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Generalized modular-value based scheme and its generalized modular value

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We consider a generalized modular-value based scheme based on the standard von Neumann measurement. We model the scheme as an interaction between a quantum system and a discrete quantum pointer where the pointer operator is a projection operator onto one of the states of the basis of the pointer Hilbert space. The interaction strength is made arbitrarily large. After postselection onto the system, the results of the pointer measurement are so-called the conditional probabilities. We first explicitly derive the analytical expressions of the conditional probabilities, the expectation value, and the average displacement in the measured value of a pointer observable, that we name as the pointer quantities. We also provide an expression for a generalized modular value and discuss the relationship between the generalized modular value and generalized weak values. The study then shows that the generalized modular value can characterize these pointer quantities. Then we give applications of our proposal to the cases of a spin-s particle pointer and a semiclassical pointer state. One of the key results is that the amplification effect, similar to the weak-value case, is also observed in the case of the generalized modular value. Our study can also apply to the cases of nonclassical pointer states.

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I. INTRODUCTION

In 2010, Kedem and Vaidman [1] treated the von Neumann measurement process performed between the preparation of an initial state and the postselection of a final state. Different from the weak-value case, the interaction strength between the system and the qubit pointer is kept arbitrarily large [1]. The expectation value of the outcomes of such measurement process is so-called "*modular value*". The modular value for a quantum observable $\hat{A}$ of the system is given by:

$$(A)_{\rm m} = \frac{\langle\phi|e^{-ig\hat{A}}|\psi\rangle}{\langle\phi|\psi\rangle}\ , \tag{1}$$

where $|\psi\rangle$ is the initially prepared state, $|\phi\rangle$ is the postselected state, and g is the coupling constant, which is not assumed to be small but can be arbitrarily large. The subscript "m" stands for "modular".

Apparently, a weak value of an observable is related to the corresponding modular value through $\langle\hat{A}\rangle_{\rm w} = i[\partial_g(\hat{A})_{\rm m}]_{g\to 0}$. For an arbitrarily large of g, the absolute value of the modular value Eq. (1) naturally behaves a periodic property, where its value is periodically repeated respect to g [2]. The study in Ref. [3] has also indicated a generalized relation between weak and modular values in the sense that the weak value can be obtained from the corresponding modular value even for an arbitrarily large g, with an arbitrarily given operator, for arbitrarily pre- and postselected states. The theoretical studies also clarified that the essence of some quantum paradoxes could be attributed to the complex nature of the modular values [2, 3]. We also proposed that the change in the detection probability could be used to calculate the amplitude of a modular value [2], instead of using the state tomography as in Ref.[1]. We also connected the phase of a modular value to the Pancharatnam phases. Moreover, Cormann et al. demonstrated that the modulus and argument of a modular value could be measured by using the interference in quantum eraser [4]. One of the possible applications of the modular-value method is that experimentally measured weak and strong values in sequential measurements [5].

So far, weak values have been studied in the context of weak measurements, and in such weak measurements, not only classical pointer states (such as Gaussian states [6–9] or Hermite-Gaussian/Laguerre-Gaussian beams [10, 11] but also nonclassical pointer states (squeezed states, and Schrödinger cat states) are used [12, 13]. Nonclassical pointer states have an advantage in controlling the precision of weak measurements, which have been recently demonstrated [11–13].

On the contrary, in the case of modular-value measurement, such generalization has not been studied yet. Therefore, the usage of various pointer states has not been investigated either. Originally, the *modular-value based scheme* was described by the interaction between a given system and a qubit pointer via an interaction Hamiltonian $\hat{H} = g(t)\hat{A}\otimes\hat{P}$, where the projection operator is chosen as $\hat{P} = |1\rangle\langle 1|$, and the initial qubit pointer is prepared in state $\gamma|0\rangle + \bar{\gamma}|1\rangle$, with $|\gamma|^2 + |\bar{\gamma}|^2 = 1$. This scheme limits the usage of modular values.

In the present work, we consider the pointer is not a qubit but a qudit (i.e., multi-level system), which we call *generalized modular-value based scheme*. The analysis of such scheme results in a generalization of modular value, which we call the *generalized modular value*. Of course, the standard modular value is derived as a special (pure states) case of the generalized modular value. The inter-

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action between the system and the pointer is given by the Hamiltonian $\hat{H} = g(t)\hat{A} \otimes \hat{P}$ with an arbitrarily large coupling constant g, which is the same as the standard modular-value case. What is different is that the pointer projection operator $\hat{P}$ is not necessarily $|1\rangle\langle 1|$ but can be any of the eigenstates $|\eta_\mu\rangle\langle\eta_\mu|$ ($\mu = 0, 1, \cdots, d-1$, where d is the dimension of the qudit pointer). Through the analysis of this generalized modular-value scheme, we derive a formula for the generalized modular value. We also illustrate a relationship between the generalized modular value and generalized weak values, which allows us to measure the generalized weak value from the generalized modular value. Then we derive the expressions for some conditional probabilities using the generalized modular value. As an application, we give an illustration of this theory to a spin-s particle pointer. We also provide an application to a semiclassical pointer state, where we take the coherent state of bosons as an initial pointer state, which suggests the implementation of the present theory to the cases of nonclassical pointer states.

This paper is organized as follows. Sec. II introduces a generalized modular-value based scheme. Sec. III represents a generalized modular value. The applications to a spin-s particle pointer and a semiclassical pointer state are presented in the Sec. IV. Finally, we summarize the results in Sec. V.

II. GENERALIZED MODULAR-VALUE BASED SCHEME

We analyze a system-pointer interaction under the standard von Neumann paradigm [14]. Therein, an operator $\hat{A}$ on the system Hilbert space $\mathcal{H}_\mathrm{s}$ couples to a projection operator $\hat{P}$ on the finite pointer Hilbert space $\mathcal{H}_\mathrm{p}$ via an interaction Hamiltonian $\hat{H}_\mathrm{sp} = g(t)\hat{A} \otimes \hat{P}$. Then the corresponding unitary operator can be written as $\hat{U}_\mathrm{sp} = e^{-ig\hat{A}\otimes\hat{P}}$, where $g = \int g(t)\mathrm{d}t$ is the coupling constant. The subscript s(p) denotes the system(pointer). The coupling constant g is important, and most of the equations below are g-dependent due to the g-dependent of the interaction Hamiltonian $\hat{H}_\mathrm{sp}$.

We also assume that the system is initially prepared in state $\hat{\rho}_i$ while the pointer state is $|\xi\rangle$. The total density operator of the system-pointer, $\hat{\rho}_\mathrm{sp}$, is written as $\hat{\rho}_\mathrm{sp} = \hat{\rho}_i \otimes |\xi\rangle\langle\xi|$ before the interaction. This joint state will involve in the Schrödinger picture under the unitary interaction to $\hat{\rho}'_\mathrm{sp} = \hat{U}_\mathrm{sp}\hat{\rho}_\mathrm{sp}\hat{U}^\dagger_\mathrm{sp}$.

The quantum system is then postselected onto a freely chosen final state $\hat{\rho}_f (\equiv |f\rangle_\mathrm{s}\,{}_\mathrm{s}\langle f|)$. The joint probability of obtaining $\hat{\rho}_f$ and a pointer outcome $\mu (= 0, 1, \cdots, d-1)$, that corresponds to the projection onto the state $|\eta_\mu\rangle$ of the pointer, is expressed as $p(\mu, \hat{\rho}_f) = \mathrm{Tr}_\mathrm{sp}[(\hat{\rho}_f \otimes |\eta_\mu\rangle\langle\eta_\mu|)\hat{\rho}'_\mathrm{sp}]$, where $\mathrm{Tr}_\mathrm{sp}[...]$ represents the total trace over the joint Hilbert space of the system and the pointer. Tracing out the pointer Hilbert space, then the joint probability is expressed by the trace over the system [see Appendix A] as

$$p(\mu, \hat{\rho}_f) = \mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{M}_\mu \hat{\rho}_i \hat{M}^\dagger_\mu], \tag{2}$$

where the operator $\hat{M}_\mu \equiv \langle\eta_\mu|e^{-ig\hat{A}\otimes\hat{P}}|\xi\rangle$ is known as the *Kraus operator*, which operates on the system Hilbert space $\mathcal{H}_\mathrm{s}$. The completeness of the Kraus operators, $\sum_\mu \hat{M}^\dagger_\mu \hat{M}_\mu = \hat{I}_\mathrm{s}$, is easily verified.

By taking the sum over all μ's in the joint probability, one can derive the probability to obtain the postselected outcome $\hat{\rho}_f$ as

$$p(\hat{\rho}_f) = \sum_\mu p(\mu, \hat{\rho}_f) = \mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{\rho}'_i], \tag{3}$$

where $\hat{\rho}'_i \equiv \sum_\mu \hat{M}_\mu \hat{\rho}_i \hat{M}^\dagger_\mu = \mathrm{Tr}_\mathrm{p}[\hat{\rho}'_\mathrm{sp}]$ is the density matrix of the quantum system after the interaction, which is obtained by tracing out the pointer Hilbert space of the bipartite $\hat{\rho}'_\mathrm{sp}$.

We next use the Bayesian rule to obtain the conditional probability of the outcome μ, which is written as

$$p(\mu|\hat{\rho}_f) = \frac{p(\mu, \hat{\rho}_f)}{p(\hat{\rho}_f)} = \frac{\mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{M}_\mu \hat{\rho}_i \hat{M}^\dagger_\mu]}{\mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{\rho}'_i]}. \tag{4}$$

The expectation value of an arbitrary pointer operator, $\hat{O}_\mathrm{p}$, is given by $\langle\hat{O}_\mathrm{p}\rangle = \mathrm{Tr}_\mathrm{sp}[(\hat{\rho}_f \otimes \hat{O}_\mathrm{p})\hat{\rho}'_\mathrm{sp}]/\mathrm{Tr}_\mathrm{sp}[(\hat{\rho}_f \otimes \hat{I}_\mathrm{p})\hat{\rho}'_\mathrm{sp}]$, where $\langle...\rangle$ denotes the expectation value for the final pointer state throughout this paper. After tracing out the pointer Hilbert space and using the spectral decomposition, $\hat{O}_\mathrm{p} = \sum_\kappa o_\kappa |\eta_\kappa\rangle\langle\eta_\kappa|$, where o_κ denotes the κ^th eigenvalue of the operator $\hat{O}_\mathrm{p}$ with $\hat{O}_\mathrm{p}|\eta_\kappa\rangle = o_\kappa|\eta_\kappa\rangle$, we have [see Appendix A]

$$\langle\hat{O}_\mathrm{p}\rangle = \sum_\kappa o_\kappa \frac{\mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{M}_\kappa \hat{\rho}_i \hat{M}^\dagger_\kappa]}{\mathrm{Tr}_\mathrm{s}[\hat{\rho}_f \hat{\rho}'_i]} = \sum_\kappa o_\kappa p(\kappa|\hat{\rho}_f)\ , \tag{5}$$

where we have used the basis $\{|\eta_\kappa\rangle\}$ of the discrete Hilbert space $\mathcal{H}_\mathrm{p}$. We define and calculate the average displacement in the measured value of the pointer observable $\hat{O}_\mathrm{p}$ as

$$\Delta\langle\hat{O}_\mathrm{p}\rangle \equiv \langle\hat{O}_\mathrm{p}\rangle - \langle\hat{O}_\mathrm{p}\rangle_\xi = \sum_\kappa o_\kappa \Big[p(\kappa|\hat{\rho}_f) - |c_\kappa|^2\Big], \tag{6}$$

where $\langle\hat{O}_\mathrm{p}\rangle_\xi \equiv \langle\xi|\hat{O}_\mathrm{p}|\xi\rangle$ is the expectation of the pointer observable for the initial pointer state $|\xi\rangle$, and we have defined $c_\kappa \equiv \langle\eta_\kappa|\xi\rangle$. The term inside the bracket [...] is the *average displacement* of the probability, which is the difference between probabilities after and before the interaction in the pointer. So, the Eq. (6) implies that the average displacement in the measured value of the pointer observable is proportional to the average displacement of the probability. The same as weak-value case, an amplification effect appears whenever the average displacement of the pointer observable is very large. Notable that the result can be generalized to the case of the continuous spectrum of an operator, such as $\hat{O}_\mathrm{p}$ to be a momentum operator $\hat{p}$. Also, in this case, the measurement of the pointer observable implies the indirect measurement of the quantum system observable [15].

Thereafter, let us assume that the initial pointer state is expanded on the basis $\{|\eta_\kappa\rangle\}$, $(\kappa = 0, 1, ..., d-1)$ of the d-dimensional discrete Hilbert space $\mathcal{H}_{\mathrm{p}}$, such that $|\xi\rangle = \sum_\kappa c_\kappa |\eta_\kappa\rangle$, where $c_\kappa \equiv \langle\eta_\kappa|\xi\rangle$. The projection operator that we consider here is $\hat{P} \equiv |\eta_\lambda\rangle\langle\eta_\lambda|$, which is analogous to that $|1\rangle\langle 1|$ in the qubit pointer case. With the aid of the spectral decomposition theorem, the unitary operator acting on the initial pointer state is given by $\hat{U}_{\mathrm{sp}}|\xi\rangle = \sum_\kappa c_\kappa |\eta_\kappa\rangle \exp(-ig\hat{A}\delta_{\kappa\lambda})$, and the Kraus operator becomes (see Appendix B)

$$\hat{M}_\mu = c_\mu \exp(-ig\hat{A}\delta_{\mu\lambda}), \tag{7}$$

where $c_\mu = \langle\eta_\mu|\xi\rangle$, and $\delta_{\mu\lambda}$ is the Kronecker delta function. We also introduce an analytic function, which is based on the joint probability and the Kraus operator (7), defined by

$$Z(\mu,\nu|\lambda) \equiv \mathrm{Tr_s}[\hat{\rho}_f e^{-ig\hat{A}\delta_{\mu\lambda}} \hat{\rho}_i e^{ig\hat{A}\delta_{\nu\lambda}}] , \tag{8}$$

where the vertical bar "|" means "conditioned by", and ν is an extra integer suffix, which will be used to express the density matrix elements like $\rho_{\mu\nu}$. This similar (but different) characteristic function has been introduced and analyzed by Lorenzo [16]. In the present work, we will show that this analytic function, $Z(\mu,\nu|\lambda)$, is used to express a generalized modular value as below.

III. GENERALIZED MODULAR VALUE

In the previous section, we have derived the analytical expressions of the conditional probability [Eq.(4)], the expectation value [Eq.(5)] and the average displacement of the measured values of the pointer observable [Eq.(6)]. In this section, we introduce a generalized modular value and connect it to these three quantities. The generalized modular value is defined as:

$$\begin{aligned}(A)_{\mathrm{m}}^{\mu,\nu|\lambda} &\equiv \frac{Z(\mu,\nu|\lambda)}{Z(\mu' \neq \lambda, \nu' \neq \lambda|\lambda)} \\ &= \frac{\mathrm{Tr_s}[\hat{\rho}_f e^{-ig\hat{A}\delta_{\mu\lambda}} \hat{\rho}_i e^{ig\hat{A}\delta_{\nu\lambda}}]}{\mathrm{Tr_s}[\hat{\rho}_f \hat{\rho}_i]},\end{aligned} \tag{9}$$

where, same as before, $\hat{\rho}_i$ is the prepared state, $\hat{\rho}_f$ is the postselected state, μ and ν are the suffixes of density matrix components used in the below. Before describing the usage of $(A)_{\mathrm{m}}^{\mu,\nu|\lambda}$, we classify it in the following three cases:

(**i**) $\mu \neq \lambda$ and $\nu \neq \lambda$ (Here, both $\mu = \nu$ and $\mu \neq \nu$ are allowed.)

$\rightarrow$ In this case, the generalized modular value $(A)_{\mathrm{m}}^{\mu,\nu|\lambda}$ becomes unity.

(**ii**) $\mu = \lambda$ and $\nu \neq \lambda$ (or, $\nu = \lambda$ and $\mu \neq \lambda$)

$\rightarrow$ For $\mu = \lambda$ and $\nu \neq \lambda$, Eq. (9) gives

$$(A)_{\mathrm{m}}^{\lambda,\nu|\lambda} \equiv \frac{Z(\mu = \lambda, \nu \neq \lambda|\lambda)}{Z(\mu' \neq \lambda, \nu' \neq \lambda|\lambda)} = \frac{\mathrm{Tr_s}[\hat{\rho}_f e^{-ig\hat{A}} \hat{\rho}_i]}{\mathrm{Tr_s}[\hat{\rho}_f \hat{\rho}_i]}. \tag{10}$$

This expression is reduced to the original definition of the standard modular value [Eq. (1)] when both pre- and postselected states are pure states, i.e., $\hat{\rho}_i = |\psi\rangle\langle\psi|$, and $\hat{\rho}_f = |\phi\rangle\langle\phi|$. For $\nu = \lambda$ and $\mu \neq \lambda$, on the other hand, the generalized modular value becomes $(A)_{\mathrm{m}}^{\mu,\lambda|\lambda} = \mathrm{Tr_s}[\hat{\rho}_f \hat{\rho}_i e^{ig\hat{A}}]/\mathrm{Tr_s}[\hat{\rho}_f \hat{\rho}_i] = [(A)_{\mathrm{m}}^{\lambda,\mu|\lambda}]^*$.

(**iii**) $\mu = \nu = \lambda$

$$\rightarrow (A)_{\mathrm{m}}^{\lambda,\lambda|\lambda} \equiv \frac{Z(\mu = \lambda, \nu = \lambda|\lambda)}{Z(\mu' \neq \lambda, \nu' \neq \lambda|\lambda)} = \frac{\mathrm{Tr_s}[\hat{\rho}_f e^{-ig\hat{A}} \hat{\rho}_i e^{ig\hat{A}}]}{\mathrm{Tr_s}[\hat{\rho}_f \hat{\rho}_i]}, \tag{11}$$

which will reduce to the square of the modulus of the standard modular value $|(A)_{\mathrm{m}}|^2$ when the system states are pure. Using $(A)_{\mathrm{m}}^{\mu,\nu|\lambda} = [(A)_{\mathrm{m}}^{\nu,\mu|\lambda}]^*$, $(A)_{\mathrm{m}}^{\lambda,\lambda|\lambda}$ becomes real. We note that there is a similar "generalized" concept for weak values [16, 17], but the way of generalization is very different.

Here we show the meaning of the indices μ and ν more explicitly. Let us consider the final state of the pointer, which is given by

$$\hat{\rho}_{\mathrm{p}}^{\mathrm{out}} = \frac{\mathrm{Tr_s}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}{\mathrm{Tr_{sp}}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]} , \tag{12}$$

where the denominator is the normalization factor, which equals to the success probability of the postselected of $\hat{\rho}_f$. We can proceed the calculation, as is seen in Appendix C, resulting in

$$\frac{\mathrm{Tr_s}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}{\mathrm{Tr_{sp}}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]} = \frac{\sum_{\mu,\nu} c_\mu c_\nu^* (A)_{\mathrm{m}}^{\mu,\nu|\lambda} |\eta_\mu\rangle\langle\eta_\nu|}{\sum_\mu |c_\mu|^2 (A)_{\mathrm{m}}^{\mu,\mu|\lambda}} , \tag{13}$$

where the denominator is the probability of obtaining $\hat{\rho}_f$, which is $p(\hat{\rho}_f)$. If we define $\left(\rho_{\mathrm{p}}^{\mathrm{out}}\right)_{\mu\nu} \equiv \dfrac{c_\mu c_\nu^* (A)_{\mathrm{m}}^{\mu,\nu|\lambda}}{p(\hat{\rho}_f)}$, then Eqs.(12) and (13) just show $\hat{\rho}_{\mathrm{p}}^{\mathrm{out}} = \sum_{\mu,\nu} \left(\rho_{\mathrm{p}}^{\mathrm{out}}\right)_{\mu\nu} |\eta_\mu\rangle\langle\eta_\nu|$. In this form, we can see that the indicators μ and ν indicate the elements of the pointer density matrix outcome.

As we mentioned in the introduction, the standard modular value and the standard weak value are related even for an arbitrarily large coupling g, that allows us to obtain the weak value from the modular value. Here we show that this relation is still valid in the generalized case. Let us illustrate this for cases (**ii**) and (**iii**), ignoring the trivial case (**i**). Following Ref. [3], for the two-dimensional nondegenerate case, the corresponding exponential term $e^{-ig\hat{A}}$ is given by

$$e^{-ig\hat{A}} = \Lambda\hat{I} + \Lambda'\hat{A}, \tag{14}$$

where $\Lambda \equiv \frac{\lambda_1 e^{-ig\lambda_2}-\lambda_2 e^{-ig\lambda_1}}{\lambda_1-\lambda_2}$, and $\Lambda' \equiv \frac{e^{-ig\lambda_1}-e^{-ig\lambda_2}}{\lambda_1-\lambda_2}$, with λ_1 and λ_2 are two distinct eigenvalues of the operator $\hat{A}$ [3]. Inserting Eq. (14) into Eq. (10), we have

$$(A)_{\rm m}^{\lambda,\nu|\lambda} = \Lambda + \Lambda' \langle A\rangle_{\rm w}, \tag{15}$$

where we used $\langle A\rangle_{\rm w} = \frac{Tr_{\rm s}[\hat{\rho}_f \hat{A}\hat{\rho}_i]}{Tr_{\rm s}[\hat{\rho}_f\hat{\rho}_i]}$, which is the *generalized weak value* [17]. Similarly, Eq. (11) gives

$$\begin{aligned}(A)_{\rm m}^{\lambda,\lambda|\lambda} &= \Lambda\Lambda_+ + \Lambda'\Lambda_+\langle A\rangle_{\rm w} \\ &+ \Lambda\Lambda'_1(\langle A\rangle_{\rm w})^* + \Lambda'\Lambda'_+|\langle A\rangle_{\rm w}|^2,\end{aligned} \tag{16}$$

where we also used $\Lambda_+ \equiv \frac{\lambda_1 e^{ig\lambda_2}-\lambda_2 e^{ig\lambda_1}}{\lambda_1-\lambda_2}$, and $\Lambda'_+ \equiv \frac{e^{ig\lambda_1}-e^{ig\lambda_2}}{\lambda_1-\lambda_2}$. The last term in Eq. (16) can be viewed as a *generalized high-order weak value* [17].

Notable, for higher-dimensional Hilbert space, the relation between the generalized modular value and the generalized weak value is still valid, which allows us to measure the generalized weak value and generalized high-order weak values from the generalized modular value with an arbitrarily coupling constant g.

We now show the usage of the generalized modular value in the pointer, which gives the attainable outcomes. First, it can be used to express the conditional probabilities. The R.H.S. of Eq. (4) is rewritten, using Eqs.(7), (9), as

$$p(\mu|\hat{\rho}_f) = \frac{|c_\mu|^2 (A)_{\rm m}^{\mu,\mu|\lambda}}{1-|c_\lambda|^2+|c_\lambda|^2(A)_{\rm m}^{\lambda,\lambda|\lambda}}\ . \tag{17}$$

Clearly the conditional probabilities satisfy $\sum_\mu p(\mu|\hat{\rho}_f) = 1$.

Since these conditional probabilities appear in Eqs.(5) and (6), this means that the generalized modular value also characterizes the expectation value of an arbitrary operator $\hat{O}_{\rm p}$ and its average displacement $\Delta\langle\hat{O}_{\rm p}\rangle$.

IV. APPLICATIONS

In this section, we apply our proposal to the cases of a spin-s particle pointer and a semiclassical pointer state.

A. Spin-s particle pointer

We first consider the spin-s particle pointer. We deal with the three quantities explained above, i.e., the conditional probability, the expectation value, and the average displacement of an arbitrary operator of the pointer. We also examine the signal-to-noise ratio (SNR) of spin operator $\hat{S}^z$ to discuss the enhancement of the signal-to-noise ratio. We denote the eigenvalue of $\hat{S}^z$ as k (k is an integer or a half-integer with the natural unit $\hbar = 1$), and its maximum value as s. So, the spin state is denoted as usual as $|s,k\rangle$, where s takes values 0, $\frac{1}{2}$, 1, $\frac{3}{2}$,..., which corresponds to the $(2s+1)$-dimensional Hilbert space, and k takes values $-s, -s+1, \cdots, s$, for a fixed s [18]. Hereafter, we omit the trivial case of $s=0$. For example, in the qubit pointer case, where $s=\frac{1}{2}$, the initial pointer state is chosen as $|\xi\rangle = \gamma\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{1-\gamma^2}\left|\frac{1}{2},\frac{1}{2}\right\rangle$. In our case, the initial pointer state is chosen (for a fixed s) as

$$|\xi\rangle = \frac{\gamma}{\sqrt{2s}}\sum_{k=-s}^{s-1}|s,k\rangle + \sqrt{1-\gamma^2}|s,s\rangle\ . \tag{18}$$

For simplicity, we assume that γ is real ($0 \le \gamma \le 1$). Also, similar to the qubit pointer case, where the projection operator is chosen as $\hat{P} = \left|\frac{1}{2},\frac{1}{2}\right\rangle\left\langle\frac{1}{2},\frac{1}{2}\right|$, the projection operator in our case is selected to be $\hat{P} = |s,s\rangle\langle s,s|$; i.e., $|\eta_\lambda\rangle = |s,s\rangle$.

The corresponding conditional probabilities Eq. (17) for the outcomes $\mu = s$ and $\mu \ne s$ become

$$p(\mu = s|\hat{\rho}_f) = \frac{(1-\gamma^2)(A)_{\rm m}^s}{\gamma^2+(1-\gamma^2)(A)_{\rm m}^s}, \tag{19a}$$

$$p(\mu \ne s|\hat{\rho}_f) = \frac{\gamma^2}{2s[\gamma^2+(1-\gamma^2)(A)_{\rm m}^s]}, \tag{19b}$$

where we have introduced for short the symbol $(A)_{\rm m}^s \equiv (A)_{\rm m}^{s,s|s}$. For the purpose of comparison between the probabilities after and before the interaction in the pointer, we calculate the probability displacement $\Delta p(\mu|\hat{\rho}_f) \equiv p(\mu|\hat{\rho}_f) - |c_\mu|^2$, which gives

$$\Delta p(\mu = s|\hat{\rho}_f) = \frac{\gamma^2(1-\gamma^2)[(A)_{\rm m}^s - 1]}{\gamma^2+(1-\gamma^2)(A)_{\rm m}^s}, \tag{20a}$$

$$\Delta p(\mu \ne s|\hat{\rho}_f) = \frac{\gamma^2(1-\gamma^2)[1-(A)_{\rm m}^s]}{2s[\gamma^2+(1-\gamma^2)(A)_{\rm m}^s]}. \tag{20b}$$

Obviously, we can see that $\Delta p(\mu = s|\hat{\rho}_f)$(20a) is positive when $(A)_{\rm m}^s > 1$, and negative when $(A)_{\rm m}^s < 1$. The behavior for $\Delta p(\mu \ne s|\hat{\rho}_f)$ (20b) is opposite.

For simplicity, we also assume that the system is chosen to be a spin-1/2 particle and described by the pure pre- and postselected states

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big), \tag{21}$$

$$|\phi\rangle = \frac{1}{\sqrt{2\epsilon^2-2\epsilon+1}}\Big(\epsilon|\uparrow\rangle - (\epsilon-1)|\downarrow\rangle\Big), \tag{22}$$

where we have used the spin orientations $\uparrow$ (up) and $\downarrow$ (down) for z-direction. It should be noted that the choice of the system is not relevant to the pointer, where the dimension can be chosen arbitrarily. We then also choose $\hat{A} \equiv \hat{S}^z = \frac{1}{2}\hat{\sigma}_z$ to be the system observable and $g = \pi$. A straightforward calculation the modular value Eq. (11) gives $(A)_{\rm m}^s = (2\epsilon - 1)^2$. To change the modular value, e.g., from 0 to 9, we vary the parameter ϵ from 0.5 to 2.0 as shown in the Insert Fig. 1. Notable that $|\phi\rangle = |\psi\rangle$ when $\epsilon = 0.5$, and they are orthogonal when $\epsilon \to \pm\infty$.

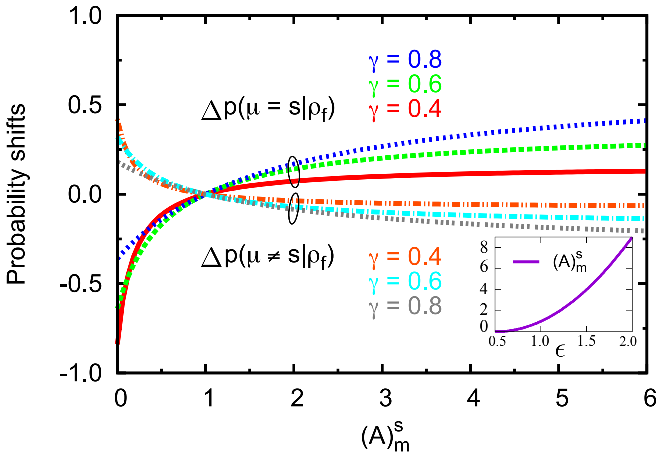

FIG. 1. (Color online) The probability displacements $\Delta p(\mu = s|\hat{\rho}_f)$ and $\Delta p(\mu \neq s|\hat{\rho}_f)$ in Eqs. (20) as functions of $(A)^s_{\rm m}$ for $\gamma = 0.4$, 0.6 and 0.8, and $s = 1$, i.e., spin-1 pointer. Insert: the modular value of the system observable $\hat{\sigma}_z/2$ varies as a function of ϵ, where the quantum system states are chosen as in Eqs. (21, 22), also we fix the value of $g = \pi$.

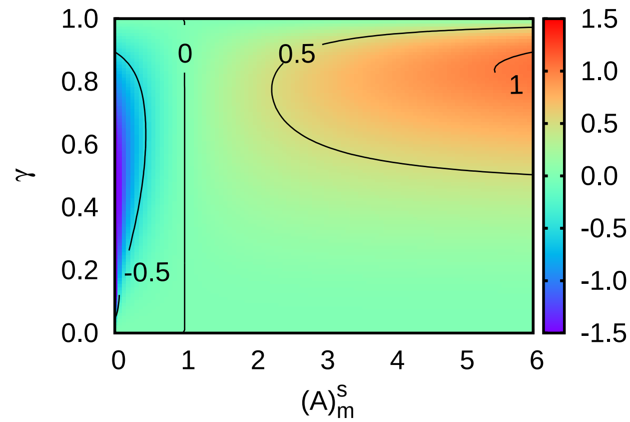

FIG. 2. (Color online) Contour plot of the average displacement in the measured value of the pointer observable $\hat{S}^z_{\rm p}$ of spin-2 ($s = 2$) particle. The vertical line at $(A)^s_{\rm m} = 1$ indicates the value zero, which means no "displacement". The quantum system is chosen the same as in Fig. 1.

The main Fig. 1 shows the results of $\Delta p(\mu = s|\hat{\rho}_f)$ and $\Delta p(\mu \neq s|\hat{\rho}_f)$ in Eqs. (20) as functions of $(A)^s_{\rm m}$ for $\gamma = 0.4$, 0.6 and 0.8. Here we assumed a three-level pointer, that is, $s = 1$. In general, with increasing $(A)^s_{\rm m}$ from 1, the probability displacements $\Delta p(\mu = s|\hat{\rho}_f)$ smoothly rise, while the probability displacements $\Delta p(\mu \neq s|\hat{\rho}_f)$ gradually descend. In other words, the modular value plays a significant role in the probability displacements. It can be used to design the measurement interaction to increase the probability of getting the desired outcome (i.e., getting $|s, s\rangle$ after the interaction).

We next examine the expectation value and the average displacement in the measured value of the pointer observable $\hat{O}_{\rm p}$. For illustration, let us choose $\hat{S}^z_{\rm p}$ as the observable whose eigenvalues are $k = -s, ..., s$. The expectation value, Eq.(5), straightforwardly gives

$$\langle \hat{S}^z_{\rm p} \rangle = \frac{2s(1-\gamma^2)(A)^s_{\rm m} - \gamma^2}{2[\gamma^2 + (1-\gamma^2)(A)^s_{\rm m}]}\ , \tag{23}$$

where we have used the expressions in Eq. (19). The average displacement in the pointer observable is given by Eq. (6), being

$$\begin{aligned}\Delta\langle \hat{S}^z_{\rm p} \rangle &= \sum_k k\Delta p(k|\hat{\rho}_f)\\ &= \frac{(2s+1)\gamma^2(1-\gamma^2)[(A)^s_{\rm m} - 1]}{2[\gamma^2 + (1-\gamma^2)(A)^s_{\rm m}]}\ .\end{aligned} \tag{24}$$

Fig. 2 presents the result of the average displacement of the pointer observable $\hat{S}^z_{\rm p}$ of spin-2 ($s = 2$) particle. It shows that, for $(A)^s_{\rm m} = 1$ (indicated by the vertical line), the average displacement of $\hat{S}^z_{\rm p}$ is 0 regardless of the γ value. The figure also shows that, by increasing (or decreasing) $(A)^s_{\rm m}$, the amount of displacement can be made large toward positive (or negative) sign direction, and the effect of the increase can be made even greater by choice of γ. Obviously, this tendency can be seen in Eqs. (6) and (24). This effect can be viewed as the amplification effect in postselected modular-value measurement. This amplification effect has been extensively studied in weak-value measurement both theoretically [9, 13, 19–22] and experimentally [23, 24], but still lack in the context of modular-value measurement. Here we first show the existence of this effect in the above example. It is worthy to note that in this example, $\hat{S}^z_{\rm p}$ does not play a major role to the amplification effect. Instead, the effect might appear for any pointer observable as we showed in Eq. (6) with a suitable choice of pre- and postselected states.

Interestingly, all the above results depend on s, which means that the amplification effect depends on the dimension of the pointer Hilbert space $2s+1$. For more investigation regarding the dimension, we next investigate the signal-to-noise ratio (SNR), which is defined by the ratio between the expectation value $\langle \hat{S}^z_{\rm p} \rangle$ and the square root of the variance $\sqrt{{\rm Var}(\hat{S}^z_{\rm p})}$, as follows [25]:

$${\rm SNR} = \frac{\langle \hat{S}^z_{\rm p} \rangle}{\sqrt{{\rm Var}(\hat{S}^z_{\rm p})}}, \tag{25}$$

where the variance ${\rm Var}(\hat{S}^z_{\rm p})$ is defined and given by

$${\rm Var}(\hat{S}^z_{\rm p}) \equiv \langle [\hat{S}^z_{\rm p}]^2 \rangle - \langle \hat{S}^z_{\rm p} \rangle^2, \tag{26}$$

where

$$\langle [\hat{S}^z_{\rm p}]^2 \rangle = \frac{\gamma^2 \left[s^2 + 2s^3(1-\gamma^2)(A)^s_{\rm m} + \displaystyle\sum_{k=-s+1}^{s-1} k^2 \right]}{2s[\gamma^2 + (1-\gamma^2)(A)^s_{\rm m}]}, \tag{27}$$

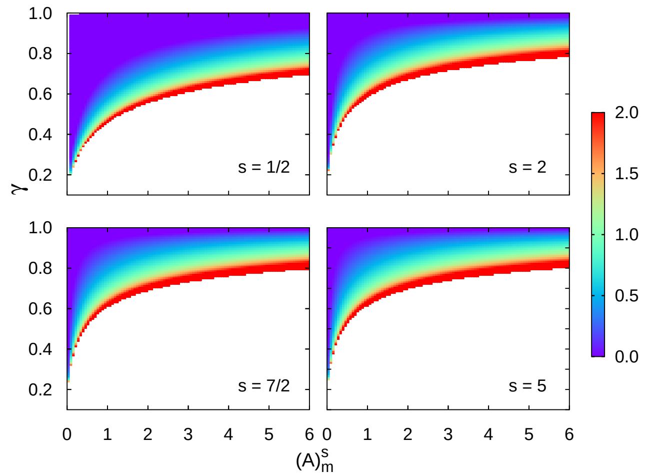

FIG. 3. (Color online) The SNRs are shown as functions of $(A)^s_{\rm m}$ and γ for $s = \frac{1}{2}, 2, \frac{7}{2}$ and 5 as can be seen in each panel. The quantum system is chosen the same as in Fig. 1.

and $\langle \hat{S}^z_{\rm p}\rangle^2$ is given in Eq. (23). We remind that $\langle ...\rangle$ denotes an expectation value for the final pointer state throughout this paper. Fig. 3 represents the signal-to-noise ratio (SNR) for $s = \frac{1}{2}, 2, \frac{7}{2}$ and 5. It shows that the SNR increase significantly for the larger s cases.

B. Semiclassical pointer state

In this subsection, we will illustrate the usage of our model to the case of the semiclassical pointer state. Here, the initial state of the pointer is a coherent state of bosons as [26]

$$|\xi\rangle \equiv |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle. \tag{28}$$

The system-pointer interaction is given as $\hat{H} = g(t)\hat{A}\otimes|n\rangle\langle n|$, where $|n\rangle$ is a specifically chosen number state for the pointer, i.e., $|\eta_\lambda\rangle \equiv |n\rangle$. After the interaction, we postselect the system state $\hat{\rho}_f$ and measure the boson number of the pointer and select the case that the final state is $|n\rangle$. So, the outcome μ is chosen to be n, which will be seen in Eq.(30).

Before the interaction, the probability of finding the number n is given by the Poisson distribution:

$$p(n) = |\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\ , \tag{29}$$

but after the interaction, the conditional probability of finding the boson number n is given in Eq. (17) as

$$p(n|\hat{\rho}_f) = \frac{p(n)(A)^n_{\rm m}}{1-p(n)+p(n)(A)^n_{\rm m}}\ , \tag{30}$$

where $(A)^n_{\rm m}$ stands for $(A)^{\mu,\nu|\lambda}_{\rm m}$ with $\mu = \nu = \lambda = n$. In this way, the conditional probability is expressed by the generalized modular value even in this semiclassical pointer-state case.

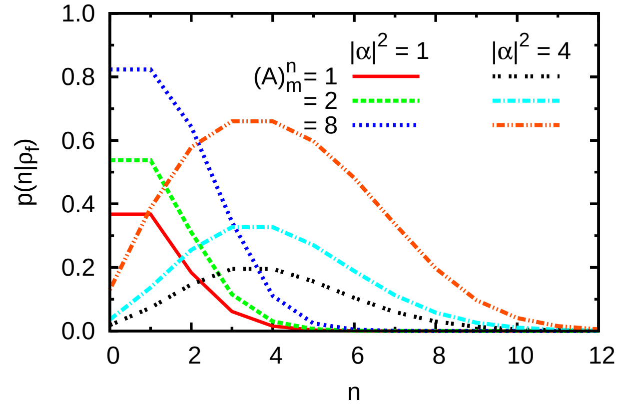

FIG. 4. (Color online) The conditional probability-versus-n curves with different modular values $(A)^n_{\rm m}$ and $|\alpha|^2$. All curves show the increasing with $(A)^n_{\rm m}$ for each $|\alpha|^2$. This phenomenon can be regarded as the amplification effect of the modular value. The quantum system can be chosen the same as in Fig. 1, i.e., interactions between spin and photon.

Eq.(30) can be used, by designing the pre-/postselected states, to increase the measurement signal, e.g., the conditional probability. When the modular value takes 1, Eq.(30) tells that $p(n|\hat{\rho}_f) = p(n)$. Now there is a possibility to increase $p(n|\hat{\rho}_f)$ by changing the value of $(A)^n_{\rm m}$ [27]. To see this, we plotted the conditional probabilities for different modular values $(A)^n_{\rm m}$ and different values of $|\alpha|^2$ in Fig. 4. The results apparently tell that we can increase the conditional probability by increasing the modular value. Furthermore, we predict that out study can be applied to various kind of nonclassical pointer states such as squeezed vacuum state, and Schrödinger cat state.

V. CONCLUSIONS

We have analyzed a proposed "generalized modular-value" through generalizing the two-level pointer to multi-level pointer in a system-pointer measurement scheme. We have shown that the conditional probabilities of the pointer outcomes are naturally expressed by the generalized modular value. We have also calculated the expectation value of an arbitrary pointer observable and the average displacement in the measured values of the pointer observable in the context of the generalized modular value. We have applied our proposal to the case of a spin-s particle pointer and a semiclassical pointer state. In the first instance, we have found that the generalized modular value can be used to analyze the amplification effect in postselected modular-value measurement. We also derived that the signal-to-noise ratio (SNR) can be increased by increasing the dimension of pointer Hilbert space. In the later case, we have also analyzed the case where the coherent state of bosons is used as the initial state of the pointer. In this case, we found the effect of modular-value amplification in the probability of finding the boson number n. It should be noted that our proposal presented here is a generalized case of the pointer state. So, it might serve as an argument for further studies on nonclassical pointer states.

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Appendix A: Joint probability and the expectation value

We first recall the definition of the joint state after the interaction:

$$\begin{aligned}\hat{\rho}'_{\mathrm{sp}} &= \hat{U}_{\mathrm{sp}}\hat{\rho}_{\mathrm{sp}}\hat{U}^{\dagger}_{\mathrm{sp}} \\ &= \hat{U}_{\mathrm{sp}}(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|)\hat{U}^{\dagger}_{\mathrm{sp}} \ . \end{aligned} \tag{A.1}$$

Substituting this expression into the joint probability, we have Eq.(2) as

$$\begin{aligned}p(\mu,\hat{\rho}_f) &= \mathrm{Tr}_{\mathrm{sp}}[(\hat{\rho}_f \otimes |\eta_\mu\rangle\langle\eta_\mu|)\hat{\rho}'_{\mathrm{sp}}] \\ &= \mathrm{Tr}_{\mathrm{sp}}[(\hat{\rho}_f \otimes |\eta_\mu\rangle\langle\eta_\mu|)\hat{U}_{\mathrm{sp}}(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|)\hat{U}^{\dagger}_{\mathrm{sp}}] \\ &= \mathrm{Tr}_{\mathrm{s}}[\hat{\rho}_f \cdot \mathrm{Tr}_{\mathrm{p}}[(\hat{I}_{\mathrm{s}} \otimes |\eta_\mu\rangle\langle\eta_\mu|)\hat{U}_{\mathrm{sp}}(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|)\hat{U}^{\dagger}_{\mathrm{sp}}]] \\ &= \mathrm{Tr}_{\mathrm{s}}[\hat{\rho}_f \cdot \langle\eta_\mu|\hat{U}_{\mathrm{sp}}|\xi\rangle\hat{\rho}_i\langle\xi|\hat{U}^{\dagger}_{\mathrm{sp}}|\eta_\mu\rangle \\ &= \mathrm{Tr}_{\mathrm{s}}[\hat{\rho}_f\hat{M}_\mu\hat{\rho}_i\hat{M}^{\dagger}_\mu] \ , \end{aligned} \tag{A.2}$$

where the operator $\hat{M}_\mu = \langle\eta_\mu|\hat{U}_{\mathrm{sp}}|\xi\rangle$ represents the Kraus operator.

It is straightforward to calculate the probability $p(\hat{\rho}_f)$ by taking the sum of all μ's for Eq.(A.2), resulting in $p(\hat{\rho}_f) = \mathrm{Tr}_{\mathrm{s}}[\hat{\rho}_f\hat{\rho}'_i]$, where $\hat{\rho}'_i$ is calculated, by tracing out the pointer Hilbert space of Eq. (A.1), to be

$$\begin{aligned}\hat{\rho}'_i &= \mathrm{Tr}_{\mathrm{p}}[\hat{U}_{\mathrm{sp}}(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|)\hat{U}^{\dagger}_{\mathrm{sp}}] \\ &= \sum_\mu \hat{M}_\mu\hat{\rho}_i\hat{M}^{\dagger}_\mu \ . \end{aligned} \tag{A.3}$$

The expectation value of an arbitrary observable in the pointer gives

$$\langle\hat{O}_{\mathrm{p}}\rangle = \frac{\mathrm{Tr}_{\mathrm{sp}}[(\hat{\rho}_f \otimes \hat{O}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}{\mathrm{Tr}_{\mathrm{sp}}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}. \tag{A.4}$$

We now insert $\hat{O}_{\mathrm{p}} = \sum_\kappa o_\kappa|\eta_\kappa\rangle\langle\eta_\kappa|$ into the numerator and $\hat{I}_{\mathrm{p}} = \sum_{\kappa'}|\eta_{\kappa'}\rangle\langle\eta_{\kappa'}|$ into the denominator, and then perform calculations as in (A.2). The result is Eq.(5).

Appendix B: Kraus operator

Consider the basis $\{|\eta_\kappa\rangle, \kappa = 0,1,2,...,d-1\}$, where d denotes the dimension of the discrete pointer Hilbert space $\mathcal{H}_{\mathrm{p}}$. The initial pointer state can be expressed in the form:

$$|\xi\rangle = \sum_\kappa c_\kappa|\eta_\kappa\rangle \ , \qquad c_\kappa = \langle\eta_\kappa|\xi\rangle \ , \tag{B.1}$$

and the projection operator $\hat{P} = |\eta_\lambda\rangle\langle\eta_\lambda|$ can be explicitly expressed as

$$\begin{aligned}\hat{P} &= \sum_\kappa \eta_\kappa|\eta_\kappa\rangle\langle\eta_\kappa| \quad \text{ with} \begin{cases}\eta_\kappa = 1 & \text{if } \kappa = \lambda \\ \eta_\kappa = 0 & \text{if } \kappa \neq \lambda\end{cases} \\ &= \sum_\kappa \delta_{kl}|\eta_\kappa\rangle\langle\eta_\kappa| \ . \end{aligned} \tag{B.2}$$

Now, the action of the unitary operator $\hat{U}_{\mathrm{sp}}$ on the initial pointer $|\xi\rangle$ can be characterized as follows [28]:

$$\begin{aligned}\hat{U}_{\mathrm{sp}}|\xi\rangle &= \sum_\kappa e^{-ig\hat{A}\delta_{\kappa\lambda}}\langle\eta_\kappa|\xi\rangle|\eta_\kappa\rangle \\ &= \sum_\kappa c_\kappa|\eta_\kappa\rangle e^{-ig\hat{A}\delta_{\kappa\lambda}} \ . \end{aligned} \tag{B.3}$$

Then, the Kraus operator is calculated as

$$\begin{aligned}\hat{M}_\mu &= \langle\eta_\mu|\hat{U}_{\mathrm{sp}}|\xi\rangle \\ &= \sum_\kappa c_\kappa\langle\eta_\mu|\eta_\kappa\rangle e^{-ig\hat{A}\delta_{\kappa\lambda}} \\ &= \sum_\kappa c_\kappa\delta_{\mu\kappa}e^{-ig\hat{A}\delta_{\kappa\lambda}} \\ &= c_\mu e^{-ig\hat{A}\delta_{\mu\lambda}}, \end{aligned} \tag{B.4}$$

where $c_\mu = \langle\eta_\mu|\xi\rangle$.

Appendix C: The final state of the pointer

The final state of the pointer is given as

$$\hat{\rho}^{\mathrm{out}}_{\mathrm{p}} = \frac{\mathrm{Tr}_{\mathrm{s}}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}{\mathrm{Tr}_{\mathrm{sp}}[(\hat{\rho}_f \otimes \hat{I}_{\mathrm{p}})\hat{\rho}'_{\mathrm{sp}}]}. \tag{C.1}$$

In the numerator, let us insert $\sum_\mu|\eta_\mu\rangle\langle\eta_\mu|(=\hat{I}_{\mathrm{p}})$ and $\sum_\nu|\eta_\nu\rangle\langle\eta_\nu|(=\hat{I}_{\mathrm{p}})$, then we have

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{s}}\Big[\Big(\hat{\rho}_f \otimes \sum_\mu|\eta_\mu\rangle\langle\eta_\mu|\Big)\hat{U}_{\mathrm{sp}}\Big(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|\Big)\hat{U}^{\dagger}_{\mathrm{sp}}\sum_\nu|\eta_\nu\rangle\langle\eta_\nu|\Big] \\ &= \sum_{\mu,\nu} c_\mu c^*_\nu \mathrm{Tr}_{\mathrm{s}}\big[\hat{\rho}_f\hat{M}_\mu\hat{\rho}_i\hat{M}^{\dagger}_\nu\big]|\eta_\mu\rangle\langle\eta_\nu|. \end{aligned} \tag{C.2}$$

Similarly, in the denominator, we insert $\sum_\mu|\eta_\mu\rangle\langle\eta_\mu|$, and then trace out the pointer Hilbert space, which leads to

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{sp}}\Big[\Big(\hat{\rho}_f \otimes \sum_\mu|\eta_\mu\rangle\langle\eta_\mu|\Big)\hat{U}_{\mathrm{sp}}\Big(\hat{\rho}_i \otimes |\xi\rangle\langle\xi|\Big)\hat{U}^{\dagger}_{\mathrm{sp}}\Big] \\ &= \sum_\mu |c_\mu|^2\mathrm{Tr}_{\mathrm{s}}\big[\hat{\rho}_f\hat{M}_\mu\hat{\rho}_i\hat{M}^{\dagger}_\nu\big]. \end{aligned} \tag{C.3}$$

After that, we divide both the numerator and denominator by the non-zero factor $\mathrm{Tr_s}[\hat{\rho}_f\hat{\rho}_i]$, and use the definition of modular value Eq. (9). Then we finally obtain

$$\hat{\rho}_{\mathrm{p}}^{\mathrm{out}} = \frac{\sum_{\mu,\nu} c_\mu c_\nu^* (A)_{\mathrm{m}}^{\mu,\nu|\lambda} |\eta_\mu\rangle\langle\eta_\nu|}{\sum_\mu |c_\mu|^2 (A)_{\mathrm{m}}^{\mu,\mu|\lambda}} , \tag{C.4}$$

which is Eq.(13).

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