Coherence-generating power of quantum unitary maps and beyond

Paolo Zanardi, Georgios Styliaris, and Lorenzo Campos Venuti

Phys. Rev. A **95**, 052306 — Published 2 May 2017

DOI: 10.1103/PhysRevA.95.052306

Coherence generating power of quantum unitary maps and beyond

Paolo Zanardi, Georgios Styliaris, and Lorenzo Campos Venuti
Department of Physics and Astronomy, and Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, CA 90089-0484

Given a preferred orthonormal basis B in the Hilbert space of a quantum system we define a measure of the coherence generating power of a unitary operation with respect to B. This measure is the average coherence generated by the operation acting on a uniform ensemble of incoherent states. We give its explicit analytical form in any dimension and provide an operational protocol to directly detect it. We characterize the set of unitaries with maximal coherence generating power and study the properties of our measure when the unitary is drawn at random from the Haar distribution. For large state-space dimension a random unitary has, with overwhelming probability, nearly maximal coherence generating power with respect to any basis. Finally, extensions to general unital quantum operations and the relation to the concept of asymmetry are discussed.

I. INTRODUCTION

One of the most fundamental attributes of quantum dynamical systems is their ability to exist in linear superpositions of different physical states. In fact any pure quantum state can be regarded, in infinitely many different ways, as a linear superposition of a *basis* of distinguishable quantum states. The experimental signature of such a superposition structure (in the given basis) is known as *quantum coherence* [1]. The latter is also known as one the basic ingredients for quantum information processing [2] and its protection e.g., by decoherence-free subspaces [3–5], is one of the fundamental challenges in the field.

Over the last few years we have witnessed a strong renewal of interest in the *quantitative* theory of coherence [6, 7]. This is partly practically motivated by the role that quantum coherence plays in quantum metrological protocols (see e.g., discussion in [8]) and, on a more conceptual ground, by its relation to the general resource theory of asymmetry [9–11]. Quantum coherence is also believed to play a role in some fundamental biological process [12–14] as well as in quantum thermodynamics [15, 16]. The general idea is that one can quantify quantum coherence by introducing a real-valued function over the quantum state-space, a *coherence measure*, such that it vanishes for all the states that are deemed to be *incoherent* and cannot increase under some class of operations that preserve incoherence [17]. Even if a preferred basis is chosen, the choice of the coherence measure is not unique and different options have been discussed in the literature [6, 8, 18, 19].

In this paper we address a closely related problem, which was first tackled in [20]: the quantification of the power of a quantum operation to generate coherence. Again, even when an underlying coherence measure is assumed, the definition of the *coherence generating power* (CGP) of a Completely Positive (CP)-map is not unique and different lines of attack are possible [20–22] (see Sect IV C of [7] for a comprehensive list of references). All of these approaches, however, are cast in terms of an optimization problem that is extremely hard to handle for generic channels in arbitrary dimensions.

Following the spirit of Ref. [23] in entanglement theory, we shall here pursue a different strategy based on probabilistic averages. We define the CGP of a map as the average coherence that is generated when the corresponding quantum operation is performed over a suitable input ensemble of incoherent states. We shall here firstly focus on unitary maps and introduce a definition of CGP based on a uniform ensemble (see below for a precise definition) of incoherent states.

Our measure of CGP is analytically computable for arbitrary unitary map in any dimension. It also enjoys several natural and desirable properties e.g., invariance under pre- and post-processing by incoherent unitaries. We shall present a simple operational protocol for the direct detection of the CGP of a given map which does not involve the ensemble generation or quantum process tomography [24, 25]. The set of unitary operations with maximal CGP is easily characterized and some universal statistical properties of our measure over the group of unitaries can be established rigorously. We will also provide some numerical study of the distribution of CGP in various dimensions d (for $d = 2$ analytical form is available). Finally, extensions of CGP to arbitrary unital operations are discussed as well as the connection to the broader concept of asymmetry generating power of a map. The proofs of the Propositions can be found in the Appendix.

II. PRELIMINARIES

Let $B = \{|i\rangle\}_{i=1}^{d}$ be an orthonormal basis in the Hilbert space $\mathcal{H} \cong \mathbf{C}^d$. Given B one has the associated B-dephasing map over $L(\mathcal{H})$ given by $X \mapsto \mathcal{D}_B(X) = \sum_{i=1}^{d} |i\rangle\langle i| \langle i|X|i\rangle$

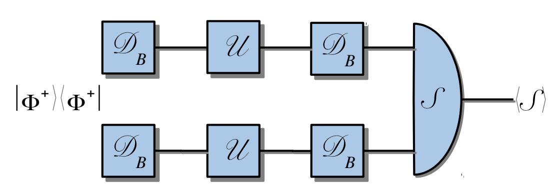

Figure 1. Protocol for the direct detection of the Coherence Generating Power (CGP) Eq. (2) of the unitary CP map $\mathcal{U}$ based on Eq. (7). Here $\mathcal{D}_B$ is the dephasing super-operator for the preferred basis B, the measurement of the swap operator is denoted by $\mathcal{S}$ and $|\Phi^+\rangle := d^{-1/2} \sum_{i=1}^{d} |i\rangle^{\otimes 2}$.

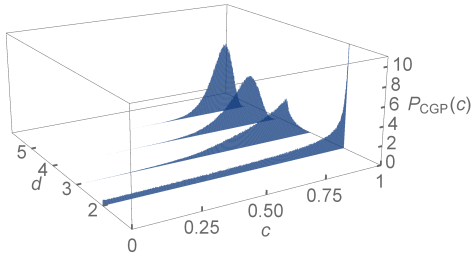

Figure 2. Probability distribution densities (PDD) of the normalized $\tilde{C}_B(U)$ for $d = 2, \ldots, 5$. An ensemble of Haar-distributed U's has been generated numerically. For $d = 2$ the analytical form of the PDD is $P_{CGP}(c) = \frac{1}{2}(1-c)^{-1/2}$ (see H).

[26]. The dephasing map $\mathcal{D}_B$ can be realized physically as the measurement CP map associated to any non degenerate observable H diagonal in the basis B. For any B, the dephasing map is an orthogonal projection over $L(\mathcal{H})$ equipped with the standard Hilbert-Schmidt scalar product $\langle X, Y\rangle := \mathrm{tr}(X^\dagger Y)$. We will denote by $\mathcal{Q}_B := \mathbb{I} - \mathcal{D}_B$ the complementary projection of $\mathcal{D}_B$. Naturally, one defines B-incoherent operators (states) as operators (states) that are diagonal in the preferred basis B.

Definition 1.– The set of B-incoherent operators is the range of the B-dephasing map, i.e., $\mathrm{Im}\, \mathcal{D}_B$. We will denote the set of B-incoherent states ρ ($\rho \geq 0$, $\mathrm{tr}\, \rho = 1$) by I_B.

From the point of view of this definition one can say that $\mathcal{D}_B$ ($\mathcal{Q}_B$) projects an operator onto its incoherent (coherent) component. The set I_B is clearly isomorphic to a $(d-1)$-dimensional simplex spanned by convex combinations of the $|i\rangle\langle i|$, $i = 1, \ldots, d$. CP maps (all such maps are assumed to be trace preserving in this paper) $\mathcal{T}$ mapping I_B into itself will play a distinguished role in this paper. A necessary and sufficient condition for the invariance of I_B under $\mathcal{T}$ is given by $\mathcal{T}\mathcal{D}_B = \mathcal{D}_B\mathcal{T}\mathcal{D}_B$ [8]. However, in this paper we will adopt a slightly stronger invariance condition.

Definition 2.– A CP map $\mathcal{T}$ on $L(\mathcal{H})$ will be called B-incoherent iff $[\mathcal{T}, \mathcal{D}_B] = 0$. We will write $\mathcal{T} \in CP_B$.

Note that B-incoherent maps leave both the subspace of B-incoherent operators *and* its orthogonal complement ($\cong \mathrm{Ker}\, \mathcal{D}_B = \mathrm{Im}\, \mathcal{Q}_B$) invariant. Let us first establish the following, almost obvious, fact.

Proposition 1.– A unitary CP map $\mathcal{U}(X) = UXU^\dagger$ (with U unitary) is B-incoherent iff $U|i\rangle = \eta_i|\sigma_U(i)\rangle$ where σ_U is a (U-dependent) permutation of $\{1, \ldots, d\}$ and the η_i's are $U(1)$-phases. B-incoherent unitary maps form a subgroup of CP_B.

III. MEASURES OF COHERENCE GENERATING POWER

Loosely speaking a coherence measure is a way to quantify how far a given state is from being incoherent, moreover this quantification is requested to fulfill some natural properties. More precisely, let us consider the function $\tilde{c}_B(\rho) := \|\rho - \mathcal{D}_B(\rho)\|_1 = \|\mathcal{Q}_B(\rho)\|_1$ ($\|X\|_1$ denotes the 1-norm of X, i.e. the sum of the singular values of X); this is vanishing iff ρ is B-incoherent. Moreover if $\mathcal{T} \in CP_B$ then $\tilde{c}_B(\mathcal{T}(\rho)) = \|\mathcal{Q}_B\mathcal{T}(\rho)\|_1 = \|\mathcal{T}\mathcal{Q}_B(\rho)\|_1 \leq \|\mathcal{Q}_B(\rho)\|_1 = \tilde{c}_B(\rho)$, where we have used Definition 2 and the monotonicity of the 1-norm under general CP maps. These remarks show that $\tilde{c}_B$ is a good *coherence measure* with respect to B-incoherent operations [8]. Unfortunately the 1-norm is hard to handle, therefore in this paper we will adopt the Hilbert-Schmidt 2-norm $\|X\|_2 = \sqrt{\langle X, X\rangle}$. We define the function

$$c_B(\rho) := \|\mathcal{Q}_B(\rho)\|_2^2. \tag{1}$$

Again, it is immediate to see that c_B vanishes iff $\rho \in I_B$ and $\tilde{c}_B(\rho) \leq \sqrt{d\, c_B(\rho)}$. On the other hand it is now not true that c_B is necessarily non-increasing under general B-incoherent CP maps (as the 2-norm does not have that property either). However if $\mathcal{T}$ is unital i.e., $\mathcal{T}(\mathbb{I}) = \mathbb{I}$, then $\|\mathcal{T}(X)\|_2 \leq \|X\|_2$ [27]. Thereby the desired monotonicity property is recovered if one restricts to the set of unital B-incoherent CP maps.

Let us now introduce the main novel concept of this paper.

Definition 3.– The coherence generating power (CGP) $C_B(U)$ of a unitary CP map $X \mapsto \mathcal{U}(X) := UXU^\dagger$, ($U \in U(\mathcal{H})$) with respect the basis B is defined as

$$C_B(U) := \langle c_B(\mathcal{U}_{off}(|\psi\rangle\langle\psi|))\rangle_\psi \tag{2}$$

where $\mathcal{U}_{off} := \mathcal{Q}_B\mathcal{U}\mathcal{D}_B$ and the average over ψ is taken according to the Haar measure.

The operational idea behind our definition (2) of CGP is simple: the power of a unitary U to generate coherence (in a preferred basis B) is given by the average coherence, as measured by the function (1), obtained by U acting over an ensemble of incoherent states. The latter is prepared by a stochastic process that involves first the generation of (Haar) random quantum states, and then their B-dephasing e.g., by performing a non-selective measurement of any non-degenerate B-diagonal observable. Note that the ensemble so generated coincides with the uniform one over the simplex I_B (see B). Of course other definitions are possible. For example, besides the freedom of choosing a coherence measure different from (1), one might have resorted to a different ensemble of B-diagonal states or even replace the average by a supremum over the ensemble [20–22]. However, our choice, thanks to the high symmetry of the Haar measure, will allow us to establish properties of CGP on general grounds as well as to compute it in an explicit analytic fashion. The most basic properties of the CGP can be derived directly from Eq. (2).

Proposition 2.– a) $C_B(U) \geq 0$ and $C_B(U) = 0$ iff $\mathcal{U} \in CP_B$. **b)** If W is a unitary such that $\mathcal{W} \in CP_B$ then $C_B(UW) = C_B(WU) = C_B(U)$. **c)** Let $\{|\tilde{i}\rangle := V|i\rangle\}_{i=1}^d$ be a new basis $\tilde{B} := BV$ obtained from B by the (right) action of the unitary V then: $C_{BV}(U) = C_B(V^\dagger UV)$.

Part b) shows that CGP does not change if the ensemble is pre- or post-processed by incoherent unitaries. Moreover, Part c) shows that computing the CGP for a single *given* basis B_0 is in principle sufficient for obtaining it for *any* basis B (for, given any pair of bases, there is always a unitary connecting them). It also implies, as we will see, that the statistical properties of the CGP over the unitary group are *universal* in

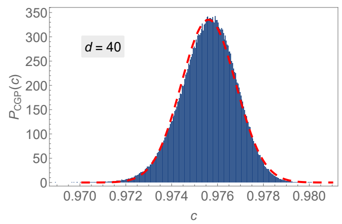

Figure 3. Probability distribution density (PDD) of the normalized $\tilde{C}_B(U)$ for $d = 40$. A Gaussian fit is superimposed on the numerically generated PDD to highlight the central-limit type behavior.

the sense of being basis independent: just the Hilbert space dimension d matters.

It is important to stress that Prop. 2 holds for a more general choice of c_B than Eq. (1) e.g., for $\tilde{c}_B$ [28]. The choice of the Hilbert-Schmidt norm in the definition of CGP, on the other hand, while imposing the somewhat severe unitality constraint, has the great advantage of allowing one for an *explicit* computation of $C_B(U)$.

Proposition 3.– Let $|\Phi^+\rangle = 1/\sqrt{d}\sum_{i=1}^{d} |i\rangle^{\otimes 2}$ be the maximally entangled $d \times d$ singlet, then: **a)**

$$C_B(U) = \frac{1}{d+1}\left[1 - \mathrm{tr}\left(S\omega_B(U)\right)\right], \tag{3}$$

where $\omega_B(U) := (\mathcal{D}_B \mathcal{U} \mathcal{D}_B)^{\otimes 2}(|\Phi^+\rangle\langle\Phi^+|)$ and $S = \sum_{i,j=1}^{d} |ij\rangle\langle ji|$ is the swap operator over $\mathcal{H}^{\otimes 2}$; **b)** $\mathrm{tr}\left(S\omega_B(U)\right) = 1/d\sum_{i,j=1}^{d} |\langle i|U|j\rangle|^4$; **c)** $C_B(U) \leq \frac{1-1/d}{d+1} =: C_d$. The upper bound is saturated iff $|\langle i|U|j\rangle|^2 = 1/d\,(\forall i,j)$.

Part c) of Prop. 3 above shows the fact that for U to be a unitary with maximal CGP the base B and the base $BU := \{U|i\rangle\}_{i=1}^{d}$ have to be *mutually* unbiased [29, 30]. For example the unitary U such that $\langle h|U|m\rangle = 1/\sqrt{d}\exp(i\frac{2\pi}{d}hm)$, $(h, m = 1, \ldots, d)$ has maximal CGP. We also remark that from a) and b) above it follows easily that $C_B(U) = C_B(U^\dagger)$.

Eq. (3) naturally leads to an operational protocol for the detection of the CGP of a unitary U which does not require the generation of a Haar distributed ensemble of states or quantum process tomography [24, 25].

Protocol for CGP detection: **1)** Prepare $|\Phi^+\rangle$; **2)** B-dephase both subsystems; **3)** Apply $\mathcal{U}$ to both subsystems; **4)** B-dephase again both subsystems; **5)** Measure the expectation value of the observable S (see e.g., [31]); **6)** Plug the obtained value in Eq. (3). This protocol is depicted in Fig. (1). Since

$$\mathcal{D}_B^{\otimes 2}(|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{d}\sum_{i=1}^{d} |i\rangle\langle i|^{\otimes 2} =: \rho_B, \tag{4}$$

steps 1) and 2) above can be replaced by 1') Prepare the maximally classically B-correlated state ρ_B (for which it is enough to B-dephase one subsystem). This shows that entanglement is not really needed in the detection of $C_B(U)$. However, in 5) one is required to measure S which involves non-trivial interactions between the two d-dimensional subsystems. This is the experimentally more challenging part of the protocol. Notice, however, that for two-qubits, this amounts to a standard Bell's basis measurement.

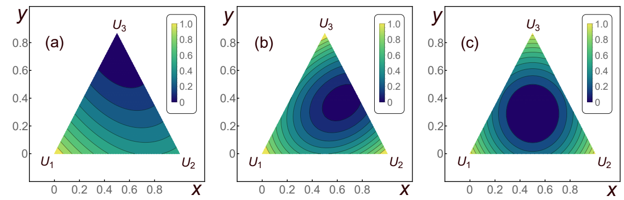

Figure 4. Coherence generating power for convex combinations of unitaries of the form $\mathcal{E}(\cdot) = \sum_{k=1}^{3} p_k U_k \cdot U_k{}^\dagger$. The simplex $\sum_k p_k = 1$ is represented as an equilateral triangle with one vertex on the origin. The (dimensionless) coordinates are $x = p_2 + p_3/2$ and $y = (\sqrt{3}/2)p_3$. One always has the convexity inequality $\tilde{C}_B(\mathcal{E}) \leq \sum_{k=1}^{3} p_k \tilde{C}_B(U_k)$ as noted in the main text. **(a)** Here $d = 3$ and we fixed U_k such that $\tilde{C}_B(U_1) = 1, \tilde{C}_B(U_2) = 1/2$ and $\tilde{C}_B(U_3) = 0$. **(b)** For this example ($d = 10$) U_1 is the Discrete Fourier Transform matrix $\langle l|U_1|m\rangle = d^{-1/2}\exp\left(ilm2\pi/d\right)$ while U_2 (U_3) is obtained by interchanging the first (last) 2 rows of U_1. All U_k have maximal CGP (simplex vertices) but the CGP of mixtures can drop significantly. **(c)** This is a typical case for randomly chosen unitaries U_k of large dimension (here $d = 40$). One observes that $\tilde{C}_B(U_k)$ is nearly maximal consistently with the concentration phenomenon.

IV. CGP AS A RANDOM VARIABLE OVER THE UNITARY GROUP

We now investigate some of the properties of the CGP of Eq. (3) seen as a random variable over the unitary group $U(\mathcal{H})$ equipped with the Haar measure $d\mu(U)$.

Proposition 4.– a) The probability distribution density $P_{CGP}(c) := \int_{U(\mathcal{H})} d\mu(U)\,\delta(c - C_B(U))$ for the CGP Eq. (3) is independent of B. **b)** The first moment is given by

$$\langle C_B(U)\rangle_U = \int dc\, c P_{CGP}(c) = \frac{d-1}{(d+1)^2}. \tag{5}$$

c) Let us define the normalized CGP $\tilde{C}_B(U) := C_B(U)/C_d \leq 1$ then $\langle \tilde{C}_B(U)\rangle_U = (1 + 1/d)^{-1}$. Using Levy's lemma for unitaries [32] one obtains

$$\mathrm{Prob}\left(\tilde{C}_B(U) \geq 1 - 2/d^{1/3}\right) \geq 1 - \exp\left(-d^{1/3}/256\right). \tag{6}$$

Eq. (6) shows that in high-dimension a random unitary will have, with overwhelming probability, nearly maximal CGP. In Fig. 2 are reported numerical simulations of the probability distribution function of $\tilde{C}_B(U)$ for Haar distributed U in different dimensions. In particular numerics shows that the variance of $\tilde{C}_B(U)$ is $O(1/d^3)$ (see 5). Moreover, Fig. 3 shows that for large Hilbert space dimension d a central-limit type behavior emerges and the P_{CGP} can be well approximated by a normal distribution.

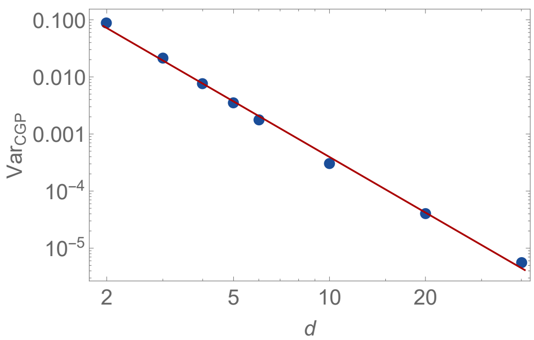

Figure 5. This log-log plot shows the numerically computed variance of the random variable $\tilde{C}_B(U)$ (where U is distributed according to the Haar measure) for different values of the dimension d of the Hilbert space. A power-law A/d^α least square fitting (taking into account only the points $d = 6, 10, 20, 40$) gives $\alpha = 3.01$, suggesting that the variance of $\tilde{C}_B(U)$ is $O(1/d^3)$.

V. BEYOND UNITARITY AND FINITE DIMENSIONS

In this section we will briefly discuss how our approach extends to CP maps that are not necessarily unitary and how one might extend our formalism to infinite dimensions e.g., optical modes. Since we still would like to employ the Hilbert-Schmidt norm we will focus here on *unital* maps. We can still adopt Eq. (1) for the definition of a coherence measure. Moreover, incoherent (according to Def. 2) $\mathcal{E}$ will not increase it. We can now define the CGP of $\mathcal{E}$ by the same Eq. (2) (with $\mathcal{E}$ replacing $\mathcal{U}$). It is still true that $C_B(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E}_{off} := \mathcal{Q}_B\mathcal{E}\mathcal{D}_B = 0$ but this is now a weaker property than incoherence as it does not imply $[\mathcal{E}, \mathcal{D}_B] = 0$. The corresponding measure of CGP is therefore not *faithful* i.e., $C_B(\mathcal{E}) = 0 \Rightarrow \mathcal{E} \in CP_B$ doesn't hold (just the converse does) [33]. The following proposition shows how Prop. 3 generalizes to unital maps more general than unitaries.

Proposition 5– Let $\mathcal{E}(\cdot) = \sum_k A_k \cdot A_k^\dagger$, $(\sum_k A_k^\dagger A_k = \mathbb{1})$ be a unital CP-map over $\mathrm{L}(\mathcal{H})$. If we define its CGP by Eq. (2) (replacing $\mathcal{U}$ with $\mathcal{E}$) then it follows that **a)** $C_B(\mathcal{E}) \geq 0$ and it vanishes if $\mathcal{E}$ is B-incoherent. **b)** If $\mathcal{T}$ is B-incoherent then $C_B(\mathcal{T}\mathcal{E}) \leq C_B(\mathcal{E})$. **c)**

$$C_B(\mathcal{E}) = \frac{1}{d+1}\left[\mathrm{tr}\,(S\tilde{\omega}_B(\mathcal{E})) - \mathrm{tr}\,(S\omega_B(\mathcal{E}))\right] \leq C_d, \quad (7)$$

where $\omega_B(\mathcal{E}) := (\mathcal{D}_B\mathcal{E})^{\otimes 2}(\rho_B)$ and $\tilde{\omega}_B(\mathcal{E}) := \mathcal{E}^{\otimes 2}(\rho_B)$. **d)** $C_B(\mathcal{E}) = [d(d+1)]^{-1}\sum_{i,l\neq m=1}^d |\sum_k (A_k)_{li}(A_k)^*_{mi}|^2$

Property b) in Prop. 5 is the analog of Eq. (3) and can be similarly interpreted by an operational protocol involving the measurement of S over the states $\omega_B(\mathcal{T})$ and $\tilde{\omega}_B(\mathcal{T})$. Point d) above gives the CGP explicitly as a function of the matrix elements of the Kraus operators of $\mathcal{E}$; it corresponds to b) in Prop. 3. We also note that the function $\mathcal{E} \mapsto C_B(\mathcal{E})$ is *convex* (since it is a convex combination of the convex functions $\mathcal{E} \mapsto c_B(\mathcal{E}_{off}(|\psi\rangle\langle\psi|))\,\forall|\psi\rangle$). It follows that the maximum CGP of a convex set of maps will be achieved over extremal points. This phenomenon can be seen in the in Fig. 4.

Remarkably, Eq. (7) seems to suggest a natural way in which our results can be extended to infinite dimensions. Let us consider, for simplicity, the unitary case and normalize Eq. (3) by dividing by C_d. Now sending $d \to \infty$ the d-dependent pre-factor of CGP disappears and one is led to consider the expression $\tilde{C}_B^{(\infty)}(U) = 1 - \mathrm{tr}\,(S\omega_B^{(\infty)}(U))$ with $\omega_B^{(\infty)}(U) = (\mathcal{D}_B\mathcal{U})^{\otimes 2}(\rho_B^{(\infty)})$ where $\rho_B^{(\infty)}$ is *some* infinite-dimensional generalization of the maximally classically B-correlated state Eq. (4). For example, for any $\lambda \in (0, 1)$, one could choose $\rho_B^{(\infty)} := (1-\lambda^2)\sum_{i=0}^\infty \lambda^{2i}|i\rangle\langle i|^{\otimes 2}$ [34]. With this choice it is immediate to check that $\tilde{C}_B^{(\infty)}(U) = 0$ iff U is incoherent and that post-processing with incoherent unitaries leaves the CGP invariant [35]. Developments in the infinite-dimensional case will be presented elsewhere [36].

VI. ASYMMETRY

Closely related to the theory of coherence is the notion of *asymmetry* [9–11]. Given an observable H one says that a state ρ (a CP map $\mathcal{E}$) is H-symmetric (H-covariant) iff $[H, \rho] =: \mathcal{H}(\rho) = 0$ ($[\mathcal{H}, \mathcal{E}] = 0$). An *asymmetry measure* is a real valued function $a_H(\rho)$ that vanishes over symmetric states and is non-increasing under covariant CP maps i.e., $a_H(\mathcal{E}(\rho)) \leq a_H(\rho)$ [11]. Following the main idea of this paper one could define the asymmetry generating power (AGP) of a CP map $\mathcal{E}$ by $A_H(\mathcal{E}) := \langle a_H(\mathcal{E}(\omega))\rangle_\omega$ where the average is performed over a suitable ensemble of H-symmetric states ω.

In order to directly connect Asymmetry Generating Power (AGP) and and CGP we assume from here on that the Hamiltonian H is *non degenerate* and that $B = \{|i\rangle\}_{i=1}^d$ is the associated basis of eigenvectors. In this case the notion of H-symmetric state and the one of B-incoherent collapse. It is indeed immediate to see that $\mathcal{H}(\rho) =: [H, \rho] = 0 \Leftrightarrow \mathcal{D}_B(\rho) = \rho$ (in the degenerate case incoherence implies symmetry). At the CP map level, however, one has just that H-covariance implies B-incoherence but not the converse. For example unitaries in Prop. 1 realizing a non-trivial permutation of B are incoherent but not covariant. As a consequence the set of coherence measures is smaller than the set of asymmetry measures [8, 37]. We introduce the following notion of AGP for unital maps $\mathcal{E}$

$$A_H(\mathcal{E}) = \langle\|\mathcal{H}\mathcal{E}\mathcal{D}_B(|\psi\rangle\langle\psi|)\|_2^2\rangle_\psi \quad (8)$$

where once again the average is taken with respect to the Haar measure. As the CGP Eq. (7) in the main text (see comment after Prop. 5) also the AGP Eq. (8) is a convex function of its argument. Furthermore, if $H = \sum_{i=1}^d \epsilon_i\,|i\rangle\langle i|$, $\delta(H) := \min_{l\neq m}|\epsilon_l - \epsilon_m| > 0$ (non-degeneracy) and $\|\mathcal{H}\| := \max_{l\neq m}|\epsilon_l - \epsilon_m|$, then the AGP (8) fulfills the following properties:

Proposition 6.– a) $A_H(\mathcal{E}) = 0$ for all incoherent maps $\mathcal{E}$. In particular all H-covariant maps have vanishing AGP. **b)** if $\mathcal{T}$ is a unital H-covariant map $A_H(\mathcal{T}\mathcal{E}) \leq A_H(\mathcal{E})$. For unitary H-covariant $\mathcal{T}$ the inequality becomes an equality. **c)** $A_H(\mathcal{E}) = [d(d+1)]^{-1}\sum_{i,l\neq m}(\epsilon_l - \epsilon_m)^2|\langle l|\mathcal{E}(|i\rangle\langle i|)|m\rangle|^2$. **d)** $\delta^2(H)\,C_B(\mathcal{E}) \leq A_H(\mathcal{E}) \leq \|\mathcal{H}\|^2 C_B(\mathcal{E})$. **e)** If $\mathcal{E}(\cdot) = U \cdot$

$U^\dagger$ and the unitary U's are Haar distributed then the induced distribution of $A_H(U)$ depends just on the gap spectrum $\{\epsilon_l - \epsilon_m\}_{l\neq m}$.

Proof.– a) Follows from $\mathcal{H}\mathcal{D}_B = 0$ and $\mathcal{E}\mathcal{D}_B = \mathcal{D}_B\mathcal{E}\mathcal{D}_B$ which holds for incoherent maps. b) Use $[\mathcal{T}, \mathcal{D}_B] = 0$ for H-covariant maps and the non-increasing property of the Hilbert-Schmidt norm under unital maps. c) Following the same steps in the proof of a) in Prop. 3 one arrives at $A_H(\mathcal{E}) = [d(d+1)]^{-1}\sum_{i=1}^d \|\mathcal{H}\mathcal{E}(|i\rangle\langle i|)\|_2^2$. Expanding the norms in this equation and using $\mathcal{H}(|l\rangle\langle m|) = (\epsilon_l - \epsilon_m)|l\rangle\langle m|)$ one completes the proof. d) From c) using $\delta(H) \leq |\epsilon_l - \epsilon_m| \leq \|\mathcal{H}\|, (\forall l, m)$. e) If the Hamiltonian eigenbasis is changed by $|i\rangle \mapsto W|i\rangle$ (W unitary) then from the result in c) one sees that $\mathcal{E} \mapsto \mathcal{W}^\dagger\mathcal{E}\mathcal{W}$ ($\mathcal{W}(\cdot) = W \cdot W^\dagger$). If $\mathcal{E}(\cdot) = U \cdot U^\dagger$ the last equation implies $U \mapsto W^\dagger U W$. The proof can be now completed following the same reasoning of point c) in the proof of Prop. 2 and observing that H enters now, having modded the basis away, just through the differences $\epsilon_l - \epsilon_m$ $(l \neq m = 1, \ldots d)$. □

VII. CONCLUSIONS

In this paper we have discussed a way to quantify the coherence generating power (CGP) of a quantum operation. As a coherence measure we have conveniently adopted the Hilbert-Schmidt norm of the coherent part of a quantum state. Our approach is to look at the average coherence produced when the operation is performed over a uniform ensemble of input incoherent states. The input ensemble is obtained by dephasing, with respect to the chosen basis, an ensemble of pure states distributed according to the Haar measure.

Under these assumptions one obtains an analytically computable measure of CGP for arbitrary unital operations in any dimension. Operational protocols for the direct detection of CGP have been described. Neither the ability to generate the Haar distributed input ensemble nor quantum process tomography are required. We focused on unitary maps, characterized those with maximal CGP, and studied the distribution of this measure over the unitary group, both analytically and numerically. For unitary maps this distribution is universal (basis independent) and for large Hilbert space dimension a central-limit type phenomenon emerges. A random unitary has, with overwhelming probability, nearly maximal CGP. Finally, we extended our approach to quantify the power of an operation to generate a more general type of asymmetry.

The analytical framework here established is particularly suited for unital quantum maps. Going beyond unitality, finite dimensionality, and extending to general resource theories represent challenging tasks for future investigations.

ACKNOWLEDGMENTS

This work was partially supported by the ARO MURI grant W911NF-11-1-0268. P.Z. thanks I. Marvian for introducing him to asymmetry measures and F.G.S. L. Brandão for pointing out the right form of the Levy's Lemma.

[1] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 1st ed. (Cambridge University Press, Cambridge ; New York, 2000).
[3] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
[4] P. Zanardi, Phys. Rev. A **57**, 3276 (1998).
[5] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
[6] T. Baumgratz, M. Cramer, and M. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[7] A. Streltsov, G. Adesso, and M. B. Plenio, arXiv:1609.02439 [cond-mat, physics:hep-th, physics:math-ph, physics:quant-ph] (2016), arXiv: 1609.02439.
[8] I. Marvian and R. W. Spekkens, arXiv:1602.08049 [cond-mat, physics:nucl-th, physics:quant-ph] (2016), arXiv: 1602.08049.
[9] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 014102 (2014).
[10] I. Marvian and R. W. Spekkens, Nature Communications **5** (2014), 10.1038/ncomms4821.
[11] I. Marvian Mashhad, *Symmetry, Asymmetry and Quantum Information*, Ph.D. thesis, University of Waterloo (2012).
[12] G. S. Engel, T. R. Calhoun, E. L. Read, T.-K. Ahn, T. Manal, Y.-C. Cheng, R. E. Blankenship, and G. R. Fleming, Nature **446**, 782 (2007).
[13] E. Collini, C. Y. Wong, K. E. Wilk, P. M. G. Curmi, P. Brumer, and G. D. Scholes, Nature **463**, 644 (2010).
[14] N. Lambert, Y.-N. Chen, Y.-C. Cheng, C.-M. Li, G.-Y. Chen, and F. Nori, Nat Phys **9**, 10 (2013).
[15] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[16] M. Lostaglio, D. Jennings, and T. Rudolph, Nature Communications **6**, 6383 (2015).
[17] F. Levi and F. Mintert, New J. Phys. **16**, 033007 (2014).
[18] D. Girolami, Phys. Rev. Lett. **113**, 170401 (2014).
[19] Y. Yao, G. H. Dong, X. Xiao, and C. P. Sun, Scientific Reports **6**, 32010 (2016).
[20] A. Mani and V. Karimipour, Phys. Rev. A **92**, 032331 (2015).
[21] T. R. Bromley, M. Cianciaruso, and G. Adesso, Phys. Rev. Lett. **114**, 210401 (2015).
[22] M. Garca-Daz, D. Egloff, and M. B. Plenio, arXiv:1510.06683 [quant-ph] (2015), arXiv: 1510.06683.
[23] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301 (2000).
[24] I. L. Chuang and M. A. Nielsen, Journal of Modern Optics **44**, 2455 (1997).
[25] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **78**, 390 (1997).
[26] One could consider more general dephasing maps as $\mathcal{D}(X) = \sum_i \Pi_i X \Pi_i$, where the Π_i are a complete set of projections not necessarily one-dimensional. This would lead to the theory of coherence with respect to the family of subspaces $\mathrm{Im}\Pi_i$ [8].
[27] One has that $\|\mathcal{T}(X)\|_2^2 = \langle \mathcal{T}^*\mathcal{T}(X),\, X\rangle \leq \lambda_M \|X\|_2^2$ where λ_M is the largest eigenvalue of $\mathcal{T}^*\mathcal{T}$. The latter operator is a trace-preserving CP-map for unital $\mathcal{T}$ and therefore $\lambda_M \leq 1$.

Since the argument of the 2-norm in Eq. (1) is always traceless, for the monotonicy property to hold suffices that $\mathcal{P}_0\mathcal{T}^*\mathcal{T}\mathcal{P}_0$ ($\mathcal{P}_0$ projection over the space of traceless operators) has eigenvalues smaller than one. This property is weaker than unitality.

[28] In fact Prop. 2 holds true by replacing Eq. (2) with $\tilde{C}_B(U) := \langle D(\mathcal{U}\mathcal{D}_B(|\psi\rangle\langle\psi|),\, \mathcal{D}_B\mathcal{U}\mathcal{D}_B(|\psi\rangle\langle\psi|))\rangle_\psi$, where D is any non-negative function over pairs of states such that i) vanishes iff the two arguments are identical, ii) is unitary invariant $D(\mathcal{U}(\sigma), \mathcal{U}(\rho)) = D(\sigma, \rho)$. For example one could take D to be the trace distance or the quantum relative entropy.

[29] J. Schwinger, PNAS **46**, 570 (1960).

[30] I. D. Ivanovic, J. Phys. A: Math. Gen. **26**, L579 (1993).

[31] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, Phys. Rev. Lett. **88**, 217901 (2002).

[32] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*, 1st ed. (Cambridge University Press, New York, 2009).

[33] If one insists that C_B should be faithful then the set of incoherent maps $\mathcal{E}$ should be defined by $\mathcal{E}_{off} = 0$. However, in this case the monotonicity property might be violated. Notice that for normal maps $[\mathcal{E}, \mathcal{E}^*] = 0$ the two different notions of incoherence coincide (see part a) in Proof of Prop. 2).

[34] This is the B-dephased of the 2-mode EPR pair $|\psi_{EPR}\rangle = \sum_{i=0}^{\infty} \sqrt{1-\lambda^2}(-\lambda)^n |i\rangle^{\otimes 2}$ with squeezing parameter $\tanh^{-1}(\lambda)$.

[35] Invariance under pre-processing by incoherent unitaries does not hold because, for $\lambda < 1$ it is not true that $[U^{\otimes 2}, \rho_B] = 0$ for all such U's. One has to consider the non-trivial limit $\lambda \to 1$.

[36] P. Zanardi, G. Styliaris, and L. Campos Venuti, arXiv:1612.08139 [quant-ph], Accepted for publication on Phys. Rev. A (2016).

[37] I. Marvian, R. W. Spekkens, and P. Zanardi, Phys. Rev. A **93**, 052331 (2016).

[38] M. Keyl, Physics Reports **369**, 431 (2002).

Appendix A: Proof of Proposition 1

This condition is clearly a sufficient for B-incoherence as commutativity of $\mathcal{U}$ and $\mathcal{D}_B$ can be explicitly checked in a straightforward fashion. It is also necessary. Indeed if $\mathcal{U}$ is B-incoherent then $\mathcal{U}(|i\rangle\langle i|)$ must be B-diagonal for all i; because of unitarity, it also must be a one-dimensional projector whence $\mathcal{U}(|i\rangle\langle i|)$ has necessarily the form $|j\rangle\langle j|$ where $|j\rangle$ is a uniquely defined element $j =: \sigma_U(i)$ of B. The only degrees of freedom of U left are then $U(1)$-phases. The last statement of the proposition is evident. □

Appendix B: Equivalence of ensembles

Here we show that the ensemble constructed in the main text coincides in fact with the uniform distribution over the simplex spanned by the states $|i\rangle\langle i|$, $i = 1, \ldots, d$. For any (measurable) function f the expectation value over the ensemble is given by $\langle f(\mathcal{D}_B(|\psi\rangle\langle\psi|)\rangle_\psi$. Calling $\psi_i = \langle i|\psi\rangle$ and $p_i = |\langle i|\psi\rangle|^2$ we can write it as

$$\langle f(\mathcal{D}_B(|\psi\rangle\langle\psi|)\rangle_\psi = M \int d\psi_1 \cdots \int d\psi_d \times \times f(p_1, \ldots, p_d)\delta(1 - \sum_{i=1}^d p_i) \quad \text{(B1)}$$

where M is a normalization constant and $d\psi_i = d\mathrm{Re}(\psi_i)\, d\mathrm{Im}(\psi_i)$. Switching to polar coordinates one has $d\psi_i = r_i dr_i\, d\vartheta_i = dp_i\, d\vartheta_i/2$. Performing the integration over the angles ϑ_i we obtain

$$\langle f(\mathcal{D}_B(|\psi\rangle\langle\psi|)\rangle_\psi = M' \int dp_1 \cdots \int dp_d \times \times f(p_1, \ldots, p_d)\delta(1 - \sum_{i=1}^d p_i)\,, \quad \text{(B2)}$$

that is, the uniform measure over the simplex (M' is another normalization constant).

Appendix C: Proof of Proposition 2

a) By definition the CGP is non-negative, moreover $C_B(U) = 0$ implies $\mathcal{U}_{off}(|\psi\rangle\langle\psi|) = 0, \forall|\psi\rangle$, which in turn implies that $\mathcal{U}_{off} = \mathcal{U}\mathcal{D}_B - \mathcal{D}_B\mathcal{U}\mathcal{D}_B = 0$. This equation, as remarked in the above, shows that $\mathrm{Im}\,\mathcal{D}_B$ is invariant under $\mathcal{U}$ but, since $\mathcal{U}$ is normal, also the orthogonal complement $\mathrm{Ker}\,\mathcal{D}_B$ is invariant. It follows that $[\mathcal{U}_B, \mathcal{D}_B] = 0$ that is what we wanted to prove. b) $C_B(WU) = C_B(U)$ ($C_B(UW) = C_B(U)$) follows from the commutativity of $\mathcal{W}$ and $\mathcal{Q}_B$ ($\mathcal{D}_B$) and the unitary invariance of the Hilbert-Schmidt norm (Haar measure). c) By definition of $\tilde{B} = BV$ one finds $\mathcal{D}_{\tilde{B}} = \mathcal{V}\mathcal{D}_B\mathcal{V}^\dagger$, ($\mathcal{V}(\cdot) = V \cdot V^\dagger$) inserting this relation in Eq. (2) in the main text and using again unitary invariance of the Hilbert-Schmidt norm and of the Haar measure one completes the proof. □

Appendix D: Lemma

If S is the swap operator over $\mathcal{H}^{\otimes 2}$ ($S = \sum_{i,j=1}^d |ij\rangle\langle ji|$, $\mathcal{H} = \mathrm{span}\{|i\rangle\}_{i=1}^d$) then: a) $\|X\|_2^2 = \mathrm{tr}\,(SX \otimes X)$; b) $\langle |\psi\rangle\langle\psi|^{\otimes 2}\rangle_\psi = [d(d+1)]^{-1}(\mathbb{1} + S)$ where the average is taken over Haar distributed ψ in $\mathcal{H}$, (see e.g., [38]).

Appendix E: Proof of Proposition 3

a) Using the *Lemma* and Definition 3 one can immediately write $C_B(U) = [d(d+1)]^{-1}\mathrm{tr}\left[S\,\mathcal{U}_{off}^{\otimes 2}(\mathbb{1} + S)\right]$. The first term in this expression is vanishing; indeed $\mathcal{U}_{off}^{\otimes 2}(\mathbb{1}) = \mathcal{Q}_B^{\otimes 2}(\mathbb{1}) = 0$ (the identity is a diagonal operator for any B). Using the fact that $\forall Y$ one has

$\mathrm{tr}\left[S\mathcal{Q}_B^{\otimes 2}Y\right] = \mathrm{tr}\left[S(\mathbb{1}-\mathcal{D}_B^{\otimes 2})Y\right]$ the second term can be written as $\mathrm{tr}\left[S\,\mathcal{U}_{off}^{\otimes 2}(S)\right] = \mathrm{tr}\left[S\,(\mathbb{1}-\mathcal{D}_B^{\otimes 2})(\mathcal{U}\mathcal{D}_B)^{\otimes 2}(S)\right]$ Moreover, $\mathcal{D}_B^{\otimes 2}(S) = \sum_{i=1}^d |i\rangle\langle i|^{\otimes 2} =: d\rho_B$ therefore the first term in the last equation can be now written as $(d+1)^{-1}\mathrm{tr}\left(S\mathcal{U}^{\otimes 2}(\rho_B))\right) = (d+1)^{-1}$. The last equality follows from the fact that $\mathcal{U}^{\otimes 2}(\rho_B)$ is entirely supported in the eigenvalue one subspace of S (symmetric subspace). Observing now that is also true that $\mathcal{D}_B^{\otimes 2}(S) = d\mathcal{D}_B^{\otimes 2}(|\Phi^+\rangle\langle\Phi^+|)$ completes the proof of part a). Let us now move to part b). One has $\mathrm{tr}\left(S(\mathcal{D}_B\mathcal{U}^{\otimes 2})(\rho_B)\right) = 1/d\sum_{i=1}^d \mathrm{tr}\left(S(\mathcal{D}_B\mathcal{U})^{\otimes 2}(|i\rangle\langle i|^{\otimes 2})\right) = 1/d\sum_{i=1}^d \|(\mathcal{D}_B\mathcal{U}(|i\rangle\langle i|)\|_2^2$. But $\mathcal{D}_B\mathcal{U}(|i\rangle\langle i|) = \sum_{j=1}^d |\langle i|U|j\rangle|^2|j\rangle\langle j|$. Bringing together the last two equations completes the Proof of part b). Now part c). From the above one sees that $d\mathrm{tr}\,(S\omega_B(U))$ is the sum of d purities $\|\mathcal{D}_B\mathcal{U}(|i\rangle\langle i|)\|_2^2,\ (i=1,\ldots,d)$. Therefore the minimum of this quantity occurs when they are all their minimum i.e., $1/d$. Adding over i one finds $\langle S\rangle_{\omega_B(U)} \geq 1/d$ from which the desired upper bound c) follows, This bound is achieved iff $\mathcal{D}_B\mathcal{U}(|i\rangle\langle i|) = \mathbb{1}/d,\ (\forall i)$. This, in turn, from the expression a few lines above, implies $|\langle i|U|j\rangle|^2 = 1/d$. Notice that this conclusion can be also derived directly from the formula c). $\square$

Appendix F: Proof of Proposition 4

a) Given a fixed basis B_0 and any other base B one has that there exists a $V \in U(\mathcal{H})$ such that $B = B_0V$ (see comment after Prop. 2 in the main text). Therefore $P_B(c) = P_{B_0V}(c)dc = \int d\mu(U)\,\delta(c - C_{B_0V}(U)) = \int d\mu(U)\,\delta(c - C_{B_0}(V^\dagger UV)) = \int d\mu(U)\,\delta(c - C_{B_0}(V^\dagger UV)) = \int d\mu(VWV^\dagger)\,\delta(c - C_{B_0}(W)) = P_{B_0}(c)dc$. Where we have used c) of Prop. 2 and the unitary invariance of the Haar measure i.e., $d\mu(VWV^\dagger) = d\mu(V)$. b) Let us consider the terms $|\langle i|U|j\rangle|^4$ from part b) of Prop. 3 and perform average with respect a Haar distributed U. Denoting by $|\psi\rangle = U|j\rangle$ this amounts to average with respect $|\psi\rangle$ the following quantity $(\langle i|\psi\rangle\langle\psi|i\rangle)^2 = \mathrm{tr}\left(|i\rangle\langle i|^{\otimes 2}|\psi\rangle\langle\psi|^{\otimes 2}\right)$. Using now the Lemma one finds $\langle|\langle i|U|j\rangle|^4\rangle_U = \langle|\langle i|\psi\rangle|^4\rangle_\psi = [d(d+1)]^{-1}\mathrm{tr}\left(|i\rangle\langle i|^{\otimes 2}(\mathbb{1}+S)\right) = 2[d(d+1)]^{-1}$. Adding over i and j and using Eq. (3) in the main text one obtains Eq. (5). c) Here we need a version of the Levy Lemma formulated for Haar distributed $d\times d$ unitaries: $\mathrm{Prob}\{X(U) - \langle X(U)\rangle_U \geq \epsilon\} \leq \exp\left[-\frac{d\epsilon^2}{4K^2}\right]$ where K is a Lipschitz constant of $X\colon U(d)\mapsto \mathbf{R}$ i.e., $|X(U) - X(V)| \leq K\|U-V\|_2$ [32]. Let us set $X(U) := 1 - \tilde{C}_B(U)$ then $X(U) - \langle X(U)\rangle_U = 1 - \tilde{C}_B(U) - 1/(d+1)$ from which $\mathrm{Prob}\{\tilde{C}_B(U) \leq 1 - \epsilon - 1/d\} \leq \exp(-d\epsilon^2/(4K^2))$. If we now set $\epsilon = d^{-\alpha}$ with $\alpha \in (0,\,1/2)$ we get

$$\mathrm{Prob}\{\tilde{C}_B(U) \leq 1 - 2/d^\alpha\} \leq \exp(-d^{1-2\alpha}/(4K^2)).$$

To complete the proof we have to estimate the Lipschitz constant K. For this, from Eq. (3), and the definitions above, is clearly enough to consider the function $f(U) = 1/d\sum_{i=1}^d \mathrm{tr}\left(S(\mathcal{D}_B^{\otimes 2}(|i_U\rangle\langle i_U|^{\otimes 2})\right) =: 1 - d/(d-1)\tilde{C}_B(U)$ where $|i_U\rangle := U|i\rangle$. Let us consider each of the d terms, called $f_i(U)$, separately: $|f_i(U) - f_i(V)| \leq |\mathrm{tr}\left(S\mathcal{D}_B^{\otimes 2}(|i_U\rangle\langle i_U|^{\otimes 2} - |i_V\rangle\langle i_V|^{\otimes 2})\right)| \leq \||i_U\rangle\langle i_U|^{\otimes 2} - |i_V\rangle\langle i_V|^{\otimes 2}\|_1$, where we have used $\mathrm{tr}(AB) \leq \|A\|_\infty\|B\|_1$, $\|S\|_\infty = 1$ and, since B-dephasing is a CP map, $\|\mathcal{D}_B^{\otimes 2}(X)\|_1 \leq \|X\|_1$. Now, the last trace-norm distance can be upper bounded by twice the Hilbert space distance $\||i_U\rangle^{\otimes 2} - |i_V\rangle^{\otimes 2}\| \leq \|U^{\otimes 2} - V^{\otimes 2}\|_\infty = \|1 - (U^\dagger V)^{\otimes 2}\|_\infty$. if $\Delta := U - V$ and $K := U^\dagger\Delta$ has $U^\dagger V = 1 - K$ and then the last norm becomes $\|1 - (1-K)^{\otimes 2}\|_\infty = \|K\otimes\mathbb{1} + \mathbb{1}\otimes K + K\otimes K\|_\infty \leq \|K\|_\infty(2 + \|K\|_\infty) \leq 4\|K\| \leq 4\|U-V\|_\infty \leq 4\|U-V\|_2$ where we have used standard operator norm inequalities. Bringing all together $\|f(U) - f(V)\| \leq 8\|U-V\|_2$ showing that one can take $K = 8$. Setting $\alpha = 1/3$ and considering he complementary inequality one obtains Eq. (6) in the main text. $\square$

Appendix G: Proof of Proposition 5

Proceed exactly as in the unitary case. The only difference is that, for general $\mathcal{E}$ the state $\mathcal{E}^{\otimes 2}(\rho_B) = \tilde{\omega}_B$ is not entirely supported in the eigenvalue one eigenspace of S. $\square$

Appendix H: PDD for CGP in $d=2$

Using Eq. (3) for a $SU(2)$ matrix one finds $C_B(U) = \frac{1}{3}(1 - |a|^4 - |b|^4)$, where $a = \langle 0|U|0\rangle = \langle 1|U|1\rangle^*$, $b = \langle 1|U|0\rangle = -\langle 0|U|1\rangle^*$. Since $|a|^2 + |b|^2 = 1$ one can use the Bloch sphere parametrization $|a| = \cos(\theta/2)$, $|b| = \sin(\theta/2)$ from which it follows $\tilde{C}_B(U) = C_B(U)/C_{d=2} = \sin^2(\theta)$. The distribution density of $c = \tilde{C}_B(U) \in [0,\,1]$ is given by

$$P_{CGP}(c) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^\pi d(\cos\theta)\delta(c - \sin^2\theta)$$
$$= \frac{1}{2}\int_{-1}^1 dx\,\delta(c - 1 + x^2) = \frac{1}{2\sqrt{1-c}},$$

where we have used $\int dx\,\delta(f(x)) = \sum_{x_0\,:\,f(x_0)=0} |f'(x_0)|^{-1}$.