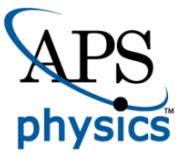

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Verifiable fault tolerance in measurement-based quantum computation

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Verifiable fault-tolerance in measurement-based quantum computation

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Quantum systems, in general, cannot be simulated efficiently by a classical computer, and hence is useful for solving certain mathematical problems and simulating quantum many-body systems. This also implies, unfortunately, that verification of the output of the quantum systems is not so trivial, since predicting the output is exponentially hard. As another problem, quantum system is very delicate for noise and thus needs error correction. Here we propose a framework for verification of the output of FTQC in the measurement-based model. Contrast to existing analyses on fault-tolerance, we do not assume any noise model on the resource state, but an arbitrary resource state is tested by using only single-qubit measurements to verify whether the output of measurement-based quantum computation on it is correct or not. Verifiability is equipped by a constant time repetition of the original meausrement-based quantum computation in appropriate meausrement bases. Since full characterization of quantum noise is exponentially hard for large-scale quantum computing systems, our framework provides an efficient way of practical verification of experimental quantum error correction.

Introduction.— Quantum computation provides a new paradigm of information processing offering both fast and secure information processing [1]. Recently, a lot of experimental efforts have been paid to realize quantum computation [2–4]. There, fault-tolerant quantum computation (FTQC) with quantum error correction [1, 5] is inevitable to obtain quantum advantage using noisy quantum devices.

Due to the recent rapid progresses on experimental quantum error correction techniques [6–9], there is an increasing demand on an efficient performance analysis of FTQC. There are three categories for this purpose, characterization, validation and verification of quantum systems (QCVV) [10, 11]. In the majority of existing performance analyses of FTQC, a specific noise model, such as stochastic Pauli errors, is assumed apriori [12–20]. By characterizing the elementary quantum operations experimentally, these could serve as validation of quantum computing devices [21]. However, in actual experiments, more general noise might occur including various correlation between qubits [22, 23]. Since full tomographic approch does not work efficiently, we need a novel scheme for the third category, verification, to guarantee correctness of the output of a quantum computer without assuming the underlying noise model. Unfortunately, existing FTQCs have not equipped such an efficient verification scheme yet.

The aim of this paper is to develop FTQCs being equipped with a verification scheme without assuming the underlying noise model. As requirements of verifiable fault-tolerance, we define the following two concepts. One is *detectability* which means that if the error of a quantum computer is not correctable, such a faulty output of the quantum computation is detected with high probability. In this stage, any assumption on the underlying noise model should not be made. The other is *acceptability* which means that an appropriately constructed quantum computer can pass the verification with high probability. In other words, under a realistic noise model, the test accepts the quantum computation with high probability. Both properties are important to characterize performance of test in statistical hypothesis testing [24].

In this paper, we develop verifiable fault-tolerance in measurement-based quantum computation (MBQC) [27, 28], which satisfies both detectability and acceptability. We take a rather different approach to fault-tolerance than conventional one. We do not assume any noise model underlying, but define a correctable set of errors on a resource state of MBQC and test whether the error on a given resource state belongs to such a set or not. To this end, we employ the stabilizer test proposed in Ref. [29, 30], where MBQC is efficiently verified by testing the graph state. However, this method is not fault-tolerant lacking acceptability; any small amount of noise on the graph state causes rejection regardless whether or not it is correctable. Therefore, we crucially extend the stabilizer test [29] for a noisy situation, so that we can decide whether the given resource states belong to a set of fault-tolerant resource states or not. Under the condition of a successful pass of the test, the accuracy of fault-tolerant MBQC is guaranteed to be arbitrarily high (i.e., contraposition of detectability). Our verification scheme works quite efficiently by simply repeating fault-tolerant MBQC for a constant time in appropriate measurement bases. Therefore, we do not need any special resource state nor entangling operation for verification. The total overhead is only factored by a constant to the original fault-tolerant MBQC. Moreover, our framework is applicable to any fault-tolerant measurement-based quantum

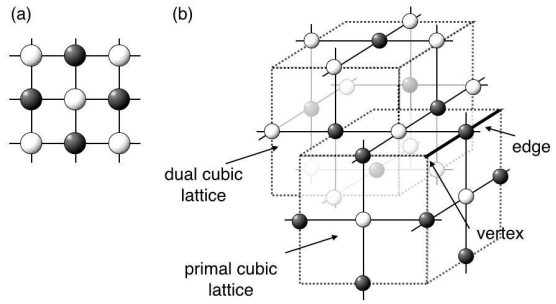

FIG. 1. (a) A two-colorable graph state. (b) The three dimensional two-colorable graph state for topologically protected measurement-based quantum computation.

computation. As a concrete example, we explicitly define a set of correctable errors on the resource state for topologically protected MBQC [5, 14, 16], where we can show acceptability by calculating the acceptance probability concretely under a realistic noise model.

A general setup for fault-tolerant MBQC.— Let us consider a generic scenario of fault-tolerant MBQC on a two-colorable graph state composed of the black system $\mathcal{H}_B$ and the white system $\mathcal{H}_W$, consisting of n_B and n_W qubits, respectively (See Fig. 1(a)). Then, we have two kinds of operators $\mathsf{X}^z := \bigotimes_{i=1}^n X^{z_i}$ and $\mathsf{Z}^x := \bigotimes_{i=1}^n Z^{x_i}$, on $\mathcal{H}_B \otimes \mathcal{H}_W = (\mathbb{C}^2)^{\otimes n}$, where $n := n_B + n_W$. When we restrict them to the black system $\mathcal{H}_B$ (the white system $\mathcal{H}_W$), we denote X^z and Z^x by X^z_B and Z^x_B (X^z_W and Z^x_W). By using the binary-valued adjacency matrix A (i.e., (i,j) element is 1 iff vertices i and j are connected) corresponding to the graph, the graph state $|G\rangle$ is characterized as

$$\mathsf{X}^{z_B}_B \otimes \mathsf{Z}^{Az_B}_W |G\rangle = |G\rangle, \quad \mathsf{X}^{z_W}_W \otimes \mathsf{Z}^{A^T z_W}_B |G\rangle = |G\rangle \quad (1)$$

for $z_B \in \mathbb{F}_2^{n_B}$ and $z_W \in \mathbb{F}_2^{n_W}$. Then, the total space $\mathcal{H}_B \otimes \mathcal{H}_W$ is spanned by $\{\mathsf{Z}^x|G\rangle\}_{x\in\mathbb{F}_2^n}$. Suppose we execute a fault-tolerant MBQC on the two-colorable graph state. Then a set of correctable errors on the two-colorable graph state is defined such that an ideal state $|G\rangle$ and erroneous one $\mathsf{Z}^x|G\rangle$ result in the same computational outcome under error correction. Such a set of errors is specified as a subset S of $\mathbb{F}_2^n = \mathbb{F}_2^{n_B} \times \mathbb{F}_2^{n_W}$. The projection to the subspace is written by Π_S. We assume that the subset S is written as $S_B \times S_W$ by using two subsets $S_B \subset \mathbb{F}_2^{n_B}$ and $S_W \subset \mathbb{F}_2^{n_W}$.

Test for verification of fault-tolerance.— Similar to Ref. [29], we employ the following sampling protocol to verify whether the error is correctable. Our protocol runs as follows:

1. Honest Bob generates $|G\rangle^{\otimes 2k+1}$. Bob sends each qubit of it one by one to Alice.

2. Alice divides $2k+1$ blocks of n qubits into three groups, two k blocks and signle block, by random choice.

3. Alice uses the third group for her computation. Other blocks are used for the test, which will be explained later.

4. If Alice passes the test, she accepts the result of the computation performed on the third group.

For each block of the first and second groups, Alice performs the following test:

T_B For each block of the first group, Alice measures qubits of W (B) in the Z (X) basis, respectively. Then, she obtain Z_W and X_B. If $X_B + A^T Z_W \in S_B$, then the test is passed.

T_W For each block of the second group, Alice measures qubits of B (W) in the Z (X) basis. Then, she obtain Z_B and X_W. If $X_W + AZ_B \in S_W$, then the test is passed.

Detectability and acceptability.— To show detectability, taking account into unexpected errors, we obtain the following theorem in the same way as Ref. [29]:

Theorem 1 *Assume that $\alpha > \frac{1}{2k+1}$. If the test is passed, with significance level α [24, 25], we can guarantee that the resultant state σ of the third group satisfies*

$$\mathrm{Tr}\,\sigma\Pi_S \geq 1 - \frac{1}{\alpha(2k+1)}. \quad (2)$$

The previous study [29] considers the case with $S_B = \{0\}$, $S_W = \{0\}$, and proves this special case by discussing the two kinds of binary events $X_B + A^T Z_W =$ or $\neq 0$ and $X_W + AZ_B =$ or $\neq 0$. Replacing these two events by the two kinds of events $X_B + A^T Z_W \in$ or $\notin S_B$ and $X_W + AZ_B \in$ or $\notin S_W$ in the proof given in [29], we can show Theorem 1 with the current general form.

From the theorem and the relation between the fidelity and trace norm [31, (6.106)], we can conclude the verifiability: If Alice passes the test, she can guarantee that

$$\left|\mathrm{Tr}\,(C_x\sigma) - \mathrm{Tr}\,(C_x \frac{\Pi_S\sigma\Pi_S}{\mathrm{Tr}\,\sigma\Pi_S})\right| \leq \frac{1}{\sqrt{\alpha(2k+1)}}$$

for any POVM $\{C_x\}$ with the significance level α. That is, the property of FTQC guarantees that the probability that the obtained computation outcome is different from the true computation outcome is less than $\frac{1}{\sqrt{\alpha(2k+1)}}$. If we take $\alpha = \frac{1}{\sqrt{2k+1}}$, for example, this error probability is $\frac{1}{(2k+1)^{1/4}} \to 0$ if $k \to \infty$, and therefore the verifiability is satisfied. Note that the lower bound, $\alpha > \frac{1}{2k+1}$, of the significance level α is tight, since if Bob generates $2k$ copies of the correct state $|G\rangle$ and a single copy of a wrong state, Bob can fool Alice with probability $\frac{1}{2k+1}$, which corresponds to $\alpha = \frac{1}{2k+1}$. The above theorem on detectability holds without any assumption on the underlying noise. Noise in the measurements can also be taken as noise on the resource state, if it does not depend on the measurement bases. Even if it is not the

case, each qubit in the resource state can be randomly rotated such that Alice's measurement bases also become random. In such a case, the proposed verification works if the noises at Alice's and Bob's sides are independent, which are physically plausible.

Next, we consider acceptability. Contrast to detectability, the requirement of acceptability is unique for the verification of FTQC. Indeed, if a quantum computer is assumed to be ideal without any error as is in Ref. [29], we can verify whether or not the quantum computer actually does what one commands to operate with probability 1, i.e., acceptability of the test is trivially satisfied. On the other hand, in verification of FTQC consisting of many elementary parts, each of which cannot be checked directly, we have to judge whether the output of the computation is correct or not carefully under an expected error model, which imposes the second requirement, acceptability.

To calculate acceptability, we assume a specific application of Pauli channel on $\mathcal{H}_B \otimes \mathcal{H}_W$ [26]. That is, the error given as the distribution P on the set $\mathbb{F}_2^{n_B+n_W} \times \mathbb{F}_2^{n_B+n_W}$ of X-basis errors and Z-basis errors. Then, we denote the marginal distribution with respect to the pair of X-basis errors on B and Z-basis errors on W (Z-basis errors on B and X-basis errors on W) by P_B (P_W). Hence, the probability that Alice passes the test T_B (T_W) with one round is $P_B(S_B)$ ($P_W(S_W)$). Since we apply them $2k$ rounds, the probability to be passed is $P_B(S_B)^k P_W(S_W)^k$. Hence, when the probabilities $P_B(S_B)$ and $P_W(S_W)$ are close to 1, Alice can accept the correct computation result on the third group with high probability.

Case study.— To show acceptability, below we will explain how to define a correctable set of the errors on a graph state. Then, for a concrete example, we will calculate the acceptance probability $P_B(S_B)^k P_W(S_W)^k$ under a realistic noise model.

In the theory of FTQC, it is conventional that we translate fault-tolerance in the circuit model into fault-tolerance in MBQC [32–34] as follows. In the circuit model, we can define a set of correctable (sparse) fault paths [35–37]. Then, translating the correctable (sparse) fault paths in the circuit model into MBQC, we can define a correctable set of the errors on the graph state in general. For example, the schemes in Refs [38, 39] and Refs [14, 16] can be viewed as measurement-based versions of circuit-based FTQC using the concatenated Steane 7-qubit code [5, 40–42] and the surface code with the concatenated Reed-Muller 15-qubit code [15, 43], respectively.

Let us see a concrete example by using topologically protected MBQC [5, 14, 16], which has been employed as a standard framework for fault-tolerant MBQC recently [44–48]. For simplicity of explanations, we here focus on the original scheme proposed in Ref. [14], where the surface code and the concatenated Reed-Muller code are employed to perform two-qubit Clifford gate and single-qubit non-Pauli-basis measurements, respectively.

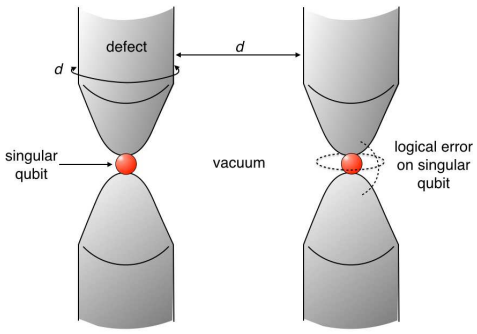

FIG. 2. The tubes indicate defect regions, in which the qubits are measured in the Z-basis. Singular qubits are located in-between two defect regions, which are measured in the $(X+Y)/\sqrt{2}$-basis for a transversal logical $(X+Y)/\sqrt{2}$-basis measurement. Other regions are vacuum, where qubits are measured in the X-basis to obtain the error syndrome.

Of course, more efficient distillation protocols as in Refs. [15, 16, 43] can also be employed.

In the following we will briefly sketch how the correctable set S_B and S_W are defined. A detailed description is shown in Appendix. The errors specified by the set S_B, which correspond to X basis (the Pauli-Z operator) on black qubits and Z basis (the Pauli-X operator) on white qubits, are detected on the priaml cubic lattice consisting of the edges on which the black qubits are located as shown in Fig. 1(b). Then, the error configuraion $x_B \in S_B$ can be associated with a set of edges on the primal cubic lattice. Similarly, the errors in the set S_W is detected on the dual cubic lattice and the error configuration x_W is associated with a set of edges on the dual cubic lattice.

Depending on quantum computation that Alice wants to do fault-tolerantly, a measurement pattern is determined. Specifically, from an analogy of topological quantum computation [49], the sets of qubits measured in X, Z, and $(X+Y)/\sqrt{2}$-bases [14] are called defect, vacuum, and singular qubits, respectively (see Fig. 2). For the surface code, minimum distance decoding (MDD) can be done by finding a shortest path connecting the boundary of the error chain on the cubic lattice. Then, if MDD results in a logical operator of a weight (distance) larger than the code distance d by wrapping around a defect or connecting two different defects, such an error is uncorrectable (see Appendix for the detail). Accordingly, we can define $S_C^{\rm sf}$ for $C = B, W$ as the complement of them. The code distance d is chosen to be polylog(n') with n' being the size of the quantum computation that Alice wants to do fault-tolerantly. Therefore, the number of qubits of the graph state is given by $n = \text{poly}(n')$.

Around the singular qubits, we still have a logical error of a weight lower than d as shown in Fig. 2. Such a logical error is corrected by using another code, the concatenated Reed-Muller code. To this end, the fault-tolerant Clifford gates using the surface code are further employed to encode the logical qubits into concatenated

Reed-Muller codes, on which we can implement all Pauli-bases and $(X+Y)$-basis measurements transversally. The corresponding physical $(X+Y)$-basis measurements, i.e., measurements on the singular qubits, are depicted by red circles in Fig. 2. Then we can define the correctable set $S_C^{\rm rm}$ of the errors for the concatenated Reed-Muller code recursively for $C = B, W$ as done in Ref. [35] (see Appendix for the detail).

Since we employ two types of error correction codes as seen above, the correctable set of the errors are defined as an intersection of the correctable sets $S_C^{\rm sf}$ and $S_C^{\rm rm}$ for the surface code and the concatenated Reed-Muller code, respectively, for both colors $C = B, W$. Since both decording can be done efficiently, we can efficiently decide whether a given error pattern $X_B + A^T Z_W$ ($X_W + A^T Z_B$) are in S_B (S_W) or not.

Acceptance probability under a typical error model.— To calculate the acceptance probability, we assume, for simplicity, the errors Z^x ($x \in \mathbb{F}^n$) are distributed independently and identically for each qubit with probability p. It is straightforward to generalize the following argument to any local noise model [50]. Then the standard counting argument of the self-avoiding walk for the surface code [42] tells us that

$$P(S_C^{\rm sf}) > 1 - {\rm poly}(n)(10p^{1/2})^d \tag{3}$$

for $C = B, W$. Apparently, if p is sufficiently smaller than a certain constant value, $P(S_C^{\rm sf})$ converges to 1 for $C = B, W$. By considering a recursive decoding of the concatenated code, we obtain

$$P(S_C^{\rm rm}) > [1 - (105^2 p_0^{\rm fault})^{2^l}/105^2]^m, \tag{4}$$

for $C = B, W$ where $p_0^{\rm fault}$ is a logical error probability of a weight lower than d, which occurs around the singular qubits. Such a logical probability is also calculated as a function of the physical error probaility p by counting the number of self-avoiding walk [42] as show in Appendix. The integer $m = {\rm poly}(n')$ and $l = {\rm poly}\log d)$ are the numbers of the logical $(X+Y)$-basis measurements and the number of concatenation, respectively. Again by using counting the number of self-avoiding walks [42, 50] we can evaluate $p_0^{\rm fault}$. By choosing p smaller than a certain constant value, $p_0^{\rm fault}$ becomes sufficiently small so that $P(S_C^{\rm rm})$ converges to 1 for $C = B, W$. Since

$$P(S_C) = P(S_C^{\rm sf} \cap S_C^{\rm rm}) > P(S_C^{\rm sf}) + P(S_C^{\rm rm}) - 1 \tag{5}$$

for $C = B, W$, the probability $P(S_C)$ also converges to 1 exponentially in the large d limit, if the physical error probability p is smaller than a certain constant threshold value (see Appendix). Since d can be chosen independently of k, the acceptance probability $P_B(S_B)^k P_W(S_W)^k$ converges to 1.

Verifiable blind quantum computation.— Finally, we address an application of the proposed verification scheme in blind quantum computation (BQC) [51–57]. A promising application of the proposed framework is verification of measurement-only BQC [55]. Suppose a quantum server generates two-colorable graph states and sends them to a client who execute MBQC with proposed verification. As same as the original measurement-only BQC [55], the blindness is guaranteed by the no-signaling principle, which contrasts to verifiable BQC [52, 57] of BFK (Broadbent-Fitzsimons-Kashefi) type [51]. According to Theorem 1 (detectability) under the condition of acceptance the accuracy of the output is guaranteed. Contrast to the earlier verifiable BQC [29, 52], by virtue of acceptability, the proposed verification scheme can accept the delegated quantum computation even under quantum server's deviation or quantum channel noise as long as they are correctable. It would be interesting to apply the proposed framework to quantum interactive proof systems [58].

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Appendix A: Test for topological protection

The error detection on the black vacuum qubits (edges of the primal cubic lattice) is executed as follows. If there is no error on the graph state, the outcome m_b of the X-basis measurements satisfies the condition: $s_v \equiv \bigoplus_{b \in \delta v} m_b = 0$, where δv indicates a set of black qubits adjacent to the vertex v. Depending on a given error Z^{x_B} ($x_B \in \mathbb{F}_2^{n_B}$) on the graph state, we can obtain the error syndrome $\{s_v(x_B)\}$ in the defect region. From the error syndrome, the most likely location of the errors is estimated using the minimum-weight-perfect-matching (MWPM) algorithm [42]. Let $\bar{x}_B \equiv \arg\min_{x|\{s_v(x)=s_v(x_B)\}} |x|$ be the estimated error location, where $|x|$ indicates the number of 1s in a bit string x. If a chain of edges specified by $x_B + \bar{x}_B$ have a nontrivial cycle in the sense of the relative homology [14–16], the error correction fails. At the defect region far from the singular qubits, a nontrivial cycle have at least length d determined as follows. Let n' be the size of the quantum computation that Alice wants to do fault-tolerantly. To guarantee the accuracy of the output, it is enough to choose $d = {\rm polylog}(n')$.

Now we can define the correctable set of errors as follow: an error location x_B belongs to the correctable set $S_B^{\rm sf} \subset \mathbb{F}^{n_B}$ of the errors iff there exists a connected component of length d in the chain of edges specified by $x_B + \bar{x}_B$. The error detection and definition of the cor-

rectable error set $S_W^{\rm sf}$ on the white vacuum qubits are done in the same way but on the dual lattice.

From the test T_B, we know the error location x_B. Since the MWPM algorithm works in polynomial time in the number of vertices with $s_v = 1$, we can decide whether or not x_B belongs to the correctable error set $S_B^{\rm sf}$. The same argument also holds for the error location x_W on the white vacuum qubits tested by T_W. Therefore, we can efficiently check whether or not the errors on a given resource belong to $S_B^{\rm sf} \times S_W^{\rm sf}$.

Appendix B: Test for the logical $(X+Y)/\sqrt{2}$-basis measurement

We here, for simplicity, do not employ magic state distillaion [15, 16] but encodes each logical qubit into the Reed-Muller 15-qubit code. Then we perform a fault-tolerant logical $(X+Y)/\sqrt{2}$-basis measurement by transversal physical $(X+Y)/\sqrt{2}$-basis measurements on the singular qubits as done in Ref [14]. Thereby, Alice can fix her strategy of quantum computation, which makes easy to define the correctable set of errors for the test. Let l and $m = {\rm poly}(n')$ be the number of concatenation levels and the number of the logical $(X+Y)$-basis measurements, respectively. Then we need $15^l m$ physical $(X+Y)/\sqrt{2}$-basis measurements, on the singular qubits. Note that $l = O({\rm poly}\log\log n')$ is enough to reduce the logical error sufficiently. In the following, we the error on the graph state is specified by $x \in \mathbb{F}_2^n$ by converting it into Z operators on the graph state, $\mathsf{Z}^x|G\rangle$.

The logical $(X+Y)/\sqrt{2}$-basis measurement is done by physical transversal $(X+Y)/\sqrt{2}$-basis measurements by encoding each qubit into a concatenated Reed-Muller 15-qubit codes [14]. This is also the case for all Pauli-basis measurements. In the vacuum region near the singular qubits, we have a logical error of length smaller than d as shown in Fig. 2, since they are not topologically protected. Correctable error for the fault-tolerant logical $(X+Y)/\sqrt{2}$-basis measurement is defined for a given error $(x_B, x_W) \in S_B^{\rm rm} \times S_W^{\rm rm}$ recursively as follows: At physical level, which we call level-0, if $x_B + \bar{x}_B$ or $x_W + \bar{x}_W$ becomes a logical error for a singular qubit, the level-0 (singular) qubit is labeled to be faulty. At l'th concatenation level, if the level-l' logical qubit consisting of 15 level-$(l'-1)$ logical qubits encoded in the Reed-Muller 15-qubit code has two or more faulty level-$(l'-1)$ logical qubits, the level-l' logical qubit is labeled to be faulty. At the highest level $l' = l$, if no level-l logical qubit is faulty, the given error (x_B, x_W) belongs to the correctable set $S_B^{\rm rm} \times S_W^{\rm rm}$.

Appendix C: Acceptance probability

Let us first consider the pass probability of the test for topological protection. The error x_B is rejected if

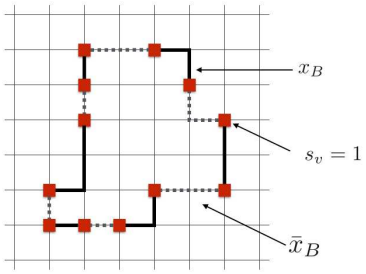

FIG. 3. Actual error x_B and estimated one $\bar{x}_B$ are denoted by solid and dotted lines, respectively. The vertices (error syndrome) of $s_v = 1$ are denoted by red squares. The 3D lattice is depicted as if it is two dimensional.

$x_B + \bar{x}_B$ contains a connected component of length at least d. Such a probability is calculated [42] to be

$$\sum_{\nu=d}^{} \sum_{\mu=\nu/2}^{\nu} \frac{6}{5} n \cdot 5^\nu \begin{pmatrix} \nu \\ \mu \end{pmatrix} p^\mu (1-p)^{n-\mu} < {\rm poly}(n)(10p^{1/2})^d.$$

Therefore, if p is sufficiently smaller than a constant value, the rejection probability is exponentially suppressed.

Next we consider the test for the logical $(X+Y)/\sqrt{2}$-basis measurement. Let $p_0^{\rm fault}$ be the probability that a level-0 (singular) qubit is faulty. $p_0^{\rm fail}$ is evaluated in a similar way to the previous case for the topological protection but we have to count logical errors consist of the chains of length lower than d:

$$p_0^{\rm fault} = \sum_{\nu=1}^{d} \sum_{\mu=\nu/2}^{\nu} C_\nu \begin{pmatrix} \nu \\ \mu \end{pmatrix} p^\mu (1-p)^{n-\mu}, \quad \text{(C1)}$$

where C_ν is the number of chains of length ν that contribute to the logical error of length ν. C_ν is counted in Ref. [50] rigorously up to $\nu = 14$, which indicates that we can reduce $p_0^{\rm fault}$ by decreasing p sufficiently.

The probability $p_{l'}^{\rm fault}$ of obtaining the level-l' faulty qubit is given recursively by

$$p_{l'}^{\rm fault} < \sum_{r=2}^{15} (p_{l'-1}^{\rm fault})^r (1 - p_{l'-1}^{\rm fault})^{15-r} = (7 \cdot 15)^2 (p_{l'-1}^{\rm fault})^2.$$

The we obtain $p_l^{\rm fault} = (105^2 p_0^{\rm fault})^{2^l}/105^2$. The probability to obtain no faulty level-l logical qubit at the highest level is given by $(1 - p_l^{\rm fault})^m$.

Accordingly, if $p_0^{\rm fault}$ is sufficiently smaller than $1/(7 \cdot 15)^2$, we can reduce the rejection probability of the test for the fault-tolerant logical $(X+Y)\sqrt{2}$-basis measurement.

Since $m = {\rm poly}(n')$, it is sufficient to chose $d = {\rm poly}(\log n')$ and $l = {\rm poly}(\log d)$, which are independent of $2k+1$, the number of the samples of the graph state. Therefore, in the large d limit for a given n', we can reduce the logical error probability polynomially, and hence amplify the acceptance probability $P_B(S_B)^k P_W(S_w)^k$ arbitrarily close to 1.

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