hat{g}_j\rangle - \langle \hat{g}_i\rangle\langle \hat{g}_j\rangle\right), \tag{7}$$

where i, j takes s and d. We can calculate F_{dd}, F_{ss}, F_{ds}, and F_{sd} explicitly (see Appendix A for the derivation. Note that F_{dd} corresponds to Eq. (2)) and then inserting these into (5), we get

$$\Delta^2\phi_d \geq \frac{1}{4T(1-T)\bar{n}_\chi}, \tag{8}$$

where the minimum of the right hand side is obtained with $T = 1/2$ as $1/\bar{n}_\chi$, which is the SNL, as it should be. That is, no matter how highly nonclassical the input state $\hat{\rho}_{\text{in}}$ is, and no matter what POVM you deploy, the SNL cannot be surpassed for $\hat{g}_d$ so long as the other input to the interferometer is the vacuum state. Thus, this result establishes the no-go theorem in its most general form, which includes the beam splitter transmissivity as a free parameter.

B. Phase shift in one arm ($\hat{g}_1$)

The $\hat{g}_1$ model is a single-parameter estimation problem. Thus, (3) is directly applied to the QCRB, which suggests sub-SNL sensitivity with input states of high V_χ, i.e., states with high photon number fluctuation, e.g., squeezed vacuum. Then, as mentioned in Sec. II, the QFI-only approach may have the pitfall that the optimal POVM attaining the QCRB might contain huge amount of hidden resources as pointed out by Jarzyna and Demkowicz-Dobrzański [11]. In other words, one might fool oneself into thinking, via the QFI-only approach, that there is some quantum metrological advantage, where none actually exists.

There are two remedies. The first is to rule out any external resource that might give some phase information to the measurement device in implementing the optimal POVM. Such a “rule-out” protocol was introduced by Jarzyna and Demkowicz-Dobrzański [11] where the issue is resolved by introducing the phase-averaging of the two-mode input state via a common phase shift. The QFI of the phase-averaged input gives the proper phase-sensing limit without any external phase reference.

The second remedy, the one we recommend, is that if one wishes to claim a quantum metrological advantage from a QFI-only calculation, then a detection scheme that actually hits the related QCRB must be provided, so that all resources hidden in the associated POVM may be laid bare for all to see. This allows one to fairly count all the resources used in the interferometer. (For example in Ref. [14], the QFI calculation is backed up by providing a detection scheme, the parity operator, hitting the QCRB).

Here we apply these two remedies separately. First, we employ the phase-averaging approach [11] to eliminate any hidden resource in the POVM. We briefly sketch the calculation in the following and describes the details in Supplementary Material 3. Consider the input state of $\hat{\rho}_{\text{in}} \otimes |0\rangle\langle 0|$ where $\hat{\rho}_{\text{in}} = \sum_{n,m}^{\infty} c_{nm}|n\rangle\langle m|$ is an arbitrary state and $|n\rangle$ is the n-photon number basis. The phase-averaging operation drops off its non-diagonal terms. After the phase-averaging, the first beamsplitter

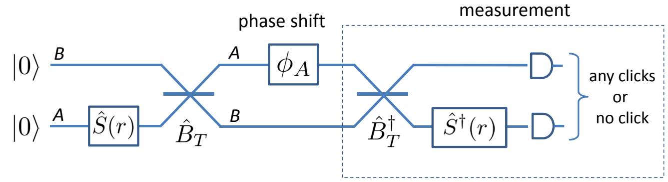

FIG. 2. Mach-Zehnder interferometer sensing with the input-phase sensitive measurement. $\hat{S}(r)$ is the squeezing operator with squeezing parameter r and $\hat{B}_T$ is the beam-splitting operator with transmissivity T.

and the ϕ phase shifting, the input is transformed to $\Psi^{\phi}_{\text{avg}} = \sum_{n=0}^{\infty} p_n |\psi_n(\phi)\rangle\langle\psi_n(\phi)|_{AB}$, where $|\psi_n(\phi)\rangle$ and $|\psi_{n'}(\phi)\rangle$ are orthogonal for $n \neq n'$.

By using the convexity of the QFI and the above orthogonality, we have $F_Q^{(1)}\left(\Psi^{\phi}_{\text{avg}}\right) = \sum_{n=0}^{\infty} p_n F_Q^{(1)}\left(|\psi_n(\phi)\rangle\right)$, and after some calculations, we see $F_Q^{(1)}\left(|\psi_n(\phi)\rangle\right) = 4nT\left(1-T\right)$, which is maximized at $T = 1/2$, and is equal to n, as it should be (see Appendix C). Consequently, the QFI for Ψ^{ϕ}_{ave} is given as

$$F_Q\left(\Psi^{\phi}_{\text{avg}}\right) = \sum_{n=0}^{\infty} n p_n = \bar{n}, \tag{9}$$

where $\bar{n}$ is the average photon number of $\hat{\rho}_{\text{in}}$, and thus we find that the phase sensitivity is lower bounded as $\Delta^2\phi_A \geq 1/\bar{n}$. That is, if the optimal POVM is not allowed to have external phase information, the estimation precision is limited by the shotnoise limit.

Secondly, we ask the question,"if one is allowed to use some additional resource at the measurement, is it possible to surpass the SNL with respect to the total number of resources used at the input and the detection?". We prove that the answer is affirmative by showing an example of the concrete measurement scheme (Fig. 2). The input state is a single-mode squeezed vacuum, which is generated from vacuum by applying the squeezing operation $\hat{S}(r)$, where r is the squeezing parameter. The measurement is a time-reversed process, that is, it consists of the complex-conjugate beam splitter $\hat{B}_T^\dagger$, anti-squeezing operation $\hat{S}^\dagger(r)$, and photon detectors that discriminate zero and non-zero photons, $\{|0\rangle\langle 0| \otimes |0\rangle\langle 0|, I - |0\rangle\langle 0| \otimes |0\rangle\langle 0|\}$. A similar mirror-image-like strategy has been considered in the state discrimination scenarios [22, 23] and the phase estimation with coherent state [24]. Since we need the phase information of the input state at the anti-squeezing process, this is a *phase-sensitive* measurement, and so the POVM has access to the input phase information. The input average photon number is given by $\bar{n} = \sinh^2 r$. Since the measurement device also uses the same amount of squeezing, the average photon number of the all resources are counted as $\bar{n}_{\text{tot}} = 2\bar{n}$.

The attainable precision limit (in the asymptotic limit, $m \to \infty$) is specified by calculating its CFI [13, 25]:

$$F(\phi_A) = \mathrm{E}\left[-\frac{d^2}{d\phi_A^2}\log p_{\phi_A}\right], \tag{10}$$

where $p_{\phi_A} = P(x|\phi_A)$ is the conditional probability of obtaining the measurement outcome x for given ϕ_A and $x = 0, 1$ represent the photon detection outcome $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$ and $I - |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, respectively.

$F(\phi_A)$ is calculated by the characteristic function approach (e.g. see Refs. [26, 29]), and the derived analytical expression of $F(\phi_A)$ is complicated (see Appendix D). Taking the limit of $\phi_A \to 0$, we get

$$F(\phi_A) = 2\bar{n}_{\text{tot}}T(1 + T + \bar{n}_{\text{tot}}T), \tag{11}$$

where we remind the reader that $\bar{n}_{\text{tot}} = 2\bar{n}$ is the *total* resource used for the input state and the detection process. Thus we get the (classical) CRB around $\phi_A = 0$ as

$$\Delta^2\phi_A \geq \frac{1}{2\bar{n}_{\text{tot}}T(1 + T + \bar{n}_{\text{tot}}T)}, \tag{12}$$

which surpasses the SNL of the total resources for any $T \neq 0$. This example shows how a QFI-only calculation could contain hidden resources in the unknown optimal POVM, that are unfairly not counted. Here, by taking all resources into account, we conclude that it is possible to beat the SNL for the $\hat{g}_1$ estimation if one uses additional energy and phase resources at the detection.

IV. CONCLUSION

In this paper, we revisited the ultimate limit of the MZI sensing precision when an input into one port is vacuum. We showed a full statement of the problem with a rigorous proof: the statement depends on the choice of the phase shift unitary operator, i.e. the physical setup of the sensing application.

First, if both arms of the MZI have different unknown phase shifts in the application and the input to one of the two ports is vacuum, then no matter what the input in the other port is, and no matter the detection scheme, one can never better the SNL in phase sensitivity. This statement holds even if the first beamsplitter of the MZI is non-50:50. The proof is based on the fact that it is intrinsically a two-parameter estimation problem. That is we cannot ignore the phase sum ϕ_s in the analysis though it is often treated as a "global phase" and ignored in real experiments. This type of sensing includes gravitational wave detection [6, 7], long-baseline interferometry [27], and differential interference contrast microscopy [16, 17], for example. In these applications, if one input is vacuum, our result rules out the possibility of doing something "quantum" at the detector (such as putting in a squeezer or doing photon addition or subtraction) to beat the SNL.

Second, if only one of the MZI arms has an unknown phase shift in the application, then the ultimate precision limit depends on the detector restriction. If one does not allow the detector to use any external phase reference and power resource, then the precision is limited by the SNL. However, if the detector is allowed to use such resources, then one can beat the SNL in terms of the total resource used at the input and detector. The explicit sensing scheme which uses squeezers for both input and detector is given. This type of sensing includes simple MZI devices measuring sample's density, pressure, temperature, etc, and also LIDAR-type sensing [28]. In these applications, only if nonclassical light is introduced into at least one input port, is there a hope to beat the SNL by doing something quantum at the detector, even if the other port is vacuum.

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Appendix A: Quantum Fisher information for the Mach-Zehnder interferometer phase sensing with a vacuum input

Here we derive Eqs. (3) and (2) in the main text. Consider $|\chi\rangle \otimes |0\rangle$ as an input to the MZ interferometer. For the calculation, it is useful to expand $|\chi\rangle$ in a coherent state basis:

$$|\chi\rangle = \int \mathrm{d}^2\alpha\, f(\alpha)|\alpha\rangle, \tag{A1}$$

where $|\alpha\rangle$ is a coherent state with complex quadrature amplitude α. Then the average photon number and the variance of the state are given by

$$\begin{aligned}\bar{n}_\chi &= \langle\chi|\hat{n}|\chi\rangle \\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\langle\alpha|\hat{n}|\beta\rangle \\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\alpha^*\beta\langle\alpha|\beta\rangle \\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\alpha^*\beta \\ &\quad\times \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-2\alpha^*\beta\right)\right],\end{aligned} \tag{A2}$$

and

$$\begin{aligned}V_\chi &= \langle\chi|\hat{n}^2|\chi\rangle - \bar{n}_\chi^2 \\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\left\{(\alpha^*\beta)^2+\alpha^*\beta\right\} \\ &\quad\times \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2-2\alpha^*\beta\right)\right] - \bar{n}_\chi^2,\end{aligned} \tag{A3}$$

where we use the fact that $\hat{n}^2 = \hat{a}^{\dagger\,2}\hat{a}^2 + \hat{a}^\dagger\hat{a}$.

The state after the beam splitter with transmittance T is given by

$$|\Phi\rangle_{AB} = \int \mathrm{d}^2\alpha\, f(\alpha)\left|\sqrt{T}\alpha\right\rangle_A\left|\sqrt{R}\alpha\right\rangle_B, \tag{A4}$$

where $R = 1-T$.

QFI with $\hat{g}_1$ [Eqs. (3)]

The quantum Fisher information (QFI) is calculated from

$$F_Q(|\chi\rangle, \hat{g}_1, T) = 4(\langle\Phi|\hat{g}_1^2|\Phi\rangle - \langle\Phi|\hat{g}_1|\Phi\rangle^2). \tag{A5}$$

We have

$$\begin{aligned}\langle\Phi|\hat{g}_1^2|\Phi\rangle &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\left\langle\sqrt{T}\alpha\right|_A\left\langle\sqrt{R}\alpha\right|_B\left(\hat{a}^{\dagger\,2}\hat{a}^2+\hat{a}^\dagger\hat{a}\right)\left|\sqrt{T}\beta\right\rangle_A\left|\sqrt{R}\beta\right\rangle_B \\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\left\{(T\alpha^*\beta)^2+T\alpha^*\beta\right\}\left\langle\sqrt{T}\alpha\middle|\sqrt{T}\beta\right\rangle\left\langle\sqrt{R}\alpha\middle|\sqrt{R}\beta\right\rangle \\ &= T^2\langle\chi|\hat{n}^2|\chi\rangle + T(1-T)\langle\chi|\hat{n}|\chi\rangle,\end{aligned} \tag{A6}$$

and

$$\begin{aligned}\langle\Phi|\hat{g}_1|\Phi\rangle^2 &= \left(\int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\left\langle\sqrt{T}\alpha\right|_A\left\langle\sqrt{R}\alpha\right|_B\left(\hat{a}^\dagger\hat{a}\right)\left|\sqrt{T}\beta\right\rangle_A\left|\sqrt{R}\beta\right\rangle_B\right)^2 \\ &= T^2\langle\chi|\hat{n}|\chi\rangle^2.\end{aligned} \tag{A7}$$

In total, we have

$$F_Q(|\chi\rangle, \hat{g}_1, T) = 4(\langle\Phi|\hat{g}_1^2|\Phi\rangle - \langle\Phi|\hat{g}_1|\Phi\rangle^2) = 4\left\{T^2 V_\chi + T(1-T)\bar{n}_\chi\right\}. \tag{A8}$$

For $T = 1/2$, it is $V_\chi + \bar{n}_\chi$ and thus we get Eq. (3).

QFIM for $\hat{g}_d$ and $\hat{g}_s$ [Eqs. (2)]

For pure states, the elements of the QFIM are given by

$$F_{ij} = 4\left(\langle\hat{g}_i\hat{g}_j\rangle - \langle\hat{g}_i\rangle\langle\hat{g}_j\rangle\right), \tag{A9}$$

where i, j takes s and d.

Recall that $\hat{g}_d = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2$ and $\hat{g}_s = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2$. Then we have

$$\begin{aligned}
4\langle\Phi|\hat{g}_d^2|\Phi\rangle &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta) \\
&\quad\times \left\langle\sqrt{T}\alpha\right|_A \left\langle\sqrt{R}\alpha\right|_B \left(\hat{a}^{\dagger\,2}\hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{b}^{\dagger\,2}\hat{b}^2 + \hat{b}^\dagger\hat{b} - 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\right) \left|\sqrt{T}\beta\right\rangle_A \left|\sqrt{R}\beta\right\rangle_B \\
&= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta) \\
&\quad\times \left\{(T\alpha^*\beta)^2 + T\alpha^*\beta + (R\alpha^*\beta)^2 + R\alpha^*\beta - 2RT(\alpha^*\beta)^2\right\} \left\langle\sqrt{T}\alpha\middle|\sqrt{T}\beta\right\rangle \left\langle\sqrt{R}\alpha\middle|\sqrt{R}\beta\right\rangle \\
&= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta) \left\{\alpha^*\beta + (T-R)^2(\alpha^*\beta)^2\right\} \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta\right)\right] \\
&= \langle\chi|\hat{n}|\chi\rangle + (1-2T)^2\left(\langle\chi|\hat{n}^2|\chi\rangle - \langle\chi|\hat{n}|\chi\rangle\right).
\end{aligned} \tag{A10}$$

Similarly, we have

$$4\langle\Phi|\hat{g}_s^2|\Phi\rangle = \langle\chi|\hat{n}^2|\chi\rangle, \tag{A11}$$

$$4\langle\Phi|\hat{g}_d\hat{g}_s|\Phi\rangle = 4\langle\Phi|\hat{g}_s\hat{g}_d|\Phi\rangle = -(1-2T)\langle\chi|\hat{n}^2|\chi\rangle. \tag{A12}$$

Also

$$\begin{aligned}
2\langle\Phi|\hat{g}_d|\Phi\rangle &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta) \left\langle\sqrt{T}\alpha\right|_A \left\langle\sqrt{R}\alpha\right|_B \left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right) \left|\sqrt{T}\beta\right\rangle_A \left|\sqrt{R}\beta\right\rangle_B \\
&= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\, (T\alpha^*\beta - R\alpha^*\beta) \left\langle\sqrt{T}\alpha\middle|\sqrt{T}\beta\right\rangle \left\langle\sqrt{R}\alpha\middle|\sqrt{R}\beta\right\rangle \\
&= (1-2T) \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta\, f^*(\alpha)f(\beta)\alpha^*\beta \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta\right)\right] \\
&= (1-2T)\langle\chi|\hat{n}|\chi\rangle,
\end{aligned} \tag{A13}$$

and similarly,

$$2\langle\Phi|\hat{g}_s|\Phi\rangle = \langle\chi|\hat{n}|\chi\rangle. \tag{A14}$$

By using the above results, we have

$$\begin{aligned}
F_{dd} &= F_Q(|\chi\rangle, \hat{g}_d, T) \\
&= \langle\chi|\hat{n}|\chi\rangle + (1-2T)^2\left(\langle\chi|\hat{n}^2|\chi\rangle - \langle\chi|\hat{n}|\chi\rangle\right) \\
&\quad -(1-2T)^2\langle\chi|\hat{n}|\chi\rangle^2 \\
&= \left\{1 - (1-2T)^2\right\}\bar{n}_\chi + (1-2T)^2 V_\chi,
\end{aligned} \tag{A15}$$

$$\begin{aligned}
F_{ss} &= \langle\chi|\hat{n}^2|\chi\rangle - \langle\chi|\hat{n}|\chi\rangle^2 \\
&= V_\chi,
\end{aligned} \tag{A16}$$

$$\begin{aligned}
F_{ds} &= F_{sd} = -(1-2T)\langle\chi|\hat{n}^2|\chi\rangle - (1-2T)\langle\chi|\hat{n}|\chi\rangle^2 \\
&= -(1-2T)V_\chi.
\end{aligned} \tag{A17}$$

Appendix B: Convexity of quantum Fisher information matrix

For completeness, here we give a proof of the convexity of the quantum Fisher information matrix (QFIM):

$$\mathcal{F}_Q(\hat{\rho}_\varphi) \leq p\mathcal{F}_Q(\hat{\sigma}_\varphi) + (1-p)\mathcal{F}_Q(\hat{\tau}_\varphi), \tag{B1}$$

for $\hat{\rho}_\varphi = p\hat{\sigma}_\varphi + (1-p)\hat{\tau}_\varphi$. Here $\hat{\rho}_\varphi$, $\hat{\sigma}_\varphi$, and $\hat{\tau}_\varphi$ are (maybe mixed) quantum states where $\varphi = \{\varphi_1, \ldots, \varphi_M\}$ is a set of M unknown parameters.

To begin with, we briefly review the definition and the structure of the QFIM that we will use in the proof. Detailed review on the QFI and QFIM can be found for example in Ref. [2, 13, 20]. The QFIM for $\hat{\rho}_\varphi$ is given by an $M \times M$ matrix $\mathcal{F}_Q(\hat{\rho}_\varphi) = [F_{ij}(\hat{\rho}_\varphi)]_{ij}$ $(i, j = 1, \ldots, M)$ where each entry is defined as

$$F_{ij}(\hat{\rho}_\varphi) = \frac{1}{2}\mathrm{Tr}\left[\hat{\rho}_\varphi \hat{L}_i \hat{L}_j + \hat{\rho}_\varphi \hat{L}_j \hat{L}_i\right], \tag{B2}$$

and $\hat{L}_i$, called the symmetrized logarithmic derivative, is a Hermitian operator satisfying

$$\frac{\partial}{\partial \varphi_i}\hat{\rho}_\varphi = \frac{1}{2}\left(\hat{L}_i \hat{\rho}_\varphi + \hat{\rho}_\varphi \hat{L}_i\right). \tag{B3}$$

Let $\hat{\rho}_\varphi = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|$ be the spectral decomposition of $\hat{\rho}_\varphi$. Then we can explicitly describe $\hat{L}_i$ as

$$\hat{L}_i = 2\sum_{k,l} \frac{\langle\lambda_k|\hat{\rho}_\varphi^{(i)}|\lambda_l\rangle}{\lambda_k + \lambda_l}|\lambda_k\rangle\langle\lambda_l|, \tag{B4}$$

where $\hat{\rho}_\varphi^{(i)} = \partial\hat{\rho}_\varphi/\partial\varphi_i$. Combining it with Eq. (B2), the QFIM is expressed as

$$F_{ij}(\hat{\rho}_\varphi) = 2\sum_{k,l} \frac{\langle\lambda_k|\hat{\rho}_\varphi^{(i)}|\lambda_l\rangle\langle\lambda_l|\hat{\rho}_\varphi^{(j)}|\lambda_k\rangle}{\lambda_k + \lambda_l}. \tag{B5}$$

We also use an important property of the QFIM: monotonicity under completely positive trace preserving (CPTP) map $\mathcal{L}$ [19, 20],

$$\mathcal{F}_Q(\hat{\rho}_\varphi) \geq \mathcal{F}_Q(\mathcal{L}(\hat{\rho}_\varphi)). \tag{B6}$$

The proof of the convexity of the QFIM is basically given by extending the proof for the QFI (i.e. single-parameter case) in Ref. [21]. Consider the bipartite state $\tilde{\rho}_\varphi^{AB} = p|e_0\rangle\langle e_0|^A \otimes \hat{\sigma}_\varphi^B + (1-p)|e_1\rangle\langle e_1|^A \otimes \hat{\tau}_\varphi^B$, where $|e_k\rangle$ is an orthonormal basis in A. Note that $\mathrm{Tr}_A[\tilde{\rho}_\varphi^{AB}] = \hat{\rho}_\varphi^B$. Then we have

$$\mathcal{F}_Q(\tilde{\rho}_\varphi^{AB}) = p\mathcal{F}_Q(\hat{\sigma}_\varphi^B) + (1-p)\mathcal{F}_Q(\hat{\tau}_\varphi^B). \tag{B7}$$

This is justified by the following observation. Since $|e_k\rangle$ is independent of the unknown parameters φ_i, $\tilde{\rho}_\varphi^{(i)} = p|e_0\rangle\langle e_0| \otimes \hat{\sigma}_\varphi^{(i)} + (1-p)|e_1\rangle\langle e_1| \otimes \hat{\tau}_\varphi^{(i)}$, for any i. Also the spectral decomposition of $\tilde{\rho}_\varphi$ is described as $p|e_0\rangle\langle e_0| \otimes \sum_i \lambda_i^\sigma |\lambda_i^\sigma\rangle\langle\lambda_i^\sigma| + (1-p)|e_1\rangle\langle e_1| \otimes \sum_i \lambda_i^\tau |\lambda_i^\tau\rangle\langle\lambda_i^\tau|$, where $\sum_i \lambda_i^\sigma |\lambda_i^\sigma\rangle\langle\lambda_i^\sigma|$ and $\sum_i \lambda_i^\tau |\lambda_i^\tau\rangle\langle\lambda_i^\tau|$ are the spectral decompositions of $\hat{\sigma}_\varphi$ and $\hat{\tau}_\varphi$, respectively. Plugging them into the expression of QFI in Eq. (B5), we get

$$F_{ij}(\tilde{\rho}_\varphi^{AB}) = pF_{ij}(\hat{\sigma}_\varphi^B) + (1-p)F_{ij}(\hat{\tau}_\varphi^B). \tag{B8}$$

Since this holds for all i and j, we get Eq. (B7).

By using Eq. (B7), the monotonicity (B6), and the fact that partial trace is a CPTP map, we have

$$\begin{aligned}\mathcal{F}_Q(\hat{\rho}_\varphi^B) &\leq \mathcal{F}_Q(\tilde{\rho}_\varphi^{AB}) \\ &= p\mathcal{F}_Q(\hat{\sigma}_\varphi^B) + (1-p)\mathcal{F}_Q(\hat{\tau}_\varphi^B),\end{aligned} \tag{B9}$$

which completes the proof of the convexity of the QFIM.

Appendix C: Quantum Fisher information for $\hat{g}_1$ with phase randomizing

Here we give a complete calculation of the QFI for the generator $\hat{g}_1 = \hat{a}^\dagger\hat{a}$ with phase randomizing. The two input states are a vacuum and an arbitrary quantum state with the density matrix of

$$\hat{\rho}_{\mathrm{in}} = \sum_{n,m=0}^{\infty} c_{nm}|n\rangle\langle m|, \tag{C1}$$

where $|n\rangle$ is the n-photon number state. Then the phase-averaged input is given by

$$\begin{aligned}\Psi_{\mathrm{avg}} &= \int \frac{d\theta}{2\pi}\hat{V}_\theta^A \hat{V}_\theta^B \left(\hat{\rho}_{\mathrm{in}}^A \otimes |0\rangle\langle 0|^B\right)\hat{V}_\theta^{A\,\dagger}\hat{V}_\theta^{B\,\dagger} \\ &= \sum_{n,m=0}^{\infty}\int \frac{d\theta}{2\pi} e^{i\theta(n-m)} c_{nm}|n\rangle\langle m|^A \otimes |0\rangle\langle 0|^B \\ &= \sum_{n=0}^{\infty} p_n |n\rangle\langle n|^A \otimes |0\rangle\langle 0|^B,\end{aligned} \tag{C2}$$

where $\hat{V}_\theta^A = e^{i\theta\hat{a}^\dagger\hat{a}}$, $\hat{V}_\theta^B = e^{i\theta\hat{b}^\dagger\hat{b}}$, and $p_n = c_{nn}$ is a real positive number satisfying $\sum_n p_n = 1$. The state after the first beamsplitter of the MZI and the phase shifting is given by

$$\begin{aligned}\Psi_{\mathrm{avg}}^\phi &= \hat{U}_\phi^{(1)\,AB}\hat{B}_T^{AB}\Psi_{\mathrm{avg}}\hat{B}_T^{\dagger AB}\hat{U}_\phi^{(1)\dagger\,AB} \\ &= \sum_{n=0}^{\infty} p_n |\psi_n(\phi)\rangle\langle\psi_n(\phi)|_{AB},\end{aligned} \tag{C3}$$

where

$$\begin{aligned}|\psi_n(\phi)\rangle_{AB} = \sum_{j=0}^{n} & e^{-ij\phi}\binom{n}{j}^{1/2} \\ & \times T^{j/2}(1-T)^{(n-j)/2}|j\rangle_A \otimes |n-j\rangle_B.\end{aligned} \tag{C4}$$

By using the convexity of the QFI and noticing that $|\psi_n(\phi)\rangle$ and $|\psi_{n'}(\phi)\rangle$ are orthogonal for $n \neq n'$, we have

$$F_Q^{(1)}\left(\Psi_{\mathrm{avg}}^\phi\right) = \sum_{n=0}^{\infty} p_n F_Q^{(1)}\left(|\psi_n(\phi)\rangle\right). \tag{C5}$$

Thus our remaining task is to calculate $F_Q^{(1)}(|\psi_n(\phi)\rangle)$ explicitly. For $|\psi_n(\phi)\rangle$, we find

$$\langle \hat{a}^\dagger \hat{a}\rangle = \sum_{j=0}^{n} j \binom{n}{j} T^j (1-T)^{n-j} = nT, \tag{C6}$$

$$\langle \hat{b}^\dagger \hat{b}\rangle = n(1-T), \tag{C7}$$

$$\begin{aligned}\langle \hat{a}^{\dagger 2} \hat{a}^2\rangle &= \sum_{j=0}^{n} j(j-1) \binom{n}{j} T^j (1-T)^{n-j} \\ &= n(n-1)T^2,\end{aligned} \tag{C8}$$

and the QFI evaluated as $4\left(\langle \hat{g}_1^2\rangle - \langle \hat{g}_1\rangle^2\right)$ is found to be

$$\begin{aligned}F_Q^{(1)}(|\psi_n(\phi)\rangle) &= 4\left(\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle \hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2\right) \\ &= 4\left\{n(n-1)T^2 + nT - n^2T^2\right\} \\ &= 4nT(1-T).\end{aligned} \tag{C9}$$

Then averaging over n, we get

$$F_Q^{(1)}\left(\Psi_{\text{avg}}^{\phi}\right) = 4\bar{n}T(1-T), \tag{C10}$$

The maximum is attained at $T = 1/2$ and is equal to $\bar{n}$.

Appendix D: Derivation of $F(\phi_A)$

The calculation of Fisher information can be performed by the characteristic function approach. For the details of the characteristic function formalism in quantum optics, see Ref. [29] for example. Here we follow the definition and the methodology developed in Ref. [26]. Then the covariance matrix of the two-mode vacuum is given by

$$\gamma_{\text{in}} = I(4), \tag{D1}$$

where $I(4)$ is the four-by-four identity matrix. The beam splitter unitary transformation is represented by the symplectic transformation:

$$S_{\text{BS}} = \begin{bmatrix} \sqrt{T} & 0 & \sqrt{1-T} & 0 \\ 0 & \sqrt{T} & 0 & \sqrt{1-T} \\ -\sqrt{1-T} & 0 & \sqrt{T} & 0 \\ 0 & -\sqrt{1-T} & 0 & \sqrt{T} \end{bmatrix} \tag{D2}$$

Similarly, the unknown phase shift is given by

$$S_{\text{PS}} = \begin{bmatrix} \cos\phi_A & \sin\phi_A & 0 & 0 \\ -\sin\phi_A & \cos\phi_A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{D3}$$

and the squeezing in the first arm is given by

$$S_{\text{SQ}}(r) = \begin{bmatrix} e^{-r} & 0 & 0 & 0 \\ 0 & e^{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{D4}$$

where $e^{-r} = \sqrt{\bar{n}+1} - \sqrt{\bar{n}}$ and $e^{r} = \sqrt{\bar{n}+1} + \sqrt{\bar{n}}$ (remember $\bar{n} = \sinh^2 r$).

Then the covariance matrix of the state before the photo detectors is calculated to be

$$\begin{aligned}\gamma_{\text{out}} = {} & S_{SQ}(-r) S_{\text{BS}}^T S_{\text{PS}} S_{\text{BS}} S_{SQ}(r) \gamma_{\text{in}} \\ & \times S_{SQ}^T(r) S_{\text{BS}}^T S_{\text{PS}}^T S_{\text{BS}} S_{SQ}^T(-r)\end{aligned} \tag{D5}$$

where the superscript T denotes the matrix transpose.

The probability of having no-clicks at both detector (i.e. the projection onto $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$) is given by [26],

$$P_{00} = \frac{4}{\sqrt{\det(\gamma_{\text{out}} + I(4))}}. \tag{D6}$$

Then the Fisher information for ϕ_A is calculated by

$$F(\phi_1) = \frac{1}{P_{00}} \left(\frac{dP_{00}}{d\phi_A}\right)^2 + \frac{1}{1-P_{00}} \left(\frac{d(1-P_{00})}{d\phi_A}\right)^2. \tag{D7}$$

The calculation is performed by Mathematica. Since the expression of $F(\phi_A)$ is quite complicated, we consider the limit of small ϕ_A. Then we get

$$\lim_{\phi_A \to 0} F = 4\bar{n}T(1 + T + 2\bar{n}T). \tag{D8}$$

Replacing $\bar{n}$ with $\bar{n}_{\text{tot}} = 2\bar{n}$, we get

$$\lim_{\phi_A \to 0} F = 2\bar{n}_{\text{tot}}T(1 + T + \bar{n}_{\text{tot}}T), \tag{D9}$$

which implies that in the limit of small phase shifts, the Fisher information of our protocol can surpass the SNL in terms of the total resource for any $T \neq 0$, and particularly for $T = 1/2$.

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