Hadronic vacuum polarization in true muonium

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Hadronic Vacuum Polarization in True Muonium

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In order to reduce the theoretical uncertainty in the prediction, the leading-order hadronic vacuum polarization contribution to the hyperfine splitting of true muonium is reevaluated in two ways. A more complex pionic form factor and better estimates of the perturbative QCD contributions are used to study the model-dependence of the previous calculation. The second, more accurate method directly integrates the Drell ratio $R(s)$ to obtain $C_{1,\mathrm{hvp}} = -0.04874(9)$. This corresponds to an energy shift in the hyperfine splitting of $\Delta E^{\mu}_{\mathrm{hfs,hvp}} = -8202(16)$ MHz, and represents a factor of 50 reduction in the theoretical uncertainty from hadronic sources. We also compute the contribution in positronium, which is too small at present to detect.

I. INTRODUCTION

True muonium is the yet unidentified $(\mu\bar{\mu})$ bound state. The bound states have lifetimes between ps to ns [1]. QED dominates the characteristics of true muonium, while QCD effects appear at $\mathcal{O}(m_\mu\alpha^5)$ [2, 3]. Electroweak effects appear at $\mathcal{O}(m_\mu\alpha^7)$ [4]. Measurements of Lamb shift, $1s-2s$ splitting, and the hyperfine splitting (hfs) will occur in the future. These experiments are motivated by the existing discrepancies in muon physics [5–9]. Numerous new physics models have been suggested to explain these discrepancies [10–31]. True muonium can produce competitive constraints on most models if standard model predictions are known to the 100 MHz level, corresponding to $\mathcal{O}(m_\mu\alpha^7)$ [32].

Beyond new physics, a further motivation for considering true muonium comes from the anomalous magnetic moment of the muon (a_μ). There exists a discrepancy between the measurement at BNL and theory, $\Delta a_\mu = a_{\mu,exp} - a_{\mu,th} = 288(80)\times 10^{-11}$ [5, 33]. Hadronic contributions dominate the theoretical uncertainty, and hadronic vacuum polarization (hvp) is the largest term. One way to reduce the theoretical uncertainty would be consistency checks from other systems. By its particle/antiparticle nature, the annihilation channel contributes to true muonium, leading to an enhancement of hvp contributions to the hfs. These contributions are measurable in true muonium unlike positronium were they are mass-suppressed.

The theoretical expression for the hfs corrections to true muonium from QED can be written

$$\Delta E_{\mathrm{hfs}} = m_\mu\alpha^4\left[C_0 + C_1\frac{\alpha}{\pi} + C_{21}\alpha^2\ln\left(\frac{1}{\alpha}\right) + C_{20}\left(\frac{\alpha}{\pi}\right)^2 + C_{32}\frac{\alpha^3}{\pi}\ln^2\left(\frac{1}{\alpha}\right) + C_{31}\frac{\alpha^3}{\pi}\ln\left(\frac{1}{\alpha}\right) + C_{30}\left(\frac{\alpha}{\pi}\right)^3 + \cdots\right], \quad (1)$$

where C_{ij} indicate the coefficient of the term proportional to $(\alpha)^i\ln^j(1/\alpha)$. All dependence on mass scales other than m_μ is in C_{ij}. The coefficients of single flavor QED bound states, used in positronium, are known up to $\mathcal{O}(m_e\alpha^6)$ and partial results for $\mathcal{O}(m_e\alpha^7)$ (For an updated review of the coefficients see [34, 35]). The exchange $m_e \to m_\mu$ translates these results to true muonium.

True muonium has extra contributions that must be considered. The lighter electron allows for large loop contributions. The relative smallness of $m_\tau/m_\mu \approx 17$ and $m_\pi/m_\mu \approx 1.3$ produce contributions to true muonium much larger than analogous contributions to positronium. Of these true muonium specific contributions, which we denote by C^{μ}_{ij}, only a few terms are known. The $\mathcal{O}(m_\mu\alpha^5)$ contributions from electron loops were found to be $C^{\mu}_{1,e} = 1.684$ [2]. The $\mathcal{O}(m_\mu\alpha^6)$ contribution from leptonic loops to the two-photon annihilation channel $C^{\mu}_{20,2\gamma} = -2.031092873$ was recently computed exactly [35], and the electron loop in three-photon annihilation at $\mathcal{O}(m_\mu\alpha^7)$ is $C^{\mu}_{30,3\gamma} = -5.86510(20)$ [36]. For a $\mathcal{O}(m_\mu\alpha^7)$ prediction of the hfs, contributions from Z-bosons must be considered [4].

The hvp first contributes at $\mathcal{O}(m_\mu\alpha^5)$ through the single-photon annihilation channel. It was previously calculated to be $C_{1,\mathrm{hvp}} = -0.047(5)$ [2]. The reported error is an estimate of model-dependence. We will refer to this result as JSIK throughout, after the authors of that paper. This result mixed a Gounaris-Sakurai form factor for the π and ρ contributions, a simple pole approximation for the ω and ϕ, and a two-constant perturbative contribution above 1 GeV.

Together, these contributions predict $\Delta E^{1s}_{\mathrm{hfs}} = 42329730(800)(700)$ MHz where the first, dominant, uncertainty is from hadronic model dependence and the second is an estimate of uncalculated $\mathcal{O}(m_\mu\alpha^6)$ contributions. If an estimate via quadrature is made, the theoretical uncertainty is $\mathcal{O}(1000$ MHz). This value is an order of magnitude too large to allow discrimination between new physics models. The goal of this work is to recalculate $C_{1,\mathrm{hvp}}$ via different approaches to reduce the model dependence and the theoretical uncertainty.

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The $\mathcal{O}(m\alpha^5)$ hvp contribution is given by

$$\Delta E_{1,\text{hvp}} = \left[m_\mu^2 \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\rho(s)}{4m_\mu^2 - s}\right] \frac{m_\mu \alpha^5}{n^3 \pi} = C_{1,\text{hvp}} \frac{m_\mu \alpha^5}{n^3 \pi} \tag{2}$$

where $\rho(s)$ is the spectral function.

II. JSIK CALCULATION

Before discussing our improved calculations, it is useful to briefly review the JSIK calculation. This will allow us to highlight the differences between the methods and introduce some notation. $C_{1,\text{hvp}}^{\text{JSIK}}$ is given by the sum of four terms [2],

$$C_{1,\text{hvp}}^{\text{JSIK}} = C_{1,\pi} + C_{1,\omega} + C_{1,\phi} + C_{1,>} \tag{3}$$

where $C_{1,\pi}$ is the contribution from the pion form factor, $C_{1,\omega}$ and $C_{1,\phi}$ are simple-pole terms, and $C_{1,>}$ is the contribution from $E > 1$ GeV were they neglect resonances and use only perturbative QCD (pQCD).

The main contribution is from the pionic loop given by [2, 37]

$$\rho(s) = \frac{(s - 4\,m_\pi^2)^{3/2}}{12\,s^{5/2}} \left|F_\pi(s)\right|^2 . \tag{4}$$

JSIK chose to use the simple Gounaris-Sakurai form factor [38]. This choice of $F_\pi(s)$ is

$$F_\pi(s) = F_{\rho,\text{GS}}(s) = \frac{N}{D_1 + D_2 - i\,D_3} . \tag{5}$$

In this decomposition, N, D_1, D_2 and D_3 are given by

$$N = m_\rho^2 + d m_\rho \Gamma_\rho , \qquad D_1 = m_\rho^2 - s,$$
$$D_2 = \Gamma_\rho \frac{m_\rho^2}{k_\rho^3}\left[k(s)^2 (h(s) - h_\rho) + k_\rho^2 h'_\rho (m_\rho^2 - s)\right],$$
$$D_3 = \frac{m_\rho^2 \Gamma_\rho}{\sqrt{s}} \left(\frac{k(s)}{k_\rho}\right)^3 , \tag{6}$$

with the parameter d defined via

$$d = \frac{3}{\pi}\frac{m_\pi^2}{k_\rho^2} \ln\frac{m_\rho + 2\,k_\rho}{2\,m_\pi} + \frac{m_\rho}{2\,\pi\,k_\rho} - \frac{m_\pi^2\,m_\rho}{\pi\,k_\rho^3} \approx 0.48. \tag{7}$$

The functions $k(s)$ and $h(s)$ are defined as

$$k(s) = \frac{1}{2}\sqrt{s - 4m_\pi^2}, \quad h(s) = \frac{2}{\pi}\frac{k(s)}{\sqrt{s}} \ln\left(\frac{\sqrt{s} + 2\,k(s)}{2\,m_\pi}\right) . \tag{8}$$

Where h' denoted the derivative of $h(s)$ with respect to s and the subscript ρ indicated evaluation of the function at m_ρ^2. This form factor includes only ρ-mesons contributions. The values used by JSIK were $\Gamma_\rho = 150.7(1.2)\,\text{MeV}$, and $m_\rho = 768.5(6)\,\text{MeV}$. Integrating these expressions yields $C_{1,\pi} = -0.032$.

To include other meson resonances, a simple pole approximation was taken. The spectral function contribution from a vector meson is given by $\rho(s) = 4\pi^2/f_V^2 \delta(s - m_V^2)$ [39] where f_V are coupling constants. These were estimated in [39] to be $f_\omega^2/4\pi = 18(2)$ and $f_\phi^2/4\pi = 11(2)$. The masses of the vector mesons are $m_\omega = 782.71(8)$ MeV and $m_\phi = 1019.461(19)$ MeV. JSIK obtained $C_{1,\omega} = -0.004$ and $C_{1,\phi} = -0.003$.

The final contribution, $C_{1,>}$ was obtained by applying the relation between the spectral function and the Drell ratio,

$$\rho(s) = \frac{R}{3s}, \text{ where } R = \frac{\sigma(e^+e^- \to h)}{\sigma(e^+e^- \to \mu^+\mu^-)}. \tag{9}$$

In pQCD, the leading order is given by $R_{LO} = N_c \sum q_i^2$ where N_c is the number of colors and q_i is the charge of quark i. Below the c threshold at ≈ 4 GeV, $R_{LO} = 2$. Between 4 GeV and 10 GeV, the value rises to $R_{LO} = 10/3$. Above 10 GeV the b quark becomes relevant yielding $R_{LO} = 11/3$. At present, perturbative calculations exist up to $\mathcal{O}(\alpha_s^4)$. JSIK estimated from the experimental results in [40] that $R_{2\text{GeV}<s<4\text{GeV}} \approx 2$ and $R_{s>4GeV} \approx 4$ (See Fig. 1 for a comparision to experiment). With these values, they obtained $C_{1,>} = -0.008$.

Putting these together and including a 11% estimate of the model-dependent uncertainties, their final result was $C_{1,\text{hvp}}^{\text{JSIK}} = -0.047(5)$

III. INVESTIGATING THE PIECES

One way to reduce the uncertainty in $C_{1,\text{hvp}}$ would be to improve the calculations of the pieces of the JSIK value. Since JSIK, experimental measurements of the pion form factor lead to the development of an improved Gounaris-Sakurai parameterization. We improve the estimate of $C_{1,>}$ by computing the numerical averages of $R(s)$ in the regimes and accounting for non-constant terms.

A. Improved Gounaris-Sakurai Parameterization

Instead of the simple Gounaris-Sakurai form factor, a more complex form exists that features two improvements. The improved form [46] includes $\rho - \omega$ mixing. Additionally this improved parameterization takes into account the wide ρ' and ρ'' states which cannot be treated by simple-pole approximations. The form factor is given by

$$F_{\pi,\text{IGS}} = \frac{1}{1 + \beta + \gamma}\left[F_{\rho,\text{GS}}(s)\left(1 + \delta\frac{s}{m_\omega^2}F_{\omega,\text{BW}}(s)\right) + \beta F_{\rho',\text{GS}}(s) + \gamma F_{\rho'',\text{GS}}(s)\right] \tag{10}$$

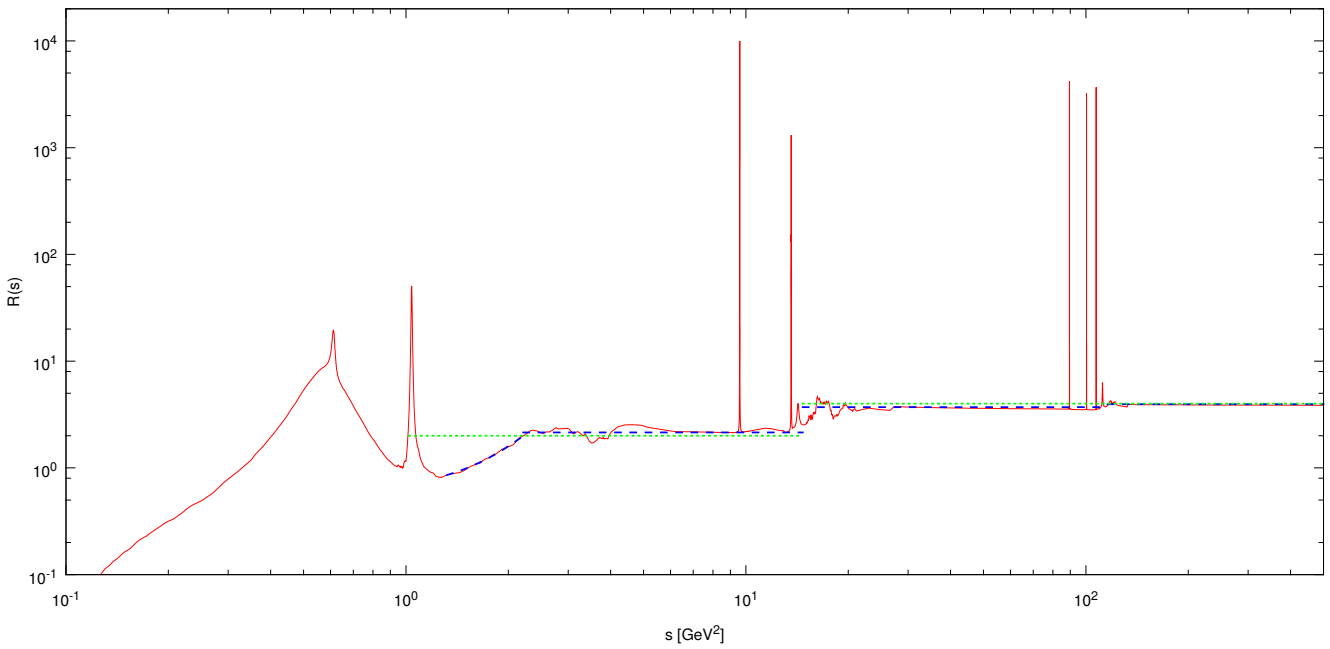

FIG. 1. $R(s)$ vs. s. The solid line indicates the compilation of experimental data produced by F. Jegerlehner available with the ALPHAQED package [41–44]. This compilation is the weighted average of many experiments through different ranges for the years up to 2012. These are supplemented by $\mathcal{O}(\alpha_s^4)$ pQCD calculations from RHAD [45] for the ranges (27.25,88.55) GeV^2 and (132.5,∞) GeV^2. The dotted line are the estimates used in the JSIK calculations of $C_{1,>}$ [2], and the dashed line indicates the estimates of this work.

where $F_{i,\mathrm{GS}}(s)$ are given by Eq.(5) with the additional masses and decay constants: $m_{\rho'} = 1409(12)$ MeV, $\Gamma_{\rho'} = 501(37)$ MeV, $m_{\rho''} = 1740(21)$ MeV, $\Gamma_{\rho''} = 235(1)$ MeV, and $\Gamma_\omega = 8.68$ MeV [46]. Further the parameters $\delta = 2.03(10)e^{0.2269(401)i}$, $\beta = -0.166(6)$, and $\gamma = 0.071(6)$ determine the mixing and relative strengths [46]. For the ω meson, a Breit-Wigner form factor is used

$$F_{\omega,\mathrm{BW}}(s) = \frac{m_\omega^2}{m^2 - s + i\Gamma_\omega m_\omega}. \tag{11}$$

Integrating, we compute a coefficient $C_{1,\mathrm{IGS}} = -0.0377(5)$ which should include the same physics as $C_{1,\pi} + C_{1,\omega}$ as the well previously uncalculated higher-order terms from ρ', ρ''. The error on $C_{1,\mathrm{IGS}}$ is estimated from parameter variation.

B. pQCD Regime

Interest in using experimental cross sections to obtain both $a_{\mu,\mathrm{had}}$ and the running of α has lead to a number of groups to compile $R(s)$ data. This data can be used to improve the estimates of $C_{1,>}$. For this work, we use the data for R(s) compiled by F. Jegerlehner in 2012 and available with the software packages ALPHAQED [41–44]. In the ranges (27.25,88.55) GeV^2 and (132.5,∞) GeV^2, data is sparse and supplemented by $\mathcal{O}(\alpha_s^4)$ pQCD calculations via RHAD [45]. These packages may be found at [47]. In Fig. 1, one sees that the JSIK value of $C_{1,>}$ leaves something to be desired. For (1.2,2.3) GeV^2, the JSIK estimate overestimates the contribution, and ignores the s-dependence. We find that R(s) here is well fit to $R(s) = 0.0895(9)s^{3.43(11)} + 0.63(2)$. We integrate from 1.2 GeV^2 instead of 1 GeV^2 like JSIK because: the ϕ resonance appears relevant up to 1.2 GeV^2, and the wide ρ' and ρ'' are included in the form factor. Above this, we take R to be a constant, and fit to the average value without resonances. Between $s = 2.3$ GeV^2 and the $s = 16$ GeV^2 $R \approx 2.15(1)$. In the region $s = [16, 120]$ GeV^2, we find $R \approx 3.71(1)$ and above this we take $R \approx 3.95(1)$. Together, these choices give a value of $C_{1,>} = -0.00574(4)$

IV. HIGHER RESONANCES

We can improve the piecewise calculations be including higher resonances not treated by JSIK. Continuing to use the simple-pole approximation, we include charmonium states. We use $m_{J/\psi(1S)} = 3096.916(11)$ MeV, $m_{\psi(2S)} = 3686.09(4)$ MeV, $f^2_{J/\psi(1S)}/4\pi = 11.5(1.4)$, and $f^2_{\psi(2S)}/4\pi = 31.2(4.5)$. Where the masses are from Ref. [48] and the coupling constants from Ref. [39]. These yield a new contribution $C_{1,\psi} = -0.00039(4)$.

The $\Upsilon(nS)$ states $n = 1, 2, 3$ are also narrow and therefore can be included as simple poles. We estimate these by three contribution at $m_\Upsilon = 2m_b \approx 10$ GeV and

TABLE I. $C^R_{1,\mathrm{hvp}}$ from directly integrating $R(s)$ for both true muonium and positronium.

States	Range (GeV)	$C^{R,\mu}_{1,\mathrm{hvp}} \times 10^{-2}$	$C^{R,e}_{1,\mathrm{hvp}} \times 10^{-6}$
ρ, ω	(0.00,0.98)	−3.943(8)	−0.814(2)
ϕ	(0.98,1.06)	−0.283(2)	−0.0633(3)
had	(1.06,3.05)	−0.477(5)	−0.109(2)
J/ψ	(3.05,3.15)	−0.0422(6)	−0.0098(2)
had	(3.15,3.63)	−0.021(2)	−0.0049(3)
$\psi(2S)$	(3.63,3.73)	−0.0126(3)	−0.00294(7)
had	(3.73,5.22)	−0.0452(5)	−0.0105(2)
pQCD	(5.22,9.41)	−0.034393(7)	−0.008036(2)
$\Upsilon(1S)$	(9.41,9.51)	−0.000623(5)	−0.000146(1)
had	(9.51,9.95)	−0.00125(2)	−0.000293(4)
$\Upsilon(2S)$	(9.95,10.07)	−0.00043(3)	−0.000102(6)
had	(10.07,10.30)	−0.00057(1)	−0.000133(3)
$\Upsilon(3S)$	(10.30,10.40)	−0.000315(4)	−0.0000737(9)
had	(10.40,11.50)	−0.00245(3)	−0.000572(7)
pQCD	(11.50,∞)	−0.01094(1)	−0.002559(3)
Tot.		−4.874(9)	−1.027(2)

$f^2_\Upsilon/4\pi = \mathcal{O}(10)$. These yields $C_{1,\Upsilon} = -0.00012(12)$. We neglect the higher n-states, which are much wider and will be approximately accounted for by the constant fits.

Adding our values of $C_{1,\mathrm{IGS}}$, $C_{1,>}$, C_ψ, and C_Υ to the JSIK value of $C_{1,\phi}$, our final results for the improved piecewise coefficient is $C^{\mathrm{imp}}_{1,\mathrm{hvp}} = -0.0472(5)$. This value represents an improvement on the JSIK value, but we note that it still has model-dependence which is difficult to estimate, and doesn't encapsulate the full effect of resonances.

V. NUMERICAL INTEGRATION OF $R(s)$

Another way to obtain $C_{1,\mathrm{hvp}}$ is numerically integrating $R(s)$ with Simpson's method. This method has negligible model-dependence and theoretical uncertainties. We numerically integrate the full $R(s)$ data from Ref. [41–45, 47] seen in Fig. 1 using Eq.(2) and Eq.(9). The results for $C_{1,\mathrm{hvp}}$ are found in Table I, split into energy ranges. Summing these, we obtain our final results of $C_{1,\mathrm{hvp}} = -0.04874(9)$. To obtain an uncertainty, we take 10^5 samples of each data point with a gaussian distribution given by the sum of its statistical and systematic uncertainty. By replacement of $m_\mu \to m_e$ we can also compute the correction to positronium. We find that value to be $C^{R,e}_{1,\mathrm{hvp}} = -1.027(2) \times 10^{-6}$, which is too small to be relevant in the near-future.

Integrating Eq.(2) and Eq.(9) only above $s = 1.2$ GeV^2, we obtain a more exact value for $C^R_{1,>} = -0.00623(6)$. Comparing this to the value obtained from pQCD, $C^{\mathrm{imp}}_{1,>} + C_{1,\psi} + C_{1,\Upsilon} = -0.00625(13)$, we see that our improved piecewise calculation has reproduced well the high energy region, albeit with larger uncertainty. Comparing to $C^{\mathrm{JSIK}}_{1,>} = -0.008$, it can be seen that the integration down to 1 GeV^2 with the value $R \approx 2$ leads to a larger prediction of $C_{1,\mathrm{hvp}}$.

If we instead consider only range below $s = 1.2$ we see that $C_{1,<} = -0.04251(9)$ which is slightly larger than the results from JSIK and our improved method, indicating the small discrepancy between the piecewise methods and direct numerical integration is likely the form factor parameterization. This re-enforces the danger of model-dependence.

VI. SUMMARY AND CONCLUSION

In this work, we have computed the coefficient $C_{1,\mathrm{hvp}}$ in two ways with greatly reduced uncertainty. The first improved upon the work of [2] through the use of a more complex pionic form factor and better modeling of the perturbative regime and resonances. The final calculation in this technique was $C^{\mathrm{imp}}_{1,\mathrm{hvp}} = -0.0472(5)$. The error was estimated by parameter variation and experimental uncertainty. While more precisely accounting for some of the features of the full spectral function, it still has drawbacks. The mixing between the $\phi - \omega$ is treated by phenomenological fits to data. It treats the ϕ meson as a simple pole, which will underestimate its contribution.

In order to avoid these problems, we computed $C_{1,\mathrm{hvp}} = -0.04874(9)$ directly from experimental $R(s)$ in analogy to methods used for a_μ. This value is in agreement with JSIK, but with an almost two orders of magnitude smaller uncertainty. It disagrees with the improved method at 3σ. We attribute this to the pion form-factor parameterization. Therefore, $C^R_{1,\mathrm{hvp}}$ is our final value for coefficient.

We can now reevaluate the prediction for hfs. Our result reduces the leading-order hadronic error estimate of JSIK [2] from 800 MHz to 16 MHz. This is a factor of 50 reduction in the hadronic uncertainty. With this improvement, the current value is $\Delta E^{1s}_{\mathrm{hfs}} = 42329437(16)_{\mathrm{had}}(700)_{\mathrm{miss}}$ MHz, where the first uncertainty is our reduced hadronic error, and the second is an estimate of missing $\mathcal{O}(m_\mu\alpha^6)$ terms. With this reduction in the leading order hadronic uncertainty, missing corrections now dominates and are the remaining step to obtaining $\mathcal{O}(100$ MHz) predictions for use in new physics searches.

Of the $\mathcal{O}(m_\mu\alpha^6)$ corrections, additional hvp corrections arise. These arise from inserting hvp loops into $\mathcal{O}(m_\mu\alpha^5)$ diagrams in analogy to the lepton loops (e.g [35, 49]). The uncertainty from missing $C_{2,\mathrm{hvp}}$ can be estimated by assuming it $2 \times C_{1,\mathrm{hvp}}$. The factor of 2 is included because the average number of photon lines at $\mathcal{O}(m_\mu\alpha^5)$ is 2. This yields $2 \times C_{1,\mathrm{hvp}} \frac{m_\mu\alpha^6}{\pi} = 30$ MHz, which is sub-dominant compared to the 700 MHz arising from missing electron loops.

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