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Quantum voting and violation of Arrow's Impossibility Theorem

Ning Bao[1,2] and Nicole Yunger Halpern[1]

[1] *Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA 91125, USA*

[2] *Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125*

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We propose a quantum voting system, in the spirit of quantum games such as the quantum Prisoner's Dilemma. Our scheme enables a constitution to violate a quantum analog of Arrow's Impossibility Theorem. Arrow's Theorem is a claim proved deductively in economics: Every (classical) constitution endowed with three innocuous-seeming properties is a dictatorship. We construct quantum analogs of constitutions, of the properties, and of Arrow's Theorem. A quantum version of majority rule, we show, violates this Quantum Arrow Conjecture. Our voting system allows for tactical-voting strategies reliant on entanglement, interference, and superpositions. This contribution to quantum game theory helps elucidate how quantum phenomena can be harnessed for strategic advantage.

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Today's voting systems are classical. Societies hold elections to determine which of several candidates will win an office. Each voter ranks the candidates, forming a *preference*. Voters' preferences are combined deterministically according to some rule set, or constitution. What if citizens could entangle, superpose, and interfere preferences? We formulate a quantum voting system, in the tradition of quantum games, that highlights the power of quantum resources.

Quantum game theory has flourished over the past several years [1**?**]. In a classical game, players can perform only local operations and classical communications. Each player can prepare and measure only systems in his/her lab. Players can communicate only via classical channels (e.g., by telephone), if at all.

Examples include the Prisoner's Dilemma. Suppose that the police arrest two suspected criminals. The suspects are isolated in separate cells. If neither suspect confesses, each will receive a lenient sentence (e.g., one year in jail). If both suspects confess, both will receive moderate sentences (e.g., two years). If just one suspect confesses, s/he will receive no sentence. The other suspect will suffer a heavy penalty (e.g., three years). Unable to communicate with the other prisoner, each suspect can optimize his/her future by confessing. Both suspects would benefit more if they could agree to remain silent. The Prisoner's Dilemma consists of the tension between (i) the optimal strategy attainable and (ii) the optimal strategy that the prisoners could attain if they could communicate.

Quantizing the game diminishes the tension [2]. Eisert *et al.* associate each prisoner with a Hilbert space. They translate each prisoner's options (to cooperate with the police and to defect) into basis elements ($|C\rangle$ and $|D\rangle$). The game becomes a quantum circuit. Measuring the prisoners' joint state determines their penalties. This quantization alters the landscape of possible outcomes and strategies.

Similar insights result from quantizing the penny-flipping game [3], the Monty Hall problem [4, 5], and Conrad's Game of Life [6, 7]. A game elucidates the canonical demonstration of entanglement's power: Clauser, Holt, Shimony, and Hauser (CHSH) reformulated Bell's Theorem in terms of a protocol cast as "the CHSH game" [8, 9].

Elections have been cast in game-theoretic terms [10]. Elections therefore merit generalization with quantum theory. Upshots of quantization, we show, include a violation of a quantum analog of Arrow's Impossibility Theorem, as well as quantum voting strategies.

Arrow's Theorem is a result derived, in economics, from deductive logic and definitions [11]. According to the theorem, every constitution that has three innocuous-seeming properties (transitivity, unanimity, and independence of irrelevant alternatives, defined below) is a dictatorship. Arrow's Theorem is surprisingly deep and has fundamentally impacted game theory and voting theory (e.g., [12]). Yet Arrow's Theorem derives from classical logic. A quantum version, we find, is false.

Classical constitutions disobey Arrow's Theorem under certain conditions. For example, Black supplements Arrow's three postulates with extra assumptions [13]. The extra assumptions, he argues, are properties as reasonable as Arrow's for a constitution to have. No constitution, he shows, can satisfy Arrow's postulates and the extras while being a nondictatorship. Some sets of votes, however, prevent constitutions that satisfy Black's extra assumptions from satisfying all of Arrow's postulates. Probabilistic mixtures of votes, too, evade Arrow's Theorem. Suppose that a voter can pledge 40% of his/her support to candidate Alice, 40% to Bob, and 20% to Charlie. Such voters can form a constitution not subject to Arrow's Theorem. Yet such constitutions do not satisfy all of Arrow's postulates [14–16]. We cleave to Arrow's postulates and avoid restricting voters' preferences. Rather, we recast Arrow's scheme in quantum terms.

Like Black's extra postulates and like probabilistic votes, alternative classical voting schemes evade Arrow's

Theorem. Engaging in *range voting*, a voter assigns each candidate a number of points independently of the other candidates [17, 18]. Whichever candidate receives the most points wins. Voters behave identically when using *majority judgment* [19]. The majority-judgment winner has the highest median number of points. Under *approval voting*, each voter assigns each candidate a thumbs-up or a thumbs-down [20]. Range voting, majority judgment, and approval voting contrast with *ordinal voting*. Ordinal-voting citizens rank candidates. Arrow's Theorem governs just ordinal voting. Yet a generalization of Arrow's Theorem, the Gibbard-Satterthwaite (GS) Theorem [21, 22], governs majority judgment and approval voting. (Range voting is not a scheme of the class governed by the GS Theorem, just as range voting is not a scheme of the class governed by Arrow's Theorem.) Like the GS Theorem, our Quantum Arrow Conjecture generalizes Arrow's Theorem. Yet classicality does not constrain our generalization as it constrains the GS Theorem. Like range voting, majority judgment, and approval voting, our voting scheme is not precisely ordinal. Yet ordinal rankings form the basis for our quantum votes' Hilbert spaces, as discussed below.

In addition to disproving a Quantum Arrow Conjecture, we present four quantum strategic-voting tactics. How one should vote is not always clear, even to opinionated citizens. You might favor a candidate unlikely to win, for example. Voting for a more likely candidate whose policies you could tolerate can optimize the election's outcome. Strategic voting is the submission of a preference other than one's opinion, in a competition amongst three or more candidates [23]. Quantizing voting unlocks new voting strategies. We exhibit three tactics reliant on entanglement and one reliant on interference and superpositions.

Earlier work on quantum voting has focused on privacy, security, and cryptography [24–26]. These references answer questions such as "How can voters and election officials hinder cheaters?" In contrast, we draw inspiration from game theory.

The paper is organized as follows. First, we introduce our quantum voting system. We define quantum analogs of properties of classical constitutions. Four of these properties appear in Arrow's Theorem, which we review and quantize. We disprove the conjecture by a counterexample. The counterexample relies on the quantization of a fifth property available to classical constitutions: majority rule. Finally, we present three strategic-voting strategies based on entanglement and one strategy based on interference.

I. QUANTUM VOTING SYSTEM

A voting system involves a *society* that consists of *voters*. *Candidates* $a, b, \ldots, m$ vie for office. Each voter ranks the candidates, forming a *preference*. A preference is a transitive ordered list. Each candidate is ranked above, ranked below, or tied with each other candidate: $a > b$, $a < b$, or $a = b$. A list is transitive if $a \geq b$ and $b \geq c$, together, imply $a \geq c$.

The voters' preferences form a *profile*. The profile serves as input to a *constitution* during an *election*. We focus on elections that feature at least three candidates. The constitution combines the voters' preferences, forming *society's preference*. Society's preference implies which candidate wins.

We quantize this classical election scheme. Our strategy resembles that of Eisert *et al.* [2]. Their quantum game consists of a general quantum process: a preparation procedure, an evolution, and a measurement [27]. So does our quantum voting scheme. We introduce a Hilbert-space formalism for quantum preferences. Elections are formulated as quantum circuits [27]. We define quantum constitutions and five properties that constitutions can have.

A. Hilbert-space formalism for quantum voters

Let $\mathcal{S}$ denote a society of voters. The voters are indexed by $i = 1, 2, \ldots N$. We associate with voter i the i^{th} copy of a Hilbert space $\mathcal{H}$. The space of density operators (unit-trace linear positive-semidefinite operators) defined on $\mathcal{H}$ is denoted by $\mathcal{D}(\mathcal{H})$.

Society is associated with a *joint quantum state* $\sigma_{\text{soc}} \in \mathcal{D}\left(\mathcal{H}^{\otimes N}\right)$. The joint state encodes all the information in the voters' preferences. This information may include correlations, such as entanglement, between votes. Consider tracing out every subsystem except the i^{th}. The result is voter i's *quantum preference*, $\rho_i := \text{Tr}_{\neq i}(\sigma_{\text{soc}})$. We sometimes denote a pure quantum preference by $|\rho_i\rangle$. The set of all voters' quantum preferences forms society's *quantum profile*, $\mathcal{P} := \{\rho_1, \ldots, \rho_N\}$.

Processing $\mathcal{P}$ must lead to the identification of a winner. More generally, the quantum society must generate a transitive ordered list of the candidates. We call such a list a *classical preference*. Each classical preference corresponds to a state in $\mathcal{H}$. For example, $c > a = b > d$ corresponds to $|c{>}a{=}b{>}d\rangle$. We denote by $|\gamma\rangle$ the γ^{th} classical-preference state and by $\chi_i^\gamma := |\gamma\rangle\langle\gamma|$ the associated density operator. The set $\{|\gamma\rangle\}$ forms the *preference basis* $\mathcal{B}$ for $\mathcal{H}$.

Consider any pair (a, b) of candidates. $\mathcal{H}$ decomposes into subspaces associated with the possible relationships between a and b. By $\mathcal{G}^{a>b}$, we denote the subspace spanned by the $\mathcal{B}$ elements that encode $a > b$ (e.g., $|a{>}b{=}c\rangle$, $|c{>}a{>}b\rangle$, etc.). The subspaces $\mathcal{G}^{b>a}$ and $\mathcal{G}^{a=b}$ are defined analogously. For example, $|a{>}b{>}c\rangle$ occupies the intersection of three subspaces: $|a{>}b{>}c\rangle \in \mathcal{G}^{a>b} \cap \mathcal{G}^{a>c} \cap \mathcal{G}^{b>c}$. The $a > b$, $b > a$, and $a = b$ subspaces are disjoint. For example, $\mathcal{G}^{a>b} \cap \mathcal{G}^{b>a} = \emptyset$. $\Pi^{a>b}$ denotes the projector onto the subspace $\mathcal{G}^{a>b}$. The projector $\Pi^{a=b}$ is defined analogously.

Consider measuring projectively a quantum preference ρ_i with respect to $\mathcal{B}$. The measurement yields a classical preference. If ρ_i is a nontrivial linear combination or mixture of $\mathcal{B}$ elements, the measurement is probabilistic. A voter's ability to superpose classical preferences resembles a prisoner's ability to superpose classical tactics in the quantum Prisoner's Dilemma [2].

During a *quantum election*, society's joint state is transformed into *society's quantum preference*: $\sigma_{\rm soc} \mapsto \rho_{\rm soc} \in \mathcal{D}(\mathcal{H})$. This $\rho_{\rm soc}$ is measured with respect to $\mathcal{B}$. generating society's classical preference. The quantum election can be formulated as a quantum circuit [27]. A quantum constitution, which we now introduce, implements the transformation.

B. Quantum constitutions

A *classical constitution* $\mathcal{C}$ is a map from the profile of the voters' classical preferences to society's classical preference. We define quantum constitutions analogously. Having completed the definition of quantum elections, we define their classical limit. The classical constitutions that obey Arrow's Theorem have four properties. We review and quantize these properties.

1. Definition of "quantum constitution"

Quantum constitutions have the form of general quantum evolutions, as does the Quantum Prisoner's Dilemma [2]. A general quantum evolution is a convex-linear completely positive trace-preserving (CPTP) map [27]. A map $\mathcal{E}$ is convex-linear if, given a probabilistic combination $\sum_i p_i \rho_i$ of states ρ_i, $\mathcal{E}$ transforms the component states independently: $\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$, wherein $p_i \geq 0 \; \forall i$ and $\sum_i p_i = 1$ [27]. Every CPTP map is equivalent to the tensoring on of an ancilla, a unitary transformation of the system-and-ancilla composite, and the tracing out of a subsystem [27].

Each quantum constitution accepts, as input, society's joint state, $\sigma_{\rm soc}$, and an ancilla. The ancilla is initialized to a fiducial state $|0\rangle\langle 0|$. When outputted by the constitution, the ancilla holds society's quantum preference, $\rho_{\rm soc}$.

Definition 1 (Quantum constitution)**.** *A* quantum constitution *is a convex-linear CPTP map*

$$\mathcal{E} : \mathcal{D}\left(\mathcal{H}^{\otimes(N+1)}\right) \to \mathcal{D}(\mathcal{H})$$

that transforms society's joint state and an ancilla into society's quantum preference:

$$\mathcal{E}(\sigma_{\rm soc} \otimes |0\rangle\langle 0|) = \rho_{\rm soc} \, . \tag{1}$$

Having defined constitutions, we can define the classical limit.

Definition 2. *The* classical limit *of a quantum election is the satisfaction of the following conditions:*

1. *Every quantum voter preference ρ_i is an element of the preference basis $\mathcal{B}$.*

2. *The quantum constitution $\mathcal{E}$ consists of classical probabilistic logic gates.*

In the classical limit, $\mathcal{E}$ can output only elements of $\mathcal{B}$ and probabilistic combinations thereof.

Classical and quantum constitutions can have various properties. Four properties appear in Arrow's Theorem. We review these classical properties, then quantize them.

2. The four constitutional properties in Arrow's Theorem and quantum analogs

Arrow's Theorem features four properties available to classical constitutions: transitivity, respecting of unanimity, respecting of independence of irrelevant alternatives, and being a dictatorship. We review these properties and define quantum analogs.

Two principles guide the quantization strategy. First, each quantum definition should preserve the corresponding classical definition's spirit. Second, each quantum definition should make sense in the context of entanglement and superpositions—should be able to characterize a quantum circuit.

A classical constitution $\mathcal{C}$ is *transitive* if every classical preference in its range is transitive. Suppose that society prefers candidate a to b and prefers b to c. $\mathcal{C}$ outputs a societal preference in which a ranks above c: $a \geq b$ and $b \geq c$, together, imply $a \geq c$.

Definition 3 (Quantum transitivity)**.** *A quantum constitution $\mathcal{E}$ respects* quantum transitivity *if every possible output $\rho_{\rm soc}$, upon being measured in the preference basis $\mathcal{B}$, collapses to a state $|a \ldots m\rangle$ associated with a transitive classical preference $(a \ldots m)$.*

Every $\mathcal{E}$ obeys quantum transitivity by definition: Given any input, $\mathcal{E}$ outputs a $\rho_{\rm soc}$ that is a linear combination or a mixture of preference-basis elements. A $\mathcal{B}$ measurement of $\rho_{\rm soc}$ yields a $\mathcal{B}$ element. Every $\mathcal{B}$ element corresponds to a transitive classical preference.

Classical *unanimity* is defined as follows. Let $\mathcal{C}$ denote a classical constitution that respects unanimity. Suppose that every voter ranks a candidate a strictly above a candidate b: $a > b$. The constitution outputs a societal preference in which a ranks strictly above b: $a > b$.

Definition 4 (Quantum unanimity)**.** *A quantum constitution $\mathcal{E}$ respects* quantum unanimity *if it has the following two subproperties:*

1. *Suppose that every voter's quantum preference has support on the $a > b$ subspace:* $\mathrm{Tr}\left(\Pi^{a>b}\, \rho_i\right) > 0 \;\; \forall i = 1, 2, \ldots, N$*. $\mathcal{E}$ outputs a societal quantum preference $\rho_{\rm soc}$ that has support on that subspace:* $\mathrm{Tr}\left(\Pi^{a>b}\, \rho_{\rm soc}\right) > 0$.

2. *Suppose that every voter's quantum preference has support only on the* $a > b$ *subspace.* $\mathcal{E}$ *outputs a societal quantum preference* $\rho_{\rm soc}$ *that has support only on that subspace:*

$$\text{supp}(\rho_i) \subseteq \Pi^{a>b} \quad \forall i = 1, 2, \ldots, N \quad \Rightarrow \\ \text{supp}(\rho_{\rm soc}) \subseteq \Pi^{a>b} \, , \tag{2}$$

wherein $\text{supp}(\rho)$ *denotes the support of the quantum state* ρ.

Subproperty 2 might appear extraneous, seeming to lack a classical counterpart. But classical unanimity satisfies the classical analog of 2 implicitly, as the following argument shows.

(A) Suppose that every voter's preference satisfies the classical analog of having support only on $\Pi^{a>b}$: Every voter prefers $a > b$ strictly.

(B) Society prefers $a > b$ strictly, by the definition of classical unanimity.

(C) Hence society ranks a and b neither as $a = b$ nor as $b > a$.

(D) Hence society's preference satisfies the classical analog of having support only on $\Pi^{a>b}$.

Definition 4 must contain subproperty 2 explicitly because the quantum analog of step (B) does not imply the quantum analog of step (C). Even if $\rho_{\rm soc}$ has support on $\Pi^{a>b}$, $\rho_{\rm soc}$ can have support on $\Pi^{a=b}$: $\rho_{\rm soc}$ can be a linear combination of elements of $B_{\rm soc}$ or can be a mixture. The generality of quantum states necessitates the articulation of subproperty 2.

Classical *independence of irrelevant alternatives* (IIA) is defined as follows. In every classical preference, the candidates a and b have some *relative ranking*. Either $a > b$, $b > a$, or $a = b$. Suppose that society's relative ranking of a and b depends only on every voter's relative ranking of a and b. Whether society prefers a to b (or prefers b to a, etc.) depends only on whether each voter prefers a to b (or prefers b to a, etc.). How any voter ranks candidate c fails to influence society's relative ranking of a and b.

Definition 5 (Quantum independence of irrelevant alternatives)**.** *A quantum constitution respects* quantum independence of irrelevant alternatives *(QIIA) if whether* $\rho_{\rm soc}$ *has support on* $\mathcal{G}^{a>b}$*, on* $\mathcal{G}^{a<b}$*, and/or on* $\mathcal{G}^{a=b}$ *depends only on whether each* ρ_i *has support on* $\mathcal{G}^{a>b}$*, on* $\mathcal{G}^{a<b}$*, and/or on* $\mathcal{G}^{a=b}$*.*

A classical dictatorship has a dominant voter. Suppose that society prefers a strictly to b if and only if some voter i prefers a strictly to b, for all pairs (a, b) of candidates:

$$\exists i \, : \, a > b \, , \text{ according to } i, \quad \Leftrightarrow \\ a > b \, , \text{ according to society}, \quad \forall a, b \, . \tag{3}$$

The classical constitution $\mathcal{C}$ that outputs society's preference is a classical *dictatorship*.

Definition 6 (Quantum dictatorship)**.** *A quantum constitution is a* quantum dictatorship *if there exists a voter* i *who has the following two characteristics:*

1. *Society's quantum preference has support on the* $a > b$ *subspace if and only if voter* i*'s has:*

$$\text{Tr}\left(\Pi^{a>b}\rho_i\right) > 0 \quad \Leftrightarrow \quad \text{Tr}\left(\Pi^{a>b}\rho_{\rm soc}\right) > 0. \tag{4}$$

2. *Society's quantum preference has support only on the* $a > b$ *subspace if and only if voter* i*'s has:*

$$\text{supp}(\rho_i) \subseteq \Pi^{a>b} \quad \Leftrightarrow \quad \text{supp}(\rho_{\rm soc}) \subseteq \Pi^{a>b} \, . \tag{5}$$

Subproperty 2 plays a role analogous to subproperty 2 in the definition of "quantum unanimity."

We have constructed quantum analogs of the four properties in Arrow's Theorem. A quantum version of Arrow's Theorem, we show, is violated by a quantum version of majority rule.

C. Majority rule

Majority rule is a fifth property that constitutions can have. We review classical majority rule, then introduce a quantum analog. *Cyclic* voting preferences prevent classical majority rule from satisfying Arrow's assumptions. Quantum majority rule is more robust.

1. Classical majority rule

Let $\mathcal{P}_{\rm cl}$ denote a classical society's voter profile. Let $\mathcal{C}$ denote a classical constitution that respects majority rule. $\mathcal{C}$ reflects the wishes shared by most voters. Suppose that over half the voters agree on the relative ranking of candidates a and b. $\mathcal{C}$ outputs a classical societal preference that has the same relative ranking of a and b.

A subtlety arises if $\mathcal{P}_{\rm cl}$ involves a cycle. Let $T = \{a, b, \ldots, k\}$ denote a set of candidates. Suppose that a and b participate, in $\mathcal{P}_{\rm cl}$, in pairwise preferences that violate transitivity. Suppose that every pair of candidates in T does. T forms a classical *cycle*.

For example, let $\mathcal{P}_{\rm cl} = \{(a > b > c), (c > a > b), (b > c > a)\}$. A naïve application of majority rule implies $a > b$ and $b > c$. Transitivity implies $a > c$. But a naïve application of majority rule implies also $c > a$. But $c > a$, combined with the previously derived $a > c$, violates transitivity. The constitution may be defined as outputting $a = b = c$ or as outputting an error message.[1]

[1] One profile can contain multiple cycles. For example, $\{(a > b > c), (b > a > c), (a > c > b)\}$ contains a cycle over (a, b) (because voters 1 and 3 rank $a > b$, whereas voter 2 ranks $b > a$) and a cycle over (b, c) (because voters 1 and 2 rank $b > c$, whereas voter 3 ranks $c > b$).

Cycles prevent classical majority rule from satisfying IIA and transitivity simultaneously. Classical majority rule fails to satisfy Arrow's assumptions. Hence classical majority rule cannot contradict Arrow's Theorem. A quantum analog of majority rule can.

2. Quantum majority rule

First, we introduce quantum cycles. We then define the Quantum Majority-Rule (QMR) constitution $\mathcal{E}_{\text{QMR}}$. This constitution, we show, respects quantum transitivity, quantum unanimity, and QIIA. These properties will enable $\mathcal{E}_{\text{QMR}}$ to violate a quantum analog of Arrow's Theorem.

Quantum cycles: Let $\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu$ be a product of preference-basis elements. Suppose that at least two χ_i^γ's are pure states labeled by classical preferences that form a classical cycle. The product will be said to contain a *quantum cycle*.

Operation of the Quantum Majority-Rule constitution: $\mathcal{E}_{\text{QMR}}$ performs the following sequence of steps. First, $\mathcal{E}_{\text{QMR}}$ decoheres each quantum preference ρ_i with respect to the preference basis:

$$\rho_i \mapsto \sum_\gamma |\gamma\rangle\langle\gamma|\rho_i|\gamma\rangle\langle\gamma| = \sum_\gamma p_i^\gamma \chi_i^\gamma =: \rho_i' \,, \tag{6}$$

wherein $\sum_\gamma p_i^\gamma = 1$. Society's quantum profile evolves as

$$\sigma_{\text{soc}} \mapsto \rho_1' \otimes \ldots \otimes \rho_N' \tag{7}$$

$$= \sum_{\alpha,\ldots,\mu} (p_1^\alpha \ldots p_N^\mu)\, (\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu) \,. \tag{8}$$

Recall that χ_1^α denotes the element, labeled by the classical preference α, of the preference basis $\mathcal{B}$ for voter 1's Hilbert space $\mathcal{H}$.

$\mathcal{E}_{\text{QMR}}$, being a quantum constitution, obeys convex linearity. To specify how $\mathcal{E}_{\text{QMR}}$ transforms the right-hand side of Eq. (8), we must specify just how $\mathcal{E}_{\text{QMR}}$ transforms each factor $\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu$.

For each factor, $\mathcal{E}_{\text{QMR}}$ constructs a directed graph, or digraph. One vertex is formed for each candidate. The edges are governed by $\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu$. If more classical preferences γ correspond to $a > b$ than to $b > a$, an edge points from a to b. If exactly as many γ's correspond to $a > b$ as to $b > a$, an edge points from a to b and from b to a.

$\mathcal{E}_{\text{QMR}}$ inputs the digraph into *Tarjan's algorithm* [28]. Tarjan's algorithm finds a digraph's strongly connected components. A *strongly connected component* (SCC) is a subgraph. Every vertex in the subgraph can be accessed from every other vertex via edges. Every vertex appears in exactly one SCC. Every SCC in the QMR graph represents a cycle or a set of interlinked cycles. For example, let $\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu = |b{>}a{>}c{>}d\rangle\langle b{>}a{>}c{>}d| \otimes |a{>}c{>}b{>}d\rangle\langle a{>}c{>}b{>}d|$. Candidates a and b participate in a cycle, as do b and c. The a, b, and c vertices form an SCC. The d vertex forms another SCC. The digraph appears in Fig. 1.

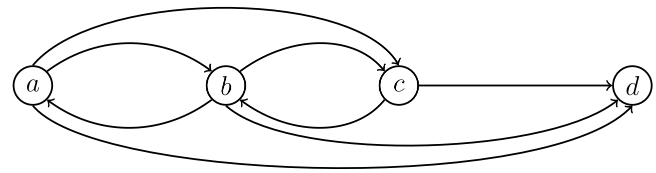

FIG. 1: Example digraph formed by the Quantum Majority Rule (QMR) constitution $\mathcal{E}_{\text{QMR}}$: Consider a society that consists of two voters. Suppose that they submit the quantum preferences (votes) $|b{>}a{>}c{>}d\rangle$ and $|a{>}c{>}b{>}d\rangle$. Voter 1 prefers candidate b strictly to candidate a, etc. Voter 2 prefers a strictly to c, etc. $\mathcal{E}_{\text{QMR}}$ maps this set of preferences to a digraph. Each candidate is associated with a vertex. The voters' preferences determine the edges. For example, most voters prefer candidate a to candidate d. Hence an edge points from vertex a to vertex d. Exactly as many voters prefer a to b as prefer b to a. Hence a doubly directed edge connects a with b. Vertices a and b form one cycle, while b and c form another. a, b, and c form a strongly connected component (SCC). So does vertex d. Tarjan's algorithm identifies digraphs' SCCs. $\mathcal{E}_{\text{QMR}}$ uses Tarjan's algorithm to form society's quantum preference, to determine who wins the election.

Tarjan's algorithm returns a list of the SCCs. The later an SCC appears in the list, the more popular the SCC's candidates, roughly speaking. More precisely, let i and j label SCCs such that $i < j$. Every vertex in the j^{th} SCC is preferred to every vertex in the i^{th}. For example, Tarjan's algorithm maps Fig. 1 to $(\{d\}, \{a, b, c\})$.

Consider the strict classical preferences in which every candidate in the j^{th} SCC ranks above every candidate in the i^{th} SCC, for all $j > i$. $\mathcal{E}_{\text{QMR}}$ forms a maximally mixed state ρ_{soc}' over the corresponding preference-basis elements. In our example,

$$\rho_{\text{soc}}' = \frac{1}{6}(|abcd\rangle\langle abcd| + |acbd\rangle\langle acbd| + |bacd\rangle\langle bacd| + |bcad\rangle\langle bcad| + |cabd\rangle\langle cabd| + |cbad\rangle\langle cbad|) \,. \tag{9}$$

$\mathcal{E}_{\text{QMR}}$ then "gives the minority a shot." For any candidate pair (a, b), suppose that at least one χ_i^γ corresponds to $a > b$. The constitution spreads an amount $\delta \in (0, 1)$ of weight across the $a > b$ subspace:[2]

$$\rho_{\text{soc}}' \mapsto \rho_{\text{soc}}'' = (1 - \delta)\rho_{\text{soc}}' + \delta\, \Pi^{a>b} \,. \tag{10}$$

This δ serves as a parameter inputted to the constitution. We omit δ from the notation $\mathcal{E}_{\text{QMR}}$ for conciseness.

[2] "Giving the minority a shot" resembles the action of the United States electoral college. Whichever candidate receives the most popular votes usually wins the presidential election. But a candidate who receives a minority can win the presidency if favored by enough of the electoral college.

Next, $\mathcal{E}_{\rm QMR}$ enforces unanimity. Suppose that every χ_i^γ corresponds to $a > b$, for any candidate pair (a, b): $\text{supp}(\chi_i^\gamma) \subseteq \Pi^{a>b} \quad \forall i = 1, 2, \ldots, N$. The constitution projects $\rho''_{\rm soc}$ onto the $a > b$ subspace:

$$\rho''_{\rm soc} \mapsto \rho'''_{\rm soc} = \Pi^{a>b}\, \rho''_{\rm soc}\, \Pi^{a>b} \,. \tag{11}$$

We have seen how $\mathcal{E}_{\rm QMR}$ calculates the $\rho'''_{\rm soc}$ associated with each term in Eq. (8). Each $(\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu)$ in Eq. (8) is replaced with the corresponding $\rho'''_{\rm soc}$. This replacement yields $\rho_{\rm soc}$. The $\rho_{\rm soc}$ is measured with respect to $\mathcal{B}$. The measurement yields society's classical preference.

Three properties of QMR: $\mathcal{E}_{\rm QMR}$ has three of the properties introduced in Sec. I B 2. These properties will enable $\mathcal{E}_{\rm QMR}$ to violate a quantum analog of Arrow's Theorem.

Lemma 1. *The Quantum Majority-Rule constitution $\mathcal{E}_{\rm QMR}$ respects quantum transitivity, quantum unanimity, and quantum independence of irrelevant alternatives.*

Proof. Every quantum constitution respects quantum transitivity, as explained below Definition 3. $\mathcal{E}_{\rm QMR}$ is a quantum constitution. Therefore, $\mathcal{E}_{\rm QMR}$ respects quantum transitivity.

Quantum unanimity involves two subproperties (see Definition 4). $\mathcal{E}_{\rm QMR}$ respects subproperty 1 due to Tarjan's algorithm and Eq. (9). Suppose that every voter's quantum profile has support on the $a > b$ subspace. Most quantum profiles have support on that subspace. Hence a appears in the b SCC or in an SCC "preferred to" the b SCC. Hence $\rho'_{\rm soc}$ contains preference-basis elements associated with $a > b$.

Equation (11) ensures that $\mathcal{E}_{\rm QMR}$ respects subproperty 2 of quantum unanimity. Suppose that every voter's quantum preference has support only on the $a > b$ subspace. Every χ_i^γ has support only on the $a > b$ subspace. $\mathcal{E}_{\rm QMR}$ projects $\rho''_{\rm soc}$ onto $\mathcal{G}^{a>b}$, not onto $\mathcal{G}^{b>a}$ or onto $\mathcal{G}^{a=b}$. Therefore, $\rho'''_{\rm soc}$ has support only on $\mathcal{G}^{a>b}$.

According to QIIA, whether $\rho_{\rm soc}$ has support on $\mathcal{G}^{a>c}$ on $\mathcal{G}^{c>a}$ and/or on $\mathcal{G}^{a=c}$ depends only on whether each voter's quantum preference, ρ_i, has support on these subspaces—not on whether any ρ_i has support on, e.g., $\mathcal{G}^{a>b}$. To check that $\mathcal{E}_{\rm QMR}$ respects QIIA, we must analyze three cases:

1. a does not participate in a cycle with c.
    (a) a participates in a cycle with at least one candidate that participates in a cycle with c. For example, a may participate in a cycle with b, while b participates in a cycle with c.
    (b) a participates in no cycle with any candidate that participates in a cycle with c.
2. a participates in a cycle with c.

Case 1a requires the most thought. Considering the example illustrated in Fig. 1 suffices. a does not participate in a cycle with c. Yet a participates in a cycle with b, which participates in a cycle with c. Therefore, a appears in the same SCC as c. According to Eq. (9), $\rho'_{\rm soc}$ has support on $\mathcal{G}^{c>a}$. Yet every quantum voter preference ρ_i has support only on $\mathcal{G}^{a>c}$. How voters rank b seems to influence how society ranks a relative to c. $\mathcal{E}_{\rm QMR}$ seems to violate QIIA.

Equation (11) rectifies this seeming violation. $\rho''_{\rm soc}$ is projected onto the $a > c$ subspace, because every χ_i^γ corresponds to $a > c$.

But suppose that not every χ_i^γ corresponded to $a > c$. Suppose that only a majority of χ_i^γ's did. $\rho''_{\rm soc}$ would not be projected onto $\mathcal{G}^{a>c}$. How voters ranked b would again seem to influence how society ranked a relative to c. $\mathcal{E}_{\rm QMR}$ would again seem to violate QIIA. $\mathcal{E}_{\rm QMR}$ would not because of Eq. (10). Some χ_i^γ's have support on the $c > a$ subspace. The "give the minority a shot" step therefore gives $\rho''_{\rm soc}$ support on $\mathcal{G}^{c>a}$. Society's quantum preference would have support on $\mathcal{G}^{c>a}$ regardless of whether a participated in a cycle with a b that participated in a cycle with c. How voters rank b therefore does not affect how society ranks a relative to c.

In case 1b, a does not participate in a cycle with any b that participates in a cycle with c. Therefore, a appears in an SCC that "is preferred" to the c SCC. $\rho'_{\rm soc}$ therefore has support on just the $a > c$ subspace, regardless of any b's.

In case 2, a participants in a cycle with c. $\rho'_{\rm soc}$ has support on the $a > c$ and $c > a$ subspaces, regardless of any b's.

□

Because QMR satisfies the quantum analogs of three properties in Arrow's Theorem, QMR can violate a quantum analog of Arrow's Theorem.

II. ARROW'S IMPOSSIBILITY THEOREM

Transitivity, unanimity, and IIA have innocent-sounding definitions. They seem unlikely to buttress authoritarianism. Yet possessing these properties, Arrow shows, renders a classical constitution a dictatorship [11].

Theorem 1 (Arrow's Impossibility Theorem)**.** *Consider any (classical) constitution used, with ranked voter preferences, to select from amongst at least three candidates. If the constitution respects transitivity, unanimity, and independence of irrelevant alternatives, the constitution is a dictatorship.*

Multiple proof exist [11, 29, 30]. Some involve a *pivotal voter* v [29, 30]. If v changes his/her mind while all other preferences remain constant, society's preference changes. One proves first that the postulates imply the existence of a voter slightly weaker than v. This voter, one then shows, is pivotal and is a dictator. No other dictator, one concludes, can exist.

We quantize Arrow's Theorem in the following conjecture.

Conjecture 1 (Quantum Arrow Conjecture)**.** *Every quantum constitution that respects quantum transitivity, quantum unanimity, and quantum independence of irrelevant alternatives is a quantum dictatorship.*

Theorem 2. *The Quantum Arrow Conjecture is false.*

Proof. We disprove the conjecture by counterexample. The QMR constitution is combined with a societal joint state $\sigma_{\rm soc}$ that encodes a cycle. This combination, we show, lacks a dictator. We have shown that QMR satisfies quantum transitivity, quantum unanimity, and QIIA. Satisfying the conjecture's assumptions but not its conclusion, QMR and cyclic voting disprove the conjecture.

For simplicity, we focus on strict pairwise preferences. We consider, e.g., $a > b$, ignoring $a = b$. This focus frees us to drop binary-relation symbols: $|abc\rangle := |a{>}b{>}c\rangle$.

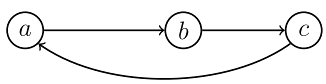

FIG. 2: Digraph representation of the quantum votes used to disprove the Quantum Arrow Conjecture: A quantum analog of majority rule, acting on the quantum votes in Eq. (12), violates a quantum analog of Arrow's Theorem. All three candidates—a, b, and c—form one cycle, as depicted by the pattern of arrows. Cycles prevent classical majority-rule constitutions from satisfying all of Arrow's postulates. Quantum majority rule is more robust.

Suppose that society's joint state is a product that involves a cycle:

$$\sigma_{\rm soc} = |abc\rangle\langle abc| \otimes |cab\rangle\langle cab| \otimes |bca\rangle\langle bca|\,. \tag{12}$$

Decoherence relative to the preference basis preserves the state. $\mathcal{E}_{\rm QMR}$ constructs the digraph in Fig. 2. One edge points from a to b (because two voters prefer $a > b$, whereas one prefers $b > a$), one edge points from b to c, and one edge points from c to a. The digraph consists of one SCC. $\mathcal{E}_{\rm QMR}$ therefore constructs the linear combination

$$\rho'_{\rm soc} \propto |abc\rangle\langle abc| + |cab\rangle\langle cab| + |bca\rangle\langle bca| + |cba\rangle\langle cba| + |bac\rangle\langle bac| + |acb\rangle\langle acb|\,. \tag{13}$$

The "give the minority a shot" step preserves the state: $\rho''_{\rm soc} = \rho'_{\rm soc}$. The voters do not unanimously prefer any candidate to any other: For every voter i, there exists a voter j such that $\mathrm{supp}(\rho_i)$ and $\mathrm{supp}(\rho_j)$ occupy subspaces labeled by distinct classical preferences. $\mathcal{E}_{\rm QMR}$ therefore does not project $\rho''_{\rm soc}$ onto any subspace: $\rho'''_{\rm soc} = \rho''_{\rm soc}$. Equation (8) consists of only one term, so $\rho_{\rm soc} = \rho'''_{\rm soc}$. Society's quantum preference appears in Eq. (13).

$\rho_{\rm soc}$ has support on multiple subspaces, e.g., $\mathcal{G}^{a>b}$ and $\mathcal{G}^{b>a}$. No quantum voter preference ρ_i has. No voter is a quantum dictator, by definition 6. Yet $\mathcal{E}$ respects quantum transitivity, quantum unanimity, and QIIA, by Lemma 1. $\mathcal{E}_{\rm QMR}$ satisfies the assumptions, but violates the conclusion, of the Quantum Arrow Conjecture. The conjecture is therefore false. □

One can understand as follows why our scheme violates the Quantum Arrow Conjecture. The successes of quantum game theory motivate the generalization of voting to accommodate entangled and superposed preferences. To introduce entanglement and superpositions, one must formulate an election as a general quantum process—a preparation procedure, an evolution, and a measurement. Classical constitutional properties must be quantized faithfully. The quantum translations enable the Quantum Majority-Rule constitution to respect quantum transitivity and QIIA simultaneously. Classical majority-rule constitutions cannot respect transitivity and IIA simultaneously, due to cyclic votes. But QMR satisfies all the assumptions in the Quantum Arrow Conjecture. QMR, with a cyclic voter profile, violates the conjecture.

Disproofs simpler than ours exist. Ours offers interpretational advantages, however. For instance, let $\mathcal{K}$ denote a quantum constitution that outputs a superposition over all inputs. $\mathcal{K}$ violates the conjecture. But imposing $\mathcal{K}$ on society—choosing society's classical preference totally randomly—makes little economic sense. Also, our disproof elucidates how quantization invalidates Arrow's idea. Classical majority rule fails to satisfy Arrow's postulates, due to cycles. Quantum Majority Rule is more resilient. Quantization elevates a classically inadequate disproof attempt to a quantum disproof.

III. QUANTUM VOTING TACTICS

Imagine that Alice, Bob, and Charlie vie for the presidency of the American Physical Society. Suppose that Alice and Bob have greater chances of winning than Charlie has. Suppose that Charlie agrees more with Alice than with Bob. Charlie's supporters might vote for Alice. They would be trying to elect a president whom they neither prefer most nor mind most. Charlie's supporters would be practicing strategic voting. *Strategic voting* is the submission of a preference other than one's opinion, to secure an unobjectionable outcome, in an election amongst three or more candidates [23].

We introduce *quantum strategic voting.* Voters leverage entanglement, superpositions, and interference. We present three tactics reliant on entanglement and one tactic reliant on interference and superpositions. Other quantum tactics may exist and merit exploration.

To highlight the basic physics, we focus on strict preferences, as in the proof of Theorem 2. For example, we consider $a > b$ to the exclusion of $a = b$. We also focus on pure joint quantum states $\sigma_{\rm soc}$.

The strict-preferences assumption lets us compactify notation. The classical preference $a > b > \ldots > k > \ell > m$ has the even permutations $m > a > b > \ldots > k > \ell$,

$\ell > m > a > b > \ldots > k$, etc. These preferences are labeled $\alpha, \ldots, \mu$. Each preference γ corresponds to one anticycle. We denote the anticycle with a bar: $\bar{\gamma}$. For example, the cycle $\alpha := a > b > \ldots > m$ corresponds to the anticycle $\bar{\alpha} := m > \ldots > b > a$.

Every pure quantum preference has the form

$$\sum_{\gamma} (c_\gamma |\gamma\rangle + c_{\bar{\gamma}} |\bar{\gamma}\rangle), \quad \text{wherein} \quad \sum_{\gamma} \left(|c_\gamma|^2 + |c_{\bar{\gamma}}|^2 \right) = 1. \tag{14}$$

Society's joint quantum state has the form

$$|\sigma_{\text{soc}}\rangle = (c_{\alpha_1} \ldots c_{\alpha_N}) |\alpha \ldots \alpha\rangle + \ldots + (c_{\bar{\mu}_1} \ldots c_{\bar{\mu}_N}) |\bar{\mu} \ldots \bar{\mu}\rangle. \tag{15}$$

A. Three entanglement-dependent voting tactics

Let us simplify our quantum analog of majority rule, now that QIIA, etc. need not concern us. We introduce the variation *QMR2*, labeled $\mathcal{E}^{(2)}_{\text{QMR}}$. QMR2 is defined as follows.

$\mathcal{E}^{(2)}_{\text{QMR}}$ processes σ_{soc} as in Eqs. (7) and (8). Society's joint quantum state σ_{soc} is decohered with respect to the product of the voters' $\mathcal{B}$'s. $\mathcal{E}^{(2)}_{\text{QMR}}$ processes each term in Eq. (8) as follows. The j^{th} term has the form

$$(p_1^\alpha \ldots p_N^\mu)_j \, (\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu)_j \equiv p_j (\chi_1^\alpha \otimes \ldots \otimes \chi_N^\mu)_j \, . \tag{16}$$

The term is labeled by a list $L_j = (\alpha, \ldots, \mu)_j$ of classical preferences. If most of the preferences are identical—if most equal γ, say—the j^{th} term in Eq. (8) is associated with (p_j, γ). If no majority favors any γ, $\mathcal{E}^{(2)}_{\text{QMR}}$ chooses uniformly randomly from amongst the classical preferences that appear with the highest frequency in L_j.

$\mathcal{E}^{(2)}_{\text{QMR}}$ has assembled a list (p_j, γ_j). Society's classical preference is selected from amongst the γ_j's according to the probability distribution $\{p_j\}$.

Entanglement can help one voter obstruct another. Imagine that the Supreme Court justices vote via QMR2. Suppose that Justice Alice wants to diminish Justice Bob's influence. However Bob votes, Alice should vote oppositely. Alice should entangle her quantum preference with Bob's. (Given how opinionated Supreme Court justices are, Bob might not mind broadcasting his quantum preference.) If Bob votes as in Eq. (14), Alice should form

$$\sum_{\gamma} (c_\gamma |\gamma \, \bar{\gamma}\rangle + c_{\bar{\gamma}} |\bar{\gamma} \, \gamma\rangle) \, . \tag{17}$$

Insofar as γ represents Bob's preference, Alice votes oppositely, with $\bar{\gamma}$. Even if Bob changes his mind seconds before everyone votes, Alice need not scramble to alter her vote.

Entanglement also facilitates party-line voting, if society uses QMR2. Suppose that Alice leads the Scientists' Party, to which Bob and Charlie belong. However Alice votes, Bob and Charlie wish to vote identically. The voters should form the entangled state

$$\sum_{\gamma} (c_\gamma |\gamma \, \gamma \, \gamma\rangle + c_{\bar{\gamma}} |\bar{\gamma} \, \bar{\gamma} \, \bar{\gamma}\rangle) \, , \tag{18}$$

whose weights Alice chooses. This state generalizes the GHZ state: If the weights equal each other and only two candidates run, (18) reduces to $\frac{1}{\sqrt{2}}(|\alpha\alpha\alpha\rangle + |\bar{\alpha}\bar{\alpha}\bar{\alpha}\rangle)$.

Finally, entangling voters' quantum preferences can pare down society's possible classical preferences. Suppose that Alice, Bob, and Charlie separately favor α twice as much as they prefer β. Each voter plans to submit $\sqrt{\frac{2}{3}} \, |\alpha\rangle + \sqrt{\frac{1}{3}} \, |\beta\rangle$. Society's joint state would be

$$\begin{aligned} |\sigma_{\text{soc}}\rangle = & \left(\frac{2}{3}\right)^{3/2} |\alpha\alpha\alpha\rangle + \left(\frac{1}{3}\right)^{3/2} |\beta\beta\beta\rangle \\ & + \frac{2}{3^{3/2}} (|\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle + |\alpha\alpha\beta\rangle) \\ & + \frac{\sqrt{2}}{3^{3/2}} (|\alpha\beta\beta\rangle + |\beta\alpha\beta\rangle + |\beta\beta\alpha\rangle). \end{aligned} \tag{19}$$

If the constitution is QMR2, society might adopt α or β as its classical preference.

Suppose that Alice, Bob, and Charlie misunderstand entanglement. Eve can take advantage of their ignorance to eliminate β from society's possible classical preferences. Suppose that Eve convinces the three citizens to submit the W state

$$|\sigma'_{\text{soc}}\rangle = \frac{1}{\sqrt{3}} (|\beta\alpha\alpha\rangle + |\alpha\beta\alpha\rangle + |\alpha\alpha\beta\rangle). \tag{20}$$

This entangled analog of $|\sigma_{\text{soc}}\rangle$, Eve might claim, represents the voters' opinion: $|\sigma'_{\text{soc}}\rangle$ contains twice as many α's as β's. But QMR2 cannot map $|\sigma'_{\text{soc}}\rangle$ to β. Entangled states lead to different possible election outcomes than product states.

B. Quantum strategic voting via interference

Like entanglement, interference and relative phases facilitate quantum strategic voting. Consider a society $\mathcal{S}$ whose voters submit pure quantum preferences. Let $\mathcal{S}$ use a variation *QMR3*, denoted by $\mathcal{E}^{(3)}_{\text{QMR}}$, on QMR.

To illustrate QMR3 and the role of interference, we consider the voter profile

$$\mathcal{P} = \left\{ |abc\rangle, \frac{1}{\sqrt{2}}(|bac\rangle + |acb\rangle), \frac{1}{\sqrt{2}}(|bac\rangle + |cba\rangle) \right\} . \tag{21}$$

Society's joint state has the form

$$\begin{aligned} |\sigma_{\text{soc}}\rangle = & \frac{1}{2} (|abc\rangle |bac\rangle |bac\rangle + |abc\rangle |bac\rangle |cba\rangle \\ & + |abc\rangle |acb\rangle |bac\rangle + |abc\rangle |acb\rangle |cba\rangle) \, . \end{aligned} \tag{22}$$

Similarly to $\mathcal{E}_{\text{QMR}}$, $\mathcal{E}_{\text{QMR}}^{(3)}$ forms a digraph from each $|\sigma_{\text{soc}}\rangle$ term. Each digraph is inputted into Tarjan's algorithm, which returns a list of the SCCs. Just as $\mathcal{E}_{\text{QMR}}$ maps each list to a mixed state ρ'_{soc}, $\mathcal{E}_{\text{QMR}}^{(3)}$ maps the i^{th} list to a superposition $|\rho_{\text{soc}}^{(i)}\rangle$. Society's quantum preference becomes

$$|\rho_{\text{soc}}\rangle \propto \sum_{i=1}^{4} |\rho_{\text{soc}}^{(i)}\rangle . \tag{23}$$

In our example,

$$|\rho_{\text{soc}}\rangle = \frac{1}{\sqrt{6}}(|bac\rangle + |bac\rangle + |abc\rangle + |acb\rangle) \tag{24}$$

$$= \sqrt{\frac{2}{3}}\,|bac\rangle + \frac{1}{\sqrt{6}}(|abc\rangle + |acb\rangle) . \tag{25}$$

$|\rho_{\text{soc}}\rangle$ may vanish: The QMR3 quantum circuit may fail to output any quantum state. If $|\rho_{\text{soc}}\rangle = 0$, society can hold a revote. (Because QMR3 is defined on just pure states and does not preserve all inputs' norms, QMR3 does not satisfy Definition 1. QMR3 can be regarded as belonging to an extension of quantum constitutions.)

Suppose that voter 3 wishes to eliminate *bac* from society's possible classical preferences. Eliminating $|bac\rangle$ from voter 3's quantum preference, $|\rho_3\rangle$, will not suffice. Voter 3 should introduce a relative phase of -1 into $|\rho_3\rangle$. (Alternatively, voter 3 could submit a superposition of $|abc\rangle$ and $|acb\rangle$.) Society's quantum profile becomes

$$\mathcal{P}' = \left\{ |abc\rangle, \frac{1}{\sqrt{2}}(|bac\rangle + |acb\rangle), \frac{1}{\sqrt{2}}(-|bac\rangle + |cba\rangle) \right\}. \tag{26}$$

Tarjan's algorithm leads to $|\rho_{\text{soc}}\rangle \propto \frac{1}{2}(-|bac\rangle + |bac\rangle - |abc\rangle + |acb\rangle)$. Hence $|\rho_{\text{soc}}\rangle = \frac{1}{\sqrt{2}}(|abc\rangle - |acb\rangle)$. Keeping the undesired $|bac\rangle$ in voter 3's quantum preference contradicts our intuitions. Yet interfering the new $|\rho_3\rangle$ with the other votes eliminates *bac* from society's possible classical preferences.

IV. CONCLUSIONS

We have quantized elections, in the tradition of quantum game theory. The quantization obviates a quantum analog of Arrow's Theorem about the impossibility of a nondictatorship's having three simple properties. Entanglement, superpositions, and interference expand voters' arsenals of manipulation strategies. Whether other quantum strategies, unavailable to classical voters, exist merits investigation. So does whether monogamy of entanglement [31] limits one voter's influence on others' quantum preferences. If creating entanglement is difficult (as in many labs), the resource theory of multipartite entanglement [32] might illuminate how voters can optimize their influence.

Additionally, other voting schemes could be quantized. Examples include proportional representation (in which the percentage of voters who favor Party a dictates the number of government seats won by Party a) and cardinal voting (in which voters grade, rather than rank, candidates).

Finally, counterstrategies may be formulated. Consider our first entanglement-dependent voting example: Justice Bob of the Supreme Court prepares his vote. Justice Alice blocks Bob's effort using entanglement. How should Justice Bob parry? Can entanglement assist him? This problem mirrors quantum-cryptographic problems: A sender wishes to communicate with a receiver securely. An eavesdropper attacks. The eavesdropper may access quantum or only classical resources, depending on the problem. How can the sender and receiver parry? We have illustrated how our "eavesdropper," Justice Alice, might wield entanglement. How Justice Bob should counter merits thought.

These opportunities can help illuminate how quantum theory changes the landscape of possible outcomes and strategies in games.

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