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# Enhancing the optical cross section of quantum antenna

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**Abstract:** The classical radio frequency antenna theory indicates that large cross sections can be realized through directional radiation. In this paper, a similar principle is applied in quantum systems, in which quantum antennas, constructed by a cluster of quantum two-level systems, explore the collective excitation of two-level systems to realize large directivity. Both the optical cross section and the coherent time can be dramatically enhanced in free space, far exceeding the case of a single two-level system.

Emerging applications such as quantum information and low-power optical switching require strong interactions between single photons and quantum two-level systems (TLSs). One common technique to realize strong interactions is to confine photons in low dimensional spaces such as cavities or waveguides [1-3]. It leads to the tremendous success of cavity (waveguide) quantum electrodynamics [2,4-10]. On the other hand, it has been more difficult to achieve strong interactions in free space [11,12]. In the three-dimensional (3D) free space, the interaction can be characterized by the optical cross section [13-17]. The larger the cross section is, the easier it is for a photon to couple to the TLS [18,19]. However, it can be easily shown that the maximum cross section of a TLS with an electric dipole transition is  $\sigma_d = 3\lambda^2/2\pi$  with  $\lambda$  being the transition wavelength [13,15,20]. The small size limits effective coupling in free space.

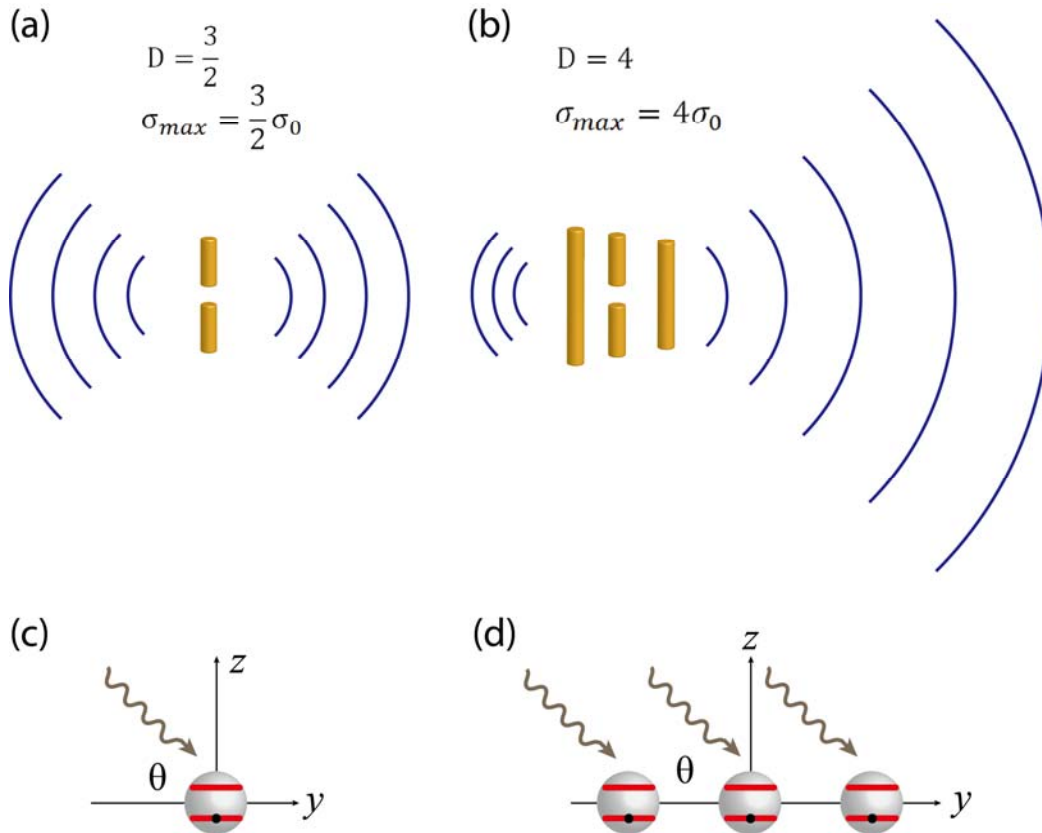


Fig.1(color online) (a) A classical dipole antenna made of metallic rods (b) A Yagi-Uda antenna with three elements. The cross section increases as a result of the increased directivity of the radiation pattern. (c) The simplest quantum antenna is a TLS in the free space. (d) Three TLSs form an antenna with enhanced directivity and cross section.

Large cross sections have been pursued for decades in the field of radio-frequency (RF) communication [17,20-22]. The goal is for a small circuit to capture free-space electromagnetic waves of an area as large as possible. The concept of RF antenna

introduces a key idea: the cross section can be enhanced by directional radiation [23-25]. The directivity of an antenna is defined by the normalized radiation pattern  $D(\theta, \phi) = 4\pi U(\theta, \phi) / \iint U(\theta, \phi) d\Omega$ , where  $U(\theta, \phi)$  is antenna's power radiated to the direction of the polar  $\theta$  and the azimuthal  $\phi$  angles. Classical antenna theory [26] shows that the maximum cross section  $\sigma_{max}$  is directly related to the maximum directivity  $D_{max} \equiv \max(D(\theta, \phi))$  by:

$$\sigma_{max} = \frac{\lambda^2}{\pi} D_{max} . \quad (1)$$

For an isotropic antenna,  $D_{max} = 1$  and the maximum cross sections is  $\sigma_0 = \lambda^2/\pi$ . A dipole antenna has zero emission along the axial direction of the dipole and thus has a higher directivity of  $D_{max} = 3/2$  (Fig. 1a) [26]. Consequently, its maximum cross section is slightly enhanced to  $3\sigma_0/2$ . In practice, sophisticated antennas [27-32] are designed to further enhance the directivity, such as the Yagi-Uda antenna [33-35] shown in Fig. 1b. It emits more strongly in one direction than the others. The maximum cross section can be further enhanced by several times than the dipole antenna.

Here we show that theorems and techniques developed in classical antenna [36-39] are equally effective in quantum antenna. A quantum antenna can be as simple as a single TLS as shown in Fig. 1c, or a complex cluster of TLSs as shown in Fig. 1d. Enhanced directivity allows quantum antennas to gain cross section as well as the coherent time, often orders of magnitudes larger than those of a single TLS. Most importantly, all the enhancement is realized in free space. Large cross sections combined with long coherent time could be used for efficient interfacing of flying and stationary qubits in free-space network.

We start by using quantum scattering theory to prove Eq. (1) for a quantum antenna. The antenna is represented by a nondegenerate resonance with a ground state  $|g\rangle$  and an excited state  $|e\rangle$ . These states can be realized in single atoms, quantum dots, or in complex systems such as the collective excitation states [40-42] in a cluster of atoms and quantum dots. The antenna's full Hamiltonian [43] can be casted as

$$H = \hbar\omega_0\sigma^\dagger\sigma + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + H_I . \quad (2)$$

The first two terms are the free Hamiltonians of the antenna and photons, respectively. Here  $\sigma^\dagger = |e\rangle\langle g|$  and  $\sigma = |g\rangle\langle e|$  are the usual raising and lowering atomic operators. The creation (annihilation) operator for photons is  $c_{\mathbf{k}}^\dagger$  ( $c_{\mathbf{k}}$ ).  $\omega_{\mathbf{k}}$  and  $\mathbf{k}$  are the angular frequency and the wavevector, respectively.  $\hbar$  is the reduced Planck constant. The third term  $H_I$  describe the photon-antenna interaction, the specific

form of which is not needed for proving Eq. (1).

The optical cross section can be calculated from a scattering process. The initial state is  $|g, \mathbf{k}_i\rangle$ , with the antenna in its ground state  $g$  and a single photon incident from a particular direction  $\mathbf{k}_i(\theta_i, \phi_i)$ . During the transient process of photon-antenna interaction, the antenna is partially excited while generating emission at the same time. After the interaction, the antenna eventually relaxes back to the ground state and the photon is scattered to many directions, represented by the final state  $|g, \mathbf{k}_f\rangle$ .

The details of our quantum scattering theory is available in Appendix A. Unlike many approach that treat the incident light as a semi-classical field [44], we use Fock states as the basis to solve the transport properties of light scattered by a cluster of TLSs in free space. A non-perturbative method [45] is adopted from waveguide quantum electrodynamics [46-49] to calculate the exact eigenstates of the open system, from which the scattering matrix and the cross sections can be obtained. Based on this theory, we show that the cross section follows a conservation law:

$$\int \int \sigma(\theta, \phi) d\Omega = 4\lambda^2, \quad (3)$$

It shows that the total optical cross section is fundamentally limited. But it also indicates that the cross section can be enhanced for certain directions if other directions can be suppressed. We can define the directivity of the cross section as  $\mathbb{D}(\theta, \phi) = 4\pi\sigma(\theta, \phi) / \int \int \sigma(\theta, \phi) d\Omega$ . Then Eq. (3) can be written as  $\sigma(\theta, \phi) = \frac{\lambda^2}{\pi} \mathbb{D}(\theta, \phi)$ , which is almost identical to Eq. (1) except that the directivity in Eq. (1) is defined for the radiation pattern. In Appendix B, we prove that the directivity is the same for the radiation pattern, the cross section and differential cross section.

We can easily verify Eq. (1) by considering the directivity and cross section of a dipole-transition TLS. However, in order to illustrate Eq. (1) in complex quantum antennas, we will study a cluster of TLSs. An example of three identical TLSs spaced by  $0.075\lambda$  is shown in Fig. 1d. These TLSs are closely packed within a subwavelength space. The small spacing induces strong interactions, resulting in collective excitation that gives rise to complex directivity unattainable in single TLS. We will explicitly calculate the directivity, cross section, and the lifetime in these quantum antennas. The results not only verify Eq. (1) but also illustrate important ways of enhancing the light-TLS interaction in free space.

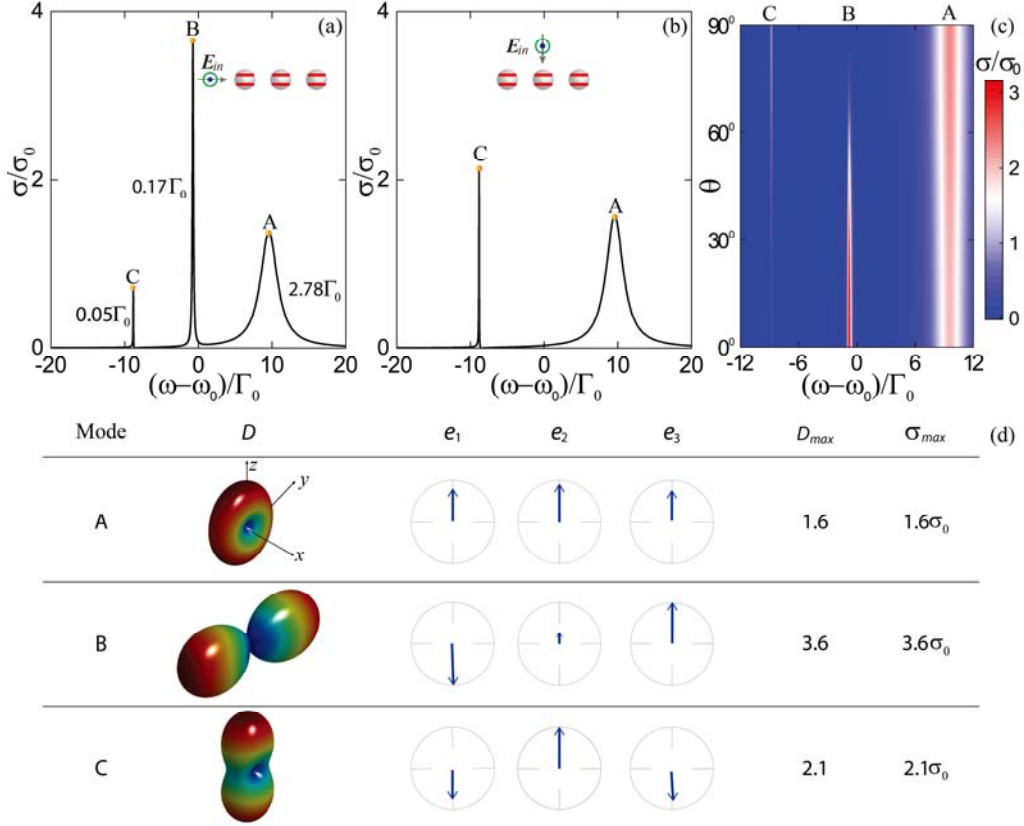


Fig.2(color online) The cross section of the photon incident angles with (a)  $\theta = 0$  and (b)  $\theta = 90$ . (c) The cross section spectra for incident angles from 0 to 90. (d) The cross section  $\sigma(\theta, \phi)$  and directivities of three collective modes. The central column shows the complex excitation amplitudes of the constituent TLSs. The relative magnitude and phase of the complex excitation amplitudes  $e_i$  are represented by the length and the direction of the blue arrows.

The free Hamiltonian for three TLSs of the antenna is  $\omega_0 \sum_{m=1}^3 \sigma_m^\dagger \sigma_m$ . The interaction Hamiltonian is  $H_I = i\hbar \sum_{m=1}^3 \sum_{\mathbf{k}} [V_{\mathbf{k}}(\mathbf{r}_m) c_{\mathbf{k}}^\dagger \sigma_m - V_{\mathbf{k}}^*(\mathbf{r}_m) c_{\mathbf{k}} \sigma_m^\dagger]$ , where  $\mathbf{r}_m$  is the position of the  $m^{\text{th}}$  TLS. We assumed dipole transitions for all TLSs and the coupling constant is given by  $V_{\mathbf{k}}(\mathbf{r}_m) = \omega_0 \sqrt{1/(2\hbar\epsilon_0\omega_{\mathbf{k}}L^3)} \mathbf{d}_m \cdot \mathbf{e}_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{r}_m)$ , where  $u_{\mathbf{k}}(\mathbf{r}_m) = e^{i\mathbf{k}\cdot\mathbf{r}_m}$  and  $\mathbf{e}_{\mathbf{k}}$  is the phase and unit polarization vector of electric field of the wave. We have incorporated the polarizations in the  $\mathbf{k}$  summation.  $L^3$  is the normalization volume.

In order to solve the Hamiltonian, we use the scattering theory developed in waveguide quantum electrodynamics [46,47] where the Hamiltonian is converted to a real-space representation. The boundary condition of this open system is specified by the condition of the incident photon and the eigenstates can be explicitly expressed.

Here we further develop this approach for three-dimensional free space. The eigenstate can be written as the summation of photon wave functions in different directions as  $|\phi\rangle = (\sum_{\mathbf{k}_i} \int d\xi \phi_{\mathbf{k}_i}(\xi) c_{\mathbf{k}_i}^\dagger(\xi) + \sum_{m=1}^3 e_m \sigma_m^\dagger) |0, g_1, g_2, g_3\rangle$ , where  $e_m$  is the excitation amplitude of the  $m^{\text{th}}$  TLS.  $\phi_{\mathbf{k}_i}(\xi)$  is the wave-function of the photon in the direction specified by  $\mathbf{k}_i$  with  $\xi$  being the spatial coordinate along the direction.  $c_{\mathbf{k}_i}^\dagger(\xi)$  is the spatial creation operator of a propagating photon at position  $\xi$  in the direction  $\mathbf{k}_i$ . In Appendix A, we show how to use the time-independent Schrodinger equation to solve this scattering problem, from which we can obtain the scattering matrix, photon's wavefunctions, TLSs' excitation amplitudes  $e_m$ , and the cross section.

Figure 2a shows the spectrum of the cross section of the 3-TLS antenna when single photons are incident from  $\theta = 0^\circ$  direction. It shows three collective modes with resonant frequencies shifted away from  $\omega_0$ . The linewidths are also quite different from the natural bandwidth  $\Gamma_0$ . This spectrum strongly depends on the incident angle  $\theta$ : when  $\theta = 90^\circ$ , the central peak completely disappears (Fig. 2b). The angular dependence of the cross section is displayed in Fig. 2c. Three stripes can be identified, corresponding to three collective modes. In Appendix C, we draw the relation between the spherical angles and the dipole moment.

The broadest mode, labeled by A in Fig. 2, is the superradiant mode [50-52]. Its spectral bandwidth is about three times natural linewidth  $\Gamma = 2.78\Gamma_0$ . Since all three TLSs oscillate in phase with nearly equal amplitudes  $e_1 \approx e_2 \approx e_3$  (Fig. 2d), the directivity  $\mathbb{D}(\theta, \phi)$  of the superradiant mode closely resembles that of a single TLS, i.e. the donut-shape directivity, as shown in Fig. 2d. The maximum directivity of  $D_{max} = 1.6$  and the maximum cross section of  $\sigma_{max} = 1.6\lambda^2/\pi$  exactly follow the relation dictated by Eq. 1. As limited by its dipolar directivity, the superradiant mode could not significantly enhance the cross section, regardless of the number of TLSs in the cluster.

In contrast to the superradiant mode-A, the directivities of mode-B and -C are drastically different from the donut shape of a dipolar transition (second and third rows in Fig. 2d). A maximum directivity of  $D_{max} = 3.6$  is observed in mode-B. The increased directivity is caused by the out-of-phase oscillation of three TLSs, whose amplitudes are displayed in Fig. 2d. Destructive interference occurs for radiation in

most directions, resulting in highly directional radiation. Strong directivity directly leads to an enhanced cross section of  $\sigma_{max} = 3.6\lambda^2/\pi$ .

Modes-B and C are also referred to as the sub-radiant modes [50] because the destructive interference significantly decreases the spontaneous emission rates. Consequently, the lifetime is greatly enhanced. For example, a 20-fold enhancement of the lifetime is observed in mode-C as shown by the bandwidth narrowing in Fig. 2a. The longest lifetime generally increases with the number of TLSs in a quantum antenna. An example of 5-TLS antenna to be shown later has a lifetime enhanced by 1143 times. Such super-long lifetime could provide long coherent storage of qubit[53-56].

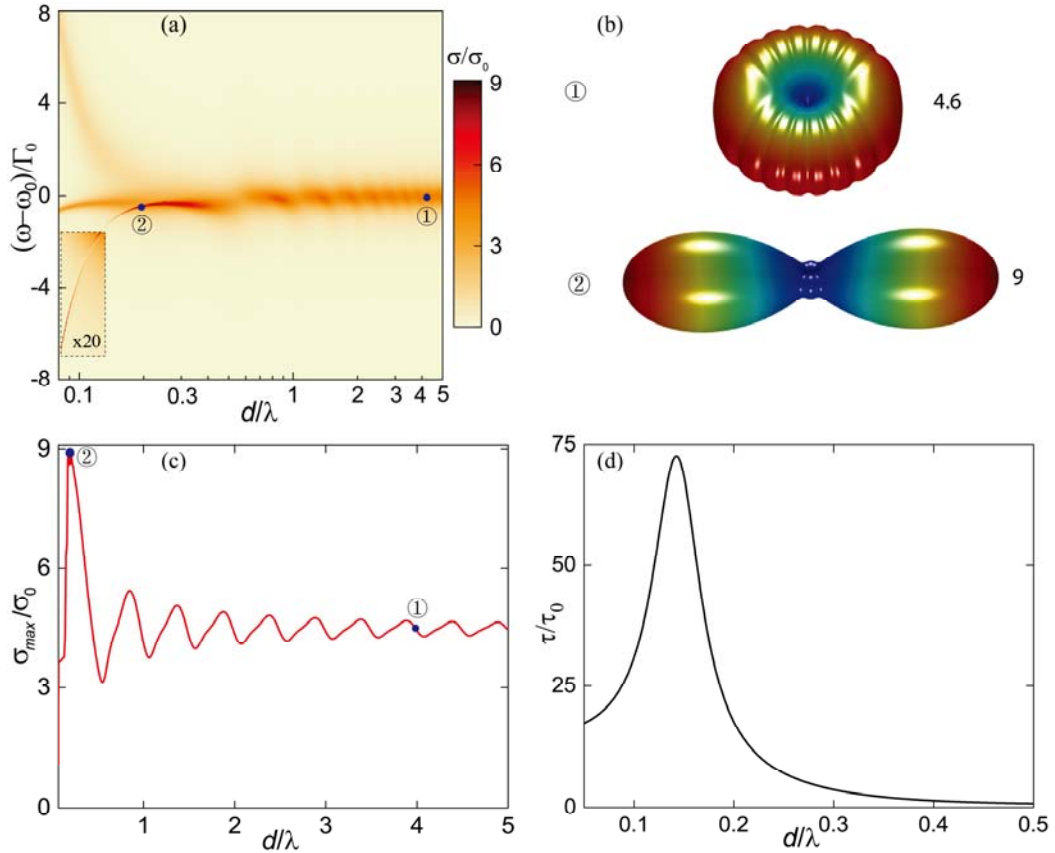


Fig.3(color online) (a) The cross section spectra for different inter-TLS spacing  $d$ . The incident angle is  $\theta = 0$ . The values in the dashed box are multiplied by 20 times for easy visualization. (b) The cross sections  $\sigma(\theta, \phi)$  for inter-TLS distances of  $0.2\lambda$  and  $4\lambda$ , respectively. (c) The maximum cross section  $\sigma_{max}$  at different inter-TLS spacing  $d$ . It converges to the incoherent addition of  $4.5\sigma_0$  when  $d \gg \lambda$ . (d) The lifetime of mode C as a function of the inter-TLS distance  $d$ .

From the analysis of the collective modes, we see that the coupling among TLSs plays a key role for enhancing directivity. A single factor that has the strongest effect on the



coupling is the inter-TLS distance  $d$ . On one hand,  $d$  must be smaller than the wavelength to ensure adequate radiative coupling, a prerequisite for collective modes. On the other hand, when  $d$  is too small, radiative coupling becomes too strong such that it splits the energies of collective modes. The result is only moderate enhancement at three different frequencies.

Figure 3a shows the effect of  $d$  on cross section spectra. At a deep subwavelength  $d < 0.1\lambda$ , three collective modes split far apart, as shown by three dark stripes. Each only has a moderately enhanced cross section in the range of  $1.5\sigma_0 < \sigma_{max} < 4\sigma_0$ . On the other hand, at a large distance  $d \gg \lambda$  (label ① in Fig. 3a), the lack of the radiative coupling makes the TLSs independent of each other. The directivity returns to a donut shape as shown in Fig. 3b. The maximum cross section converges to a constant of  $3 \times 1.5\sigma_0$ , which is simply the incoherent addition of three independent dipolar TLSs (Fig. 3c).

An optimal antenna realizes the highest cross section when three collective modes just start to merge. It occurs when the inter-TLS distance  $d$  is around  $0.2\lambda$ , as labelled by ② in Fig. 3a. This distance strikes the optimal balance: the radiative coupling is strong enough to induce collective excitations but not too strong to split resonant frequencies far apart. The maximum cross section reaches  $\sigma_{max} = 9\sigma_0$  with a directivity drastically different the donut shape (Fig. 3b).

In a similar way, the inter-TLS distance also has a dominant effect on the lifetime of the modes in quantum antennas. Figure 3d shows the lifetime of the collective mode with largest cross section. A lifetime of  $73\tau_0$  is observed at  $d = 0.14\lambda$ . The lifetime quickly converges to the natural lifetime  $\tau_0$  when the distance increases as shown in Fig. 3d.

Lastly, we discuss another way commonly used to increase the directivity of classical antennas: increasing the number of emission elements. Figure 4 shows the cross section and coherent time of antennas consisting of a linear array of  $N$  identical TLSs. For each antenna, an optimized inter-TLS distance  $d$  is used, which all happens to be all around  $0.2\lambda(1 \pm 5\%)$ . Indeed, the maximum cross section increases as  $N$  increases, with a super-linear scaling. It reaches  $\sigma_{max} = 22.8\sigma_0$  for  $N = 5$ . In comparison, the dashed line shows the incoherent addition of  $N$  independent TLSs, i.e.  $N \times 1.5\sigma_0$ . Quantum antennas perform far better than  $N$  independent TLSs. The insets of Fig. 4a show the cross section  $\sigma(\theta, \phi)$  for  $N = 4$  and 5. The maximum lifetime also increases with  $N$  as shown in Fig. 4c. It becomes remarkably long  $\tau = 1143\tau_0$  when  $N = 5$ .

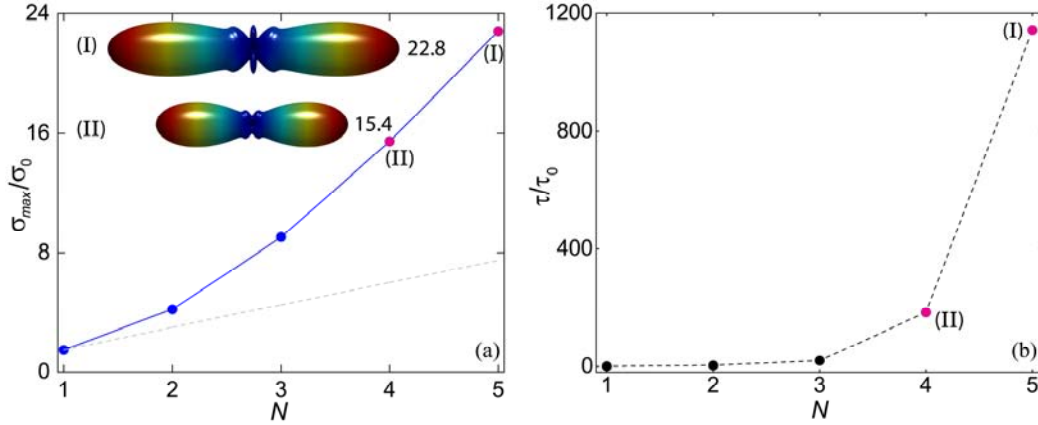


Fig. 4(color online) (a) The maximum cross section of a quantum antenna increases with the number of TLSs. The insets show  $\sigma(\theta, \phi)$  for 4-TLS and 5-TLS antennas (marked with red dots), respectively. (b) The maximum coherent time also increases with  $N$ , reaching over 1000 times of natural lifetime  $\tau_0$ .

Scattering cross section describes a quantum antenna's capability in transferring photons from one far field to another. In practice, one might also want to transfer free-space photons to certain localized energy. The cross section for such transfer process can be easily calculated from the maximum scattering cross section, the rates of energy transfer to the localized energy and radiation [57].

In conclusion, directional quantum antennas greatly enhance the optical cross section and lifetime of TLSs in free space. Here we only showed the operation for single photons. It would certainly be interesting to extend the theory for multi-photon cases. Mediated by TLS's Fermionic excitation, quantum antennas could induce strong photon-photon interactions, which can be useful for receiving and transmitting entanglement in free space. These efficient and yet compact receivers and transmitters could provide a scalable approach to construct quantum information network in free space.

## ACKNOWLEDGMENTS

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## APPENDIX A: Quantum Scattering Theory for Multiple TLSs

Here we discuss the theoretical framework that we developed to solve the scattering problem of Fock-state photons scattered by multiple TLSs in free space. It is used to derive most of the analytical and numerical results shown in the main text. It uses a non-perturbative method to solve the scattering problem. Specifically, we solve the

eigenstate of the full Hamiltonian, an open system that includes both TLSs and free-space photons. It treats incident light as Fock state instead of semi-classical electric fields that drive the master equation. A similar approach has been developed in 1D space in waveguide quantum electrodynamics (waveguide-QED) [47,58,59]. The theory here works in 3D free-space.

We consider a cluster of  $M$  TLSs in free space. The TLSs' excited states are  $|e\rangle_{l=1,\dots,M}$ ; the ground states are  $|g\rangle_{l=1,\dots,M}$ ; the transition frequencies are  $\omega_{l=1,\dots,M}$ ; and transition dipole moments are  $\mathbf{d}_{l=1,\dots,M}$ . The TLSs are located at position  $\boldsymbol{\xi}_{l=1,\dots,M}$ . With the electric dipole and rotating wave approximation, the Hamiltonian of the system with rotation-wave approximation (RWA) can be written as

$$H = \sum_{m=1}^M \hbar\omega_m \sigma_m^\dagger \sigma_m + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + i\hbar \sum_{m=1}^M \sum_{\mathbf{k}} [V_{\mathbf{k}}^*(\boldsymbol{\xi}_m) c_{\mathbf{k}}^\dagger \sigma_m - V_{\mathbf{k}}(\boldsymbol{\xi}_m) c_{\mathbf{k}} \sigma_m^\dagger], \quad (\text{A1})$$

where  $\omega_{\mathbf{k}}$  is the angular frequency of the incident photon with a wave vector of  $\mathbf{k}$ .  $\hbar$  is reduced Planck constant.  $c_{\mathbf{k}}^\dagger$  and  $c_{\mathbf{k}}$  are the bosonic creation and annihilation operator of the photon, respectively.  $\sigma_m^\dagger$  and  $\sigma_m$  are the raising and lowering atomic operators of the atoms, respectively.  $V_{\mathbf{k}}(\boldsymbol{\xi}_m)$  is the coupling constant between the  $m^{\text{th}}$  TLS and photons with a wave vector of  $\mathbf{k}$ .

We will explicitly solve for the eigenstates of the full Hamiltonian of the open system described by Eq. (A1). For this purpose, we need to do some preparation to transform the Hamiltonian into a real-space representation.

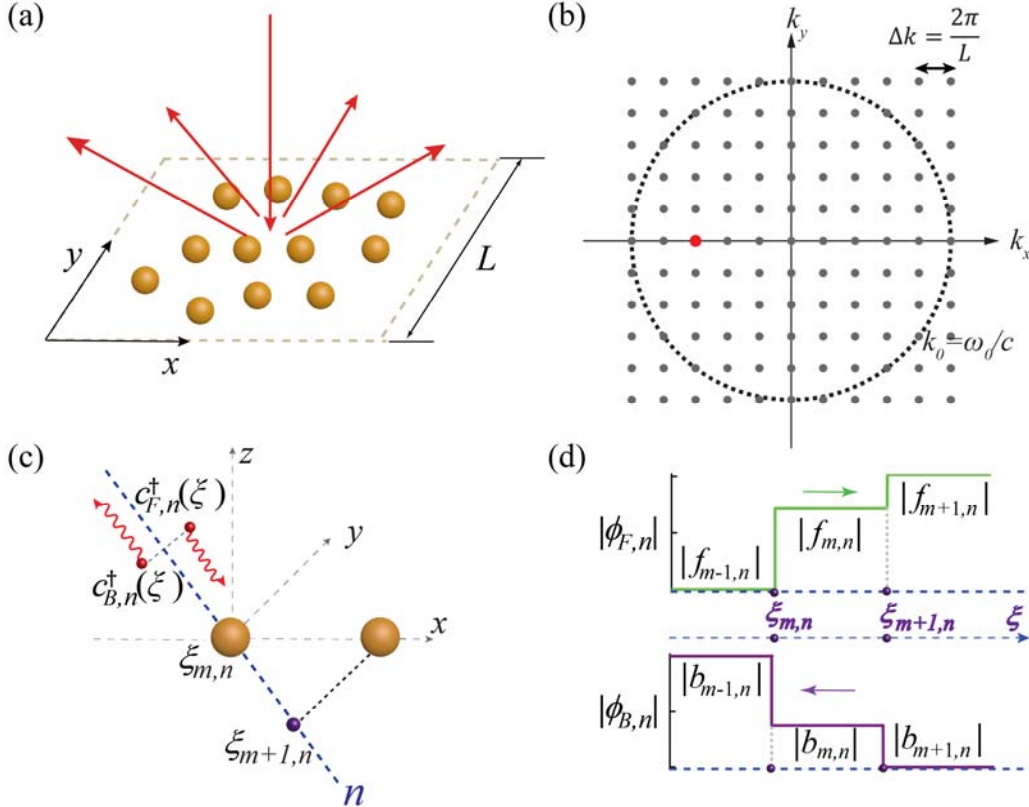


Fig. A1(color online) (a) Schematic of a cluster of TLSs in free space. (b) Distribution of channels in the  $\mathbf{k}_{xy}$ -space. The channels are discretized by  $\Delta k = 2\pi/L$ . The allowed channels are located within the circle with radius of  $k_0$ . (c) Real-space bosonic creation operators.  $c_{F/B,n}^\dagger(\xi)$  creates a forward or backward moving photon at the location  $\xi$  in the  $n^{\text{th}}$  channel. (d) Spatial single-photon wavefunctions in the  $n^{\text{th}}$  channel. The coefficient are the amplitudes of forward (green line) and backward (purple line) directions between  $m^{\text{th}}$  and  $(m+1)^{\text{th}}$  TLSs in the  $n^{\text{th}}$  channel, respectively.

**Free-space Channels** First, we introduce the method to describe the continuum radiation in 3D free space. We will discretize the space by using the box quantization in a periodic boundary condition in the  $x$ - $y$  plane (Fig. A1a). The period is  $L$ . At the end of the derivation, we will take the limit of  $L \rightarrow \infty$  to recover the case of free space. Because of the periodicity, incident photon can only be scattered to a set of discrete directions. These directions are defined by the waves' in-plane wavevectors  $\mathbf{k}_{xy} = (m_x, m_y)2\pi/L$ , where  $m_x$  and  $m_y$  are integers (Fig. A1b). We call these directions channels. As a convention of notation, the index  $(m_x, m_y)$  in the upper semi-infinite space also includes the waves in lower semi-infinite space in the direction of  $(-m_x, -m_y)2\pi/L$ . Channels are all located within the circle of  $k_0 = \omega_0/c$  for the interested frequency range around the resonant frequency  $\omega_0$ . The total number is  $N = \pi[L/\lambda_0]^2$ , where  $\lambda_0 = 2\pi/k_0$  is the resonant wavelength. The floor operator  $[A]$  gives the largest integer smaller than  $A$ . Using channels, we can convert the Hamiltonian to:

$$H = \sum_{m=1}^M \hbar\omega_m \sigma_m^\dagger \sigma_m + \sum_{n=1}^N \sum_k \hbar\omega_k c_{n,k}^\dagger c_{n,k} + i\hbar \sum_{m=1}^M \sum_{n=1}^N \sum_k [V_k^*(\xi_{m,n}) c_{n,k}^\dagger \sigma_m - V_k(\xi_{m,n}) c_{n,k} \sigma_m^\dagger]. \quad (\text{A2})$$

Here  $\xi_{m,n}$  is the projected position of  $m^{\text{th}}$  TLS along the direction of  $n^{\text{th}}$  channel. The coupling coefficient  $V_k(\xi_{m,n})$  is given by  $V_k(\xi_{m,n}) = \omega_m \sqrt{\frac{1}{2\hbar\epsilon_0\omega_k L^3}} \mathbf{d}_m \cdot \mathbf{e}_n e^{ik\xi_{m,n}}$ , where  $\epsilon_0$  is vacuum permittivity, and  $\mathbf{e}_n$  is polarization direction of the electric field of photons. By using  $\sum_k = \Delta k \int dk$ , we convert the Hamiltonian to

$$H = \sum_{m=1}^M \hbar\omega_m \sigma_m^\dagger \sigma_m + \sum_{n=1}^N \int dk \hbar\omega(k) c_n^\dagger(k) c_n(k) + i\hbar \sum_{m=1}^M \sum_{n=1}^N \int dk \sqrt{\frac{L}{2\pi}} [V_{m,n}(k) e^{-ik\xi_{m,n}} c_n^\dagger(k) \sigma_m - V_{m,n}(k) e^{ik\xi_{m,n}} c_n(k) \sigma_m^\dagger], \quad (\text{A3})$$

where  $c_n^\dagger(k) = \sqrt{L/2\pi} c_{n,k}^\dagger$  and  $c_n(k) = \sqrt{L/2\pi} c_{n,k}$ .  $V_{m,n}(k) = \omega_m \sqrt{\frac{1}{2\hbar\epsilon_0\omega_k L^3}} \mathbf{d}_m \cdot \mathbf{e}_n$ . By further defining  $\tilde{V}_{m,n} = V_{m,n}(k) \sqrt{L/2\pi}$ , we can simplify Eq. (A3) to

$$\begin{aligned}
H &= \sum_{m=1}^M \hbar\omega_m \sigma_m^\dagger \sigma_m + \sum_{n=1}^N \int dk \hbar\omega(k) c_n^\dagger(k) c_n(k) \\
&+ i\hbar \sum_{m=1}^M \sum_{n=1}^N \int dk \tilde{V}_{m,n} [c_n^\dagger(k) e^{-ik\xi_{m,n}} \sigma_m - c_n(k) e^{ik\xi_{m,n}} \sigma_m^\dagger],
\end{aligned} \tag{A4}$$

There are two propagation directions in each channel, *i.e.* the forward- and backward propagation directions. Thus, we can express the second term on the right-hand side in Eq. (A4) as

$$\begin{aligned}
&\sum_{n=1}^N \int dk \hbar\omega(k) c_n^\dagger(k) c_n(k) \\
&= \sum_{n=1}^N \int dk \hbar\omega(k) c_{F,n}^\dagger(k) c_{F,n}(k) \\
&+ \sum_{n=1}^N \int dk \hbar\omega(k) c_{B,n}^\dagger(k) c_{B,n}(k),
\end{aligned} \tag{A5}$$

and the last term as

$$\begin{aligned}
&i\hbar \sum_{m=1}^M \sum_{n=1}^N \int dk \tilde{V}_{m,n} [c_n^\dagger(k) e^{-ik\xi_{m,n}} \sigma_m - c_n(k) e^{ik\xi_{m,n}} \sigma_m^\dagger] \\
&= i\hbar \sum_{m=1}^M \sum_{n=1}^N \int dk \tilde{V}_{m,n} \{ [c_{F,n}^\dagger(k) + c_{B,n}^\dagger(k)] e^{-ik\xi_{m,n}} \sigma_m \\
&\quad - [c_{F,n}(k) + c_{B,n}(k)] e^{ik\xi_{m,n}} \sigma_m^\dagger \}.
\end{aligned} \tag{A6}$$

*Convert to Real-space Hamiltonian*--Now that we have expressed the free space radiation in terms of channels, we begin to convert the Hamiltonian to a real-space representation. Each channel is effectively a one-directional waveguide, which allows us to apply an important technique developed in waveguide QED [46]. Specifically, we will convert the k-space operators  $c_{F,n}^\dagger(k)$ ,  $c_{B,n}^\dagger(k)$ ,  $c_{F,n}(k)$  and  $c_{B,n}(k)$  to real space operators by applying the following Fourier transformation [46]

$$c_{F/B,n}^\dagger(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_{F/B,n}^\dagger(\xi) e^{ik\xi} d\xi, \tag{A7a}$$

$$c_{F/B,n}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_{F/B,n}(\xi) e^{-ik\xi} d\xi, \tag{A7b}$$

where  $c_{F/B,n}^\dagger(\xi)$  and  $c_{F/B,n}(\xi)$  (illustrated in Fig. A1c) create and annihilate a forward- and backward- propagation photon at position  $\xi$  in the  $n^{\text{th}}$  channel, respectively. Using above Fourier transformation, the Hamiltonian can be expressed in real space as

$$\begin{aligned}
H = & \sum_{m=1}^M \hbar\omega_m \sigma_m^\dagger \sigma_m \\
& + \sum_{n=1}^N \int_{-\infty}^{\infty} d\xi (-i\hbar c) [c_{F,n}^\dagger(\xi) \frac{d}{d\xi} c_{F,n}(\xi) \\
& - c_{B,n}^\dagger(\xi) \frac{d}{d\xi} c_{B,n}(\xi)] \\
& + i\hbar \sum_{m=1}^M \sum_{n=1}^N \int_{-\infty}^{\infty} d\xi \delta(\xi - \xi_{m,n}) V_{m,n} \{ [c_{F,n}^\dagger(\xi) + c_{B,n}^\dagger(\xi)] \sigma_m \\
& - [c_{F,n}(\xi) + c_{B,n}(\xi)] \sigma_m^\dagger \},
\end{aligned} \tag{A8}$$

where  $\xi_{m,n}$  is the projected location of the  $m^{\text{th}}$  TLS in the  $n^{\text{th}}$  channel. Here we have used the dispersion relation  $\omega(k) = ck$  and  $\omega(k) = -ck$  for the forward- and backward- propagating waves, respectively. The coupling constant is  $V_{m,n} = \sqrt{2\pi} \tilde{V}_{m,n}$ .

*Eigenstates of real-space Hamiltonian.*-- The general form of the eigenstates of Eq. (A8) is given by

$$\begin{aligned}
|\phi\rangle = & \left\{ \sum_{n=1}^N \int d\xi [ \phi_{F,n}(\xi) c_{F,n}^\dagger(\xi) + \phi_{B,n}(\xi) c_{B,n}^\dagger(\xi) ] \right. \\
& \left. + \sum_{m=1}^M e_m \sigma_m^\dagger \right\} |0, g_1, \dots, g_M\rangle,
\end{aligned} \tag{A9}$$

Here we only consider the single photon case.  $\phi_{F,n}(\xi)$  and  $\phi_{B,n}(\xi)$  are the spatial wave-functions of forward- and backward- propagating photons in the  $n^{\text{th}}$  channel, respectively.  $|0, g_1, \dots, g_M\rangle$  is the zero photon state with all the TLSs in the ground states.  $e_m$  is the excitation amplitude of  $m^{\text{th}}$  TLS. By solving the time-independent Schrödinger equation  $H|\phi\rangle = E|\phi\rangle$ , where  $E = \hbar\omega$  is the eigen energy of the system, we get

$$-ic \frac{d}{d\xi} \phi_{F,n}(\xi) + i \sum_{m=1}^M V_{m,n} \delta(\xi - \xi_{m,n}) e_m = \omega \phi_{F,n}(\xi), \tag{A10}$$

$$ic \frac{d}{d\xi} \phi_{B,n}(\xi) + i \sum_{m=1}^M V_{m,n} \delta(\xi - \xi_{m,n}) e_m = \omega \phi_{B,n}(\xi), \tag{A11}$$

$$\omega_m e_m - i \sum_{n=1}^N V_{m,n} [\phi_{F,n}(\xi_{m,n}) + \phi_{B,n}(\xi_{m,n})] = \omega e_m, \tag{A12}$$

To be specific, we consider a photon incident from the  $p^{\text{th}}$  channel. The wave function in the  $n^{\text{th}}$  channel can be written as

$$\phi_{F,n}(\xi) = e^{ik\xi} \left[ \theta(\xi_{1,n} - \xi) \delta_{np} + \sum_{m=1}^{M-1} f_{m,n} \theta(\xi - \xi_{m,n}) \theta(\xi_{m+1,n} - \xi) + f_{M,n} \theta(\xi - \xi_{M,n}) \right], \quad (\text{A13})$$

$$\phi_{B,n}(\xi) = e^{-ik\xi} \left[ b_{0,n} \theta(\xi_{1,n} - \xi) + \sum_{m=1}^{M-1} b_{m,n} \theta(\xi - \xi_{m,n}) \theta(\xi_{m+1,n} - \xi) \right]. \quad (\text{A14})$$

Here  $\theta(\xi)$  is the step function,  $\delta_{np}$  is Kronecker delta. The coefficients  $f_{m,n}$  and  $b_{m,n}$  are the amplitudes of forward- and backward- propagating waves between  $m^{\text{th}}$  and  $(m+1)^{\text{th}}$  TLSs in the  $n^{\text{th}}$  channel, respectively. They are schematically shown in Fig. A1d.

Substituting Eq. (A13) into Eq. (A10), we can get

$$-ice^{ik\xi_{1,n}}(-\delta_{np} + f_{1,n}) + iV_{1,n}e_1 = 0, \quad (\text{A15a})$$

$$-ice^{ik\xi_{m,n}}(-f_{m-1,n} + f_{m,n}) + iV_{m,n}e_m = 0, \quad (\text{A15b})$$

$$-ice^{ik\xi_{M,n}}(-f_{M-1,n} + f_{M,n}) + iV_{M,n}e_M = 0, \quad (\text{A15c})$$

Substituting Eq. (A14) into Eq. (A11), we also can get

$$ice^{-ik\xi_{1,n}}(b_{1,n} - b_{0,n}) + iV_{1,n}e_1 = 0, \quad (\text{A16a})$$

$$ice^{-ik\xi_{m,n}}(b_{m,n} - b_{m-1,n}) + iV_{m,n}e_m = 0, \quad (\text{A16b})$$

$$ice^{-ik\xi_{M,n}}(-b_{M-1,n}) + iV_{M,n}e_M = 0, \quad (\text{A16c})$$

Next, we can directly solve Eq. (A15) and get the amplitudes of the forward-propagating waves

$$f_{1,n} = \frac{V_{1,n}}{c} e^{-ik\xi_{1,n}} e_1 + \delta_{np}, \quad (\text{A17a})$$

$$f_{m,n} = \frac{V_{m,n}}{c} e^{-ik\xi_{m,n}} e_m + \dots + \frac{V_{1,n}}{c} e^{-ik\xi_{1,n}} e_1 + \delta_{np}, \quad (\text{A17b})$$

$$f_{M,n} = \frac{V_{M,n}}{c} e^{-ik\xi_{M,n}} e_M + \dots + \frac{V_{1,n}}{c} e^{-ik\xi_{1,n}} e_1 + \delta_{np}, \quad (\text{A17c})$$

Similarly, we solve Eq. (A16) and get the amplitudes of backward-propagating waves

$$b_{0,n} = \frac{V_{M,n}}{c} e^{ik\xi_{M,n}} e_M + \dots + \frac{V_{m,n}}{c} e^{ik\xi_{m,n}} e_m + \dots + \frac{V_{1,n}}{c} e^{ik\xi_{1,n}} e_1, \quad (\text{A18a})$$

$$b_{m-1,n} = \frac{V_{M,n}}{c} e^{ik\xi_{M,n}} e_M + \dots + \frac{V_{m,n}}{c} e^{ik\xi_{m,n}} e_m, \quad (\text{A18b})$$

$$b_{M,n} = \frac{V_{M,n}}{c} e^{ik\xi_{M,n}} e_M. \quad (\text{A18c})$$

Now, we use these results to calculate the summation in Eq. (A12) and get

$$\begin{aligned} \omega_m e_m - i \left( V_{m,p} e^{ik\xi_{m,p}} + \sum_{n=1}^N \frac{V_{m,n}^2}{2c} e_m \right. \\ \left. + \sum_{l=1, l \neq m}^M \sum_{n=1}^N \frac{V_{m,n} V_{l,n}}{2c} e^{ik|\xi_{m,n} - \xi_{l,n}|} e_l \right) = \omega e_m, \end{aligned} \quad (\text{A19})$$

Equation (A19) can be written in a more compact form. For this purpose, we define  $\Gamma_{m,n} = \frac{V_{m,n}^2}{c}$  and  $\Gamma_m = \sum_{n=1}^N \frac{V_{m,n}^2}{2c} = \sum_{n=1}^N \Gamma_{m,n}$ .  $\Gamma_m$  is the spontaneous emission rate of the  $m^{\text{th}}$  scatterer. Sum over all the channels, then Eq. (A19) can be written as

$$(\omega - \omega_m + i \frac{\Gamma_m}{2}) e_m + \sum_{l=1, l \neq m}^M \left( -\Xi_{ml} + i \frac{\Gamma_{ml}}{2} \right) e_l = -i V_{m,p} e^{ik\xi_{m,p}}, \quad (\text{A20})$$

Here we also use the collective spontaneous rate  $\Gamma_{ml}$  and the induced dipole-dipole interaction  $\Xi_{ml}$  arising from the coupling between the  $m^{\text{th}}$  and the  $l^{\text{th}}$  TLSs through the vacuum field [41,60]:  $\Gamma_{ml} = \sqrt{\Gamma_m \Gamma_l} \frac{3}{2} F_{ml}(k\xi_{ml})$  and  $\Xi_{ml} = \sqrt{\Gamma_m \Gamma_l} \frac{3}{4} \mathbb{Q}_{ml}(k\xi_{ml})$ , where  $F_{ml}(k\xi_{ml}) = \sum_{n=1}^N \frac{e^{ik|\xi_{m,n} - \xi_{l,n}|}}{2} = [\mathbf{d}_m \cdot \mathbf{d}_l - (\mathbf{d}_m \cdot \hat{\mathbf{r}})(\mathbf{d}_l \cdot \hat{\mathbf{r}})] \frac{\sin(k\xi_{ml})}{k\xi_{ml}} + [\mathbf{d}_m \cdot \mathbf{d}_l - 3(\mathbf{d}_m \cdot \hat{\mathbf{r}})(\mathbf{d}_l \cdot \hat{\mathbf{r}})] (\frac{\cos(k\xi_{ml})}{(k\xi_{ml})^2} - \frac{\sin(k\xi_{ml})}{(k\xi_{ml})^3})$ , and  $\mathbb{Q}_{ml}(k\xi_{ml}) = -[\mathbf{d}_m \cdot \mathbf{d}_l - (\mathbf{d}_m \cdot \hat{\mathbf{r}})(\mathbf{d}_l \cdot \hat{\mathbf{r}})] \frac{\cos(k\xi_{ml})}{k\xi_{ml}} + [\mathbf{d}_m \cdot \mathbf{d}_l - 3(\mathbf{d}_m \cdot \hat{\mathbf{r}})(\mathbf{d}_l \cdot \hat{\mathbf{r}})] (\frac{\sin(k\xi_{ml})}{(k\xi_{ml})^2} + \frac{\cos(k\xi_{ml})}{(k\xi_{ml})^3})$ . Here  $\xi_{ml}$  and  $\hat{\mathbf{r}}$  are the distance and the unit vector between the  $m^{\text{th}}$  and the  $l^{\text{th}}$  TLSs, respectively.

*Cross sections*-- Using Eqs(A17c) and (A18a), we can derive the scattering cross section[61]

$$\begin{aligned} \sigma(\omega) &= \frac{\sum_{n=1}^N (f_{M,n}^\dagger) f_{M,n} + \sum_{n=1}^N (b_{0,n}^\dagger) b_{0,n}}{1/L^2} \\ &= \frac{3\lambda^2}{2\pi} \left[ \sum_{m=1}^M \frac{\Gamma_m}{2} |\tilde{e}_m|^2 + \sum_{m=1, m \neq l}^M \sum_{l=1}^M \frac{\Gamma_{ml}}{2} \tilde{e}_l^* \tilde{e}_m \right]. \end{aligned} \quad (\text{A21})$$

Here we use the relation  $e_m = \sqrt{\frac{3\lambda^2}{2\pi} \frac{1}{2L}} \tilde{e}_m$ . Analytical solutions for one and two TLSs can be directly derived[45]. But for more than two TLSs, we resort to numerical calculation in main text.

The rotation-wave approximation is applied in the interaction part of our quantum scattering theory in Eq. (A1). Here we also calculate the results without using rotation-wave approximation [62]. The results are shown by black solid curves in Fig. A2. Compared to the red curves obtained from RWA, the magnitudes of the cross



sections remain is not affected significantly despite slight frequency shift of the resonant peaks.

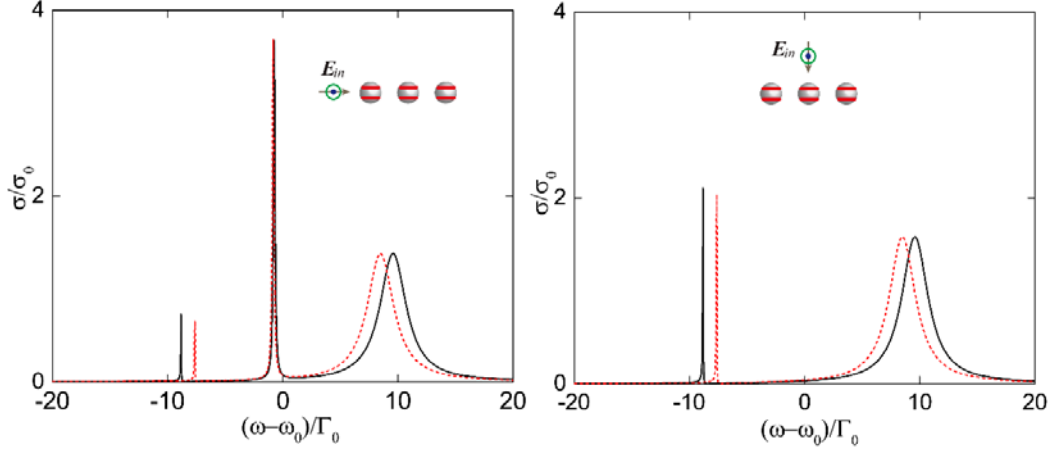


Fig. A2. The scattering cross section calculated with (red lines) and without (black solid lines) rotation-wave approximation for light incident from two different directions.

*Conservation law*--Here we prove that the total cross section of a single non-degenerate TLS is fundamentally constrained, regardless of specific implementation of the resonance.

The Hamiltonian  $H$  of the system under study can be written as two parts

$$H = H_0 + H_I \quad (\text{A22})$$

where  $H_0 = \hbar\omega_0\sigma^\dagger\sigma + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$  is the free Hamiltonian of the TLSs and photons, and  $H_I$  is the interaction between them. Here, we consider the initial and final states of the scattering process are eigenstates of the free Hamiltonian. For scattering of single photons, the initial state and final state of the scattering process is represented by  $|g, \mathbf{k}_i\rangle$  and  $|g, \mathbf{k}_f\rangle$ , respectively.

The scattering cross section is given by  $\sigma = \frac{\sum_{\mathbf{k}_f} T_{fi}}{\Phi}$ , where  $\Phi = \frac{c}{L^3}$  is the density of photon flux and  $T_{fi} = \frac{2\pi}{\hbar} |\mathfrak{S}_{fi}|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_{\mathbf{k}_i})$  is the transition rate from the initial state to the final state [44]. Here  $\mathfrak{S}_{fi}$  is the transition matrix. The period  $L$ , which is larger than wavelength, is used for normalization purpose. In general, we consider the resonant scattering, *i.e.*, the incident photon frequency  $\omega_{\mathbf{k}_i}$  equals to the transition frequency  $\omega_0$  of the TLS.

The transition matrix can be obtained from the time evolution of the system[63], which can be written as

$$\mathfrak{S}_{fi} = \frac{\langle g, \mathbf{k}_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, \mathbf{k}_i \rangle}{\hbar\omega - \hbar\omega_0 - \hbar\Delta + i\pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0)}, \quad (\text{A23})$$

where  $\Delta = \frac{1}{\hbar} \wp \sum_{\mathbf{k}_f} \frac{|\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2}{\hbar\omega_0 - \hbar\omega_{\mathbf{k}_f}}$  is the frequency shift and  $\wp$  indicates the Cauchy principal value. The transition matrix element  $\mathfrak{S}_{fi}$  is a complex number which exhibits a resonance for  $\omega = \omega_0 + \Delta$ . At resonance,  $\mathfrak{S}_{fi}$  is purely imaginary and its amplitude is maximal.

The scattering cross section now can be written as

$$\begin{aligned} & \sigma(\theta_i, \phi_i) \\ &= \frac{2L^3}{\hbar c} \frac{\pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, \mathbf{k}_i \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0)}{(\hbar\omega - \hbar\omega_0 - \hbar\Delta)^2 + \left[ \pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0) \right]^2}, \end{aligned} \quad (\text{A24})$$

We can further simplify Eq. (A) to

$$\begin{aligned} \sigma(\theta_i, \phi_i) &= \frac{2L^3}{\pi \hbar c} \frac{[|\langle e, 0 | H_I | g, \mathbf{k}_i \rangle|^2]}{\left[ \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0) \right]} \\ &\times \frac{\left[ \pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0) \right]^2}{(\hbar\omega - \hbar\omega_0 - \hbar\Delta)^2 + \left[ \pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0) \right]^2}, \end{aligned} \quad (\text{A25})$$

At the resonance scattering peak for  $\omega = \omega_0 + \Delta$ , the cross section is

$$\sigma(\theta_i, \phi_i) = \frac{2L^3}{\hbar \pi c} \frac{\pi |\langle e, 0 | H_I | g, \mathbf{k}_i \rangle|^2}{\pi \sum_{\mathbf{k}_f} |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 \delta(\hbar\omega_{\mathbf{k}_f} - \hbar\omega_0)} \quad (\text{A26})$$

The denominator summation over scattered state in Eq.(A26) can be converted into an integration over frequency and solid angle  $d\Omega$  by the relation  $\sum_{\mathbf{k}_f} \rightarrow \frac{L^3}{(2\pi)^3} \int \frac{\omega^2}{\hbar c^3} d\omega d\Omega$ . Integrating Eq. (A26) over  $4\pi$  solid angle, we can easily get the integration as

$$\int \sigma(\theta_i, \phi_i) d\Omega = \frac{2L^3}{\hbar \pi c} \frac{\pi \int |\langle e, 0 | H_I | g, \mathbf{k}_i \rangle|^2 d\Omega}{\pi \int |\langle g, \mathbf{k}_f | H_I | e, 0 \rangle|^2 d\Omega \rho(\omega_0)} = \frac{2L^3}{\hbar \pi c \rho(\omega_0)}, \quad (\text{A27})$$

where  $\rho(\omega_0) = \frac{L^3}{(2\pi)^3} \frac{\omega_0^2}{\hbar c^3}$  is the density of the photon[63]. Therefore, the total cross section is given by

$$\int \sigma(\theta_i, \phi_i) d\Omega = 4\lambda_0^2, \quad (\text{A28})$$

Equation (A28) indicates the conservation law of total cross section. Note we do not need the specific form of  $H_I$  to obtain Eq. (A28). It holds true for any non-degenerate TLS.

## APPENDIX B: The directivity of radiation, cross section, and differential cross section

In this appendix, we prove that the directivities of radiation pattern, cross section

and differential cross section are the same.

The radiation pattern can be obtained by considering the evolution of an excited state without incident photons, i.e.  $|e, 0\rangle$ . The power radiated to a direction  $(\theta_i, \phi_i)$  is related to the transition probability amplitude  $\langle g, k_i | H_I | e, 0 \rangle$ . The directivity of the radiation pattern is given by

$$D(\theta_i, \phi_i) = 4\pi \frac{\langle g, k_i | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle}{\sum_i \langle e, 0 | H_I | g, k_i \rangle \langle g, k_i | H_I | e, 0 \rangle} ,$$

Here  $|k_i\rangle$  represents a photon in the direction of  $k_i$ . The numerator is the probability of the excited state decaying to a photon in the direction  $k_i$ . The denominator is the total probability of decaying to all directions

The total cross section for a particular incident direction  $k_i$  is proportional to the summation of the transition probability to all final states as  $\sum_f |\langle g, k_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle|^2$ . Thus the directivity of the cross section can be calculated as

$$\mathbb{D}(\theta_i, \phi_i) = 4\pi \frac{\sum_f |\langle g, k_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle|^2}{\sum_{f,i} |\langle g, k_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle|^2} ,$$

where the denominator accounts for the summation over all incident directions. We can easily show that

$$\begin{aligned} \mathbb{D}(\theta_i, \phi_i) &= 4\pi \frac{\sum_f |\langle g, k_f | H_I | e, 0 \rangle|^2 |\langle e, 0 | H_I | g, k_i \rangle|^2}{\sum_f |\langle g, k_f | H_I | e, 0 \rangle|^2 \sum_i |\langle e, 0 | H_I | g, k_i \rangle|^2} \\ &= 4\pi \frac{|\langle e, 0 | H_I | g, k_i \rangle|^2}{\sum_i |\langle e, 0 | H_I | g, k_i \rangle|^2} = D . \end{aligned}$$

The directivity of the differential cross section can also be calculated from the transition probability as

$$\mathbb{D}(\theta_i, \phi_i) = 4\pi \frac{|\langle g, k_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle|^2}{\sum_f |\langle g, k_f | H_I | e, 0 \rangle \langle e, 0 | H_I | g, k_i \rangle|^2} .$$

We can further simply as

$$\mathbb{D}(\theta_i, \phi_i) = 4\pi \frac{|\langle e, 0 | H_I | g, k_i \rangle|^2 |\langle g, k_f | H_I | e, 0 \rangle|^2}{|\langle e, 0 | H_I | g, k_i \rangle|^2 \sum_f |\langle g, k_f | H_I | e, 0 \rangle|^2} .$$

It is straight forward to verify that  $D = \mathbb{D} = \mathbb{D}$ . The radiation pattern, cross section, and differential cross section have the same directivity.

### APPENDIX C: Definition of the angles used in the derivation.

The spectrum of the cross section shown in Figure 2(a)-2(c) depend on the

incident and polarization direction of the single photons. Without loss of generality, here we arrange the TLSs along  $z$ -axis and define the dipole directions along the  $x$ -axis directions.  $\theta$  and  $\varphi$  are the polar and azimuthal angles of the incident photon.  $\mathbf{e}_k$  is the polarization direction of the incident photon which perpendicular to the incident direction. In figure 2(a)-2(c), we define  $\varphi = 0$ . The polarization of the incident photon is parallel with the dipole direction.

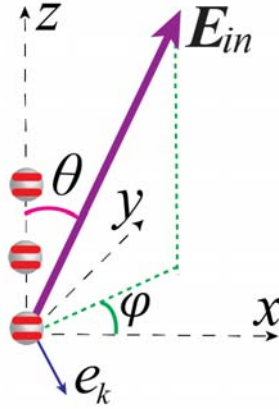


Fig. C(color online) The TLSs arrangement along the  $z$ -axis and the dipole moment  $\mathbf{d}$  along the  $x$ -axis.  $\mathbf{e}_k$  is the polarization of the incident direction and is perpendicular to the incident direction.  $\theta$  and  $\varphi$  are the polar and azimuthal angles of the incident photon.

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