Conditional mutual information and quantum steering

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Conditional Mutual Information and Quantum Steering

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Quantum steering has recently been formalized in the framework of a resource theory of steering, and several quantifiers have already been introduced. Here, we propose an information-theoretic quantifier for steering called *intrinsic steerability*, which uses conditional mutual information to measure the deviation of a given assemblage from one having a local hidden-state model. We thus relate conditional mutual information to quantum steering and introduce monotones that satisfy certain desirable properties. The idea behind the quantifier is to suppress the correlations that can be explained by an inaccessible quantum system and then quantify the remaining intrinsic correlations. A variant of the intrinsic steerability finds operational meaning as the classical communication cost of sending the measurement choice and outcome to an eavesdropper who possesses a purifying system of the underlying bipartite quantum state that is being measured.

I. INTRODUCTION

Quantum steering was first introduced by Schrödinger in 1935 [1] in order to formalize an argument made by Einstein, Podolsky, and Rosen in [2]. It refers to the following scenario: two parties called Alice and Bob share a bipartite quantum state. Alice measures her system, which can have the effect of steering the reduced state on Bob's system, depending on the measurement that she performs. She thus can influence Bob's subsystem without having access to it. However, Bob does not have any knowledge about the influence, nor can he detect it unless Alice communicates the measurement that she performed and the outcome of the measurement. For example, consider a maximally entangled singlet shared by Alice and Bob. Alice can measure her system in either the Pauli σ_Z basis or the Pauli σ_X basis. If she measures in the Pauli σ_Z basis, the resulting state of Bob's subsystem is represented as the ensemble $\left\{(\frac{1}{2}, |1\rangle\langle 1|), (\frac{1}{2}, |0\rangle\langle 0|)\right\}$. Alternatively, if she measures in the Pauli σ_X basis, the state of Bob's subsystem is represented as the ensemble $\left\{(\frac{1}{2}, |+\rangle\langle +|), (\frac{1}{2}, |-\rangle\langle -|)\right\}$.

The notion of steering was formalized in [3], which defines it in the context of an entanglement certification task, with Alice having access to an untrusted device and Bob to a trusted quantum system. Alice's device can be thought of as a black box, which accepts a classical input X and outputs a classical system $\overline{A}$. The mathematical description of the relation between Alice's classical input X, her output $\overline{A}$, and Bob's quantum system is called an *assemblage*, whose formal definition we recall later.

The fact that Alice's system is classical and Bob's system is quantum in the scenario of steering makes it natural to study in the context of one-sided device-independent tasks such as quantum key distribution [4] and randomness certification [5, 6]. Apart from this, Ref. [7] demonstrated the usefulness of steering in a task called sub-channel discrimination, which deals with determining the direction of the evolution of a system. Consider a state ρ that evolves according to a channel $\mathcal{N} = \sum_z p_Z(z)\mathcal{N}_z$, which is equal to a random selection of a channel $\mathcal{N}_z$ according to the probability distribution p_Z. Then the information regarding which path the system takes is known as sub-channel discrimination.

A framework for a resource theory of steering was introduced in [8], in which one-way classical communication from Bob to Alice and local operations (1W-LOCC) are taken as free operations. In this framework, Bob is also allowed to measure his system and communicate the classical measurement outcome prior to the measurement choice by Alice [8, Definition 1]. Thus, he can influence the input to her black box. See Figure 1 for a schematic representation. In the resource theory of steering, any steering monotone should be non-increasing under 1W-LOCC and equal to zero if a given assemblage is unsteerable. It is also desirable for the quantity to be convex. Several steering quantifiers, including robustness of steering [9], steerable weight [7], and relative entropy of steering [8, 10], have been defined and proven to be a steering monotone.

One contribution of our paper is to introduce *intrinsic steerability* as a measure of steering. Intrinsic steerability uses conditional mutual information to measure the deviation of a given assemblage from one having a local hidden-state model. The idea behind the quantifier is to suppress the correlations that can be explained by an inaccessible quantum system and then quantify the remaining intrinsic correlations. We prove that intrinsic steerability is monotone with respect to 1W-LOCC and also that it is convex and superadditive in general.

We also consider a simpler, restricted class of free operations in which Bob cannot influence Alice's input to her black box. In considering this restricted class, we are motivated by practical, relativistic constraints that can potentially limit the performance of Alice and Bob's quantum devices in any quantum steering protocol. Typ-

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ically, in any such protocol, Alice, Bob, and the source of their systems are spatially separated, and furthermore, their quantum devices typically have a finite coherence time. If Alice were to wait to receive a signal from Bob before taking any action on her system, the performance of her device could potentially get much worse than it would be if she were simply instead to input to her system as soon as she receives it from the source. This perspective motivates a restricted class of 1W-LOCC operations in which any classical communication from Bob reaches Alice only after she has received the output $\overline{A}$ from her black box. We refer to these free operations as *restricted 1W-LOCC*.

We define the *restricted intrinsic steerability* as a steering quantifier, which is relevant for the aforementioned restricted class of 1W-LOCC operations. We prove that, along with it being a monotone with respect to restricted 1W-LOCC and satisfying the properties mentioned above, it also satisfies additivity and monogamy. To our knowledge, this is the first measure shown to be monogamous and additive with respect to tensor products of assemblages.

Our approach to defining these steering quantifiers is inspired by the approach of [11] to quantifying non-Markovianity in Bayesian networks, which in turn bears connections to the squashed-entanglement measure [12] and the intrinsic-information quantifier from classical information theory [13]. To see this, consider that correlations in any unsteerable assemblage can be explained by a hidden variable, which implies that such an assemblage has a Markov-chain structure. Assemblages with this structure thus have zero conditional mutual information when conditioning on the shared variable [14], where we recall that the conditional mutual information of a tripartite quantum state σ_{KLM} is defined as

$$I(K;L|M)_\sigma := H(KM)_\sigma + H(LM)_\sigma - H(KLM)_\sigma - H(M)_\sigma \quad (1)$$

and $H(G)_\omega := -\operatorname{Tr}(\omega_G \log_2 \omega_G)$ denotes the quantum entropy of the state ω_G defined on system G (note that throughout this paper, we use the binary logarithm in the definition of entropy). So our primary idea is to take a non-signaling extension of an assemblage, remove the correlations which can be explained by a shared variable (by conditioning), and then quantify the remaining intrinsic correlations.

II. PRELIMINARIES

We begin by reviewing the framework of quantum steering as discussed in [8]. Let ρ_{AB} be a bipartite quantum state shared by Alice and Bob. Suppose that Alice performs a measurement labeled by $x \in \mathcal{X}$, with $\mathcal{X}$ denoting a finite set of quantum measurements, and she gets a classical output $a \in \mathcal{A}$, with $\mathcal{A}$ denoting a finite set of measurement outcomes. An *assemblage* consists of the state of Bob's subsystem and the conditional probability of Alice's outcome a (correlated with Bob's state) given the measurement choice x. This is specified as

$$\{p_{\overline{A}|X}(a|x), \rho_B^{a,x}\}_{a\in\mathcal{A},x\in\mathcal{X}}. \quad (2)$$

The sub-normalized state possessed by Bob is

$$\hat{\rho}_B^{a,x} := p_{\overline{A}|X}(a|x)\rho_B^{a,x}. \quad (3)$$

Taking $p_X(x)$ as a probability distribution over measurement choices, we can then embed the assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$ in a classical-quantum state as follows:

$$\rho_{X\overline{A}B} := \sum_{a,x} p_X(x)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \hat{\rho}_B^{a,x}, \quad (4)$$

where $\{|x\rangle_X\}_x$ and $\{|a\rangle_{\overline{A}}\}_a$ are orthonormal bases. Following the approach of [8], we work directly with an assemblage in what follows, such that the device on Alice's side is considered as a black box, accepting a classical input x and outputting a classical variable a with probability $p_{\overline{A}|X}(a|x)$, while the quantum state of Bob's system is $\rho_B^{a,x}$.

Assemblages are restricted by the no-signaling principle. That is, the reduced state of Bob's system should not depend on the input x to Alice's black box if the measurement output a is not available to him:

$$\sum_a \hat{\rho}_B^{a,x} = \sum_a \hat{\rho}_B^{a,x'} \quad \forall x, x' \in \mathcal{X}. \quad (5)$$

This is equivalent to $I(X;B)_\rho = 0$, where

$$I(X;B)_\rho := H(X)_\rho + H(B)_\rho - H(XB)_\rho \quad (6)$$

is the mutual information of the reduced state $\rho_{XB} = \operatorname{Tr}_{\overline{A}}(\rho_{X\overline{A}B})$.

An assemblage is *unsteerable* if arises from a classical, shared random variable Λ in the following sense [3]:

$$\hat{\rho}_B^{a,x} := \sum_\lambda p_\Lambda(\lambda)\; p_{\overline{A}|X\Lambda}(a|x,\lambda)\; \rho_B^\lambda, \quad (7)$$

where $p_\Lambda(\lambda)$ is a probability distribution for Λ. The above structure indicates that the correlations observed can be explained by a classical random variable Λ, a copy of which is sent to both Alice and Bob, who then take actions conditioned on the particular realization λ of Λ. The set of all unsteerable assemblages is referred to as LHS (short for assemblages having a "local-hidden-state model").

We point out that the setting considered in the resource theory of steering [8], reviewed above, is somewhat different from that in [3]. In the original paper [3], steering is considered as a property of a quantum state. That is, a quantum state is considered steerable if there exists a local measurement on Alice's system that leads to correlations that cannot be explained in terms of a local-hidden-state model. The definition considered in

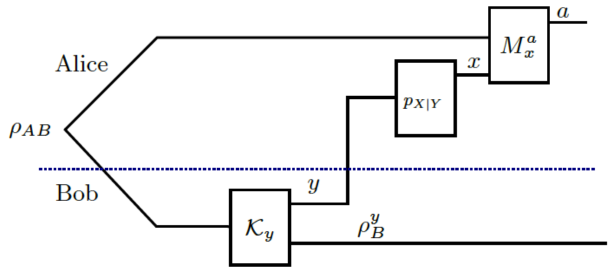

FIG. 1. This figure represents a 1W-LOCC operation acting on an assemblage realized by an underlying quantum state ρ_{AB} and measurement apparatus $\{M_x^a\}_a$. Bob is allowed to send classical information y to Alice, who chooses the input x to her black box according to $p_{X|Y}$.

[8] (and that which we consider here) is to work directly with an assemblage, i.e., such that Alice's share of the bipartite quantum state is embedded in the untrusted measurement device and the entire embedding is treated as a black box with unknown internal functioning.

As discussed above, the most general free operations allowed in the context of quantum steering are 1W-LOCC. Starting with a given assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$, it is possible for Bob to perform a quantum instrument on his system, specified as the following measurement channel acting on an input state σ_B:

$$\mathcal{M}_{B\to B'Y}(\sigma_B) := \sum_y \mathcal{K}_y(\sigma_B)\otimes |y\rangle\langle y|_Y\,, \tag{8}$$

$$\mathcal{K}_y(\sigma_B) := \sum_t K_{y,t}\sigma_B K_{y,t}^\dagger. \tag{9}$$

The sum map $\sum_y \mathcal{K}_y$ is trace preserving, i.e., $\sum_{y,t} K_{y,t}^\dagger K_{y,t} = I_B$ and each $K_{y,t}$ is a Kraus operator, taking a vector in $\mathcal{H}_B$ to a vector in $\mathcal{H}_{B'}$. Bob can then communicate the classical result y to Alice, who chooses the input x to her black box according to a classical channel $p_{X|Y}(x|y)$. The state after these operations is

$$\rho_{X\overline{A}B'Y} := \sum_{a,x,y} p_{X|Y}(x|y)\,|x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \mathcal{K}_y(\hat{\rho}_B^{a,x})\otimes |y\rangle\langle y|_Y\,. \tag{10}$$

Figure 1 depicts a 1W-LOCC operation acting on an assemblage realized by an underlying quantum state ρ_{AB} and measurement apparatus $\{M_x^a\}_a$.

III. DEFINITIONS AND SUMMARY OF RESULTS

A. Intrinsic Steerability

We allow Alice and Bob to operate on the assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$ to maximize their correlations, resulting in the following definition for intrinsic steerability:

Definition 1 (Intrinsic Steerability) *Let $\{\hat{\rho}_B^{a,x}\}_{a,x}$ denote an assemblage, and let $\rho_{X\overline{A}B'Y}$ be a state resulting from a 1W-LOCC operation as described above. Consider a non-signaling extension $\rho_{X\overline{A}B'EY}$ of $\rho_{X\overline{A}B'Y}$ of the following form:*

$$\rho_{X\overline{A}B'EY} := \sum_{x,a,y} p_{X|Y}(x|y)\,|x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \hat{\rho}_{B'E}^{a,x,y}\otimes |y\rangle\langle y|_Y\,, \tag{11}$$

where $\hat{\rho}_{B'E}^{a,x,y}$ satisfies

$$\mathrm{Tr}_E(\hat{\rho}_{B'E}^{a,x,y}) = \mathcal{K}_y(\hat{\rho}_B^{a,x}) \tag{12}$$

and the following no-signaling constraints:

$$\sum_a \hat{\rho}_{B'E}^{a,x,y} = \sum_a \hat{\rho}_{B'E}^{a,x',y}\ \forall x,x'\in\mathcal{X},\ y\in\mathcal{Y}. \tag{13}$$

We define the intrinsic steerability of a given assemblage as follows:

$$S(\overline{A};B)_{\hat{\rho}} := \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}}\ \inf_{\rho_{X\overline{A}B'EY}} I(X\overline{A};B'|EY)_\rho, \tag{14}$$

where the supremum is with respect to all quantum instruments, consisting of trace non-increasing maps $\{\mathcal{K}_y\}_y$ such that the sum map $\sum_y \mathcal{K}_y$ is trace preserving and all classical channels $p_{X|Y}$ leading to Alice's input choice x. The infimum is with respect to all non-signaling extensions of $\rho_{X\overline{A}B'Y}$. Using the no-signaling constraints, which imply that $I(X;B'|EY)_\rho = 0$, we can write

$$S(\overline{A};B)_{\hat{\rho}} := \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}}\ \inf_{\rho_{X\overline{A}B'EY}} I(\overline{A};B'|EXY)_\rho. \tag{15}$$

The idea behind the intrinsic steerability is to measure the correlations between Alice and Bob's systems after conditioning on all of the systems that an eavesdropper could have, with the worst possible scenario being that the eavesdropper possesses an arbitrary non-signaling extension of $\mathcal{K}_y(\hat{\rho}_B^{a,x})$. We take the order of optimizations to be similar to the order given for the squashed entanglement of a quantum channel [15]: Alice and Bob first pick a 1W-LOCC strategy to maximize their correlations, and Eve is allowed to react to this strategy, with the goal of minimizing their correlations. Here the only restriction on Eve's system is that it has to be no-signaling. It is possible to have other restrictions on Eve's system and have modifications of the measure accordingly. Our most fundamental result is the following theorem about intrinsic steerability.

Theorem 2 *The intrinsic steerability $S(\overline{A};B)_{\hat{\rho}}$ is a convex steering monotone. That is, it does not increase on average under deterministic 1W-LOCC, it vanishes for an assemblage having a local-hidden-state model, and it is convex.*

Our proof of Theorem 2 is given in Section V.

B. Restricted Intrinsic Steerability

Definition 1 might seem rather complicated with the number of systems involved and the number of objects involved in the optimizations. While undesirable, we note that other steering quantifiers, such as the relative entropy of steering [8, 10], feature similar complications, and this seems unavoidable, having to do with the structure of assemblages and 1W-LOCC operations.

We are thus motivated to find simpler definitions, and we can do so by considering restricted 1W-LOCC operations as discussed above.

Definition 3 (Restricted Intrinsic Steerability) *Let $\{\hat{\rho}_B^{a,x}\}_{a,x}$ denote an assemblage, and let $\rho_{X\overline{A}B}$ denote a corresponding classical–quantum state. Consider a non-signaling extension $\rho_{X\overline{A}BE}$ of $\rho_{X\overline{A}B}$ of the following form:*

$$\rho_{X\overline{A}B'E} := \sum_{a,x} p_X(x)\, |x\rangle\!\langle x|_X \otimes |a\rangle\!\langle a|_{\overline{A}} \otimes \hat{\rho}_{BE}^{a,x}, \qquad (16)$$

where $\hat{\rho}_{BE}^{a,x}$ satisfies $\mathrm{Tr}_E(\hat{\rho}_{BE}^{a,x}) = \hat{\rho}_B^{a,x}$ and the following no-signaling constraints:

$$\sum_a \hat{\rho}_{BE}^{a,x} = \sum_a \hat{\rho}_{BE}^{a,x'} \;\forall x, x' \in \mathcal{X}. \qquad (17)$$

We define the restricted intrinsic steerability of $\{\hat{\rho}_B^{a,x}\}_{a,x}$ as follows:

$$S^R(\overline{A};B)_{\hat{\rho}} := \sup_{p_X} \inf_{\rho_{X\overline{A}BE}} I(X\overline{A};B|E)_\rho, \qquad (18)$$

where the supremum is with respect to all probability distributions p_X and the infimum is with respect to all non-signaling extensions of $\rho_{X\overline{A}B}$. Using the no-signaling constraints, which imply that $I(X;B|E)_\rho = 0$, it follows that

$$S^R(\overline{A};B)_{\hat{\rho}} := \sup_{p_X} \inf_{\rho_{X\overline{A}BE}} I(\overline{A};B|EX)_\rho. \qquad (19)$$

We prove that the restricted intrinsic steerability is a steering monotone with respect to restricted 1W-LOCC and that it is convex.

Theorem 4 *The restricted intrinsic steerability $S^R(\overline{A};B)_{\hat{\rho}}$ is a convex steering monotone with respect to restricted 1W-LOCC. That is, it does not increase under restricted deterministic 1W-LOCC, it vanishes for assemblages having a local-hidden-state model, and it is convex.*

Our proof for Theorem 4 is given in Section VI.

By inspecting definitions, we can conclude that intrinsic steerability is never smaller than restricted intrinsic steerability:

$$S(\overline{A};B)_{\hat{\rho}} \geq S^R(\overline{A};B)_{\hat{\rho}}. \qquad (20)$$

This follows because the restricted intrinsic steerability involves a supremization over particular 1W-LOCC strategies that are included in the supremization in the definition of the intrinsic steerability.

By using known bounds on conditional mutual information, the expression in (15), and the fact that taking an infimum over classical extensions E does not decrease $S(\overline{A};B)_{\hat{\rho}}$, we can conclude that

$$0 \leq S(\overline{A};B)_{\hat{\rho}} \leq \log_2 |\overline{A}|. \qquad (21)$$

The lower bound follows from the strong subadditivity of quantum entropy [16] and the upper bound follows from a dimension bound (see, e.g., [17]). Similarly, using known bounds on conditional mutual information, the expression in (19), and the fact that taking an infimum over classical extensions E does not decrease $S^R(\overline{A};B)_{\hat{\rho}}$, we find that

$$0 \leq S^R(\overline{A};B)_{\hat{\rho}} \leq \min\{\log_2 |\overline{A}|, \log_2 |B|\}. \qquad (22)$$

IV. EXAMPLES

As an example, consider the following "BB84 assemblage" resulting from Pauli σ_Z or σ_X measurements on one share of a maximally entangled state

$$|\Phi\rangle_{AB} := (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}, \qquad (23)$$

consisting of the following four subnormalized states:

$$\hat{\rho}_B^{a=0,x=0} = \tfrac{1}{2}|0\rangle\langle 0|_B, \qquad (24)$$
$$\hat{\rho}_B^{a=1,x=0} = \tfrac{1}{2}|1\rangle\langle 1|_B, \qquad (25)$$
$$\hat{\rho}_B^{a=0,x=1} = \tfrac{1}{2}|+\rangle\langle +|_B, \qquad (26)$$
$$\hat{\rho}_B^{a=1,x=1} = \tfrac{1}{2}|-\rangle\langle -|_B. \qquad (27)$$

As we show in the proof of Proposition 5, the non-signaling constraint for this case imposes that any non-signaling extension of the above assemblage has the form $\hat{\rho}_B^{a,x} \otimes \omega_E$ for all $a, x \in \{0,1\}$ and for some state ω_E. ***Thus, in this sense, the BB84 assemblage is unextendible and features a certain kind of monogamy against non-signaling adversaries***. As a consequence, we find that this assemblage has exactly one bit of intrinsic steerability.

In Proposition 6, we generalize the above result to an assemblage resulting from an arbitrary pure bipartite state being measured in the Schmidt basis and the basis Fourier conjugate to this one. We find that this assemblage has the same kind of monogamy against non-signaling adversaries and that it has restricted intrinsic steerability equal to the entropy of entanglement [18] of the state being measured.

Proposition 5 *Consider a maximally entangled state*

$$|\Phi\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}). \qquad (28)$$

Let measurement $x = 0$ be Pauli σ_Z on system A, with outcomes $a = 0$ and $a = 1$. Let measurement $x = 1$ be Pauli σ_X on system A, with outcomes $a = 0$ and $a = 1$. This leads to the following assemblage:

$$\left\{ \begin{array}{ll} \hat{\rho}_B^{a=0,x=0} = \frac{1}{2}|0\rangle\langle 0|_B, & \hat{\rho}_B^{a=1,x=0} = \frac{1}{2}|1\rangle\langle 1|_B, \\ \hat{\rho}_B^{a=0,x=1} = \frac{1}{2}|+\rangle\langle +|_B, & \hat{\rho}_B^{a=1,x=1} = \frac{1}{2}|-\rangle\langle -|_B \end{array} \right\}, \tag{29}$$

which has one bit of intrinsic steerability and restricted intrinsic steerability:

$$S(\overline{A};B)_{\hat{\rho}} = S^R(\overline{A};B)_{\hat{\rho}} = 1. \tag{30}$$

Proof. Arbitrary extensions of each of the above sub-normalized states are as follows:

$$\hat{\rho}_{BE}^{a=0,x=0} = \frac{1}{2}|0\rangle\langle 0|_B \otimes \omega_E^{00}, \tag{31}$$

$$\hat{\rho}_{BE}^{a=1,x=0} = \frac{1}{2}|1\rangle\langle 1|_B \otimes \omega_E^{10}, \tag{32}$$

$$\hat{\rho}_{BE}^{a=0,x=1} = \frac{1}{2}|+\rangle\langle +|_B \otimes \omega_E^{01}, \tag{33}$$

$$\hat{\rho}_{BE}^{a=1,x=1} = \frac{1}{2}|-\rangle\langle -|_B \otimes \omega_E^{11}, \tag{34}$$

where $\omega_E^{ij} \geq 0$ and $\mathrm{Tr}(\omega_E^{ij}) = 1$ for all $i, j \in \{0,1\}$. The no-signaling constraint is as follows:

$$\hat{\rho}_{BE}^{a=0,x=0} + \hat{\rho}_{BE}^{a=1,x=0} = \hat{\rho}_{BE}^{a=0,x=1} + \hat{\rho}_{BE}^{a=1,x=1}. \tag{35}$$

Writing out the left-hand side of (35) in matrix form, we find that

$$\frac{1}{2}|0\rangle\langle 0|_B \otimes \omega_E^{00} + \frac{1}{2}|1\rangle\langle 1|_B \otimes \omega_E^{10} = \frac{1}{2}\begin{bmatrix} \omega_E^{00} & 0 \\ 0 & \omega_E^{10} \end{bmatrix}. \tag{36}$$

Writing out the right-hand side of (35) in matrix form, we find that

$$\frac{1}{2}|+\rangle\langle +|_B \otimes \omega_E^{01} + \frac{1}{2}|-\rangle\langle -|_B \otimes \omega_E^{11} \tag{37}$$

$$= \frac{1}{4}\left[|0\rangle\langle 0|_B + |1\rangle\langle 0|_B + |0\rangle\langle 1|_B + |1\rangle\langle 1|_B\right] \otimes \omega_E^{01} + \frac{1}{4}\left[|0\rangle\langle 0|_B - |1\rangle\langle 0|_B - |0\rangle\langle 1|_B + |1\rangle\langle 1|_B\right] \otimes \omega_E^{11} \tag{38}$$

$$= \frac{1}{2}|0\rangle\langle 0|_B \otimes \left(\frac{\omega_E^{01} + \omega_E^{11}}{2}\right) + \frac{1}{2}|1\rangle\langle 0|_B \otimes \left(\frac{\omega_E^{01} - \omega_E^{11}}{2}\right) + \frac{1}{2}|0\rangle\langle 1|_B \otimes \left(\frac{\omega_E^{01} - \omega_E^{11}}{2}\right) + \frac{1}{2}|1\rangle\langle 1|_B \otimes \left(\frac{\omega_E^{01} + \omega_E^{11}}{2}\right). \tag{39}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{\omega_E^{01}+\omega_E^{11}}{2} & \frac{\omega_E^{01}-\omega_E^{11}}{2} \\ \frac{\omega_E^{01}-\omega_E^{11}}{2} & \frac{\omega_E^{01}+\omega_E^{11}}{2} \end{bmatrix}. \tag{40}$$

So equating them, we find that the following equation (no-signaling constraint) should be satisfied

$$\begin{bmatrix} \omega_E^{00} & 0 \\ 0 & \omega_E^{10} \end{bmatrix} = \begin{bmatrix} \frac{\omega_E^{01}+\omega_E^{11}}{2} & \frac{\omega_E^{01}-\omega_E^{11}}{2} \\ \frac{\omega_E^{01}-\omega_E^{11}}{2} & \frac{\omega_E^{01}+\omega_E^{11}}{2} \end{bmatrix}. \tag{41}$$

This implies that $\omega_E^{01} = \omega_E^{11}$, which in turn implies that $\omega_E^{10} = \omega_E^{01} = \omega_E^{11} = \omega_E^{00}$. Thus, the only possible extension allowed in order to satisfy the no-signaling constraint is a product extension independent of a and x, meaning one of the following form:

$$\hat{\rho}_{BE}^{a=0,x=0} = \frac{1}{2}|0\rangle\langle 0|_B \otimes \omega_E,$$
$$\hat{\rho}_{BE}^{a=1,x=0} = \frac{1}{2}|1\rangle\langle 1|_B \otimes \omega_E,$$
$$\hat{\rho}_{BE}^{a=0,x=1} = \frac{1}{2}|+\rangle\langle +|_B \otimes \omega_E,$$
$$\hat{\rho}_{BE}^{a=1,x=1} = \frac{1}{2}|-\rangle\langle -|_B \otimes \omega_E, \tag{42}$$

where $\omega_E \geq 0$ and $\mathrm{Tr}(\omega_E) = 1$. We can then evaluate the restricted intrinsic steerability in terms of the following classical–quantum state:

$$\begin{bmatrix} \frac{1}{2}|0\rangle\langle 0|_X \otimes |0\rangle\langle 0|_{\overline{A}} \otimes \frac{1}{2}|0\rangle\langle 0|_B \\ +\frac{1}{2}|0\rangle\langle 0|_X \otimes |1\rangle\langle 1|_{\overline{A}} \otimes \frac{1}{2}|1\rangle\langle 1|_B \\ +\frac{1}{2}|1\rangle\langle 1|_X \otimes |0\rangle\langle 0|_{\overline{A}} \otimes \frac{1}{2}|+\rangle\langle +|_B \\ +\frac{1}{2}|1\rangle\langle 1|_X \otimes |1\rangle\langle 1|_{\overline{A}} \otimes \frac{1}{2}|-\rangle\langle -|_B \end{bmatrix} \otimes \omega_E. \tag{43}$$

The conditional mutual information of this state is as follows:

$$\begin{aligned} I(X\overline{A};B|E) &= I(X\overline{A};B) \\ &= H(B) - H(B|X\overline{A}) = H(B) = 1, \end{aligned} \tag{44}$$

so that this assemblage has *one bit of restricted intrinsic steerability*. The first equality follows because the system E is product regardless of the extension, due to the above analysis with the no-signaling constraint. The second equality follows by expanding the mutual information. The third equality follows because the state of the B system is pure when conditioned on systems $X\overline{A}$. The final equality follows because the reduced state on the B system is maximally mixed. Also, it is clear that this is the maximum value of the restricted intrinsic steerability, given that it is always bounded from above by $\log\dim(\mathcal{H}_B)$ or $\log\dim(\mathcal{H}_{\overline{A}})$. By considering the upper bound $\log\dim(\mathcal{H}_{\overline{A}})$ for intrinsic steerability, we see that this assemblage achieves the upper bound on intrinsic steerability and thus has one bit of intrinsic steerability. ■

Proposition 6 *Consider a pure bipartite state $|\varphi\rangle_{AB}$ in its Schmidt basis:*

$$|\varphi\rangle_{AB} := \sum_{j=0}^{d-1} \alpha_j |j\rangle_A \otimes |j\rangle_B, \tag{45}$$

where $|\alpha_j| \neq 0$ for all $j \in \{0,\ldots,d-1\}$. Let measurement $x=0$ be a measurement $\{|j\rangle\langle j|_A\}_j$ in the Schmidt basis on system A, with outcomes $a = j \in \{0,\ldots,d-1\}$. Let measurement $x=1$ be a measurement $\{|\widetilde{j}\rangle\langle\widetilde{j}|_A\}_j$ in the Fourier conjugate basis, where

$$|\widetilde{j}\rangle_A := \frac{1}{\sqrt{d}}\sum_k e^{2\pi ijk/d}|k\rangle_A, \tag{46}$$

on system A, with outcomes $a = j \in \{0,\ldots,d-1\}$. This leads to the following assemblage:

$$\left\{\left\{\hat{\rho}_B^{a=j,x=0} = |\alpha_j|^2\,|j\rangle\langle j|_B\right\}_j,\right.$$
$$\left.\left\{\hat{\rho}_B^{a=j,x=1} = \frac{1}{d}Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j)\right\}_j,\right\}, \tag{47}$$

where $|\psi\rangle_B := \sum_j \alpha_j|j\rangle_B$. This assemblage has

$$H(\{|\alpha_j|^2\}_j) = H(A)_\varphi \tag{48}$$

bits of restricted intrinsic steerability. Note that this is equal to the entropy of entanglement of the state $|\varphi\rangle_{AB}$. If the state $|\varphi\rangle_{AB}$ is maximally entangled so that $\alpha_j = 1/\sqrt{d}$, then the resulting assemblage has $\log_2(d)$ bits of intrinsic steerability.

Proof. It is clear that the post-measurement state for Bob $\hat{\rho}_B^{a=j,x=0}$ is as above. For the other case, consider that

$$\begin{aligned}
&\langle\widetilde{j}|_A \otimes I_B|\varphi\rangle_{AB} \\
&= \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} e^{-2\pi ijk/d}\langle k|_A \sum_{l=0}^{d-1}\alpha_l|l\rangle_A \otimes |l\rangle_B && (49)\\
&= \frac{1}{\sqrt{d}}\sum_{k,l=0}^{d-1} \alpha_k e^{-2\pi ijk/d}\langle k|l\rangle_A \otimes |l\rangle_B && (50)\\
&= \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} \alpha_k e^{-2\pi ijk/d}|k\rangle_B. && (51)
\end{aligned}$$

Now defining the unitary operator $Z(j)$ by $Z(j)|k\rangle = e^{2\pi ijk/d}|k\rangle$ for $j,k \in \{0,\ldots,d-1\}$, we can write

$$\langle\widetilde{j}|_A \otimes I_B|\varphi\rangle_{AB} = \frac{1}{\sqrt{d}}Z^\dagger(j)|\psi\rangle_B, \tag{52}$$

confirming the post-measurement subnormalized states $\hat{\rho}_B^{a=j,x=1}$. Arbitrary extensions of each of the above subnormalized states are as follows:

$$\hat{\rho}_{BE}^{a=j,x=0} = |\alpha_j|^2\,|j\rangle\langle j|_B \otimes \omega_E^j, \tag{53}$$
$$\hat{\rho}_{BE}^{a=j,x=1} = \frac{1}{d}Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j) \otimes \tau_E^j, \tag{54}$$

where $\omega_E^j, \tau_E^j \geq 0$ and $\mathrm{Tr}(\omega_E^j) = \mathrm{Tr}(\tau_E^j) = 1$ for all $j \in \{0,\ldots,d-1\}$. The no-signaling constraint is as follows:

$$\sum_{j=0}^{d-1}\hat{\rho}_{BE}^{a=j,x=0} = \sum_{j=0}^{d-1}\hat{\rho}_{BE}^{a=j,x=1}, \tag{55}$$

which is the same as

$$\begin{aligned}
&\sum_{k=0}^{d-1}|k\rangle\langle k|_B \otimes |\alpha_k|^2\,\omega_E^k \\
&= \sum_{j=0}^{d-1}\frac{1}{d}Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j) \otimes \tau_E^j && (56)\\
&= \sum_{j,k,k'=0}^{d-1}\frac{1}{d}\alpha_k\alpha_{k'}^* e^{-2\pi ij(k-k')/d}|k\rangle\langle k'|_B \otimes \tau_E^j && (57)\\
&= \sum_{k,k'=0}^{d-1}|k\rangle\langle k'|_B \otimes \frac{1}{d}\alpha_k\alpha_{k'}^*\sum_{j=0}^{d-1}e^{-2\pi ij(k-k')/d}\tau_E^j. && (58)
\end{aligned}$$

Set $k'=0$. For $k \in \{0,1,\ldots,d-1\}$, we get the following constraints from the no-signaling condition:

$$\omega_E^0 = \frac{1}{d}\sum_{j=0}^{d-1}\tau_E^j, \tag{59}$$
$$0 = \sum_{j=0}^{d-1}e^{-2\pi ijk/d}\tau_E^j. \tag{60}$$

We can conclude that τ_E^j is independent of j, so that $\tau_E^j = \omega_E^0$ for all $j \in \{0,\ldots,d-1\}$. To see this, let us solve the above equations, thinking of ω_E^0 as fixed and τ_E^j as free for all $j \in \{0,\ldots,d-1\}$. Consider that

$$\sum_{j=0}^{d-1}e^{-2\pi ijk/d} = 0 \quad \forall k \in \{1,\ldots,d-1\}. \tag{61}$$

Then we can see that $\tau_E^0 = \tau_E^1 = \cdots = \tau_E^{d-1} = \omega_E^0$ is one of the solutions of the equations in (59)–(60). Since the equations are linearly independent, it is a unique solution. Now considering the other blocks in (56) (i.e., for $k = k' = 1,\ldots,d-1$), we find that $\omega_E^1 = \cdots = \omega_E^{d-1} = \omega_E^0$. Thus, the only possible extension allowed in order to satisfy the no-signaling constraint is a product extension independent of a and x, meaning one of the following form:

$$\hat{\rho}_{BE}^{a=j,x=0} = |\alpha_j|^2\,|j\rangle\langle j|_B \otimes \omega_E, \tag{62}$$
$$\hat{\rho}_{BE}^{a=j,x=1} = \frac{1}{d}Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j) \otimes \omega_E, \tag{63}$$

where $\omega_E \geq 0$ and $\mathrm{Tr}(\omega_E) = 1$. We can then evaluate the restricted intrinsic steerability in terms of the following classical–quantum state:

$$\Big[p|0\rangle\langle 0|_X \otimes \sum_j |j\rangle\langle j|_{\overline{A}} \otimes |\alpha_j|^2\, |j\rangle\langle j|_B + (1-p)\,|1\rangle\langle 1|_X \otimes \sum_j |j\rangle\langle j|_{\overline{A}} \otimes \frac{1}{d} Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j)\Big] \otimes \omega_E, \quad (64)$$

where $(p, 1-p)$ is a probability distribution for the input x. The conditional mutual information of this state is as follows:

$$I(X\overline{A}; B|E) = I(X\overline{A}; B) = H(B) - H(B|X\overline{A}) \quad (65)$$

$$= H(B) = H(\{|\alpha_j|^2\}), \quad (66)$$

so that this assemblage has $H(\{|\alpha_j|^2\})$ *bits of restricted intrinsic steerability*. The first step follows because the system E is product regardless of the extension, due to the above analysis with the no-signaling constraint. The second step follows by expanding the mutual information. The third step follows because the state of the B system is pure when conditioned on systems $X\overline{A}$. The final step follows because the reduced state on the B system is $\sum_j |\alpha_j|^2\, |j\rangle\langle j|_B$, which can be seen from

$$\mathrm{Tr}_{X\overline{A}}\Bigg(p|0\rangle\langle 0|_X \otimes \sum_j |j\rangle\langle j|_{\overline{A}} \otimes |\alpha_j|^2\, |j\rangle\langle j|_B + (1-p)\,|1\rangle\langle 1|_X \otimes \sum_j |j\rangle\langle j|_{\overline{A}} \otimes \frac{1}{d} Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j)\Bigg)$$

$$= p \sum_j |\alpha_j|^2\, |j\rangle\langle j|_B + (1-p) \sum_j \frac{1}{d} Z^\dagger(j)|\psi\rangle\langle\psi|_B Z(j) \quad (67)$$

$$= p \sum_j |\alpha_j|^2\, |j\rangle\langle j|_B + (1-p) \sum_j |\alpha_j|^2\, |j\rangle\langle j|_B \quad (68)$$

$$= \sum_j |\alpha_j|^2\, |j\rangle\langle j|_B. \quad (69)$$

This state is independent of the input probability distribution, so that the maximum is achieved for any choice of $p \in (0, 1)$.

If the state $|\varphi\rangle_{AB}$ is maximally entangled, then

$$H(\{|\alpha_j|^2\}) = \log_2(d). \quad (70)$$

Given the upper bound $\log(\dim(\mathcal{H}_{\overline{A}})) = \log_2(d)$ on intrinsic steerability, we see that the upper bound is achieved in this case. ■

V. INTRINSIC STEERABILITY

We now give a proof for Theorem 2 and proofs for other properties of the intrinsic steerability stated earlier.

Proposition 7 *Intrinsic steerability vanishes for assemblages having an LHS model.*

Proof. To prove this, consider the following particular non-signaling extension for an assemblage with a local-hidden-state model:

$$\sum_{x,a,\lambda,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes p_{\overline{A}|X\Lambda}(a|x,\lambda)\, |a\rangle\langle a|_{\overline{A}} \otimes \sum_t K_{y,t} \hat{\rho}_B^{\lambda} K_{y,t}^\dagger \otimes p_\Lambda(\lambda)\, |\lambda\rangle\langle\lambda|_E \otimes |y\rangle\langle y|_Y\,. \quad (71)$$

For this non-signaling extension, conditioned on the values λ and y, systems $X\overline{A}$ and B' are in a product state, so that the conditional mutual information $I(X\overline{A}; B'|EY)$ vanishes. The same argument applies to all quantum instruments $\{\mathcal{K}_y\}_y$ and channels $p_{X|Y}$, so that

$$S(\overline{A}; B)_\rho = 0 \quad (72)$$

in this case. ■

Proposition 8 (1W-LOCC monotone) *Let $\{\hat{\rho}_B^{a,x}\}_{a,x}$ be an assemblage, and suppose that*

$$\left\{\hat{\rho}_{B_f,z}^{a_f,x_f} := \sum_{a,x} p(a_f|x_f, x, a, z) p(x|x_f, z) \mathcal{K}_z(\hat{\rho}_B^{a,x})/p(z)\right\}_{a_f,x_f}, \quad (73)$$

is an assemblage that arises from it by the action of a general 1W-LOCC operation, where

$$p(z) := \mathrm{Tr}\left(\mathcal{K}_z\left(\sum_a \hat{\rho}_B^{a,x}\right)\right) = \mathrm{Tr}(\mathcal{K}_z(\rho_B)). \quad (74)$$

Then the intrinsic steerability is monotone on average under deterministic 1W-LOCC, in the following sense:

$$\sum_z p(z) S(\overline{A}_f; B_f)_{\hat{\rho}_z} \le S(\overline{A}; B)_{\hat{\rho}}. \quad (75)$$

Proof. First, we give a proof sketch for the monotonicity of intrinsic steerability on average under deterministic 1W-LOCC:

$$S(\overline{A}; B)_{\hat{\rho}} \ge \sum_z p_Z(z) S(\overline{A}_f; B_f)_{\hat{\rho}_z}, \quad (76)$$

where $\hat{\rho}_z := \{\hat{\rho}_{B_f,z}^{a_f,x_f}\}_{a_f,x_f}$ is the assemblage resulting from a 1W-LOCC operation on the initial assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$ and is given as [8]

$$\hat{\rho}_{B_f,z}^{a_f,x_f} := \sum_{a,x} p(a_f|a, x, x_f, z) p(x|x_f, z) \mathcal{K}_z(\hat{\rho}_B^{a,x}). \quad (77)$$

In the above, $p(a_f|a,x,x_f,z)$ and $p(x|x_f,z)$ are local classical channels that Alice uses, respectively, to pick the output a_f of the final assemblage and the input x to her initial assemblage. The set $\{\mathcal{K}_z\}_z$ is such that the sum map $\sum_z \mathcal{K}_z$ is trace preserving and thus corresponds to a measurement of Bob's system. The definition of the intrinsic steerability involves a supremum over measurements of the system B_f of the final assemblage and classical channels for the input X_f to the final assemblage. Using data processing and when given Z, we can say that system $\overline{A}_f$ was obtained by processing systems $XX_f\overline{A}$. Then, the two successive measurements on Bob's system can be thought of as a single measurement. Since the intrinsic steerability involves a supremum over all possible measurements, the result follows.

We now give a detailed proof. To see this, consider that, in accordance with the definition of $S(\overline{A}_f;B_f)_{\hat{\rho}_z}$, the assemblages $\{\hat{\rho}^{a_f,x_f}_{B_f,z}\}_{a_f,x_f}$ can be further preprocessed by a z-dependent 1W-LOCC $\{p_{X_f|YZ=z},\{\mathcal{L}^{(z)}_y\}_y\}$, resulting in the following state:

$$\sigma^z_{X_f\overline{A}_fB'_fY} := \sum_{a_f,x_f,y} p(x_f|yz)[x_f]\otimes[a_f]\otimes\mathcal{L}^{(z)}_y(\hat{\rho}^{a_f,x_f}_{B_f,z})\otimes[y]. \tag{78}$$

Notation 9 *In the above and in what follows, we employ a shorthand* $[x]\equiv|x\rangle\langle x|_X$ *or* $[a]\equiv|a\rangle\langle a|_{\overline{A}}$*, etc.*

The state in (78) is extended by the following one:

$$\sigma^z_{X_fX\overline{A}_f\overline{A}B'_fY} := \sum_{a_f,a,x,x_f,y} p(x_f|yz)[x_f] \otimes p(x|x_f,z)[x]\otimes p(a_f|x_f,x,a,z)[a_f]\otimes[a] \otimes \frac{\mathcal{L}^{(z)}_y(\hat{\rho}^{a,x}_B)}{p(z)}\otimes[y], \tag{79}$$

which in turn are elements of the following classical–quantum state:

$$\sigma_{X_fX\overline{A}_f\overline{A}B'_fYZ} := \sum_z \sigma^z_{X_fX\overline{A}_f\overline{A}B'_fY}\otimes p(z)[z]. \tag{80}$$

An *arbitrary* non-signaling extension of the state in (78), according to that needed in the definition of $S(\overline{A}_f;B_f)_{\hat{\rho}_z}$, is as follows:

$$\sigma^z_{X_f\overline{A}_fB'_fEY} := \sum_{a_f,x_f,y} p(x_f|yz)[x_f]\otimes[a_f] \otimes \hat{\tau}^{a_f,x_f,y,z}_{B'_fE}\otimes[y], \tag{81}$$

where $\hat{\tau}^{a_f,x_f,y,z}_{B'_fE}$ satisfies

$$\mathrm{Tr}_E(\hat{\tau}^{a_f,x_f,y,z}_{B'_fE}) = \mathcal{L}^{(z)}_y(\hat{\rho}^{a_f,x_f}_{B_f,z}), \tag{82}$$

$$\sum_{a_f}\hat{\tau}^{a_f,x_f,y,z}_{B'_fE} = \sum_{a_f}\hat{\tau}^{a_f,x'_f,y,z}_{B'_fE} \quad \forall x_f,x'_f\in\mathcal{X}_f,\ y\in\mathcal{Y},\ z\in\mathcal{Z}. \tag{83}$$

A *particular* non-signaling extension of the state in (78), according to that needed in the definition of $S(\overline{A}_f;B_f)_{\hat{\rho}_z}$, is as follows:

$$\zeta^z_{X_f\overline{A}_fB'_fEY} := \sum_{a_f,x_f,y} p(x_f|yz)[x_f]\otimes[a_f] \otimes\sum_{a,x} p(a_f|x_f,x,a,z)p(x|x_f,z)\hat{\omega}^{a,x,y,z}_{B'_fE}\otimes[y], \tag{84}$$

where $\hat{\omega}^{a,x,y,z}_{B'_fE}$ satisfies

$$\mathrm{Tr}_E(\hat{\omega}^{a,x,y,z}_{B'_fE}) = \frac{\mathcal{L}^{(z)}_y(\mathcal{K}_z(\hat{\rho}^{a,x}_B))}{p(z)}, \tag{85}$$

$$\sum_a\hat{\omega}^{a,x,y,z}_{B'_fE} = \sum_a\hat{\omega}^{a,x,y,z}_{B'_fE} \quad \forall x,x'\in\mathcal{X},\ y\in\mathcal{Y},\ z\in\mathcal{Z}. \tag{86}$$

The operator $\hat{\omega}^{a,x,y,z}_{B'_fE}$ will serve as an arbitrary non-signaling extension needed in the definition of $S(\overline{A};B)_{\hat{\rho}}$. Let $\zeta_{X_f\overline{A}_fB'_fEYZ}$ denote the following state:

$$\zeta_{X_f\overline{A}_fB'_fEYZ} := \sum_z \zeta^z_{X_f\overline{A}_fB'_fEY}\otimes p(z)[z]. \tag{87}$$

This in turn is a marginal of the following state:

$$\zeta_{X_fX\overline{A}_f\overline{A}B'_fEYZ} := \sum_{a_f,a,x_f,x,y} p(x_f|yz)[x_f] \otimes p(x|x_f,z)[x]\otimes p(a_f|x_f,x,a,z)[a_f]\otimes[a] \otimes\hat{\omega}^{a,x,y,z}_{B'_fE}\otimes[y]\otimes p(z)[z]. \tag{88}$$

Consider that

$$\sum_z p(z) \inf_{\text{ext. in (81)}} I(X_f\overline{A}_f;B'_f|EY)_{\sigma^z}$$
$$\leq \sum_z p(z) I(X_f\overline{A}_f;B'_f|EY)_{\zeta^z} \tag{89}$$
$$= I(X_f\overline{A}_f;B'_f|EYZ)_\zeta \tag{90}$$
$$\leq I(X_fX\overline{A};B'_f|EYZ)_\zeta \tag{91}$$
$$= I(X\overline{A};B'_f|EYZ)_\zeta + I(X_f;B'_f|EYZX\overline{A})_\zeta \tag{92}$$
$$= I(X\overline{A};B'_f|EYZ)_\zeta. \tag{93}$$

The first inequality follows because the extension state $\zeta^z_{X_f\overline{A}_fB'_fEY}$ is a particular kind of non-signaling extension required in the definition of $S(\overline{A}_f;B_f)_{\hat{\rho}_z}$. The first equality follows because system Z is classical and thus can be incorporated as a conditioning system in the conditional mutual information. The second inequality follows from local data processing for the conditional mutual information: given Z, the system $\overline{A}_f$ arises from local processing of systems $X_fX\overline{A}$. The second equality follows from the chain rule for conditional mutual information. The final equality follows from the fact that systems B'_fE are independent of X_f when given the classical systems $YZX\overline{A}$

(one can inspect the state in (88) to see this explicitly). Since the above chain of inequalities holds for any non-signaling extension of the form in (84), we can conclude that

$$\sum_z p(z) \inf_{\text{ext. in (81)}} I(X_f\overline{A}_f; B'_f|EY)_{\sigma^z} \leq \inf_{\text{ext. in (84)}} I(X\overline{A}; B'_f|EYZ)_\zeta. \quad (94)$$

Now we can take the supremum of both sides with respect to 1W-LOCC operations $\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z$ and we find that

$$\sup_{\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z} \sum_z p(z) \inf_{\text{ext. in (81)}} I(X_f\overline{A}_f; B'_f|EY)_{\sigma^z}$$
$$\leq \sup_{\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z} \inf_{\text{ext. in (84)}} I(X\overline{A}; B'_f|EYZ)_\zeta. \quad (95)$$

Since the 1W-LOCC operation $\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z$ is a particular 1W-LOCC operation that can be performed on the original assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$, we find that

$$\sup_{\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z} \inf_{\text{ext. in (84)}} I(X\overline{A}; B'_f|EYZ)_\zeta \leq S(\overline{A}; B)_{\hat{\rho}}. \quad (96)$$

Since each z-dependent 1W-LOCC operation $\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}$ depends only on a particular value of z, we can then exchange the supremum and the sum over z in (95) to conclude that

$$\sup_{\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}_z} \sum_z p(z) \inf_{\text{ext. in (81)}} I(X_f\overline{A}_f; B'_f|EY)_{\sigma^z}$$
$$= \sum_z p(z) \sup_{\{p_{X_f|YZ=z}, \{\mathcal{L}_y^{(z)}\}_y\}} \inf_{\text{ext. in (81)}} I(X_f\overline{A}_f; B'_f|EY)_{\sigma^z} \quad (97)$$
$$= \sum_z p(z) S(\overline{A}_f; B_f)_{\hat{\rho}_z}. \quad (98)$$

Putting these last steps together, we conclude (75). ■

Proposition 10 (Convexity) *Let* $\{\hat{\rho}_B^{a,x}\}_{a,x}$ *and* $\{\hat{\sigma}_B^{a,x}\}_{a,x}$ *be assemblages, and let* $\lambda \in [0,1]$. *Let* $\{\hat{\tau}_B^{a,x}\}_{a,x}$ *be a mixture of the two assemblages, defined as*

$$\hat{\tau}_B^{a,x} := \lambda\hat{\rho}_B^{a,x} + (1-\lambda)\hat{\sigma}_B^{a,x}. \quad (99)$$

Then

$$S(\overline{A}; B)_{\hat{\tau}} \leq \lambda S(\overline{A}; B)_{\hat{\rho}} + (1-\lambda) S(\overline{A}; B)_{\hat{\sigma}}. \quad (100)$$

Proof. We first give a proof sketch for the convexity of intrinsic steerability. Let $\lambda \in [0,1]$. Let $\{\hat{\rho}_B^{a,x}\}_{a,x}$ and $\{\hat{\sigma}_B^{a,x}\}_{a,x}$ be two assemblages, and consider an assemblage $\{\hat{\tau}_B^{a,x} := \lambda\hat{\rho}_B^{a,x} + (1-\lambda)\hat{\sigma}_B^{a,x}\}_{a,x}$. Convexity of the intrinsic steerability is the following statement:

$$S(\overline{A}; B)_{\hat{\tau}} \leq \lambda S(\overline{A}; B)_{\hat{\rho}} + (1-\lambda) S(\overline{A}; B)_{\hat{\sigma}}, \quad (101)$$

whose physical interpretation is that steering cannot increase when mixing two assemblages. A proof for convexity is similar to known proofs for the convexity of squashed entanglement [12] and the squashed entanglement of a channel [19]. To prove convexity, first consider arbitrary non-signaling extensions of $\{\hat{\rho}_B^{a,x}\}_{a,x}$ and $\{\hat{\sigma}_B^{a,x}\}_{a,x}$. Embedding these in a larger classical–quantum state with a label chosen according to λ gives a particular non-signaling extension of $\hat{\tau}$. Convexity then follows from a property of conditional mutual information and because the intrinsic steerability involves an infimum over all non-signaling extensions.

We now give a detailed proof. Let $\{p_{X|Y}, \{\mathcal{K}_y\}_y\}$ denote an arbitrary 1W-LOCC operation, which leads to the following classical–quantum state:

$$\tau_{X\overline{A}B'Y} := \sum_{a,x,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \mathcal{K}_y(\hat{\tau}_B^{a,x}) \otimes |y\rangle\langle y|_Y. \quad (102)$$

An *arbitrary* non-signaling extension of this state, is as follows:

$$\tau_{X\overline{A}B'YE} := \sum_{a,x,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \hat{\tau}_{B'E}^{a,x,y} \otimes |y\rangle\langle y|_Y, \quad (103)$$

where

$$\mathrm{Tr}_E(\hat{\tau}_{B'E}^{a,x,y}) = \mathcal{K}_y(\hat{\tau}_B^{a,x}), \quad (104)$$
$$\sum_a \hat{\tau}_{B'E}^{a,x,y} = \sum_a \hat{\tau}_{B'E}^{a,x',y} \quad \forall x, x' \in \mathcal{X},\ y \in \mathcal{Y}. \quad (105)$$

Let $\hat{\rho}_{B'E}^{a,x,y}$ and $\hat{\sigma}_{B'E}^{a,x,y}$ be *arbitrary* non-signaling extensions of $\mathcal{K}_y(\hat{\rho}_B^{a,x})$ and $\mathcal{K}_y(\hat{\sigma}_B^{a,x})$, satisfying

$$\mathrm{Tr}_E(\hat{\rho}_{B'E}^{a,x,y}) = \mathcal{K}_y(\hat{\rho}_B^{a,x}), \quad (106)$$
$$\sum_a \hat{\rho}_{B'E}^{a,x,y} = \sum_a \hat{\rho}_{B'E}^{a,x',y} \quad \forall x, x' \in \mathcal{X},\ y \in \mathcal{Y}, \quad (107)$$
$$\mathrm{Tr}_E(\hat{\sigma}_{B'E}^{a,x,y}) = \mathcal{K}_y(\hat{\sigma}_B^{a,x}), \quad (108)$$
$$\sum_a \hat{\sigma}_{B'E}^{a,x,y} = \sum_a \hat{\sigma}_{B'E}^{a,x',y} \quad \forall x, x' \in \mathcal{X},\ y \in \mathcal{Y}. \quad (109)$$

These lead to the following states:

$$\rho_{X\overline{A}B'YE} := \sum_{a,x,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \hat{\rho}_{B'E}^{a,x,y} \otimes |y\rangle\langle y|_Y, \quad (110)$$
$$\sigma_{X\overline{A}B'YE} := \sum_{a,x,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \hat{\sigma}_{B'E}^{a,x,y} \otimes |y\rangle\langle y|_Y. \quad (111)$$

A *particular* non-signaling extension $\tau'_{X\overline{A}B'YEE'}$ of $\tau_{\overline{A}B'XY}$, given by

$$\tau'_{X\overline{A}B'YEE'} := \sum_{a,x,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes (\lambda \hat{\rho}^{a,x,y}_{B'E} \otimes |0\rangle\langle 0|_{E'} + (1-\lambda)\hat{\sigma}^{a,x,y}_{B'E} \otimes |1\rangle\langle 1|_{E'}) \otimes |y\rangle\langle y|_Y\,. \tag{112}$$

Then consider that

$$\inf_{\text{ext. in (103)}} I(X\overline{A};B'|EY)_\tau \leq I(X\overline{A};B'|EYE')_{\tau'} \tag{113}$$

$$= \lambda I(X\overline{A};B'|EY)_\rho + (1-\lambda) I(X\overline{A};B'|EY)_\sigma. \tag{114}$$

Since the inequality above holds for all general non-signaling extensions of the form in (110) and (111), we conclude that

$$\inf_{\text{ext. in (103)}} I(X\overline{A};B'|EY)_\tau \leq \lambda \inf_{\text{ext. in (110)}} I(X\overline{A};B'|EY)_\rho + (1-\lambda) \inf_{\text{ext. in (111)}} I(X\overline{A};B'|EY)_\sigma. \tag{115}$$

Now taking a supremum over all 1W-LOCC operations, we find that

$$S(\overline{A};B)_{\hat{\tau}} = \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}} \inf_{\text{ext. in (103)}} I(X\overline{A};B'|EY)_\tau \tag{116}$$

$$\leq \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}} \left(\lambda \inf_{\text{ext. in (110)}} I(X\overline{A};B'|EY)_\rho + (1-\lambda) \inf_{\text{ext. in (111)}} I(X\overline{A};B'|EY)_\sigma\right) \tag{117}$$

$$\leq \lambda \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}} \inf_{\text{ext. in (110)}} I(X\overline{A};B'|EY)_\rho + (1-\lambda) \sup_{\{p_{X|Y},\{\mathcal{K}_y\}_y\}} \inf_{\text{ext. in (111)}} I(X\overline{A};B'|EY)_\sigma \tag{118}$$

$$= \lambda S(\overline{A};B)_{\hat{\rho}} + (1-\lambda) S(\overline{A};B)_{\hat{\sigma}}. \tag{119}$$

This concludes the proof. ∎

We now consider a superadditivity property of assemblages, which holds for intrinsic steerability. Suppose that Alice has two quantum systems A_1 and A_2 and suppose that Bob has two quantum systems B_1 and B_2. Alice could perform a local measurement on A_1 chosen according to x_1 and with output a_1. Similarly, Alice could perform a local measurement on A_2 chosen according to x_2 and with output a_2. This process realizes a joint assemblage $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$ obeying certain no-signaling constraints, but it also realizes some local assemblages as well. One would expect that the steering available in the joint assemblage should never be smaller than the sum of the steering available in the local assemblages, and this is what the following proposition addresses:

Proposition 11 (Superadditivity) *Let $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$ be an assemblage for which the following additional no-signaling constraints hold*

$$\sum_{a_2} \hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2} = \sum_{a_2} \hat{\rho}^{a_1,a_2,x_1,x_2'}_{B_1B_2} := \hat{\theta}^{a_1,x_1}_{B_1B_2} \quad \forall x_2, x_2',$$

$$\sum_{a_1} \hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2} = \sum_{a_1} \hat{\rho}^{a_1,a_2,x_1',x_2}_{B_1B_2} := \hat{\kappa}^{a_2,x_2}_{B_1B_2} \quad \forall x_1, x_1',$$

Let $\{\mathrm{Tr}_{B_2}(\hat{\theta}^{a_1,x_1}_{B_1B_2})\}_{a_1,x_1}$ and $\{\mathrm{Tr}_{B_1}(\hat{\kappa}^{a_2,x_2}_{B_1B_2})\}_{a_2,x_2}$ be reduced, local assemblages arising from the joint assemblage $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$. Then intrinsic steerability is superadditive in the following sense:

$$S(\overline{A}_1\overline{A}_2;B_1B_2)_{\hat{\rho}} \geq S(\overline{A}_1;B_1)_{\hat{\theta}} + S(\overline{A}_2;B_2)_{\hat{\kappa}}. \tag{120}$$

Proof. The core idea behind our proof of Proposition 11 is to exploit the chain rule for conditional mutual information. First, pick a 1W-LOCC strategy where Alice's inputs X_1 and X_2 depend only on measurement outcomes Y_1 and Y_2 of B_1 and B_2, respectively. The chain rule and non-negativity of conditional mutual information imply that

$$I(X_1X_2\overline{A}_1\overline{A}_2;B_1B_2|EY_1Y_2)_\rho \geq$$
$$I(X_1\overline{A}_1;B_1|EY_1Y_2)_\rho + I(X_2\overline{A}_2;B_2|EB_1Y_1Y_2)_\rho, \tag{121}$$

where system E denotes a non-signaling extension system. The idea is then to take EY_2 as a non-signaling extension for $X_1\overline{A}_1B_1Y_1$, systems EB_1Y_1 as a non-signaling extension for $X_2\overline{A}_2B_2Y_2$, and work from there.

We now give a detailed proof. Suppose that we apply to the assemblage $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$ a general 1W-LOCC operation $\{p_{X_1X_2|Y},\{\mathcal{K}_y\}_y\}$, resulting in the following classical–quantum state:

$$\rho_{\overline{A}_1X_1\overline{A}_2X_2YB_1'B_2'} := \sum_{a_1,x_1,a_2,x_2,y} p_{X_1X_2|Y}(x_1,x_2|y)[a_1]\otimes[x_1]\otimes[a_2]\otimes[x_2]\otimes[y]\otimes\mathcal{K}_y(\hat{\rho}^{a_1,x_1,a_2,x_2}_{B_1B_2}). \tag{122}$$

Let $\hat{\rho}^{a_1,x_1,a_2,x_2,y}_{B_1'B_2'E}$ be a non-signaling extension of $\mathcal{K}_y(\rho^{a_1,x_1,a_2,x_2}_{B_1B_2})$ and consider the following extension of the above state:

$$\rho_{\overline{A}_1X_1\overline{A}_2X_2YB_1'B_2'} := \sum_{a_1,x_1,a_2,x_2,y} p_{X_1X_2|Y}(x_1,x_2|y)[a_1]\otimes[x_1]\otimes[a_2]\otimes[x_2]\otimes[y]\otimes\hat{\rho}^{a_1,x_1,a_2,x_2,y}_{B_1'B_2'E}. \tag{123}$$

A particular "product" 1W-LOCC operation has the form $\{p_{X_1|Y_1}p_{X_2|Y_2},\{\mathcal{L}_{y_1}\otimes\mathcal{M}_{y_2}\}_{y_1,y_2}\}$ and results in the following state:

$$\omega_{\overline{A}_1X_1\overline{A}_2X_2Y_1Y_2B_1'B_2'} := \sum_{a_1,x_1,a_2,x_2,y} p_{X_1|Y_1}(x_1|y_1)p_{X_2|Y_2}(x_2|y_2)[a_1]\otimes[x_1]\otimes[a_2]\otimes[x_2]\otimes[y_1]\otimes[y_2] \\ \otimes\left(\mathcal{L}_{y_1}\otimes\mathcal{M}_{y_2}\right)\hat{\rho}^{a_1,x_1,a_2,x_2}_{B_1B_2}. \tag{124}$$

Let $\hat{\omega}^{a_1,x_1,a_2,x_2,y_1,y_2}_{B_1'B_2'E}$ be a non-signaling extension of $\left(\mathcal{L}_{y_1}\otimes\mathcal{M}_{y_2}\right)(\hat{\rho}^{a_1,x_1,a_2,x_2}_{B_1B_2})$, and define the following state:

$$\omega_{\overline{A}_1X_1\overline{A}_2X_2Y_1Y_2B_1'B_2'E} := \sum_{a_1,x_1,a_2,x_2,y} p_{X_1|Y_1}(x_1|y_1)p_{X_2|Y_2}(x_2|y_2)[a_1]\otimes[x_1]\otimes[a_2]\otimes[x_2]\otimes[y_1]\otimes[y_2] \\ \otimes\hat{\omega}^{a_1,x_1,a_2,x_2,y_1,y_2}_{B_1'B_2'E}. \tag{125}$$

Let $\hat{\theta}^{a_1,x_1,y_1}_{B_1'F}$ be a non-signaling extension of $\mathcal{L}_{y_1}(\hat{\theta}^{a_1,x_1}_{B_1})$ and let $\hat{\kappa}^{a_2,x_2,y_2}_{B_2'G}$ be a non-signaling extension of $\mathcal{M}_{y_2}(\hat{\kappa}^{a_2,x_2}_{B_2})$, leading to the following classical–quantum states:

$$\theta_{X_1\overline{A}_1B_1'FY_1} := \sum_{x_1,a_1} p_{X_1|Y_1}(x_1|y_1)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1,y_1}_{B_1'F}\otimes[y_1], \tag{126}$$

$$\kappa_{X_2\overline{A}_2B_2'GY_2} := \sum_{x_2,a_2} p_{X_2|Y_2}(x_2|y_2)[x_2]\otimes[a_2]\otimes\hat{\kappa}^{a_2,x_2,y_2}_{B_2'G}\otimes[y_2]. \tag{127}$$

Consider that

$$\begin{aligned} & I(\overline{A}_1X_1\overline{A}_2X_2;B_1'B_2'|EY_1Y_2)_\omega \\ &= I(\overline{A}_1X_1\overline{A}_2X_2;B_1'|EY_1Y_2)_\omega \\ &\quad + I(\overline{A}_1X_1\overline{A}_2X_2;B_2'|EB_1'Y_1Y_2)_\omega && (128)\\ &= I(\overline{A}_1X_1;B_1'|EY_1Y_2)_\omega + I(\overline{A}_2X_2;B_1'|EY_1Y_2\overline{A}_1X_1)_\omega \\ &\quad + (\overline{A}_2X_2;B_2'|EB_1'Y_1Y_2)_\omega \\ &\quad + I(\overline{A}_1X_1;B_2'|EB_1'Y_1Y_2\overline{A}_2X_2)_\omega && (129)\\ &\geq I(\overline{A}_1X_1;B_1'|EY_1Y_2)_\omega + I(\overline{A}_2X_2;B_2'|EB_1'Y_1Y_2)_\omega && (130)\\ &\geq \inf_{\text{ext. in (126)}} I(\overline{A}_1X_1;B_1'|FY_1)_\theta \\ &\quad + \inf_{\text{ext. in (127)}} I(\overline{A}_2X_2;B_2'|GY_2)_\kappa. && (131) \end{aligned}$$

The first two equalities follow from the chain rule for conditional mutual information. The first inequality follows by dropping two of the terms and from the fact that the conditional mutual information is non-negative. To see the last inequality, consider that the state $\sum_{a_2,x_2,y_2}\hat{\omega}^{a_1,x_1,a_2,x_2,y_1,y_2}_{B_1'E}\otimes[y_2]$ is a particular non-signaling extension of $\mathcal{L}_{y_1}(\hat{\theta}^{a_1,x_1}_{B_1})$ and the state $\sum_{a_1,x_1,y_1}\hat{\omega}^{a_1,x_1,a_2,x_2,y_1,y_2}_{B_1'B_2'E}\otimes[y_1]$ is a particular non-signaling extension of $\mathcal{M}_{y_2}(\hat{\kappa}^{a_2,x_2}_{B_2})$, such that an infimization over arbitrary respective non-signaling extensions $\hat{\theta}^{a_1,x_1,y_1}_{B_1'F}$ and $\hat{\kappa}^{a_2,x_2,y_2}_{B_2'G}$ can never lead to higher values of the conditional mutual informations. Since we have shown the inequality above for an arbitrary non-signaling extension $\hat{\omega}^{a_1,x_1,a_2,x_2,y_1,y_2}_{B_1'B_2'E}$, we can conclude that

$$\inf_{\text{ext. in (125)}} I(\overline{A}_1X_1\overline{A}_2X_2;B_1'B_2'|EY_1Y_2)_\omega \\ \geq \inf_{\text{ext. in (126)}} I(\overline{A}_1X_1;B_1'|FY_1)_\theta \\ + \inf_{\text{ext. in (127)}} I(\overline{A}_2X_2;B_2'|GY_2)_\kappa, \tag{132}$$

which in turn implies that

$$\sup_{\{p_{X_1|Y_1}p_{X_2|Y_2},\{\mathcal{L}_{y_1}\otimes\mathcal{M}_{y_2}\}_{y_1,y_2}\}} \inf_{\text{ext. in (125)}} I(\overline{A}_1X_1\overline{A}_2X_2;B_1'B_2'|EY_1Y_2)_\omega \\ \geq \inf_{\text{ext. in (126)}} I(\overline{A}_1X_1;B_1'|FY_1)_\theta + \inf_{\text{ext. in (127)}} I(\overline{A}_2X_2;B_2'|GY_2)_\kappa. \tag{133}$$

The reduced 1W-LOCC operations $\{p_{X_1|Y_1}, \{\mathcal{L}_{y_1}\}_{y_1}\}$ and $\{p_{X_2|Y_2}, \{\mathcal{M}_{y_2}\}_{y_2}\}$ are arbitrary, and so we can conclude that

$$\sup_{\{p_{X_1|Y_1}p_{X_2|Y_2}, \{\mathcal{L}_{y_1}\otimes\mathcal{M}_{y_2}\}_{y_1,y_2}\}} \inf_{\text{ext. in (125)}} I(\overline{A}_1X_1\overline{A}_2X_2; B_1'B_2'|EY_1Y_2)_\omega$$

$$\geq \sup_{\{p_{X_1|Y_1}, \{\mathcal{L}_{y_1}\}_{y_1}\}} \inf_{\text{ext. in (126)}} I(\overline{A}_1X_1; B_1'|FY_1)_\theta + \sup_{\{p_{X_2|Y_2}, \{\mathcal{M}_{y_2}\}_{y_2}\}} \inf_{\text{ext. in (127)}} I(\overline{A}_2X_2; B_2'|GY_2)_\kappa \tag{134}$$

$$= S(\overline{A}_1; B_1)_{\hat{\theta}} + S(\overline{A}_2; B_2)_{\hat{\kappa}}. \tag{135}$$

Finally, since the 1W-LOCC operation $\{p_{X_1|Y_1}p_{X_2|Y_2}, \{\mathcal{L}_{y_1} \otimes \mathcal{M}_{y_2}\}_{y_1,y_2}\}$ has a particular product form, we could never achieve a lower value of the quantity on the LHS by allowing for an arbitrary 1W-LOCC operation, implying the desired superadditivity:

$$S(\overline{A}_1\overline{A}_2; B_1B_2)_{\hat{\rho}} \geq S(\overline{A}_1; B_1)_{\hat{\theta}} + S(\overline{A}_2; B_2)_{\hat{\kappa}}. \tag{136}$$

This concludes the proof. ■

VI. RESTRICTED INSTRINSIC STEERABILITY

As stated above, we also consider a steering quantifier relevant in the context of restricted 1W-LOCC. Here we give a proof of Theorem 4 and proofs of various other properties of restricted intrinsic steerability.

Proposition 12 *The restricted intrinsic steerability vanishes for an assemblage having a local-hidden state model.*

Proof. To prove this, consider the following non-signaling, classical extension of an unsteerable assemblage:

$$\rho_{X\overline{A}BE} := \sum_{a,x} p_X(x)\,|x\rangle\langle x|_X \otimes p_{\overline{A}|X\Lambda}(a|x,\lambda)\,|a\rangle\langle a|_{\overline{A}} \otimes \hat{\rho}_B^\lambda \otimes p_\Lambda(\lambda)\,|\lambda\rangle\langle\lambda|_E\,. \tag{137}$$

Then $I(X\overline{A}; B|E)_\rho = \sum_\lambda p_\Lambda(\lambda) I(X\overline{A}; B)_{\rho^\lambda}$, where

$$\rho^\lambda_{X\overline{A}B} = \sum_{a,x} p_X(x)\,|x\rangle\langle x|_X \otimes p_{\overline{A}|X\Lambda}(a|x,\lambda)\,|a\rangle\langle a|_{\overline{A}} \otimes \rho_B^\lambda, \tag{138}$$

and we have used the fact that the conditional mutual information can be written as a convex combination of mutual informations for a classical conditioning system. By inspection, we see that systems $X\overline{A}$ and B are independent when given the shared variable $\Lambda = \lambda$. By choosing system E to contain the shared random variable Λ, the result is that the systems form a Markov chain $X\overline{A} - E - B$, so that the conditional mutual information $I(X\overline{A}; B|E)_\rho$ is equal to zero. Since this argument holds for any probability distribution p_X, we conclude that $S^R(\overline{A}; B)_{\hat{\rho}} = 0$. ■

Proposition 13 (Restricted 1W-LOCC monotone) *Let $\{\hat{\rho}_B^{a,x}\}_{a,x}$ be an assemblage, and let*

$$\{p_{X|X_f}, p_{\overline{A}_f|\overline{A}XX_fZ}, \{\mathcal{K}_z\}_z\} \tag{139}$$

denote a restricted 1W-LOCC operation that results in an assemblage $\{\hat{\sigma}_{B'}^{a_f,x_f}\}_{a_f,x_f}$, defined as

$$\hat{\sigma}_{B'}^{a_f,x_f} := \sum_{a,x,z} p_{X|X_f}(x|x_f) p_{\overline{A}_f|\overline{A}XX_fZ}(a_f|a,x,x_f,z)\mathcal{K}_z(\hat{\rho}_B^{a,x}). \tag{140}$$

Then

$$S^R(\overline{A}; B)_{\hat{\rho}} \geq S^R(\overline{A}_f; B')_{\hat{\sigma}}. \tag{141}$$

Proof. Taking a distribution p_{X_f} over the black-box inputs of the final assemblage, we can embed the state of the final assemblage into the following classical–quantum state:

$$\sigma_{X_f\overline{A}_fB'} := \sum_{x_f,a_f} p_{X_f}(x_f)[x_f] \otimes [a_f] \otimes \hat{\sigma}_{B'}^{a_f,x_f}, \tag{142}$$

which is a marginal of the following state:

$$\sigma_{X_fX\overline{A}_f\overline{A}ZB'} := \sum_{x_f,a_f,a,x,z} p_{X_f}(x_f)[x_f] \otimes p_{X|X_f}(x|x_f)[x] \otimes p_{\overline{A}_f|\overline{A}XX_fZ}(a_f|a,x,x_f,z)[a_f] \otimes [a] \otimes [z] \otimes \mathcal{K}_z(\hat{\rho}_B^{a,x}). \tag{143}$$

An *arbitrary* non-signaling extension of the state in (142) is as follows:

$$\sigma_{X_f\overline{A}_fB'E} := \sum_{x_f,a_f} p_{X_f}(x_f)[x_f] \otimes [a_f] \otimes \hat{\sigma}_{B'E}^{a_f,x_f}, \tag{144}$$

where

$$\mathrm{Tr}_E(\hat{\sigma}_{B'E}^{a_f,x_f}) = \hat{\sigma}_{B'}^{a_f,x_f}, \tag{145}$$

$$\sum_{a_f} \hat{\sigma}_{B'E}^{a_f,x_f} = \sum_{a_f} \hat{\sigma}_{B'E}^{a_f,x_f'} \quad \forall x_f, x_f' \in \mathcal{X}_f. \tag{146}$$

A *particular* non-signaling extension of the state in (142) is as follows:

$$\omega_{X_f\overline{A}_fB'EZ} := \sum_{x_f,a_f} p_{X_f}(x_f)[x_f]\otimes[a_f]\otimes \sum_{x_f,a_f,a,x,z} p_{X|X_f}(x|x_f)p_{\overline{A}_f|\overline{A}XX_fZ}(a_f|a,x,x_f,z)\mathcal{K}_z(\hat{\rho}^{a,x}_{BE})\otimes[z], \tag{147}$$

where

$$\mathrm{Tr}_E(\hat{\rho}^{a,x}_{BE}) = \hat{\rho}^{a,x}_{B}, \qquad \sum_a \hat{\rho}^{a,x}_{BE} = \sum_a \hat{\rho}^{a,x'}_{BE} \quad \forall x,x'\in\mathcal{X}. \tag{148}$$

The state $\omega_{X_f\overline{A}_fB'E}$ is a marginal of the following state:

$$\omega_{X_fX\overline{A}_f\overline{A}B'EZ} := \sum_{x_f,a_f,a,x,z} p_{X_f}(x_f)[x_f]\otimes p_{X|X_f}(x|x_f)[x]\otimes p_{\overline{A}_f|\overline{A}XX_fZ}(a_f|a,x,x_f,z)[a_f]\otimes[a]\otimes\mathcal{K}_z(\hat{\rho}^{a,x}_{BE})\otimes[z]. \tag{149}$$

Let $\rho_{X\overline{A}BE}$ be the following state:

$$\rho_{X\overline{A}BE} := \sum_{x_f,a,x} p_{X_f}(x_f)[x_f]\otimes p_{X|X_f}(x|x_f)[x] \otimes [a]\otimes\hat{\rho}^{a,x}_{BE}. \tag{150}$$

Consider that

$$\inf_{\text{ext. in (144)}} I(X_f\overline{A}_f;B'|E)_\sigma$$
$$\leq I(X_f\overline{A}_f;B'|EZ)_\omega \tag{151}$$
$$\leq I(X_f\overline{A}_fX\overline{A};B'|EZ)_\omega \tag{152}$$
$$= I(X\overline{A};B'|EZ)_\omega + I(X_f;B'|EZX\overline{A})_\omega + I(\overline{A}_f;B'|EZX_fX\overline{A})_\omega \tag{153}$$
$$= I(X\overline{A};B'|EZ)_\omega \tag{154}$$
$$\leq I(X\overline{A};B'Z|E)_\omega \tag{155}$$
$$\leq I(X\overline{A};B|E)_\rho. \tag{156}$$

The first inequality follows because the non-signaling extension in (147) is a particular kind of non-signaling extension. The second inequality follows from data processing. The first equality follows from the chain rule for conditional mutual information. The second equality follows from various Markov-chain structures when inspecting (149): X_f is independent of $B'E$ when given $ZX\overline{A}$, and $\overline{A}_f$ is independent of $B'E$ when given $ZX_fX\overline{A}$, so that $I(X_f;B'|EZX\overline{A})_\omega = I(\overline{A}_f;B'|EZX_fX\overline{A})_\omega = 0$. The third inequality follows by applying the chain rule for and non-negativity of conditional mutual information. The last inequality follows again from data processing. Since the inequality holds for all non-signaling extensions of the form in (150), we can conclude that

$$\inf_{\text{ext. in (144)}} I(X_f\overline{A}_f;B'|E)_\sigma$$
$$\leq \inf_{\text{ext. in (150)}} I(X\overline{A};B|E)_\rho \tag{157}$$
$$\leq \sup_{p_X}\inf_{\text{ext. in (150)}} I(X\overline{A};B|E)_\rho. \tag{158}$$

Since the inequality above holds for an arbitrary choice of p_{X_f}, we can finally conclude that

$$\sup_{p_{X_f}}\inf_{\text{ext. in (144)}} I(X_f\overline{A}_f;B'|E)_\sigma \leq \sup_{p_X}\inf_{\text{ext. in (150)}} I(X\overline{A};B|E)_\rho, \tag{159}$$

which is equivalent to the statement of the proposition. ■

The proof of convexity of the restricted intrinsic steerability is along the same lines as that for intrinsic steerability, given already in the proof of Proposition 10. We summarize the result as the following proposition:

Proposition 14 (Convexity) *Let $\{\hat{\rho}^{a,x}_B\}_{a,x}$ and $\{\hat{\sigma}^{a,x}_B\}_{a,x}$ be assemblages, and let $\lambda\in[0,1]$. Let $\{\hat{\tau}^{a,x}_B\}_{a,x}$ be a mixture of the two assemblages, defined as*

$$\hat{\tau}^{a,x}_B := \lambda\hat{\rho}^{a,x}_B + (1-\lambda)\hat{\sigma}^{a,x}_B. \tag{160}$$

Then

$$S^R(\overline{A};B)_{\hat{\tau}} \leq \lambda S^R(\overline{A};B)_{\hat{\rho}} + (1-\lambda)S^R(\overline{A};B)_{\hat{\sigma}}. \tag{161}$$

Proposition 15 (Superadditivity and Additivity) *Let $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$ be an assemblage for which the following additional no-signaling constraints hold*

$$\sum_{a_2}\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2} = \sum_{a_2}\hat{\rho}^{a_1,a_2,x_1,x_2'}_{B_1B_2} := \hat{\theta}^{a_1,x_1}_{B_1B_2} \quad \forall x_2,x_2', \tag{162}$$

$$\sum_{a_1}\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2} = \sum_{a_1}\hat{\rho}^{a_1,a_2,x_1',x_2}_{B_1B_2} := \hat{\kappa}^{a_2,x_2}_{B_1B_2} \quad \forall x_1,x_1', \tag{163}$$

Let $\{\mathrm{Tr}_{B_2}(\hat{\theta}^{a_1,x_1}_{B_1B_2})\}_{a_1,x_1}$ and $\{\mathrm{Tr}_{B_1}(\hat{\kappa}^{a_2,x_2}_{B_1B_2})\}_{a_2,x_2}$ be reduced assemblages arising from the joint assemblage $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$. Then the restricted intrinsic steerability is superadditive in the following sense:

$$S^R(\overline{A}_1\overline{A}_2;B_1B_2)_{\hat{\rho}} \geq S^R(\overline{A}_1;B_1)_{\hat{\theta}} + S^R(\overline{A}_2;B_2)_{\hat{\kappa}}. \tag{164}$$

If the assemblage $\{\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2}\}_{a_1,a_2,x_1,x_2}$ has a tensor-product form, so that $\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2} = \hat{\theta}^{a_1,x_1}_{B_1} \otimes \hat{\kappa}^{a_2,x_2}_{B_2}$ for assemblages $\{\hat{\theta}^{a_1,x_1}_{B_1}\}_{a_1,x_1}$ and $\{\hat{\kappa}^{a_2,x_2}_{B_2}\}_{a_2,x_2}$, then the restricted intrinsic steerability is additive:

$$S^R(\overline{A}_1\overline{A}_2;B_1B_2)_{\hat{\rho}} = S^R(\overline{A}_1;B_1)_{\hat{\theta}} + S^R(\overline{A}_2;B_2)_{\hat{\kappa}}. \quad (165)$$

Proof. The superadditivity of restricted intrinsic steerability is similar to the proof above for intrinsic steerability. Thus, to prove the additivity of intrinsic steerability with respect to product assemblages, it is sufficient to prove the following subadditivity inequality:

$$S^R(\overline{A}_1\overline{A}_2;B_1B_2)_{\hat{\rho}} \leq S^R(\overline{A}_1;B_1)_{\hat{\theta}} + S^R(\overline{A}_2;B_2)_{\hat{\kappa}}. \quad (166)$$

Our proof of the above inequality has some similarities to the proof of the additivity of the squashed entanglement of a channel [15] (there are, however, some key differences). Let $\hat{\theta}^{a_1,x_1}_{B_1E_1}$ and $\hat{\kappa}^{a_2,x_2}_{B_2E_2}$ be non-signaling extensions of $\hat{\theta}^{a_1,x_1}_{B_1}$ and $\hat{\kappa}^{a_2,x_2}_{B_2}$, respectively, and suppose that $|\hat{\theta}^{a_1,x_1}\rangle_{B_1E_1F_1}$ and $|\hat{\kappa}^{a_2,x_2}\rangle_{B_2E_2F_2}$ purify $\hat{\theta}^{a_1,x_1}_{B_1E_1}$ and $\hat{\kappa}^{a_2,x_2}_{B_2E_2}$, respectively. Consider the following states:

$$\rho_{X_1X_2\overline{A}_1\overline{A}_2B_1B_2E} := \sum_{x_1,x_2,a_1,a_2} p_{X_1X_2}(x_1,x_2)[x_1]\otimes[x_2]\otimes[a_1]\otimes[a_2]\otimes\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2E}, \quad (167)$$

$$\omega_{X_1X_2\overline{A}_1\overline{A}_2B_1B_2E_1E_2F_1F_2} := \sum_{x_1,x_2,a_1,a_2} p_{X_1X_2}(x_1,x_2)[x_1]\otimes[x_2]\otimes[a_1]\otimes[a_2]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\hat{\kappa}^{a_2,x_2}_{B_2E_2F_2}, \quad (168)$$

where $p_{X_1X_2}(x_1,x_2)$ is some probability distribution and $\mathrm{Tr}_E(\hat{\rho}^{a_1,a_2,x_1,x_2}_{B_1B_2E}) = \hat{\theta}^{a_1,x_1}_{B_1}\otimes\hat{\kappa}^{a_2,x_2}_{B_2}$. Consider that

$$\begin{aligned}
&\inf_{\rho_{\overline{A}_1\overline{A}_2X_1X_2B_1B_2E}} I(\overline{A}_1\overline{A}_2X_1X_2;B_1B_2|E)_\rho \\
&\leq I(\overline{A}_1\overline{A}_2X_1X_2;B_1B_2|E_1E_2)_\omega && (169)\\
&= H(B_1B_2|E_1E_2)_\omega - H(B_1B_2|E_1E_2\overline{A}_1X_1\overline{A}_2X_2)_\omega && (170)\\
&= H(B_1B_2|E_1E_2)_\omega + H(B_1B_2|F_1F_2\overline{A}_1X_1\overline{A}_2X_2)_\omega && (171)\\
&\leq H(B_1|E_1)_\omega + H(B_2|E_2)_\omega + H(B_1|F_1\overline{A}_1X_1)_\omega + H(B_2|F_2\overline{A}_2X_2)_\omega && (172)\\
&= H(B_1|E_1)_\omega + H(B_2|E_2)_\omega - H(B_1|E_1\overline{A}_1X_1)_\omega - H(B_2|E_2\overline{A}_2X_2)_\omega && (173)\\
&= I(X_1\overline{A}_1;B_1|E_1)_\omega + I(X_2\overline{A}_2;B_2|E_2)_\omega. && (174)
\end{aligned}$$

The first inequality follows because $\omega_{X_1X_2\overline{A}_1\overline{A}_2B_1B_2E_1E_2}$ is a particular non-signaling extension whereas $\rho_{X_1X_2\overline{A}_1\overline{A}_2B_1B_2E}$ is an arbitrary non-signaling extension. The first equality follows from the chain rule for conditional mutual information. Conditioned on $\overline{A}_1\overline{A}_2X_1X_2$, the state on $B_1E_1B_2E_2F_1F_2$ is pure, and so the second equality follows from the duality of conditional entropy. The first inequality is a consequence of the strong subadditivity of quantum entropy [16]. The third equality follows again from the duality of conditional entropy as well as the no-signaling condition. To see this for the entropy $H(B_1|F_1\overline{A}_1X_1)_\omega$, consider that this entropy is evaluated with respect to the following reduced state:

$$\begin{aligned}
&\mathrm{Tr}_{X_2\overline{A}_2B_2E_2F_2}\left(\sum_{x_1,x_2,a_1,a_2} p_{X_1X_2}(x_1,x_2)[x_1]\otimes[x_2]\otimes[a_1]\otimes[a_2]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\hat{\kappa}^{a_2,x_2}_{B_2E_2F_2}\right)\\
&= \sum_{x_1,x_2,a_1,a_2} p_{X_1X_2}(x_1,x_2)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\mathrm{Tr}_{B_2E_2F_2}\{\hat{\kappa}^{a_2,x_2}_{B_2E_2F_2}\} && (175)\\
&= \sum_{x_1,a_1} p_{X_1}(x_1)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\mathrm{Tr}_{B_2}\left(\sum_{x_2} p_{X_2|X_1}(x_2|x_1)\sum_{a_2}\hat{\kappa}^{a_2,x_2}_{B_2}\right) && (176)\\
&= \sum_{x_1,a_1} p_{X_1}(x_1)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\mathrm{Tr}_{B_2}\left(\sum_{x_2} p_{X_2|X_1}(x_2|x_1)\kappa_{B_2}\right) && (177)\\
&= \sum_{x_1,a_1} p_{X_1}(x_1)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}\otimes\mathrm{Tr}_{B_2}(\kappa_{B_2}) && (178)\\
&= \sum_{x_1,a_1} p_{X_1}(x_1)[x_1]\otimes[a_1]\otimes\hat{\theta}^{a_1,x_1}_{B_1E_1F_1}. && (179)
\end{aligned}$$

In the above, the third equality is the critical one in which we have used the no-signaling constraint for the assemblage $\{\hat{\kappa}_{B_2}^{a_2,x_2}\}_{a_2,x_2}$, allowing for the effective removal of correlation between X_1 and X_2. Thus, the above analysis allows for seeing that the remaining state on $B_1E_1F_1$ conditioned on $\overline{A}_1$ and X_1 is independent of any of the second system. For the last equality, we employ the definition of conditional mutual information. Since the above development holds for all non-signaling extensions of the form in (168), we find that

$$\begin{aligned}
&\inf_{\rho_{\overline{A}_1\overline{A}_2X_1X_2B_1B_2E}} I(\overline{A}_1\overline{A}_2X_1X_2;B_1B_2|E_1)_\rho \\
&\quad\leq \inf_{\omega_{\overline{A}_1X_1B_1E_1}} I(\overline{A}_1X_1;B_1|E_1)_\omega \\
&\qquad + \inf_{\omega_{\overline{A}_2X_2B_2E_2}} I(\overline{A}_2X_2;B_2|E_2)_\omega \qquad (180)\\
&\quad\leq \sup_{p_{X_1}} \inf_{\omega_{\overline{A}_1X_1B_1E_1}} I(\overline{A}_1X_1;B_1|E_1)_\omega \\
&\qquad + \sup_{p_{X_2}} \inf_{\omega_{\overline{A}_2X_2B_2E_2}} I(\overline{A}_2X_2;B_2|E_2)_\omega. \qquad (181)
\end{aligned}$$

Since the above inequality holds for an arbitrary probability distribution $p_{X_1X_2}$, we conclude that

$$\begin{aligned}
&\sup_{p_{X_1X_2}} \inf_{\rho_{\overline{A}_1\overline{A}_2X_1X_2B_1B_2E}} I(\overline{A}_1\overline{A}_2X_1X_2;B_1B_2|E)_\rho \\
&\quad\leq \sup_{p_{X_1}} \inf_{\omega_{\overline{A}_1X_1B_1E_1}} I(\overline{A}_1X_1;B_1|E_1)_\omega \\
&\qquad + \sup_{p_{X_2}} \inf_{\omega_{\overline{A}_2X_2B_2E_2}} I(\overline{A}_2X_2;B_2|E_2)_\omega, \qquad (182)
\end{aligned}$$

which is equivalent to (166). ■

Monogamy of steering has been explored in [20, 21]. We prove here that the restricted intrinsic steerability is monogamous in the following sense: for a tripartite state ρ_{ABC}, Alice and Charlie perform measurements on their systems and steer Bob's system. We see that their ability to steer Bob's system is restricted.

Proposition 16 (Monogamy) *Let $\{\hat{\rho}_B^{a,c,x_1,x_2}\}$ be an assemblage with classical inputs x_1 and x_2 for Alice and Charlie, respectively, and classical outputs a and c for Alice and Charlie, respectively, and obeying the following additional no-signaling constraints:*

$$\sum_c \hat{\rho}_B^{a,c,x_1,x_2} = \sum_c \hat{\rho}_B^{a,c,x_1,x_2'} := \hat{\theta}_B^{a,x_1} \qquad \forall x_2,x_2', \qquad (183)$$

$$\sum_a \hat{\rho}_B^{a,c,x_1,x_2} = \sum_a \hat{\rho}_B^{a,c,x_1',x_2} := \hat{\kappa}_B^{c,x_2} \qquad \forall x_1,x_1', \qquad (184)$$

such that the reduced assemblages are $\{\hat{\theta}_B^{a,x_1}\}_{a,x_1}$ and $\{\hat{\kappa}_B^{c,x_2}\}_{c,x_2}$. Then the following monogamy inequality holds

$$S^R(\overline{AC};B)_{\hat{\rho}} \geq S^R(\overline{A};B)_{\hat{\theta}} + S^R(\overline{C};B)_{\hat{\kappa}}. \qquad (185)$$

Proof. This proof follows from an application of the chain rule for conditional mutual information, much like the proof of monogamy for the squashed entanglement [22]. First, consider the following classical–quantum state:

$$\rho_{X_1X_2\overline{AC}BE} := \sum_{x_1,x_2,a,c} p_{X_1}(x_1)p_{X_2}(x_2)[x_1]\otimes[x_2]\otimes[a]\otimes[c]\otimes\hat{\rho}_{BE}^{a,c,x_1,x_2}, \qquad (186)$$

where $\hat{\rho}_{BE}^{a,c,x_1,x_2}$ is a non-signaling extension of $\hat{\rho}_B^{a,c,x_1,x_2}$. Let

$$\theta_{X_1\overline{A}BF} := \sum_{x_1,a} p_{X_1}(x_1)[x_1]\otimes[a]\otimes\hat{\theta}_{BF}^{a,x_1}, \qquad (187)$$

$$\kappa_{X_2\overline{C}BG} := \sum_{x_2,a} p_{X_2}(x_2)[x_2]\otimes[c]\otimes\hat{\kappa}_{BG}^{c,x_2}, \qquad (188)$$

where $\hat{\theta}_{BF}^{a,x_1}$ is a non-signaling extension of $\hat{\theta}_B^{a,x_1}$ and $\hat{\kappa}_{BG}^{c,x_2}$ is a non-signaling extension of $\hat{\kappa}_B^{c,x_2}$. Then we have from the chain rule for conditional mutual information that

$$\begin{aligned}
&I(X_1X_2\overline{AC};B|E)_\rho \\
&\quad= I(X_1\overline{A};B|E)_\rho + I(X_2\overline{C};B|E\overline{A}X_1)_\rho \qquad (189)\\
&\quad\geq \inf_{\theta_{X_1\overline{A}BF}} I(X_1\overline{A};B|E)_\theta + \inf_{\kappa_{X_2\overline{C}BG}} I(X_2\overline{C};B|G)_\kappa. \qquad (190)
\end{aligned}$$

Since the above inequality holds for all non-signaling extensions $\rho_{X_1X_2\overline{AC}BE}$, we conclude that

$$\begin{aligned}
&\inf_{\rho_{X_1X_2\overline{AC}BE}} I(X_1X_2\overline{AC};B|E)_\rho \\
&\quad\geq \inf_{\theta_{X_1\overline{A}BF}} I(X_1\overline{A};B|E)_\theta + \inf_{\kappa_{X_2\overline{C}BG}} I(X_2\overline{C};B|G)_\kappa. \qquad (191)
\end{aligned}$$

Optimizing the left-hand side with respect to product distributions, we find that

$$\begin{aligned}
&\sup_{p_{X_1},p_{X_2}} \inf_{\rho_{X_1X_2\overline{AC}BE}} I(X_1X_2\overline{AC};B|E)_\rho \\
&\quad\geq \inf_{\theta_{X_1\overline{A}BF}} I(X_1\overline{A};B|E)_\theta + \inf_{\kappa_{X_2\overline{C}BG}} I(X_2\overline{C};B|G)_\kappa. \qquad (192)
\end{aligned}$$

The development holds for any choice of distributions p_{X_1} and p_{X_2}, and so we conclude that

$$\begin{aligned}
&\sup_{p_{X_1},p_{X_2}} \inf_{\rho_{X_1X_2\overline{AC}BE}} I(X_1X_2\overline{AC};B|E)_\rho \\
&\geq \sup_{p_{X_1}} \inf_{\theta_{X_1\overline{A}BF}} I(X_1\overline{A};B|E)_\theta + \sup_{p_{X_2}} \inf_{\kappa_{X_2\overline{C}BG}} I(X_2\overline{C};B|G)_\kappa \qquad (193)\\
&= S^R(\overline{A};B)_{\hat{\theta}} + S^R(\overline{C};B)_{\hat{\kappa}}. \qquad (194)
\end{aligned}$$

Finally optimizing the left-hand side with respect to all input distributions $p_{X_1X_2}$, we conclude (185). ■

VII. OPERATIONAL INTERPRETATION

Let ψ_{ABE} be a pure tripartite state, p_X a probability distribution, and $\{\Lambda_a^{(x)}\}_a$ a positive operator-valued measure (POVM) for each x. Then $\{p_X(x)\Lambda_a^{(x)}\}_{a,x}$ is a POVM as well, representing a random choice of the POVM $\{\Lambda_a^{(x)}\}_a$ according to p_X, along with keeping a record x of the choice in addition to the measurement outcome a. Consider the following state resulting from performing the POVM on ψ_{ABE}:

$$\rho_{X\overline{A}BE} := \sum_x |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes \mathrm{Tr}_A((p_X(x)\Lambda_a^{(x)} \otimes I_{BE})\psi_{ABE}). \quad (195)$$

Here we consider that Alice performs the measurement $\{p_X(x)\Lambda_a^{(x)}\}_{a,x}$ on her system A, which results in the measurement outcomes being placed in classical systems $X\overline{A}$. Suppose now that many copies of the above state ψ_{ABE} are available, and that Alice would like to perform individual measurements $\{p_X(x)\Lambda_a^{(x)}\}_{a,x}$ of her systems and send all of the outcomes to Eve, who possesses the E systems. Alice could certainly simply perform the measurements and send the outcomes to Eve, but if she shares randomness with Eve, then she can simulate the measurements in such a way as to reduce the number of classical bits she would need to send to Eve. Furthermore, the simulation can be such that no external party observing all of the systems could tell the difference between the scenario in which Alice actually performs the measurements and the one in which Alice and Eve perform a simulation of the measurements. One of the main results of [23] is that the conditional mutual information $I(X\overline{A};B|E)_\rho$ is the optimal rate of classical information that Alice needs to send to Eve in order to have a successful simulation. The protocol that achieves this task is called measurement compression with quantum side information [23]. Thus, this information-processing task gives an operational interpretation of the main quantity $I(X\overline{A};B|E)_\rho$ appearing in the restricted intrinsic steerability. We note that our setting above, regarding the classical communication cost of simulating steering, is rather different from the setting considered in [24].

VIII. OTHER POSSIBLE MEASURES

We note here that other variations of the intrinsic steerability are possible. Fix an assemblage $\{\hat{\rho}_B^{a,x}\}_{a,x}$. Let Eve have a non-signaling extension of this assemblage, and we write the extended assemblage as $\{\hat{\rho}_{BE}^{a,x}\}_{a,x}$. Bob applies the quantum instrument consisting of trace-non-increasing completely positive maps $\{\mathcal{K}_y\}_y$, gets the outcome y, and publicly announces it. Then, Alice prepares the input x based on y, and Eve performs a quantum channel κ_y on her system. The state after this scenario is given by

$$\rho_{\overline{A}XB'YE} := \sum_{x,a,y} p_{X|Y}(x|y)\, |x\rangle\langle x|_X \otimes |a\rangle\langle a|_{\overline{A}} \otimes (\mathcal{K}_y \otimes \kappa_y)(\hat{\rho}_{BE}^{a,x}) \otimes |y\rangle\langle y|_Y . \quad (196)$$

We could then define a variation of the intrinsic steerability as

$$\inf_{\rho_{\overline{A}XBYE}} \sup_{\{p(x|y),\{\mathcal{K}_y\}_y\}} I(\overline{A}X;B|EY)_\rho. \quad (197)$$

This quantity however is generally larger than the intrinsic steerability, and we suspect that the definition we provided will be more useful in future applications because the definition we gave is analogous to the squashed entanglement of a channel [15], which has found a number of applications in quantum information theory. We note that it is possible to consider other restrictions that result in a modification of the measure accordingly.

IX. CONCLUSION

We have introduced a quantifier for quantum steering based on conditional quantum mutual information. It exploits the Markov-chain structure of assemblages with a local hidden-state model, measuring the deviation of a given assemblage from one having a local-hidden-state model. The intrinsic steerability is a steering monotone and superadditive in general. This suggests that the intrinsic steerability should find applications in protocols where steering as a resource is relevant. Also, we looked at a restricted class of free operations. In this case, the quantity simplifies considerably and also satisfies additivity and monogamy. The restricted intrinsic steerability could find applications in protocols where it suffices to consider the restricted class of free operations.

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