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Power of one non-clean qubit

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The one-clean qubit model (or the DQC1 model) is a restricted model of quantum computing where only a single qubit of the initial state is pure and others are maximally mixed. Although the model is not universal, it can efficiently solve several problems whose classical efficient solutions are not known. Furthermore, it was recently shown that if the one-clean qubit model is classically efficiently simulated, the polynomial hierarchy collapses to the second level. A disadvantage of the one-clean qubit model is, however, that the clean qubit is too clean: for example, in realistic NMR experiments, polarizations are not high enough to have the perfectly pure qubit. In this paper, we consider a more realistic one-clean qubit model, where the clean qubit is not clean, but depolarized. We first show that, for any polarization, a multiplicative-error calculation of the output probability distribution of the model is possible in a classical polynomial time if we take an appropriately large multiplicative error. The result is in a strong contrast to that of the ideal one-clean qubit model where the classical efficient multiplicative-error calculation (or even the sampling) with the same amount of error causes the collapse of the polynomial hierarchy. We next show that, for any polarization lower-bounded by an inverse polynomial, a classical efficient sampling (in terms of a sufficiently small multiplicative error or an exponentially-small additive error) of the output probability distribution of the model is impossible unless BQP is contained in the second level of the polynomial hierarchy, which suggests the hardness of the classical efficient simulation of the one non-clean qubit model.

To show a supremacy of quantum computing over classical one is one of the most central research subjects in physics and computer science. Although several quantum advantages have been shown in terms of the communication complexity [1, 2] and the query complexity [3, 4], the ultimate question “is $BPP \neq BQP$?” remains open.

One good strategy to study the gap between quantum and classical is restricting the quantum side. It is also important from the experimental point of view given the high technological demands for the realization of a universal quantum computer. For example, quantum computing that uses only Clifford gates [5, 6] or Fermionic linear optical gates (or the matchgates) [7–10] is classically efficiently simulatable. On the other hand, restricted models that do not seem to be classically efficiently simulatable do exist [11–15]. For example, if quantum computing that uses only non-interacting Bosons [12] or commuting gates [13–15] (so called the IQP model) is classically efficiently simulated, then the polynomial hierarchy collapses to the third level (or the second level [16]). Since a collapse of the polynomial hierarchy is not believed to happen, these results suggest the hardness of the classical efficient simulation of these restricted models.

The one-clean qubit model (or the DQC1 model) [17] is another restricted model of quantum computing that is believed to be stronger than classical computing. The model was originally motivated by NMR, which has over half a century history and matured control schemes [18–20]. An NMR spin ensemble system has several physical advantages: for example, molecules consisting of wide varieties of nuclear and electron spins can be chemically synthesized. Furthermore, the macroscopic signals are obtained by virtue of the huge number of copies in the ensemble with less backaction. Finally, each spin is highly isolated from external degrees of freedom, which is favorable to avoid decoherence. Because of these reasons, an NMR spin ensemble system is a useful experimental setup to probe quantum many-body dynamics, and in fact, it has been applied to several quantum information processing tasks including the quantum simulation [21]. However, a disadvantage of NMR is that the initialization (polarization) of a nuclear spin is not easy. Therefore, NMR quantum information processing has to be a highly mixed state quantum computation.

The one-clean qubit model formalizes the NMR quantum information processing in the following way: First, the initial state is $|0\rangle\langle 0| \otimes (\frac{I}{2})^{\otimes n-1}$, where $I \equiv |0\rangle\langle 0| + |1\rangle\langle 1|$ is the two-dimensional identity operator. Second, any (uniformly-generated polynomial-time) n -qubit unitary operator is applied on it. Finally, some qubits are measured in the computational basis. (Note that in some strict definitions, only a single-qubit is allowed to be measured, or only an expectation value of a single-qubit mea-

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surement is obtained.)

If the clean qubit $|0\rangle$ of the initial state is replaced with the maximally-mixed state $\frac{I}{2}$, the quantum computing is trivially simulatable with a polynomial-time classical computer, since $U(\frac{I}{2})^{\otimes n}U^\dagger = (\frac{I}{2})^{\otimes n}$ for any unitary operator U . This example suggests that the one-clean qubit model is also classically efficiently simulatable, since only a single pure qubit does not seem to cause any drastic change. However, surprisingly, the model can efficiently solve several problems whose classical efficient solutions are not known, such as the spectral density estimation [17], testing integrability [22], calculations of the fidelity decay [23], and approximations of the Jones polynomial, HOMFLY polynomial, and Turaev-Viro invariant [24–27]. Furthermore, it was recently shown that if the probability distribution of the measurement result on the single output qubit of the one-clean qubit model is classically efficiently sampled (in terms of a multiplicative error or an exponentially-small additive error), then the polynomial hierarchy collapses to the second level [16, 28]. Roles of quantumness (entanglement, discord, etc.) in the one-clean qubit model has also been intensively studied [29–37].

A disadvantage of the one-clean qubit model is, however, that the clean qubit is too clean: for example, in realistic experiments, the polarization of spins in an NMR ensemble is not high enough to obtain the perfectly pure qubit (even if the algorithmic cooling or the quantum data compression [38–40] is employed). Therefore, the following important question remains open: can we show any hardness of a classical efficient simulation of a more realistic one “non-clean” qubit model?

In this paper, we consider a modified version of the one-clean qubit model where the clean qubit of the initial state is not clean but depolarized (Eq. (1)). We first show that for any polarization, a multiplicative-error calculation of the output probability distribution of the model is possible in a classical polynomial time if we take a sufficiently large multiplicative error. Note that the result is in a strong contrast to that of the ideal one-clean qubit model where the classical efficient multiplicative-error calculation (or even the sampling) with the same amount of error causes the collapse of the polynomial hierarchy [16, 28]. We also point out that the bound of the multiplicative error is optimal by showing a counter example for errors smaller than the bound. We next consider the sampling of the output probability distribution of our model. We show that for any polarization lower-bounded by an inverse polynomial, a classical efficient sampling (in terms of a sufficiently small multiplicative error or an exponentially-small additive error) is impossible unless BQP is contained in the second level of the polynomial hierarchy. Since it is not believed to happen [41], the result demonstrates the power of one non-clean qubit. In short, the computational capability of the one non-clean qubit model with a small polarization exhibits a “phase transition” on the magnitude of the polarization: a classical simulation with a multiplicative

error larger than the polarization is possible, but it is impossible when the error is smaller than the polarization.

Note that, with similar and other motivations, noisy versions of IQP circuits have been studied recently, and shown to be hard to classically efficiently simulate [15, 42]. Moreover, quantum computing that uses a universal gate set but is too noisy to realize fault-tolerant universal quantum computing was shown to be hard to classically efficiently simulate [43].

One non-clean qubit model.— We consider the following model. The initial state is the n -qubit state

$$\rho_\epsilon^{\text{init}} \equiv \left(\frac{1+\epsilon}{2} |0\rangle\langle 0| + \frac{1-\epsilon}{2} |1\rangle\langle 1| \right) \otimes \left(\frac{I}{2} \right)^{\otimes (n-1)}, \quad (1)$$

where the first qubit corresponds to the nuclear spin to be probed whose polarization ϵ is relatively higher than the others but still very small. (This type of non-clean qubit model was also studied in Ref. [44].) The case $\epsilon = 1$ corresponds to the original one-clean qubit model. Any (uniformly-generated polynomial-time) n -qubit unitary operator U is applied on the initial state to obtain $\rho_\epsilon \equiv U \rho_\epsilon^{\text{init}} U^\dagger$. Finally some qubits are measured in the computational basis. If we measure all qubits, the probability p_z of obtaining the result $z \in \{0, 1\}^n$ is

$$p_z \equiv \langle z | \rho_\epsilon | z \rangle = \epsilon \langle z | U \left(|0\rangle\langle 0| \otimes \frac{I^{\otimes (n-1)}}{2^{n-1}} \right) U^\dagger | z \rangle + \frac{1-\epsilon}{2^n}.$$

Multiplicative-error calculation.— First, we consider calculations of the output probability distribution of the model. As is shown in Appendix A, the exact calculation is trivially $\#P$ -hard (actually GapP-complete). We therefore consider approximations, namely, multiplicative-error calculations. Here, a multiplicative-error approximation with the error $c \geq 0$ means that the target value p and the calculated value q satisfy $|p - q| \leq cp$.

Result 1: For any $0 \leq \epsilon < 1$, p_z can be approximated by the uniform distribution $q_z = \frac{1}{2^n}$ with any multiplicative error c that satisfies $c \geq \frac{\epsilon}{1-\epsilon}$.

Proof: We can show $\frac{1-\epsilon}{2^n} \leq p_z \leq \frac{1+\epsilon}{2^n}$ for any $z \in \{0, 1\}^n$. Therefore, $|p_z - \frac{1}{2^n}| \leq \frac{\epsilon}{2^n} = \frac{\epsilon}{1-\epsilon} \frac{1-\epsilon}{2^n} \leq cp_z$. \square

According to the result of Ref. [16], if the output probability distribution of the computational-basis measurement on the single output qubit of the one-clean qubit model is classically efficiently sampled with the $c = 1 - \frac{1}{2^n}$ multiplicative-error, then the polynomial hierarchy collapses to the second level. (See Appendix B.) Result 1 shows that the hardness result does no longer hold for the one non-clean qubit case [45]. In fact, from Result 1, for any $x \in \{0, 1\}$,

$$\begin{aligned} \left| \sum_{y \in \{0,1\}^{n-1}} p_{xy} - \frac{1}{2} \right| &= \left| \sum_{y \in \{0,1\}^{n-1}} p_{xy} - \sum_{y \in \{0,1\}^{n-1}} \frac{1}{2^n} \right| \\ &\leq \sum_{y \in \{0,1\}^{n-1}} \left| p_{xy} - \frac{1}{2^n} \right| \leq c \sum_{y \in \{0,1\}^{n-1}} p_{xy}, \end{aligned}$$

which means that the probability $\sum_{y \in \{0,1\}^{n-1}} p_{xy}$ of obtaining $x \in \{0,1\}$ when the first qubit of our model is measured in the computational basis is approximated to $\frac{1}{2}$ (and therefore classically efficiently sampled) with the multiplicative error c . If we take the polarization ϵ as $\epsilon \leq \frac{1}{2} - \frac{1}{2^{n+2}-2}$, for example, c can be $c = 1 - \frac{1}{2^n}$.

Optimality of the bound.— We can show that the bound $c \geq \frac{\epsilon}{1-\epsilon}$ of Result 1 is optimal in the following sense:

Result 2: For any $0 \leq \epsilon < 1$ and $c \geq 0$ such that $0 \leq c < \frac{\epsilon}{1-\epsilon}$, and for any probability distribution $q : \{0,1\}^n \ni z \mapsto q_z \in [0,1]$, there exists an n -qubit unitary operator U such that $|p_z - q_z| > cp_z$ for a certain $z \in \{0,1\}^n$.

Proof: If $q_z < \frac{1}{2^n}$ for all $z \in \{0,1\}^n$, then $\sum_{z \in \{0,1\}^n} q_z < 1$, which is contradiction. Therefore there is at least one $y \in \{0,1\}^n$ such that $q_y \geq \frac{1}{2^n}$. Let $y_1 \in \{0,1\}$ be the first bit of y . If we take $U = X^{y_1 \oplus 1} \otimes I^{\otimes n-1}$,

$$\begin{aligned} p_y &= \epsilon \langle y | U \left(|0\rangle\langle 0| \otimes \frac{I^{\otimes n-1}}{2^{n-1}} \right) U^\dagger | y \rangle + \frac{1-\epsilon}{2^n} \\ &= \epsilon \langle y | \left(|y_1 \oplus 1\rangle\langle y_1 \oplus 1| \otimes \frac{I^{\otimes n-1}}{2^{n-1}} \right) | y \rangle + \frac{1-\epsilon}{2^n} = \frac{1-\epsilon}{2^n}, \end{aligned}$$

and therefore $|p_y - q_y| \geq \frac{\epsilon}{2^n}$, while $cp_y < \frac{\epsilon}{1-\epsilon} \frac{1-\epsilon}{2^n} = \frac{\epsilon}{2^n}$. Hence we obtain $|p_y - q_y| > cp_y$. \square

Multiplicative-error sampling.— We next consider the sampling. We first show the hardness result for the multiplicative error case.

Result 3: Let us assume that for any (uniformly-generated polynomial-time) n -qubit unitary operator U , there exists a $\text{poly}(n)$ -time classical probabilistic algorithm that outputs $z \in \{0,1\}^n$ with probability q_z such that

$$|p_{0^n} - q_{0^n}| \leq cp_{0^n} \quad (2)$$

with a certain c that satisfies $0 \leq c \leq \epsilon - \frac{1}{\delta(n)}$ for a polynomial $\delta > 0$. Then BQP is contained in SBP.

Note that the value of c considered in Result 3 is always smaller than that of Result 1, since $\frac{\epsilon}{1-\epsilon} - (\epsilon - \frac{1}{\delta}) \geq 0$, and therefore there is no contradiction between these two results. More importantly, since $\frac{\epsilon}{1-\epsilon} = \epsilon + O(\epsilon^2)$ for small ϵ , the combination of Result 1 and Result 3 means that the polarization ϵ is the “phase transition point” for the multiplicative error c : if $c > \epsilon$ then the classical simulation is possible (Result 1), while if $c < \epsilon$ then it is impossible (Result 3).

There are further three remarks before the proof of Result 3. First, Result 3 implicitly assumes that ϵ is lower-bounded by an inverse polynomial, since otherwise no c can satisfy $c \leq \epsilon - \frac{1}{\delta}$. The assumption, $\epsilon \geq 1/\text{poly}$, is acceptable, since we can take such ϵ in realistic NMR experiments. (Actually, ϵ can be even a small but system-size-independent constant.) Second, the standard definition of the multiplicative-error sampling is that $|p_z - q_z| \leq cp_z$ for any $z \in \{0,1\}^n$, but in Result 3, the satisfiability only for $z = 0^n$ is enough. Finally, SBP is defined in the following way [47]: A language L is in SBP if there

exist a polynomial r and a uniformly-generated family of polynomial-size probabilistic classical circuits such that if $x \in L$ then the acceptance probability is $\geq 2^{-r(|x|)}$, and if $x \notin L$ then the acceptance probability is $\leq 2^{-r(|x|)-1}$. As is shown in Appendix C, the bound $(2^{-r}, 2^{-r-1})$ can be replaced with $(a2^{-r}, b2^{-r})$ for any $0 \leq b < a \leq 1$ such that $a - b \geq \frac{1}{\text{poly}}$. It is known that SBP is in AM [47], and therefore in the second level of the polynomial hierarchy: $\text{SBP} \subseteq \text{AM} \subseteq \Pi_2^P$. Hence $\text{BQP} \subseteq \text{SBP}$ means that BQP is in the second level of the polynomial hierarchy. Note that $\text{BQP} \subseteq \text{SBP}$ itself is also unlikely, since $\text{SBP} \subseteq \text{BPP}_{\text{path}}$ and there is an oracle such that BQP is not contained in BPP_{path} [48].

Proof: Let us assume that a language L is in BQP. This means that for any polynomial r , there exists a uniformly generated family $\{V_x\}$ of polynomial-size quantum circuits such that

$$\langle 0^n | V_x^\dagger (|0\rangle\langle 0| \otimes I^{\otimes n-1}) V_x | 0^n \rangle \begin{cases} \geq 1 - 2^{-r} & (x \in L) \\ \leq 2^{-r} & (x \notin L). \end{cases}$$

Here, $n = \text{poly}(|x|)$. Let us take $U = V_x^\dagger$. We also take r such that $\epsilon 2^{-r+1} \leq \frac{1}{2\delta}$.

If $x \in L$,

$$\begin{aligned} q_{0^n} &\geq (1-c) \left[\frac{\epsilon}{2^{n-1}} \langle 0^n | V_x^\dagger (|0\rangle\langle 0| \otimes I^{\otimes n-1}) V_x | 0^n \rangle + \frac{1-\epsilon}{2^n} \right] \\ &\geq (1-c) \left[\frac{\epsilon}{2^{n-1}} (1 - 2^{-r}) + \frac{1-\epsilon}{2^n} \right] \\ &= \frac{(1-c)}{2^n} (1 + \epsilon - \epsilon 2^{-r+1}). \end{aligned}$$

If $x \notin L$,

$$\begin{aligned} q_{0^n} &\leq (1+c) \left[\frac{\epsilon}{2^{n-1}} \langle 0^n | V_x^\dagger (|0\rangle\langle 0| \otimes I^{\otimes n-1}) V_x | 0^n \rangle + \frac{1-\epsilon}{2^n} \right] \\ &\leq (1+c) \left[\frac{\epsilon}{2^{n-1}} 2^{-r} + \frac{1-\epsilon}{2^n} \right] \\ &= \frac{(1+c)}{2^n} (1 - \epsilon + \epsilon 2^{-r+1}). \end{aligned}$$

Since

$$\begin{aligned} &(1-c)(1 + \epsilon - \epsilon 2^{-r+1}) - (1+c)(1 - \epsilon + \epsilon 2^{-r+1}) \\ &= 2(\epsilon - \epsilon 2^{-r+1} - c) \\ &\geq 2\left(\epsilon - \frac{1}{2\delta} - \left(\epsilon - \frac{1}{\delta}\right)\right) = \frac{1}{\delta}, \end{aligned}$$

L is in SBP. \square

Exponentially-small additive error sampling.— We can also show a similar hardness result for the exponentially-small additive error case.

Result 4: Let us assume that for any (uniformly-generated polynomial-time) n -qubit unitary operator U , there exists a $\text{poly}(n)$ -time classical probabilistic algorithm that outputs $z \in \{0,1\}^n$ with probability q_z such that

$$|p_{0^n} - q_{0^n}| \leq \eta \quad (3)$$

with a certain η that satisfies $0 \leq \eta \leq (\epsilon - \frac{1}{\delta})2^{-n}$ for a polynomial $\delta > 0$. Then BQP is contained in SBP.

Before giving a proof, there are two remarks: First, we again implicitly assume $\epsilon \geq 1/\text{poly}$. Second, the assumption Eq. (3) can be replaced with the more standard assumption (L_1 -norm additive-error approximation), $\sum_{z \in \{0,1\}^n} |p_z - q_z| \leq \eta$, since if it is satisfied then $|p_{0^n} - q_{0^n}| \leq \sum_{z \in \{0,1\}^n} |p_z - q_z| \leq \eta$, and therefore Eq. (3) is satisfied. Since Eq. (3) is weaker, we have used it.

Proof: Let us assume that a language L is in BQP, and let V_x be the corresponding circuit as assumed in the proof of Result 3. We take $U = V_x^\dagger$, and r such that $\epsilon 2^{-r+1} \leq \frac{1}{2\delta}$. If $x \in L$, $q_{0^n} \geq \frac{1}{2^n}(1 + \epsilon - \epsilon 2^{-r+1} - 2^n \eta)$. If $x \notin L$, $q_{0^n} \leq \frac{1}{2^n}(\epsilon 2^{-r+1} + 1 - \epsilon + 2^n \eta)$. Since

$$\begin{aligned} & (1 + \epsilon - \epsilon 2^{-r+1} - 2^n \eta) - (1 - \epsilon + \epsilon 2^{-r+1} + 2^n \eta) \\ & \geq 2\left(\epsilon - \frac{1}{2\delta} - \left(\epsilon - \frac{1}{\delta}\right)\right) = \frac{1}{\delta}, \end{aligned}$$

L is in SBP. \square

Discussion.— In this paper, we have used a multiplicative or an exponentially-small additive error in the definition of the classical samplability. It is an important open problem whether we can generalize the results to a constant or inverse-polynomial L_1 -norm error as was done for the Boson sampling [12], the IQP [14, 15], and the Fourier sampling [49]. (These results do not seem to be directly applied to the one qubit model, even in the perfect polarization case, since the one qubit model seems to be able to simulate standard quantum computing with only an exponentially small rate.) In the present case, however, using a multiplicative or an exponentially-small additive error is justified, since in our case the model itself is noisy. In other words, we consider the following sampling problem: “sample the output probability distribution of a noisy one-clean qubit model”. The problem can be, of course, exactly solvable with the noisy one-clean qubit model, but we have shown that solving the problem classically is impossible even with a multiplicative or an exponentially-small additive error. We have therefore shown the existence of a sampling problem that can be exactly solvable by a realistic non-universal quantum computer but cannot be solved by a classical computer even with a multiplicative or an exponentially-small additive error.

We have considered the output probability distribution of the measurements on all qubits. It is an open problem whether we can reduce the number of measured qubits to one. Furthermore, we want to improve our consequence, $\text{BQP} \subseteq \text{SBP}$, to more unlikely one such as the collapse of the polynomial hierarchy, but at this moment we do not know how to do it.

Finally, to conclude this paper, let us discuss roles of entanglement in NMR quantum computing. In Ref. [50], a criteria on the initial polarization, below which the system becomes a separable state, was derived, and pointed out that states used in NMR experiments are separable states. It sounds like NMR quantum information processing has no quantum power, and in fact some researchers have insisted that NMR quantum information processing is useless. The conclusion is, however, wrong. In fact, a

polynomially small purity keeps the state outside the separable ball [44]. Furthermore, as is shown in the present paper, NMR quantum computing can demonstrate the quantum supremacy for some sampling problems. Finally, in the first place, entanglement is not directly connected to the quantum speedup: recently it was shown that a larger entanglement does not necessarily mean a quantum speedup [51, 52], and that quantum computing whose register always has a small bipartite entanglement can solve any BQP problem [53]. Interestingly, even if we consider a much weaker model, which we call separable quantum computing, where the register is always separable during the computation, its classical simulatability is not so obvious. For example, even if the register is always separable, it seems to be hard to find a separable decomposition after every local unitary gate operation, since after a local unitary gate operation, some pure states in the mixture can be entangled. Furthermore, although any discord free quantum computation (with one or two-qubit gates) is classically simulatable [54], a separable state can have non-zero quantum discord.

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Appendix A.— It is easy to see that the exact calculation of the output probability distribution of our model is $\#P$ -hard (actually, GapP-complete), because the ability of the exact calculation of the output probability distribution of the model allows us to exactly calculate the output probability distribution of the one-clean qubit model, which contains (in an exponentially small rate) the output probability distribution of any (polynomial-time) quantum computing [44]. It is known that the exact calculation of the output probability distribution of (polynomial-time) quantum computing is $\#P$ -hard (actually GapP-complete) [46].

Appendix B.— Here we show that if the output probability distribution of the one-clean qubit model is classically efficiently sampled with the multiplicative error $c = 1 - \frac{1}{2^n}$ then NQP is in NP, which causes the collapse of the polynomial hierarchy to the second level. We follow the argument in Refs. [16, 55]. Let us assume that a language L is in NQP, which means that there exists a uniformly-generated family $\{V_x\}$ of polynomial-size quantum circuits such that if $x \in L$ then $0 < p < 1$, and

if $x \notin L$ then $p = 0$, where p is the acceptance probability. It was shown in Ref. [55] that from V_x , which acts on $n-1$ qubits, we can construct an n -qubit one-clean qubit circuit such that the probability \tilde{p} of obtaining 1 when the clean qubit is measured in the computational basis is $\tilde{p} = \frac{4}{2^n-1}p(1-p)$. Therefore if $x \in L$ then $\tilde{p} > 0$, and if $x \notin L$ then $\tilde{p} = 0$. Let us assume that there exists a classical polynomial-time probabilistic algorithm whose acceptance probability q satisfies $|\tilde{p} - q| \leq (1 - \frac{1}{2^n})\tilde{p}$. Then, if $x \in L$ we have $q \geq \frac{\tilde{p}}{2^n} > 0$, and if $x \notin L$ then $q \leq (2 - \frac{1}{2^n})\tilde{p} = 0$. Therefore, NQP is in NP, which causes the collapse of the polynomial hierarchy to the second level.

Appendix C.— Here we show that the bound $(2^{-r}, 2^{-r-1})$ of SBP can be replaced with $(a2^{-r}, b2^{-r})$ for any $0 \leq b < a \leq 1$ such that $a - b \geq \frac{1}{q}$, where $q > 0$ is a polynomial.

Since $a \geq b + \frac{1}{q} \geq \frac{1}{q}$, there exists a polynomial $k \geq 0$ such that $a > \frac{1}{2^k}$. Let V_x be the original circuit of SBP. We define the modified circuit V'_x in the following way: it first runs the original circuit V_x , and then accepts with probability $\frac{1}{a2^k}$ if V_x accepts. If $x \in L$, the acceptance probability of V'_x is $p_{acc} \geq \frac{a2^{-r}}{a2^k} = \frac{1}{2^{r+k}}$. If $x \notin L$, it is

$$\begin{aligned} p_{acc} &\leq \frac{b2^{-r}}{a2^k} \\ &= \frac{1}{2^{r+k}} \frac{a - (a - b)}{a} \\ &= \frac{1}{2^{r+k}} \left(1 - \frac{a - b}{a}\right) \\ &\leq \frac{1}{2^{r+k}} \left(1 - \frac{1}{q}\right). \end{aligned}$$

We run V'_x q times, and accept if all results accept. If $x \in L$, the acceptance probability is $p_{acc}^q \geq \frac{1}{2^{(r+k)q}}$. If $x \notin L$, it is

$$\begin{aligned} p_{acc}^q &\leq \frac{1}{2^{(r+k)q}} \left(1 - \frac{1}{q}\right)^q \\ &= \frac{1}{2^{(r+k)q}} \left[\left(1 + \frac{1}{q-1}\right)^q\right]^{-1} \\ &\leq \frac{1}{2^{(r+k)q}} \frac{1}{2}, \end{aligned}$$

where we have used

$$\begin{aligned} \left(1 + \frac{1}{q-1}\right)^q &= \sum_{j=0}^q \binom{q}{j} \left(\frac{1}{q-1}\right)^j \\ &\geq 1 + \binom{q}{1} \frac{1}{q-1} \\ &= 1 + \frac{q}{q-1} \\ &\geq 2. \end{aligned}$$

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